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BOUNDED DOMAINS WHICH ARE ISOSPECTRAL BUT NOT CONGRUENT

BY HAJIME URAKAWA

1. Introduction

The purpose of this paper is to give examples of bounded domains of $\mathbb{R}^n$ of dimension not less than four which are isospectral but not congruent.

Let Ω be a bounded domain in the n-dimensional Euclidean space $\mathbb{R}^n$ with the appropriately regular boundary $\partial\Omega$. For the Laplacian $\Delta_0 = -\sum \partial^2/\partial x_i^2$ on $\mathbb{R}^n$, let us consider the following problems:

Dirichlet Problem

$$\begin{cases} \Delta_0 f = \lambda f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega. \end{cases}$$

Neumann Problem

$$\begin{cases} \Delta_0 f = \lambda f & \text{in } \Omega, \\ \dfrac{\partial f}{\partial \nu} = 0 & \text{a. e. } \partial\Omega, \text{ i. e., where the exterior normal } \nu \text{ of } \partial\Omega \text{ is defined.} \end{cases}$$

It is well known that each problem has a discrete spectrum which consists of the eigenvalues with finite multiplicites. We denote by $\mathrm{Spec}_D(\Omega)$ (resp. $\mathrm{Spec}_N(\Omega)$) the spectrum of the Dirichlet problem (resp. the Neumann Problem) for the domain Ω in $\mathbb{R}^n$.

One of the important problems of the spectra is to find how the spectra $\mathrm{Spec}_D(\Omega)$ or $\mathrm{Spec}_N(\Omega)$ reflect the shape of Ω. In his paper [K], M. Kac gave the following interesting expression of this problem: thinking of Ω as a drum and its eigenvalues as its fundamental tones, *is it possible, just by listening with a perfect ear, to hear the shape of* Ω ? (*See* also [M.S.]).

Many mathematicians, e. g., Weyl [W], Carleman [C], Kac [K], McKean-Singer [M.S.] and others challenged it, so that one can hear the several geometric quantities of Ω, that is, the dimension of Ω, the volume of Ω, the area of the boundary $\partial\Omega$, etc. Moreover let us consider the following final problem:

PROBLEM (*cf.* [K]).

For two bounded domains Ω_1, Ω_2 *in* $\mathbb{R}^n (n \geqq 2)$, *assume that* $\mathrm{Spec}_D(\Omega_1)=\mathrm{Spec}_D(\Omega_2)$ *or* $\mathrm{Spec}_N(\Omega_1)=\mathrm{Spec}_N(\Omega_2)$. *Are the domains* Ω_1, Ω_2 *congruent in* $\mathbb{R}^n$? Here two domains Ω_1, Ω_2 are *congruent* in $\mathbb{R}^n$ if there exists an isometry Φ of $\mathbb{R}^n$ such that $\Phi(\Omega_1)=\Omega_2$.

It is just the problem proposed by Kac (*cf.* [K], *see* also [Yau], problem No. 67). A partial answer is known: in case of Ω_1 = a disc, due to the celebrated inequality of Faber-Krahn [F], [Kr] (resp. that of Weinberger [Wr]) related to the first eigenvalue of the Dirichlet problem (resp. the Neumann problem), $\mathrm{Spec}_D(\Omega_1)=\mathrm{Spec}_D(\Omega_2)$(resp. $\mathrm{Spec}_N(\Omega_1)=\mathrm{Spec}_N(\Omega_2)$) implies that Ω_2 is the disc with the same radius as Ω_1.

In this paper, we give an eventual answer of the problem of Kac:

THEOREM 4.4. — *There exist two domains* Ω_1, Ω_2 *in* $\mathbb{R}^n (n \geqq 4)$ *such that*

$$\mathrm{Spec}_D(\Omega_1)=\mathrm{Spec}_D(\Omega_2) \quad \textit{and} \quad \mathrm{Spec}_N(\Omega_1)=\mathrm{Spec}_N(\Omega_2),$$

but Ω_1 *and* Ω_2 *are not congruent in* $\mathbb{R}^n$.

In case of dimension two or three, the problem is still open. By the way note that one can formulate an analogous problem for compact Riemannian manifolds without boundary and the answer is negative by virtue of examples of Milnor [M], Ikeda [I] and Vignéras [V].

The proof of Theorem 4.4 is very simple. Our examples can be found among the truncated cones $D_\varepsilon (0<\varepsilon<1)$ given by $D_\varepsilon=\{r\omega;\ \varepsilon<r<1,\ \omega\in C_1\}$ where C_1 are the domains in the unit sphere S^{n-1} in $\mathbb{R}^n$. The outline of the proof is as follows:

First, for a fixed ε $(0<\varepsilon<1)$, we show by the separation of the variables, that $\mathrm{Spec}_D(D_\varepsilon)$ (resp. $\mathrm{Spec}_N(D_\varepsilon)$) is completely determined by the number ε and the spectrum $\mathrm{Spec}_D(C_1)$ (resp. $\mathrm{Spec}_N(C_1)$) of the Dirichlet problem (resp. the Neumann problem) of the spherical domain C_1 for the Laplacian of the standard unit sphere S^{n-1} (*cf.* § 4). Then we have only to answer the following problem:

(A) *Find two domains* C_1, $\tilde{C}_1$ *in* S^{n-1} *which satisfy* $\mathrm{Spec}_D(C_1)=\mathrm{Spec}_D(\tilde{C}_1)$ *and* $\mathrm{Spec}_N(C_1)=\mathrm{Spec}_N(\tilde{C}_1)$, *but are not congruent in* S^{n-1}.

Recently, Bérard-Besson [B.B.] determined the spectra $\mathrm{Spec}_D(C_1)$, $\mathrm{Spec}_N(C_1)$ of the spherical domains C_1 which are the intersections of S^{n-1} with the chambers of the Weyl groups W (i. e., the finite reflection groups). They showed that the spectra $\mathrm{Spec}_D(C_1)$, $\mathrm{Spec}_N(C_1)$ are completely determined by the set of the exponents of W. Hence due to their results, the problem (A) for these domains C_1 can be modified into the following:

(B) *Find two finite reflection groups* W, $\tilde{W}$ *acting on the same Euclidean space* $\mathbb{R}^n$ *which satisfy the conditions:* (i) *the sets of the exponents of* W, $\tilde{W}$ *coincide each other and* (ii) *the intersections* C_1, $\tilde{C}_1$ *of their chambers with* S^{n-1} *are not congruent in* S^{n-1}.

Notice that the condition (ii) is equivalent to that the Coxeter graphs of W, $\tilde{W}$ are not isomorphic (*cf.* § 3). Thus we have only to consider the following:

(C) *Does the set of the exponents of the finite reflection group* W *acting on* $\mathbb{R}^n$ *determine the Coxeter graph of* W *uniquely ?*

In case of $n \geqq 4$, the answer of the problem (C) is NO, i.e., there exist examples of finite reflection groups with the same set of the exponents and the different Coxeter graphs (*cf.* § 3). Thus we obtain Theorem 4.4 and the following:

THEOREM 3.8. — *There exist two domains* C_1, $\tilde{C}_1$ *in the unit sphere* S^{n-1} *in* $\mathbb{R}^n (n \geqq 4)$ *such that*

$$\mathrm{Spec}_D(C_1) = \mathrm{Spec}_D(\tilde{C}_1) \quad \text{and} \quad \mathrm{Spec}_N(C_1) = \mathrm{Spec}_N(\tilde{C}_1),$$

but C_1, $\tilde{C}_1$ *are not congruent in* S^{n-1}.

