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The isometry groups of riemannian manifolds admitting strictly convex functions


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THE ISOMETRY GROUPS
OF RIEMANNIAN MANIFOLDS
ADMITTING STRICTLY CONVEX FUNCTIONS

BY TAKAO YAMAGUCHI

0. Introduction

A function $f$ on a complete connected Riemannian manifold $M$ is said to be convex if for any geodesic $\gamma : \mathbb{R} \to M$, any $t_1, t_2 \in \mathbb{R}$ and any $0 < \lambda < 1$, $f$ satisfies the following inequality:

$$f \circ \gamma((1 - \lambda)t_1 + \lambda t_2) \leq (1 - \lambda)f \circ \gamma(t_1) + \lambda f \circ \gamma(t_2).$$

It is well known that a convex function is Lipschitz continuous on every compact subset. If the above inequality is strict for all $\gamma, t_1, t_2$ and $\lambda$, then $f$ is said to be strictly convex. A function is said to be locally nonconstant if it is not constant on any open subset. If $M$ admits a nontrivial convex function, then $M$ is noncompact. Clearly strict convexity induces local nonconstancy. Recently the topological structure of manifolds which admit locally nonconstant convex functions has been decided by Greene-Shiohama [4]. Since a convex function imposes a certain restriction to the Riemannian structure, it is natural to ask the influences of the existence of a convex function on the Riemannian structure. In this paper we will investigate the influences of the existence of strictly convex functions with compact levels on the isometry groups. According to [4], if a level set $f^{-1}(t)$ of a locally nonconstant convex function $f$ on $M$ is compact then all level sets are also compact. Such an $f$ is said to be with compact levels. And corresponding to each $t \in f(M)$ the diameter $\delta(t)$ of $f^{-1}(t)$, the diameter function of $f$, $\delta : f(M) \to \mathbb{R}$, is well defined and is monotone nondecreasing. We will prove the following theorems.

**Theorem A.** — If $M$ admits a strictly convex function with minimum, then each compact subgroup of the isometry group $I(M)$ of $M$ has a common fixed point.

**Theorem B.** — If $M$ admits a strictly convex function with compact levels and with no minimum, then all the isometric images of any level set intersect the level set. In particular, $I(M)$ is compact.

Cheeger-Gromoll [3] proved the following splitting theorem for complete manifolds of nonnegative sectional curvature by constructing an expanding filtration of $M$ by compact totally convex sets which are sublevel sets of a convex function.
THEOREM [3]. — A complete Riemannian manifold \( M \) of nonnegative sectional curvature splits uniquely as \( M \times \mathbb{R}^k \), where the isometry group of \( M \) is compact and \( \Gamma (M) = \Gamma (M) \times \Gamma (\mathbb{R}^k) \).

Recently S. T. Yau [9] has obtained a similar result to Theorem A for strongly convex functions, which is stronger than strict convexity. A function \( f : M \rightarrow \mathbb{R} \) is said to be strongly convex if for a given compact set \( K \) of \( M \) there exists a \( \varepsilon > 0 \) such that \( \left\{ f \circ \gamma (t) + f \circ \gamma (-t) - 2 f \circ \gamma (0) \right\} / t^2 > \varepsilon \) for any geodesic \( \gamma \) with \( \gamma (0) \in K \). Clearly \( f(t) = t^4 \) is not strongly convex but strictly convex. It will be clear from examples which we will construct later that Theorem A is a natural extension of a classical theorem due to E. Cartan which states that each compact subgroup of the isometry group of a simply connected complete Riemannian manifold of nonpositive sectional curvature has a common fixed point. We note that any manifold satisfying the hypothesis of Theorem A is diffeomorphic to \( \mathbb{R}^n (n = \dim M) \), and in the situation of Theorem B \( M \) is homeomorphic to \( N \times \mathbb{R} \), where \( N \) is a level set [4]. The key to the proof of Theorem B is to show that the metric projection onto any sublevel set is locally distance decreasing. This is done in paragraph 3.

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1. Preliminaries

Hereafter let \( M \) be a complete connected Riemannian manifold with \( \dim M \geq 2 \) and let \( \rho \) be the distance function induced from the Riemannian metric. For an \( r > 0 \) and a point \( p \) of \( M \) let \( B_r(p) \) denote the open metric ball of radius \( r \) around \( p \). It is well known as the Whitehead Theorem (see [2]) that there exists a positive continuous function \( c \) on \( M \), which is called a convexity radius function, such that for every point \( p \in M \) (1) any open ball \( B_r(p) \) contained in \( B_{c(p)}(p) \) is a strongly convex set, (2) \( \rho^2(p', \cdot) \) is \( C^\infty \)-strongly convex on \( B_{c(p)}(p') \). A set \( A \subset M \) is called to be strongly convex if for any two points \( p \) and \( q \) of \( A \) there exists a unique minimizing geodesic from \( p \) to \( q \) and it is contained in \( A \). A set \( A \subset M \) is called to be totally convex if \( A \) contains all geodesic segments which join any two points of \( A \), and a set \( C \subset M \) is called to be convex if for any point \( p \) of the closure \( \overline{C} \) of \( C \) there exists a positive number \( \varepsilon (p) \), \( 0 < \varepsilon (p) \leq c(p) \), such that \( C \cap B(p) \) is strongly convex.

PROPOSITION (cf. [4], Prop. 1.2). — If \( C \) is a closed convex set of \( M \) then there exists an open neighborhood \( U \) of \( C \) such that for any point \( p \) of \( C \) there exists a unique point \( q \) of \( C \) such that \( \rho(p, q) = \rho(p, C) \).

Then the map \( \pi \subset : U \rightarrow C \), which is called the metric projection of \( U \) onto \( C \), can be defined by \( \rho(p, \pi \subset (p)) = \rho(p, C) \) and is continuous.

For a real valued function \( f \) on \( M \) and for arbitrary real numbers \( a \) and \( b \), \( a \leq b \), we will denote \( f([a, b]) \) and \( f((\infty, a]) \) by \( M^b_a(f) \) and \( M^a_b(f) \) respectively, or briefly \( M^b_a \) and \( M^a_b \). If \( M^a_b \) (resp. \( M^b_a \)) is not empty, then it is called a level set of \( f \) (resp. a sublevel set of \( f \)). It is clear that every sublevel set of a convex function is totally convex.

Let \( C \) be a convex set of \( M \) and let \( p \in C \). A tangent vector \( v \) to \( M \) at \( p \) is normal to \( C \) at \( p \) if for any smooth curve \( \gamma \) in \( C \) emanating from \( p \) we have \( \langle \gamma'(0), v \rangle \leq 0 \). If \( \pi \subset : U \rightarrow C \) is a metric projection onto \( C \) and if \( p \in U - C \) and if \( \gamma \) is a minimizing geodesic from \( \pi \subset (p) \) to \( p \), then \( \gamma'(0) \) is normal to \( C \) at \( \pi \subset (p) \). Conversely if \( v \) is a normal vector to \( C \) at \( p \) then
\[ \pi_\ast(\exp_{p}tv/\|v\|)=p \] for any sufficiently small \( t>0 \). We note that the set of all normal vectors to \( C \) at \( p \) is a closed subset of \( M_p \).

2. Proof of Theorem A and examples

\textit{Proof of Theorem A.} — Let \( f \) be a strictly convex function with minimum on \( M \) and let \( G \) be a compact subgroup of the isometry group of \( M \). We note that \( M^a(f) \) is compact for any \( a \in f(M) \). Let \( \mu \) denote the Haar measure on \( G \) normalized by \( \int_G d\mu = 1 \). We define a function \( F \) on \( M \) by:

\[ F(x) = \int_G f(gx) d\mu(g). \]

For every element \( g \) of \( G \), \( f \circ g \) is also strictly convex, and so is \( F \). Now we will show that \( F \) has also minimum.