Remark. — The boundaries of our examples are *not* smooth, but polygons. The boundary value problems in non-smooth domains have been treated by Agmon [A], Grisvard [Gd], Brownell [B], Kac [K], p. 19 and others. But the original version of the problem of Kac was proposed for domains of smooth boundaries. In this sense, the problem of Kac is still *open* for every dimension $n \geqq 2$.

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2. Preliminaries

In this section, we will review reflection groups following Bourbaki [B.N.].

Let $(E, (\ ,\ ))$ be a finite dimensional real vector space with an inner product $(\ ,\ )$. Put $n = \dim(E)$. Let $\mathfrak{h}$ be a finite set consisting of hyperplanes of E. In this paper, we deal only *finite* reflection groups, so we always assume that each hyperplane belonging to $\mathfrak{h}$ passes through the origin o of E. Let $O(E)$ be the orthogonal group of E with respect to the inner product $(\,,)$. For $H \in \mathfrak{h}$, let $s_H \in O(E)$ be the reflection relative to H, i.e.,

$$s_H(x) = x - \frac{2(x, a)}{(a, a)} a, \qquad x \in E,$$

where a is a vector orthogonal to the hyperplane H. The subgroup W of $O(E)$ generated by $\{s_H;\ H \in \mathfrak{h}\}$ is called a *reflection group* on E (*cf.* [B.N], p. 72) if it satisfies the conditions (D1), (D2):

(D1) If $w \in W$ and $H \in \mathfrak{h}$, then $w(H) \in \mathfrak{h}$.

(D2) W is finite, so W acts properly discontinuously on E.

A connected component C of $E \setminus \cup \{H;\ H \in \mathfrak{h}\}$ is called a *chamber* of W in E and a hyperplane H of $\mathfrak{h}$ is called a *wall* of the chamber C if the intersection of the closure $\overline{C}$ of C with H includes a non-empty open subset of H. Then it is known that (1) W acts simply transitively on the set of all chambers, (2) the set of all hyperplanes H such that $s_H \in W$ coincides with $\mathfrak{h}$ and (3) for every chamber C, its closure $\overline{C}$ is a fundamental domain of W in E (*cf.* [B.N], p. 74, 75).

Let $W_i (0 \leqq i \leqq s)$ be reflection groups on the Euclidean spaces $(E_i, (\ ,\ ))$, $\mathfrak{h}_i$ the sets of their hyperplanes in $E_i (1 \leqq i \leqq s)$ and $W_0 = \{\mathrm{id}\}$. Let $E = E_0 \times E_1 \times \ldots \times E_s$ be their direct product of which inner product $(\ ,\ )$ is given by $(x, y)=\sum_{i=0}^{s} x_i y_i$ for $x=(x_0, \ldots, x_s)$, $y=(y_0, \ldots, y_s)\in \mathrm{E}$. The direct product $\mathrm{W}=\mathrm{W}_0\times\ldots\times\mathrm{W}_s$ acts on E by $w(x)=(x_0, w_1(x_1), \ldots, w_s(x_s))$ for $w=(\mathrm{id}, w_1, \ldots, w_s)\in\mathrm{W}$. Then W is a reflection group on $(\mathrm{E}, (\ ,\ ))$ generated by reflections relative to the hyperplanes all of which are of the form:

$$(2.1)\qquad \mathrm{H}=\mathrm{E}_0\times\mathrm{E}_1\times\ldots\times\mathrm{E}_{i-1}\times\mathrm{H}_i\times\mathrm{E}_{i+1}\times\ldots\times\mathrm{E}_s,$$

where H_i belong to $\mathfrak{h}_i$, $i=1, \ldots, s$. Each chamber of (W, E) is of the form:

$$(2.2)\qquad \mathrm{C}=\mathrm{E}_0\times\mathrm{C}_1\times\ldots\times\mathrm{C}_s,$$

where C_i are chambers of $(\mathrm{W}_i, \mathrm{E}_i)$, $i=1, \ldots, s$. Each reflection group W on the Euclidean space $(\mathrm{E}, (\ ,\ ))$ is decomposed as the direct product of reflection groups W_i, $i=0, 1, \ldots, s$, $\mathrm{W}_0=\{\mathrm{id}\}$, in such a way that the Euclidean space $(\mathrm{E}, (\ ,\ ))$ is decomposed as the direct product of the Euclidean spaces $(\mathrm{E}_i, (\ ,\ ))$, $i=0, 1, \ldots, s$ and W_i act irreducibly on E_i as subgroups of $\mathrm{O}(\mathrm{E}_i)$, $i=1, \ldots, s$ (*cf.* [B.N], p. 82). The subgroup $\mathrm{W}'=\mathrm{W}_1\times\ldots\times\mathrm{W}_s$ of W is called the *essential part* of W. Put $\mathrm{E}'=\mathrm{E}_1\times\ldots\times\mathrm{E}_s$ and $l=\dim(\mathrm{E}')$. For an arbitrary fixed chamber C′ of W′ in E′, let $\mathfrak{m}$ be the set of all walls of C′. For $\mathrm{H}\in\mathfrak{m}$, let e_H be the unit vector in E′ which is orthogonal to H and belongs to the one of two connected components of $\mathrm{E}'\setminus\mathrm{H}$ containing C′. Then $\{e_\mathrm{H};\ \mathrm{H}\in\mathfrak{m}\}$ is a basis of E′ (*cf.* [B.N.], p. 85). So we may put $\mathfrak{m}=\{\mathrm{H}_i\}_{i=1}^{l}$. Let $\{\omega_i\}_{i=1}^{l}$ be the dual basis of $\{e_{\mathrm{H}_i}\}_{i=1}^{l}$, i.e., $(\omega_i, e_{\mathrm{H}_j})=\delta_{ij}$. Then the chamber C′ of W′ in E′ is an open simplex cone in E′ with the vertex o given by

$$(2.3)\qquad \mathrm{C}'=\left\{\sum_{i=1}^{l} x_i\omega_i\in\mathrm{E}';\ x_i>0\ (i=1, \ldots, l)\right\}\quad (cf.\ [\mathrm{B.N.}],\ \mathrm{p.}\ 85).$$

For a chamber C′ of W′ and the set $\{\mathrm{H}_i\}_{i=1}^{l}$ of all the walls of C′, an element $c=s_{\mathrm{H}_1}\ldots s_{\mathrm{H}_l}$ of W′ is called a *Coxeter transformation* of W′. Each Coxeter transformation of W′ has the same order $h=h(\mathrm{W}')$, which is called the *Coxeter number* of W′ and the same characteristic polynomial $\mathrm{P}(\mathrm{T})=\det(\mathrm{T}\ \mathrm{id}-c)$ which can be written of the form:

$$\mathrm{P}(\mathrm{T})=\prod_{j=1}^{l}(\mathrm{T}-\exp(2\pi\sqrt{-1}\,m_j/h))\quad (\mathrm{cf.}\ [\mathrm{B.N.}].\ 116).$$

Here $m_j (j=1, \ldots, l)$ are integers which can be arranged by $0\leqq m_1\leqq m_2\leqq\ldots\leqq m_l<h$. These l non-negative integers m_j are called the *exponents* of W′ (*cf.* [B.N.], p. 118).

Then the number of all the hyperplanes of W (or W′) is given by

$$(2.4)\qquad \#\,\mathfrak{h}=\frac{1}{2}\sum_{i=1}^{s} l_i h_i=\sum_{i=1}^{l} m_i,$$

where $l_i=\dim(\mathrm{E}_i)$ and h_i is the Coxeter number of W_i, $i=1, \ldots, s$. In fact, since a chamber C′ of W′ is given by $\mathrm{C}_1\times\ldots\times\mathrm{C}_s$ where C_i is a chamber of W_i, a Coxeter transformation c of W′ relative to C′ is given as a product $c=c_1\ldots c_s$ of the ones of the irreducible reflection groups W_i relative to C_i Since the number of all the hyperplanes of W_i is $2^{-1} l_i h_i$ which coincides with the sum of all the exponents of W_i (*cf.* [B.N.], p. 119,118), we have (2.4).

Moreover the order of the group W (or W′) coincides with $(m_1+1)\ldots(m_l+1)$ (*cf.* [B.N.], p. 122).

For a chamber C′ of the essential part W′ of a reflection group W, let $\{H_i\}_{i=1}^{l}$ be the set of all the walls of C′. Let $\{e_{H_i}\}_{i=1}^{l}$ be the unit vectors in E′ defined as above. Let m_{ij}, i, $j=1, \ldots, l$, be the order of the element of $s_{H_i} s_{H_j}$ in W′. Then we have:

LEMMA 2.1. — *The positive integers m_{ij} satisfy the following conditions:*

(1) $$(e_{H_i}, e_{H_j}) = -\cos(\pi/m_{ij}),$$

(2) $$m_{ij} \geqq 2 \quad \text{if} \quad i \neq j, \text{ i. e.,} \quad (e_{H_i}, e_{H_j}) \leqq 0,$$

where $m_{ij}=2$ implies $(e_{H_i}, e_{H_j})=0$.

(3) $$m_{ij}=m_{ji}, \quad i, j=1, \ldots, l, \quad \text{and} \quad m_{ii}=1, \quad i=1, \ldots, l.$$

Proof. — *See* [B.N.], p. 77.

Due to this lemma, for a reflection group W, we give a graph Γ consisting of l vertices $\{1, \ldots, l\}$ and the numbers m_{ij}. Two vertices i, j of Γ $(i \neq j)$ are joined by an edge if $m_{ij} \geqq 3$ and the edge is labelled with the number m_{ij} if $m_{ij} > 3$. Such a graph is called a *Coxeter graph* (*cf.* [B.N.], p. 20). Two vertices a, b of the graph Γ are connected if there exist vertices $\{x_j\}_{j=0}^{r}$ of Γ such that $a=x_0$, $b=x_r$ and each x_j is joined to x_{j+1} by an edge. A maximal set of connected vertices and edges of Γ is called a connected component of Γ. For a reflection group W, let $W' = W_1 \times \ldots \times W_s$ be the decomposition of the essential part W′ of W. Then the Coxeter graph Γ corresponding to W′ consists of s connected components $\{\Gamma_i\}_{i=1}^{s}$ such that each graph Γ_i is the Coxeter graph of the irreducible subgroup W_i of W′ (*cf.* [B.N.], p. 22). Note that two reflection groups are isomorphic if and only if their Coxeter graphs coincide. Furthermore the classification of irreducible reflection groups is as follows:

LEMMA 2.2. — *The irreducible reflection group is the one of which Coxeter graph is some of the following table.*
Here h, $\#\mathfrak{h}$ and "order" in the table are the Coxeter number, the number of all the hyperplanes in $\mathfrak{h}$ and the order of the corresponding reflection group, respectively.

Proof. — *See* [B.N.], p. 193, 200-221 and 231, exercices 11), 12).

A reflection group corresponding the graph $A_l \sim G_2$ is called a *crystallographic group* which is given as a Weyl group of a root system.

3. Case of spherical domains

3.1. Let (E, (,)) be the n-dimensional Euclidean space. Let $(x_1, \ldots, x_n)$ be the coordinate of E with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^{n}$ of E. We identify E with

Coxeter graph	Exponents	h	$\# \mathfrak{h}$	Order
$A_l (l \geqq 1)$	$1, 2, \ldots, l$	$l+1$	$l(l+1)/2$	$(l+1)!$
$B_l (l \geqq 2)$	$1, 3, 5, \ldots, 2l-3, 2l-1$	$2l$	l^2	$2^l \cdot l!$
$D_l (l \geqq 4)$	$1, 3, 5, \ldots, 2l-3, l-1$	$2l-2$	$l(l-1)$	$2^{l-1} l!$
E_6	1, 4, 5, 7, 8, 11	12	36	$2^7 \cdot 3^4 \cdot 5$
E_7	1, 5, 7, 9, 11, 13, 17	18	63	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$
E_8	1, 7, 11, 13, 17, 19, 23, 29	30	120	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$
F_4	1, 5, 7, 11	12	24	$2^7 \cdot 3^2$
G_2	1, 5	6	6	12
H_3	1, 5, 9	10	15	120
H_4	1, 11, 19, 29	30	60	$2^6 \cdot 3^2 \cdot 5^2$
$I_2(p) (p=5 \text{ or } p \geqq 7)$	$1, p-1$	p	p	$2p$

$\mathbb{R}^n$ by the mapping $E \ni x = \sum_{i=1}^{n} x_i e_i \mapsto (x_1, \ldots, x_n) \in \mathbb{R}^n$. For $x = \sum_{i=1}^{n} x_i e_i \in E$, put $|x| = (x, x)^{1/2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$. Let $S^{n-1} = \{ x \in E ; |x| = 1 \}$, the unit sphere in E. For $x \in E - (0)$, let (r, ω) be the polar coordinate of x defined by

$$r = |x| \quad \text{and} \quad \omega = x / |x| \in S^{n-1}.$$

Then the Laplacian $\Delta_0 = - \sum_{i=1}^{n} \partial^2 / \partial x_i^2$ of the Euclidean space $(E, (\ ,\ ))$ is expressed relative to the polar coordinate (r, ω) as:

$$\Delta_0 = - \frac{\partial^2}{\partial r^2} - \frac{(n-1)}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta, \tag{3.1}$$

where the operator Δ is the Laplacian of the standard unit sphere (S^{n-1}, g_0) whose metric g_0 is induced from the inner product $(\ ,\ )$ of E.

Now let W be a finite reflection group of $(E, (\ ,\ ))$ defined by a finite set $\mathfrak{h}$ of hyperplanes of E passing through the origin o. Let C be a chamber of W in E. Then C is given as $E_0 \times C'$ where $W = \{ \mathrm{id} \} \times W'$, $E = E_0 \times E'$, W' is the essential part of W and C' is a chamber of E', which is an open simplex cone in E'.

DEFINITION 3.1. — *Let C_1 be the intersection of the chamber* C *with the unit sphere* S^{n-1}, *which is an open simplex of* S^{n-1}. *For* $0 < \varepsilon < 1$, *let* D_ε *be the domain "truncated cone" in* E *given by*

$$D_\varepsilon = \{ r\omega ; \varepsilon < r < 1, \omega \in C_1 \}. \tag{3.2}$$

Then we have:

LEMMA 3.2. — (1) *The boundary* ∂C_1 *of* C_1 *in* S^{n-1} *is given by*

$$\partial C_1 = \partial C \cap S^{n-1} = \cup \{ H \cap S^{n-1} ; H \in \mathfrak{m} \},$$

where ∂C is the boundary of C *in* E *and* $\mathfrak{m}$ *is the set of all the walls of the chamber* C *of* W *in* E.

(2) *The boundary ∂D_ε of D_ε in* E *is given by*

$$\partial D_\varepsilon = \overline{C}_1 \cup \varepsilon \overline{C}_1 \cup \{ r\omega ; \omega \in \partial C_1, \ \varepsilon < r < 1 \},$$

where $\overline{C}_1$ is the closure of C_1 in S^{n-1}.

(3) *The closure $\overline{C}_1$ is the fundamental domain in S^{n-1} relative to the isometry actions of* W *in* (S^{n-1}, g_0).

Proof. — (1) and (2) follow from the fact that $C = E_0 \times C'$ and C' is an open simplex cone in E'. (3) follows from that $\overline{C}$ is the fundamental domain of W in E.