\textbf{Assertion.} — For any \( a \in \mathbb{R} \) there is a \( b \in \mathbb{R} \) such that \( M^a(F) = M^b(f) \).

To prove the assertion, suppose that it is not true. Then there are some \( a \in \mathbb{R} \) and a sequence \( \{x_n\} \) in \( M^a(F) \) so that \( f(x_n) \to \infty \). It follows from the definition of \( F \) that for each \( n \) there is a \( g_n \in G \) such that \( f(g_nx_n) \leq a \). Thus it turns out that \( G \cdot M^a(f) \) is unbounded. This contradicts the compactness of \( G \) and \( M^a(f) \).

The proof of Theorem A is complete since \( F \) has a unique minimum point by the strict convexity of \( F \) and since it is \( G \)-invariant.

Q.E.D.

\textit{Examples.} — (a) Let \( H \) denote a simply connected Riemannian manifold of nonpositive sectional curvature. For a given point \( p \) of \( H \rho^2(p, .) \) is \( C^\infty \)-strongly convex with minimum.

(b) Paraboloid; \( \{(x,y,z) \in \mathbb{R}^3; \quad z=x^2+y^2\} \). \( f(x,y,z)=z \) is strictly convex with minimum. The curvature is positive everywhere.

(c) (see [8]). Let \( 0 < a < b \) and \( h : [0, \infty) \to [0, 1] \) be a \( C^\infty \)-function such that (1) \( h(v) = 0 \) for \( v \leq a \) and \( h(v) = 1 \) for \( v \geq b \), (2) if we define \( g \) by \( g(v) = v^2 + h(v) \) for \( v \geq 0 \), then \( g'(v) > 0 \) for all \( v > 0 \) and \( g''(v_0) < 0 \) for some \( v_0 \), \( a < v_0 < b \). We consider a surface of revolution; \( S = \{(v \cos u, v \sin u, g(v)); \quad 0 \leq u \leq 2\pi, v \geq 0\} \) whose curvature is negative on a neighborhood of \( \{(u, v_0); \quad 0 \leq u \leq 2\pi, v \geq a \text{ or } v \geq b\} \) and is positive on \( \{(u, v); \quad 0 \leq u \leq 2\pi, v \leq a \text{ or } v \leq b\} \). For each positive integer \( n \) we define a function \( f_n \) on \( S \) by \( f_n(u, v) = g^n(v) \). Then \( f_n \) is strongly convex with minimum for any sufficiently large \( n \).

3. The diameter functions for strictly convex functions

Let \( f \) be a locally nonconstant convex function with compact levels on \( M \) and let \( m = \inf_M f \), then the diameter function \( \delta : (m, \infty) \to \mathbb{R} \) is defined by \( \delta(t) = \max \{ p(x, y); x, y \in M^f_t \} \). \( \delta \) is monotone nondecreasing [4]. In this section we will
prove that if \( f \) is strictly convex with compact levels, then \( \delta \) is strictly increasing. Hereafter we will fix a strictly convex function \( f \) with compact levels. Let \( a, b \in (m, \infty), a \leq b, \) be fixed and \( B \) be a sufficiently large compact neighborhood of \( M^b \) and let \( r_0 = \min_b c \) where \( c \) is a convexity radius function on \( M \). There exists a neighborhood \( U \) of the zero section of \( TM \) such that \( \text{Exp}(U) \) is an embedding and \( \text{Exp}(U) \supset B_{r_0}(x) \times B_{r_0}(x) \) for any \( x \in M^b \), where \( \text{Exp} : TM \to M \times M \) is the exponential mapping defined by \( \text{Exp}(v) = (\pi(v), \exp\pi(v)) \) and \( \pi : TM \to M \) is the natural projection. For each \( x \in B \) let:

\[
L_x = \inf \left\{ L > 0; L^{-1} \leq \|d(\text{Exp}(U))^{-1}\| B_{r_0}(x) \times B_{r_0}(x) \| \leq L \right\}
\]