Let us consider the following boundary value problems for the domains C_1 and D_ε.

Case 1. — *The spherical domain C_1 of S^{n-1}*:

$$\text{(S.D.P.)} \quad \begin{cases} \Delta f = \lambda f & \text{in } C_1, \\ f = 0 & \text{on } \partial C_1. \end{cases}$$

$$\text{(S.N.P.)} \quad \begin{cases} \Delta f = \lambda f & \text{in } C_1, \\ \dfrac{\partial f}{\partial \nu} = 0 & \text{a. e. } \partial C_1, \text{ where the exterior normal } \nu \text{ of } \partial C_1 \text{ is defined.} \end{cases}$$

Case 2. — *The Euclidean domain $D_\varepsilon (0 < \varepsilon < 1)$ of* E:

$$\text{(E.D.P.)} \quad \begin{cases} \Delta_0 f = \lambda f & \text{in } D_\varepsilon, \\ f = 0 & \text{on } \partial D_\varepsilon. \end{cases}$$

$$\text{(E.N.P.)} \quad \begin{cases} \Delta_0 f = \lambda f & \text{in } D_\varepsilon, \\ \dfrac{\partial f}{\partial \nu} = 0 & \text{a. e. } \partial D_\varepsilon, \text{ where the exterior normal } \nu \text{ of } \partial D_\varepsilon \text{ is defined.} \end{cases}$$

In this section, we treat with Case 1. Case 2 will be dealt in section 4.

3.2. In this subsection, we review the works of Bérard-Besson [B.B.] who determined the spectrum $\mathrm{Spec}_D(C_1)$ (resp. $\mathrm{Spec}_N(C_1)$) of (S.D.P.) (resp. (S.N.P.)). Their results are valid in case of the reflection groups (*cf.* [B2]).

First for the above domain C_1 of S^{n-1} corresponding to the reflection group W, we define the inner product (,) on $C^\infty(C_1)$ by

$$(f_1, f_2) = \int_{C_1} f_1(x) f_2(x) \, d\omega(x), \qquad f_1, f_2 \in C^\infty(C_1),$$

where $d\omega$ is the volume element of the standard unit sphere (S^{n-1}, g_0). Let $L^2(C_1)$ be the completion of $C^\infty(C_1)$ with respect to the inner product (,).

Now consider a C^∞ function f on S^{n-1} satisfying the conditions

$$(3.3) \qquad \Delta f = \lambda f \quad \text{in } S^{n-1}$$

and

$$w \cdot f = \varepsilon(w) f, \qquad w \in W, \tag{3.4}$$

where $(w \cdot f)(x) = f(w^{-1}(x))$ for $w \in W$ and $x \in S^{n-1}$ and $\varepsilon(w)(w \in W)$ is given by

$$\varepsilon(w) = 1 \quad \text{for every} \quad w \in W, \tag{3.5}$$

or

$$\varepsilon(w) = \det(w) \quad \text{for every } w \in W. \tag{3.6}$$

Then the restriction of f to C_1 satisfies (S.D.P.) (resp. (S.N.P.)) if ε satisfies (3.6) resp. (3.5)). Furthermore the set of all restrictions of C^∞ eigenfunctions of Δ on S^{n-1} with the condition (3.4) is dense in $L^2(C_1)$ (*cf.* [B.B.], p. 239). Thus, to determine $\mathrm{Spec}_D(C_1)$ and $\mathrm{Spec}_N(C_1)$, we have only to consider the set of all C^∞ eigenfunctions of Δ on S^{n-1} satisfying the condition (3.4). Of course, every solution f_1 of (S.D.P.) or (S.N.P.) can be extended to a function f on S^{n-1} by

$$\begin{cases} f(x) = f_1(x), & x \in C_1 \\ \text{and} & \\ w \cdot f(x) = \varepsilon(w) f_1(x), & x \in S^{n-1}, \quad w \in W. \end{cases}$$

Then it is well defined on S^{n-1} due to Lemma 3.2, moreover, it can be proved by the same manner as Lemma 8 in [B1] that f is C^∞ on S^{n-1}.

Now we set:

$H_k(E)$; = the set of all harmonic (i.e. $\Delta_0 P = 0$) polynomials P in E of degree k,

$H_k^a(E)$; $= \{ P \in H_k(E);\ P(w^{-1}(x)) = \det(w) P(x)$ for all $w \in W$ and $x \in E \}$,

$H_k^i(E)$; $= \{ P \in H_k(E);\ P(w(x)) = P(x)$ for all $w \in W$ and $x \in E \}$,

$h_k^a(E)$; $= \dim(H_k^a(E))$ and $h_k^i(E)$; $= \dim(H_k^i(E))$.

Then the inclusion $i : S^{n-1} \to E$ induces a linear mapping i^* of $C^\infty(E)$ into $C^\infty(S^{n-1})$ by $P \to P \circ i$. The mapping i^* is injective and its image of the space $\sum_{k=0}^{\infty} H_k(E)$ is dense in $C^\infty(S^{n-1})$. Furthermore the image of $H_k(E)$ by i^* coincides with the eigenspace of Δ on S^{n-1} with the eigenvalue $k(k+n-2)$, $k = 0, 1, 2, \ldots$

Therefore the spectrum $\mathrm{Spec}_D(C_1)$ (resp. $\mathrm{Spec}_N(C_1)$) of the Dirichlet problem (resp. the Neumann problem) of the domain C_1 in S^{n-1} is determined as follows:

(1) The set of all the eigenvalues of the Dirichlet problem (S.D.P.) and the Neumann problem (S.N.P.) is included in the set $\{k(k+n-2);\ k = 0, 1, 2, \ldots\}$.

(2) If $h_k^a(E) \neq 0$ (resp. $h_k^i(E) \neq 0$), $k(k+n-2)$ is really the eigenvalues of (S.D.P.) (resp. (S.N.P.)) with multiplicity $h_k^a(E)$ (resp. $h_k^i(E)$).

Thus to determine $\mathrm{Spec}_D(C_1)$ and $\mathrm{Spec}_N(C_1)$, we have only to compute $h_k^a(E)$ and $h_k^i(E)$ $(k = 0, 1, 2, \ldots)$. For this purpose, consider the Poincaré series:

$$F^a(T) = \sum_{k=0}^{\infty} h_k^a(E) T^k, \tag{3.7}$$

$$F^i(T) = \sum_{k=0}^{\infty} h_k^i(E) T^k, \tag{3.8}$$

where T is an indeterminate.

Bérard-Besson [B.B.] computed the series (3.7), (3.8) making use of the Poincaré series of the subring of the polynomial ring consisting of invariant polynomials under the action of the reflection group W as follows:

PROPOSITION 3.5 (*the Neumann problem (S.N.P.)*). — *Let* W *be a reflection group of* $(E, (\ ,\ ))$ $(\dim(E) = n)$ *defined by a finite set* $\mathfrak{h}$ *of hyperplanes of* E *passing through the origin o. Let* $W = \{id\} \times W'$, $E = E_0 \times E'$ *be the decomposition of* (W, E) *such that* (W′, E′) *is the essential part of* (W, E). *Let* C_1 *be the intersection of a chamber* C *of* (W, E) *with the unit sphere* S^{n-1} *in* $(E, (\ ,\ ))$. *Then the series* (3.7) *which determines the spectrum* $\mathrm{Spec}_N(C_1)$ *of the Neumann problem* (S.N.P.) *is given as follows:*

$$F^i(T) = \frac{1-T^2}{\prod_{j=1}^{n}(1-T^{m_j+1})},$$

where $\{m_j\}_{j=1}^n$ *is the set consisting of* $\underbrace{0, \ldots, 0}_{l_0}$ $(\dim(E_0) = l_0)$ *and the exponents of the reflection group* W′.

Proof. — *See* [B.B.], p. 241, Propositions 2 and 6.

PROPOSITION 3.6 (the Dirichlet problem (S.D.P.)). — *Under the same assumptions of Proposition* 3.5, *the Poincaré series* $F^a(T)$ *which determines the spectrum* $\mathrm{Spec}_D(C_1)$ *is given by*

$$F^a(T) = T^d \cdot F^i(T),$$

where d is the number $\# \mathfrak{h}$ *of all the elements in* $\mathfrak{h}$, *which is given by* (2.4).

Proof. — *See* [B.B], p. 242, Proposition 4.

3.3. Due to Propositions 3.5, 3.6, we have:

THEOREM 3.7. — *Let* W (*resp.* $\tilde{W}$) *be a finite reflection group defined by a finite set* $\mathfrak{h}$ (*resp.* $\tilde{\mathfrak{h}}$) *of hyperplanes of the Euclidean space* $(E, (\ ,\ ))$, $\dim(E) = n$, passing through the origin *o*. *Let* $W = \{id\} \times W'$, $E = E_0 \times E'$ (*resp.* $\tilde{W} = \{id\} \times \tilde{W}'$, $E = \tilde{E}_0 \times \tilde{E}'$) *be the decomposition of* (W, E) (*resp.* ($\tilde{W}$, E)) *such that* (W′, E′) (*resp.* ($\tilde{W}'$, $\tilde{E}'$)) *is the essential part of* (W, E) (*resp.* ($\tilde{W}$, E)). *Let* $C = E_0 \times C'$ (*resp.* $\tilde{C} = \tilde{E}_0 \times \tilde{C}'$) *be a chamber of* (W, E) (*resp.* ($\tilde{W}$, E)), *where* C′ (*resp.* $\tilde{C}'$) *is a chamber of the essential part* (W′, E′) (*resp.* ($\tilde{W}'$, $\tilde{E}'$)). *Put* $C_1 = C \cap S^{n-1}$ (*resp.* $\tilde{C}_1 = \tilde{C} \cap S^{n-1}$) *where* S^{n-1} *is the unit sphere of the Euclidean space* $(E, (\ ,\ ))$. *Then we have:*

(1) *If the sets of the exponents of* W′ *and* $\tilde{W}'$ *coincide each other and* $\dim(E_0) = \dim(\tilde{E}_0)$, *then*

$$\mathrm{Spec}_D(C_1) = \mathrm{Spec}_D(\tilde{C}_1) \quad \text{and} \quad \mathrm{Spec}_N(C_1) = \mathrm{Spec}_N(\tilde{C}_1).$$

(2) *Let* $\dim(E_0)=\dim(\tilde{E}_0)$. *Then the following conditions are equivalent:*

(i) *The domains* C_1 *and* $\tilde{C}_1$ *are congruent in the unit sphere* (S^{n-1}, g_0), *i. e., there exist an isometry* Ψ *of* (S^{n-1}, g_0) *such that* $\Psi(C_1)=\tilde{C}_1$.

(ii) *The chambers* C *and* $\tilde{C}$ *are congruent in the Euclidean space* (E, (,)).

(iii) *The Coxeter graphs of* W′ *and* $\tilde{W}'$ *coincide.*

Proof. — Propositions 3.5, 3.6 and (2.4) imply the assertion (1). (2) The chamber C′ (resp. $\tilde{C}'$) is an open simplex cone in E′ (resp. $\tilde{E}'$). Combining this with the definitions of C_1 and $\tilde{C}_1$, we have the equivalence between (i) and (ii). The equivalence between (ii) and (iii) follows from (2.3), Lemma 2.1 and the definition of the Coxeter graph.

Q.E.D.

Notice that the set of the exponents does not determine the reflection group uniquely. There exist many examples of pairs of the reflection groups of which have the same set of the exponents but the different Coxeter graphs as in the table below.

Moreover, for such a pair of reflection groups (W′, E′), and ($\tilde{W}'$, E′) and an arbitrary dimensional Euclidean space (E_0, (,)), define the direct products $W=\{id\}\times W'$, $\tilde{W}=\{id\}\times\tilde{W}'$ and $E=E_0\times E'$. Then for these reflection groups (W, E) and ($\tilde{W}$, E), the spectra of (S.D.P.) and (S.N.P.) for the intersections C_1, $\tilde{C}_1$ of their chambers with the unit sphere coincide each other, but C_1 and $\tilde{C}_1$ are not congruent in the unit sphere by Theorem 3.7. Therefore we have:

THEOREM 3.8. — *There exist two domains* C_1 *and* $\tilde{C}_1$ *in the unit sphere* (S^{n-1}, g_0) $(n\geqq 4)$ *such that*

$$\mathrm{Spec}_D(C_1)=\mathrm{Spec}_D(\tilde{C}_1) \quad \text{and} \quad \mathrm{Spec}_N(C_1)=\mathrm{Spec}_N(\tilde{C}_1),$$

but C_1 *is not congruent to* $\tilde{C}_1$ *in the unit sphere* (S^{n-1}, g_0).

Remark 3.9. — There exist many examples other than the above table. For example,

		Exponents	$\#\mathfrak{h}$	Order
10)	$B_l\times I_2(l)\,(l\geqq 4)$	$1, 3, 5, \ldots, 2l-3, 2l-1, 1, l-1$	l^2+l	$2^l\cdot l!\cdot l$
	$D_l\times I_2(2l)$	$1, 3, 5, \ldots, 2l-3, l-1, 1, 2l-1$	$l(l-1)+2l$	$2^{l-1}\cdot l!\cdot 2l$
11)	$E_6\times A_1\times A_1$	1, 4, 5, 7, 8, 11, 1, 1	36+2	$2^7\cdot 3^4\cdot 5\cdot 2^2$
	$F_4\times I_2(5)\times I_2(9)$	1, 5, 7, 11, 1, 4, 1, 8	24+5+9	$2^7\cdot 3^2\cdot 2^2\cdot 5\cdot 9$

Examples 3.10. — The simplest cases in the above table are:

(1) $A_3\times A_1$ and $I_2(3)\times I_2(4)$,

(2) $B_3\times A_1$ and $I_2(4)\times I_2(6)$,

where $I_2(3)=A_2$, $I_2(4)=B_2$ and $I_2(6)=G_2$. The chambers of these reflection groups are given as follows:

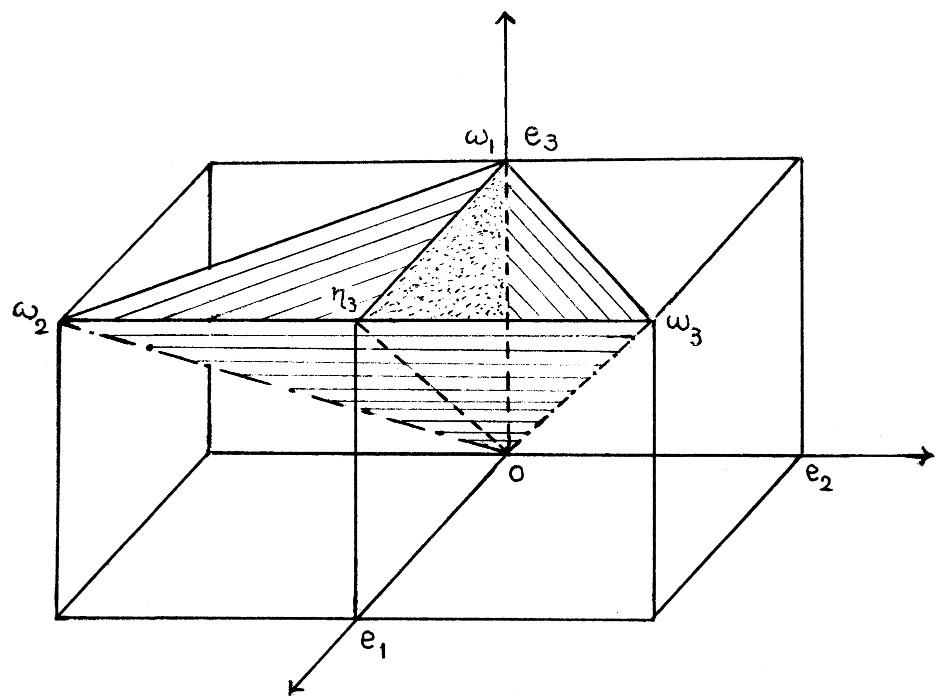
	Pairs of Coxeter graphs	Exponents	$\#\mathfrak{h}$	Order
1)	$A_l \times A_1 \underbrace{\times \ldots \times}_{l-2} A_1 \ (l \geqq 3)$	$1, 2, \ldots, l, \underbrace{1, \ldots, 1}_{l-2}$	$l(l+1)/2+(l-2)$	$(l+1)!\cdot 2^{l-2}$
	$I_2(3) \times I_2(4) \times \ldots \times I_2(l+1)$	$1, 2, 1, 3, \ldots, 1, l$	$3+4+\ldots+(l+1)$	$\prod_{i=1}^{l-1} 2(i+2)$
2)	$B_l \times A_1 \underbrace{\times \ldots \times}_{l-2} A_1 \ (l \geqq 3)$	$1, 3, 5, \ldots, 2l-1, \underbrace{1, \ldots, 1}_{l-2}$	$l^2+(l-2)$	$2^l \cdot l! \cdot 2^{l-2}$
	$I_2(4) \times I_2(6) \times \ldots \times I_2(2l)$	$1, 3, 1, 5, \ldots, 1, 2l-1$	$4+6+\ldots+2l$	$\prod_{i=2}^{l} 2(2i)$
3)	$D_l \times A_1 \underbrace{\times \ldots \times}_{l-2} A_1 \ (l \geqq 4)$	$1, 3, 5, \ldots, 2l-3, l-1, \underbrace{1, \ldots, 1}_{l-2}$	$l(l-1)+(l+2)$	$2^{l-1} \cdot l! \cdot 2^{l-2}$
	$I_2(4) \times I_2(6) \times \ldots \times I_2(2l-2) \times I_2(l)$	$1, 3, 1, 5, \ldots, 1, 2l-3, 1, l-1$	$4+6+\ldots+(2l-2)+l$	$\prod_{i=2}^{l-1} 2(2i) \times 2l$
4)	$E_6 \times A_1 \underbrace{\times \ldots \times}_{4} A_1$	$1, 4, 5, 7, 8, 11, \underbrace{1, \ldots, 1}_{4}$	$36+4$	$2^7 \cdot 3^4 \cdot 5 \cdot 2^4$
	$I_2(5) \times I_2(6) \times I_2(8) \times I_2(9) \times I_2(12)$	$1, 4, 1, 5, 1, 7, 1, 8, 1, 11$	$5+6+8+9+12$	$2^5 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12$
5)	$E_7 \times A_1 \underbrace{\times \ldots \times}_{5} A_1$	$1, 5, 7, 9, 11, 13, 17, \underbrace{1, \ldots, 1}_{5}$	$63+5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7 \cdot 2^5$
	$I_2(6) \times I_2(8) \times I_2(10) \times I_2(12) \times I_2(14) \times I_2(18)$	$1, 5, 1, 7, 1, 9, 1, 11, 1, 13, 1, 17$	$6+8+10+12+14+18$	$2^6 \cdot 6 \cdot 8 \cdot 10 \cdot 12$
6)	$E_8 \times A_1 \underbrace{\times \ldots \times}_{6} A_1$	$1, 7, 11, 13, 17, 19, 23, 29, \underbrace{1, \ldots, 1}_{6}$	$120+6$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 2^6$
	$I_2(8) \times I_2(12) \times I_2(14) \times I_2(18) \times I_2(20) \times I_2(24) \times I_2(30)$	$1, 7, 1, 11, 1, 13, 1, 17, 1, 19, 1, 23, 1, 29$	$8+12+14+18+20+24+30$	$2^7 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \times 20 \cdot 24 \cdot 30$
7)	$F_4 \times A_1 \times A_1$	$1, 5, 7, 11, 1, 1$	$24+2$	$2^7 \cdot 3^2 \cdot 2^2$
	$I_2(6) \times I_2(8) \times I_2(12)$	$1, 5, 1, 7, 1, 11$	$6+8+12$	$2^3 \cdot 6 \cdot 8 \cdot 12$
8)	$H_3 \times A_1$	$1, 5, 9, 1$	$15+1$	$120 \cdot 2$
	$G_2 \times I_2(10)$	$1, 5, 1, 9$	$6+10$	$2^2 \cdot 6 \cdot 10$
9)	$H_4 \times A_1 \times A_1$	$1, 11, 19, 29, 1, 1$	$60+2$	$2^6 \cdot 3^2 \cdot 5^2 \cdot 2^2$
	$I_2(12) \times I_2(20) \times I_2(30)$	$1, 11, 1, 19, 1, 29$	$12+20+30$	$2^3 \cdot 12 \cdot 20 \cdot 30$

(1) A chamber $C_{(1)}$ of $A_3 \times A_1$ is given by $\left\{ \sum_{i=1}^{4} x_i \omega_i ; x_i > 0, i=1, \ldots, 4 \right\}$ as a cone in the 4-dimensional Euclidean space $(\mathbb{R}^4, (\ ,\ ))$, where the vector ω_4 is orthogonal to each ω_i, $i=1, 2, 3$ which are given such as in the Figure 1. That is, let e_1, e_2, e_3 be the orthonormal basis of the 3-dimensional subspace of $(\mathbb{R}^4, (\ ,\ ))$ orthogonal to ω_4. Then $\omega_1 = e_3$, $\omega_2 = e_1 - e_2 + e_3$ and $\omega_3 = e_1 + e_2 + e_3$.

A chamber $\tilde{C}_{(1)}$ of $I_2(3) \times I_2(4)$ is given by $\left\{ \sum_{i=1}^{4} y_i \tilde{\omega}_i ; y_i > 0, i=1, \ldots, 4 \right\}$ as a cone in the 4-dimensional Euclidean space $(\mathbb{R}^4, (\ ,\ ))$, where both vectors $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are orthogonal to both vectors $\tilde{\omega}_3$ and $\tilde{\omega}_4$ and the angle between $\tilde{\omega}_1$ and $\tilde{\omega}_2$ (resp. $\tilde{\omega}_3$ and $\tilde{\omega}_4$) is $\pi/3$

(resp. $\pi/4$). On the other hand, since the angle between ω_1 and ω_2 is arc-tan$(2^{1/2})$, it is impossible that $C_{(1)}$ and $\tilde{C}_{(1)}$ are congruent in the 4-dimensional Euclidean space.

(2) A chamber $C_{(2)}$ of $B_3 \times A_1$ is given by $\left\{ \sum_{i=1}^{4} x_i \eta_i;\ x_i > 0,\ i=1, \ldots, 4 \right\}$, where $\eta_1 = \omega_1 = e_3$, $\eta_2 = \omega_2 = e_1 - e_2 + e_3$, $\eta_3 = e_1 + e_3$ and $\eta_4 = \omega_4$ in the example (1). A chamber of $\tilde{C}_{(2)}$ of $I_2(4) \times I_2(6)$ is given by $\left\{ \sum_{i=1}^{4} y_i \tilde{\eta}_i;\ y_i > 0,\ i=1, \ldots, 4 \right\}$, where both vectors $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are orthogonal to both vectors $\tilde{\eta}_3$ and $\tilde{\eta}_4$, and the angle between $\tilde{\eta}_1$ and $\tilde{\eta}_2$ (resp. $\tilde{\eta}_3$ and $\tilde{\eta}_4$) is $\pi/4$ (resp. $\pi/6$).