and let \( L = \sup \{ L_x; x \in B \} \). It is clear from compactness argument that \( 0 < L < \infty \). Let \( x \) be the maximum of the absolute values of the sectional curvature on \( B \). Let \( \mu = \min \{ \delta(a)/8, r_0/8 \} \) and let \( A = \{ (x, y) \in M^b \times M^b; \mu \leq \rho(x, y) \leq r_0/2, \beta \leq \beta \} \). For each \( x \in M \) we denote the set of all unit normal vectors to \( M^b \) at \( x \) by \( N^b_x(f) \). Now for each \( (x, y) \in A \) and for each \( v_1 \in N^b_1(f), v_2 \in N^b_1(f) \) let \( \gamma_1 \) and \( \gamma_2 \) be the geodesics emanating from \( x \) and \( y \) whose velocity vectors are \( v_1 \) and \( v_2 \) respectively. Let \( x = \gamma_1(t_1) \) and \( y = \gamma_2(t_2) \) be arbitrary fixed points on \( \gamma_1 \) and \( \gamma_2 \) so that \( t_1 > 0, \mu/4 \geq t_1, t_2 \geq 0 \). We reparametrize the subarc of \( \gamma_1 \) and \( \gamma_2 \) by \( \tau_1(s) = \gamma_1(s) \) and \( \tau_2(s) = \gamma_2(t_2 s/t_1) \), \( 0 \leq s \leq t_1 \). \( \alpha : [0, 1] \times [0, t_1] \to M \) is the rectangle such that each \( \alpha \) is a unique minimizing geodesic segment from \( x \) to \( z \). Then by the convexity of \( f, \gamma \) is contained in \( M^p \). Since \( \gamma ' \) makes an acute angle with \( v_1 \), this is a contradiction for \( v_1 \) to be a normal vector. It follows that \( L' (\alpha_s) |_{s=0} > 0 \). Now let:

\[
C_1 = \inf \{ L(\alpha_s) |_{s=0}; (x, y) \in A, v_1 \in N^b_1(f), v_2 \in N^b_1(f), x', y' \text{ as above} \}.
\]

It is easy to see that \( C_1 > 0 \). It follows from the preceding lemma that \( L'(\alpha_s) = L'(0) + sL''(\theta s) \geq C_1 - sC_2 \) for some \( \theta \geq 0 \). Hence we have obtained:

**Lemma 3.2.** For any \( (x, y) \in A \) and any \( v_1 \in N^b_1(f), v_2 \in N^b_1(f) \) and for any \( x' = \gamma(t_1), y' = \gamma(t_2) \) such that \( C_1/C_2 \geq t_1 \geq t_2 \geq 0, t_1 > 0 \) as before, \( L(\alpha_s) \) is strictly increasing on \([0, t_1] \).
For any $\beta \in [a, b]$ $M^\beta$ is a totally convex set. If we set $U = \bigcup_{x \in M} B_{r_0/8}(x)$ then the metric projection $\pi_M$ of $U$ onto $M^\beta$, which we briefly denote by $\pi_{\beta}$, can be defined as in paragraph 1.

**Lemma 3.3.** There exists a positive constant $\varepsilon_0$ such that for each $\beta \in [a, b]$ if $x \in M^{\beta+r_0} - M^\beta$ and $y \in M^{\beta+r_0}$ satisfy $2\mu \leq \rho(x, y) \leq 3r_0/8$, then we have $\rho(x, y) > \rho(\pi_{\beta}(x), \pi_{\beta}(y))$.

**Proof.** Let $\varepsilon_1 = \min \{\mu/4, C_1/2\}$ and let:

$$
\epsilon_0(\beta) = \inf \left\{ \int (\exp_x \epsilon ; v_x) ; x \in M^\beta, v_x \in N_1(f) \right\} - \beta.
$$

The required constant will be obtained by $\epsilon_0 = \inf \{\epsilon_0(\beta) ; a \leq \beta \leq b\}$. We note that $\epsilon_0 > 0$. Then for any $x$ and $y$ as in this lemma we have $\rho(\pi_{\beta}(x), x) \leq \varepsilon_1$, $\rho(\pi_{\beta}(y), y) \leq \varepsilon_1$ and $(\pi_{\beta}(x), \pi_{\beta}(y)) \in A$ by triangle inequalities. Therefore the preceding lemma completes the proof.