Chambers of A_3 and B_3.

4. Case of Euclidean domains

In this section, we consider the boundary value problems (E.D.P.), (E.N.P.) (Case 2) for the domains $D_\varepsilon (0 < \varepsilon < 1)$ (3.2) of the Euclidean space $(E, (\ ,\ ))$ of dimension n as in 3.1. We preserve the situations in 3.1. Recall that, for $0 < \varepsilon < 1$, $D_\varepsilon = \{ r\omega;\ \varepsilon < r < 1,\ \omega \in C_1 \}$, where $C_1 = C \cap S^{n-1}$, C is a chamber of a finite reflection group W in E.

Firstly, note that the volume element $dx = dx_1 \ldots dx_n$ can be expressed on E-(o) by the polar coordinate (r, ω) as

$$dx = r^{n-1}\, dr\, d\omega,$$

where $d\omega$ is the volume element of the standard unit sphere (S^{n-1}, g_0). Let $L^2(D_\varepsilon, dx)$ be the space of all square integrable functions on D_ε with respect to the measure dx, and $L^2((\varepsilon, 1)\times C_1, dr\, d\omega)$ the space of all square integrable functions on the product space $(\varepsilon, 1)\times C_1$ of the open interval $(\varepsilon, 1)$ and C_1 with respect to the product measure $dr\, d\omega$. Since $0<\varepsilon^{n-1}<r^{n-1}<1$ on the interval $(\varepsilon, 1)$, $L^2(D_\varepsilon, dx)$ can be identified with $L^2((\varepsilon, 1)\times C_1, dr\, d\omega)$ by the mapping $D_\varepsilon \ni r\omega \mapsto (r, \omega)\in(\varepsilon, 1)\times C_1$.

Now let $\{\lambda_1 \leqq \lambda_2 \leqq \ldots\}$ be the set of all the eigenvalues (counted repeatedly as many as their multiplicities) of the Dirichlet problem (S.D.P.) (resp. the Neumann problem (S.N.P.)) of the Laplacian Δ of (S^{n-1}, g_0) for the domain C_1 in S^{n-1}. Let $\{\Psi_i\}_{i=1}^{\infty}$ be a complete basis of $L^2(C_1, d\omega)$ such that

(4.1) $$\Delta\Psi_i=\lambda_i\Psi_i \quad \text{in } C_1,$$

and

(4.2) $$\Psi_i=0 \quad \text{on } \partial C_1 \left(\text{resp. } \frac{\partial\Psi_i}{\partial\nu}=0 \text{ a. e. } \partial C_1, \text{i. e., where the exterior normal } \nu \text{ of } \partial C_1 \text{ is defined}\right).$$

For each eigenvalue λ of Δ, recalling (3.1), define a differential operator L_λ on the open interval $(\varepsilon, 1)$ by

(4.3) $$L_\lambda=-\frac{d^2}{dr^2}-\frac{n-1}{r}\frac{d}{dr}+\frac{\lambda}{r^2}.$$

Let $L_1^2(\varepsilon, 1)$, $L_2^2(\varepsilon, 1)$ be the spaces of all square integrable functions on the interval $(\varepsilon, 1)$ with respect to the measure dr, $r^{n-1}dr$, respectively. Note that a C^∞ function Φ on $(\varepsilon, 1)$ is an eigenfunction of L_λ with an eigenvalue μ:

$$L_\lambda\Phi=\mu\Phi,$$

if and only if Φ satisfies the following equation of Sturm-Liouville type on $(\varepsilon, 1)$:

(4.4) $$\frac{d}{dr}\left(r^{n-1}\frac{d\Phi}{dr}\right)-\lambda r^{n-3}\Phi+\mu r^{n-1}\Phi=0.$$

LEMMA 4.1. — *Let us consider the boundary value problem of* (4.4) *with the boundary condition:*

(4.5) $$\Phi(\varepsilon)=\Phi(1)=0 \left(\text{resp. } \frac{d}{dr}\Phi(\varepsilon)=\frac{d}{dr}\Phi(1)=0\right).$$

Let $\{\mu_j^\lambda\}_{j=1}^{\infty}$ *be the set of all eigenvalues of the boundary value problem* (4.4) *and* (4.5), *and let* Φ_j^λ $(j=1, 2, \ldots)$ *be the eigenfunction with the eigenvalue* μ_j^λ. *Then* $\{\Phi_j^\lambda\}_{j=1}^{\infty}$ *is a complete basis of* $L_2^2(\varepsilon, 1)$.

Proof. — *See* [P], p. 508 or [Y], p. 109, Theorem 1.

Now, for the eigenvalues $\lambda_i (i=1, 2, \ldots)$ of the boundary value problem (S.D.P.) (resp. (S.N.P.)) for the domain C_1, consider C^∞ functions $\Phi_j^{\lambda_i}$ $(j=1, 2, \ldots)$ on the interval $(\varepsilon, 1)$. For a C^∞ function Φ, (resp. Ψ) on the interval $(\varepsilon, 1)$ (resp. C_1), we define a C^∞ function $\Phi\otimes\Psi$ on D_ε (or $(\varepsilon, 1)\times C_1$) by:

$$\Phi\otimes\Psi(r\omega)=\Phi(r)\Psi(\omega), \qquad r\omega\in D_\varepsilon [\text{or } (\varepsilon, 1)\times C_1].$$

Then, by (3.1), the C^∞ functions $\Phi_j^{\lambda_i}\otimes\Psi_i (i, j=1, 2, \ldots)$ on D_ε satisfy the equation

$$(4.6) \quad \Delta_0(\Phi_j^{\lambda_i}\otimes\Psi_i)=-\left(\frac{d^2\Phi_j^{\lambda_i}}{dr^2}+\frac{n-1}{r}\frac{d\Phi_j^{\lambda_i}}{dr}\right)\otimes\Psi_i+\frac{1}{r^2}\Phi_j^{\lambda_i}\otimes\Delta\Psi_i$$
$$=(L_{\lambda_i}\Phi_j^{\lambda_i})\otimes\Psi_i=\mu_j^{\lambda_i}\Phi_j^{\lambda_i}\otimes\Psi_i \quad \text{in } D_\varepsilon,$$

and the boundary condition

$$(4.7) \qquad \Phi_j^{\lambda_i}\otimes\Psi_i=0 \quad \text{on } \partial D_\varepsilon,$$

(resp. $\partial/\partial\nu(\Phi_j^{\lambda_i}\otimes\Psi_i)=0$ a. e. ∂D_ε, where the exterior normal of ∂D_ε is defined), since $\Phi_j^{\lambda_i}$ satisfies (4.1) and (4.2) and Ψ_i satisfies (4.4) and (4.5). In fact, $\partial/\partial\nu(\Phi_j^{\lambda_i}\otimes\Psi_i)(r\omega)$ coincides with $-(d/dr)\Phi_j^{\lambda_i}(\varepsilon)\Psi_i(\omega)$, $(d/dr)\Phi_j^{\lambda_i}(1)\Psi_i(\omega)$, or $\Phi_j^{\lambda_i}(r)(\partial/\partial\nu)\Psi_i(\omega)$ a. e. ∂D_ε. Here $(\partial/\partial\nu)\Psi_i$ is the derivation of Ψ_i with respect to the exterior normal of ∂C_1 (*cf.* Lemma 3.2(2)).

Furthermore we have the following lemma.

LEMMA 4.2. — $\{\Phi_j^{\lambda_i}\otimes\Psi_i;\ i, j=1, 2, \ldots\}$ *is a complete basis of* $L^2(D_\varepsilon, dx)$.

Proof. — It can be proved by the similar manner as Theorem 2.1 in [E]. Consider the following boundary value problem on the interval $(\varepsilon, 1)$:

$$\begin{cases} -\dfrac{d^2}{dr^2}u=\lambda u \quad \text{on } (\varepsilon, 1), \\ u(\varepsilon)=u(1)=0\left(\text{resp. } \dfrac{d}{dr}u(\varepsilon)=\dfrac{d}{dr}u(1)=0\right). \end{cases}$$

Let $\{u_l\}_{l=1}^\infty$ be a complete basis of $L_1^2(\varepsilon, 1)$ such that u_l is the eigenfunction of the above problem with the eigenvalue $\alpha_l (l=1, 2, \ldots)$. Let $\|\cdot\|_{L^2(D_\varepsilon, dx)}, \|\cdot\|_{L^2(S^{n-1}, d\omega)}, \|\cdot\|_{L_2^2(\varepsilon, 1)}$ be the L^2-norms of $L^2(D_\varepsilon, dx)$, $L^2(S^{n-1}, d\omega)$, $L_2^2(\varepsilon, 1)$, respectively. Since $\{\Phi_j^{\lambda_i}\}_{j=1}^\infty$, for each $\lambda_i (i=1, 2, \ldots)$, is a complete basis of $L_2^2(\varepsilon, 1)$, for each $\lambda_i (i=1, 2, \ldots)$ and $l=1, 2, \ldots$, there exist $a_{l,k}^{\lambda_i}\in\mathbb{R}$ $(k=1, 2, \ldots)$ such that

$$\lim_{p\to\infty}\left\|u_l-\sum_{k=1}^p a_{l,k}^{\lambda_i}\Phi_k^{\lambda_i}\right\|_{L_2^2(\varepsilon, 1)}=0.$$

On the other hand, we have

$$\left\| u_l \otimes \Psi_i - \sum_{k=1}^{p} a_{l,k}^{\lambda_i} \Phi_k^{\lambda_i} \otimes \Psi_i \right\|_{L^2(D_\varepsilon,\, dx)} = \left\| \left(u_l - \sum_{k=1}^{p} a_{l,k}^{\lambda_i} \Phi_k^{\lambda_i} \right) \otimes \Psi_i \right\|_{L^2(D_\varepsilon,\, dx)} = \left\| u_l - \sum_{k=1}^{p} a_{l,k}^{\lambda_i} \Phi_k^{\lambda_i} \right\|_{L_2^2(\varepsilon,\, 1)} \cdot \| \Psi_i \|_{L^2(S^{n-1},\, d\omega)}.$$

Thus we obtain

$$\lim_{p \to \infty} \left\| u_l \otimes \Psi_i - \sum_{k=1}^{p} a_{l,k}^{\lambda_i} \Phi_k^{\lambda_i} \otimes \Psi_i \right\|_{L^2(D_\varepsilon,\, dx)} = 0.$$

On the other hand, $\{u_l \otimes \Psi_i\}_{l,\, i=1}^{\infty}$ is a complete basis of $L^2((\varepsilon, 1) \times C_1, dr\, d\omega)$, due to the Stone-Weierstrass theorem (*cf.* [B.G.M.], p. 144). As $L^2(D_\varepsilon, dx)$ can be identified with $L^2((\varepsilon, 1) \times C_1, dr\, d\omega)$, $\{\Phi_j^{\lambda_i} \otimes \Psi_i;\ i, j=1, 2, \ldots\}$ is a complete basis of $L^2(D_\varepsilon, dx)$.

Q.E.D.

Therefore the spectrum $\mathrm{Spec}_D(D_\varepsilon)$ (resp. $\mathrm{Spec}_N(D_\varepsilon)$) of the Dirichlet problem (E.D.P.) (resp. the Neumann problem (E.N.P.)) for the domain D_ε in E is given by

$$\{\mu_j^{\lambda_i};\ i, j=1, 2, \ldots\},$$

where $\{\lambda_i\}_{i=1}^{\infty}$ is the spectrum $\mathrm{Spec}_D(C_1)$ (resp. $\mathrm{Spec}_N(C_1)$) for the domain C_1 in the unit sphere S^{n-1}. Since $\mu_j^{\lambda_i}$ depend on λ_i but not on Ψ_i, we obtain the following theorem.

THEOREM 4.3. – *For two reflection groups* W, $\tilde{W}$ *on the same Euclidean space* (E,(,)), *let* C, $\tilde{C}$ *be their chambers and* $C_1 = C \cap S^{n-1}$, $\tilde{C}_1 = \tilde{C} \cap S^{n-1}$, *where* S^{n-1} *is the unit sphere in* (E,(,)). *For each* $0<\varepsilon<1$, *define the domains* $D_\varepsilon = \{r\omega;\ \varepsilon<r<1,\ \omega \in C_1\}$, $\tilde{D}_\varepsilon = \{r\omega;\ \varepsilon<r<1,\ \omega \in \tilde{C}_1\}$ *respectively. Let* $\mathrm{Spec}_D(D_\varepsilon)$, $\mathrm{Spec}_D(\tilde{D}_\varepsilon)$ (resp. $\mathrm{Spec}_N(D_\varepsilon)$, $\mathrm{Spec}_N(\tilde{D}_\varepsilon)$) *be the spectra of the Dirichlet problems (E.D.P.) (resp. the Neumann problems (E.N.P.)) for the domains* D_ε, $\tilde{D}_\varepsilon$ *in* E. *Let* $\mathrm{Spec}_D(C_1)$, $\mathrm{Spec}_D(\tilde{C}_1)$ (resp. $\mathrm{Spec}_N(C_1)$, $\mathrm{Spec}_N(\tilde{C}_1)$) *be the spectra of the Dirichlet problems (S.D.P.) (resp. the Neumann problems (S.N.P.)) for the domains* C_1, $\tilde{C}_1$ *in* S^{n-1}. *Then we have:*

If $\mathrm{Spec}_D(C_1) = \mathrm{Spec}_D(\tilde{C}_1)$ (resp. $\mathrm{Spec}_N(C_1) = \mathrm{Spec}_N(\tilde{C}_1)$), *then* $\mathrm{Spec}_D(D_\varepsilon) = \mathrm{Spec}_D(\tilde{D}_\varepsilon)$ (*resp.* $\mathrm{Spec}_N(D_\varepsilon) = \mathrm{Spec}_N(\tilde{D}_\varepsilon)$) *for each* $0<\varepsilon<1$.

We note that, if C_1 is not congruent to $\tilde{C}_1$ in the unit sphere S^{n-1}, then D_ε is not congruent to $\tilde{D}_\varepsilon$ in the Euclidean space (E,(,)) for each $0<\varepsilon<1$. Therefore by Theorems 3.8, 4.3, we have:

THEOREM 4.4. – *There exist domains* D_ε, $\tilde{D}_\varepsilon$ $(0<\varepsilon<1)$ *in the n-dimensional Euclidean space* $\mathbb{R}^n$ $(n \geq 4)$ *such that*

$$\mathrm{Spec}_D(D_\varepsilon) = \mathrm{Spec}_D(\tilde{D}_\varepsilon) \quad \text{and} \quad \mathrm{Spec}_N(D_\varepsilon) = \mathrm{Spec}_N(\tilde{D}_\varepsilon),$$

but these domains D_ε, $\tilde{D}_\varepsilon$ $(0<\varepsilon<1)$ *are not congruent each other in the Euclidean space* $\mathbb{R}^n$.

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