Q.E.D.

**Proposition 3.4.** $\delta$ is strictly increasing.

**Proof.** For a given $c \in (m, \infty)$ let $\epsilon_0$ be the positive constant given in the preceding lemma for $a = b = c$. Fix an arbitrary $s$ such that $0 < s \leq \varepsilon_0$. Let $x_0$ and $y_0$ be two points of $M^c$ such that $\rho(x_0, y_0) = \delta(c)$, and let $t_1 \in N^1(c, f)$, $t_2 \in N^1(c, f)$ and let $x_1$ and $y_1$ be two points of $M^c + s$, at which two geodesics $\exp_{x_0} t_1\varepsilon_1$, $\exp_{x_1} t_2\varepsilon_1$, $t \geq 0$, intersect $M^c + s$ respectively. By $\sigma : [0, d] \to M$ we denote a minimizing unit speed geodesic from $x_1$ to $y_1$. We consider two cases.

**Case 1.** $\sigma([0, d]) \cap M^c = \emptyset$.

We can choose a subdivision $0 = t_0 < t_1 < \ldots < t_k = d$ of $[0, d]$ such that $2\mu \leq t_i - t_{i-1} \leq 3r_0/8$ for all $i, 1 \leq i \leq k$. Using Lemma 3.3 we have:

$$
\rho(x_1, y_1) = \sum_{i=1}^k \rho(\sigma(t_{i-1}), \sigma(t_i)) \geq \sum_{i=1}^k \rho(\pi_{\epsilon_1}(\sigma(t_{i-1})), \pi_{\epsilon_1}(\sigma(t_i))) \geq \rho(x_0, y_0).
$$

Hence $\delta(c + s) > \delta(c)$.

**Case 2.** $\sigma([0, d]) \cap M^c \neq \emptyset$.

Then there exist $s_1, s_2 \in (0, d)$, $s_1 \leq s_2$, such that $\sigma([0, s_1])$ and $\sigma([s_2, d])$ are contained in $M^{c+s} - M^c$ and $\sigma([s_1, s_2])$ is contained in $M^c$. We can choose two subdivisions, $0 = t_0 < t_1 < \ldots < t_{k_1} = s_1$ and $s_2 = u_0 < u_1 < \ldots < u_{k_2} = d$ of $[0, s_1]$ and $[s_2, d]$ which satisfy the following conditions:

$$
2\mu \leq t_i - t_{i-1} \leq 3r_0/8 \quad \text{for} \quad i = 1, \ldots, k_1 - 1, s_1 - t_{k_1-1} < 2\mu,
$$

$$
2\mu \leq u_i - u_{i-1} \leq 3r_0/8 \quad \text{for} \quad i = 2, \ldots, k_2, u_1 - s_2 < 2\mu.
$$
Since \( \rho^2(\sigma(s_1), .) \) and \( \rho^2(\sigma(s_2), .) \) are \( C^\infty \)-strongly convex on \( B_{r_n}(\sigma(s_1)) \) and \( B_{r_n}(\sigma(s_2)) \) respectively, we have \( \rho(\sigma(t_{k-1}), \sigma(s_1)) > \rho(\pi, \sigma(t_{k-1})), \sigma(s_1)) \) and \( \rho(\sigma(s_2), \sigma(t_i)) > \rho(\sigma(s_2), \pi, \sigma(t_i)) \). It follows from the same argument as in case 1 that \( \rho(x_1, \sigma(s_1)) > \rho(\pi, (x_1), \sigma(s_1)) \) and \( \rho(\sigma(s_2), y_1) > \rho(\sigma(s_2), \pi, (y_1)) \). It follows that \( \rho(x_1, y_1) > \rho(\pi, (x_1), \pi, (y_1)) \). Therefore \( \delta(c + s) > \delta(c) \).

Q.E.D.

4. Proof of Theorem B

Let \( f \) be a strictly convex function on \( M \) with compact levels and with no minimum, and let \( m = \inf_M f \). The proof of Theorem B is achieved by supposing that it is not true and then by deriving a contradiction. The contradiction, roughly speaking, comes as follows. By the fact that \( M \) is homeomorphic to \( N \times \mathbb{R} \) where \( N \) is any level set (see [4], Theorem C), the isometric image of a level set must always separate \( M \) into two unbounded components. But by the diameter increasing property this is not possible if a low level set is moved to a higher level, where a larger diameter would be required.

Suppose that \( M^c \cap \psi(M^c) = \emptyset \) for some \( c \in f(M) \) and some \( \psi \in \Gamma(M) \). It follows that \( \psi(M^c) \cap M^c = \emptyset \) or \( \psi(M^c) \subseteq M^c \). We consider two cases.

Proof of Theorem B in the case \( \psi(M^c) \cap M^c = \emptyset \). — Let \( a = \min \{ f(x); x \in \psi(M^c) \} \) and \( b = \max \{ f(x); x \in \psi(M^c) \} \). Notice that \( c < a \). Let \( e_0 \) denote the constant obtained in Lemma 3.3 for these \( a \) and \( b \). We choose subdivision \( a = t_0 < t_1 < \ldots < t_k = b \) of \([a, b] \) such that \( t_i - t_{i-1} \leq e_0 \) for all \( i, 1 \leq i \leq k \). For each \( i, 1 \leq i \leq k - 1 \), let \( \pi_{t_i}: M^{t_i+1} \to M^{t_i} \) be the metric projection and let \( H = \pi_{t_i} \circ \ldots \circ \pi_{t_0}: M^b \to M^a \).

Assertion. — \( d(H \circ \psi(M^c)) \leq \delta(c) \), where \( d(H \circ \psi(M^c)) \) is by definition the diameter of \( H \circ \psi(M^c) \).

Proof of Assertion. — We suppose that \( d(H \circ \psi(M^c)) > \delta(c) \) and take two points \( x \) and \( y \) of \( H \circ \psi(M^c) \) such that \( \rho(x, y) = d(H \circ \psi(M^c)) \). Let \( x' \) and \( y' \) be such points of \( \psi(M^c) \) that \( H(x') = x \) and \( H(y') = y \). We may assume that \( t_i \leq f(x') < t_{i+1} \) and \( t_{j_0} \leq f(y') < t_{j_0+1} \) for \( i_0 \leq j_0 \). Let \( x_i = \pi_{t_i} \circ \ldots \circ \pi_{t_0}(x') \) for each \( i \leq i_0 \) and let \( y_j = \pi_{t_j} \circ \ldots \circ \pi_{t_0}(y') \) for each \( j \leq j_0 \). In the proof of Proposition 3.4 if we replace \( \mu = \min \{ \delta(c)/8, r_0/8 \} \) by \( \min \{ \delta(c)/8, r_0/8 \} \) then we have \( \rho(x, y) < \rho \left( x_i, y_i \right) < \ldots < \rho \left( x_{j_0}, y_{j_0} \right) < \rho \left( x_{j_0+1}, y' \right) \). Let \( \eta: [0, d] \to M \) be a unit speed minimizing geodesic from \( x' \) to \( y' \). For each \( i, j_0 + 1 \leq i \leq i_0 \), let \( z_i \) be the point of intersection of \( \eta \) with \( M^c \). In the same way as Proposition 3.4 we have \( \rho(x', z_i) \leq \rho(x', z_{i-1}) \). It follows that

\[
\rho(x', z_{i-1}) \geq \rho(x_i, z_{i-1}) + \rho(z_{i-1}, z_i) \geq \rho(x_i, z_i).
\]

Iterating this, we have:

\[
\rho(x', z_{i-2}) \geq \rho(x_{i-1}, z_{i-2}), \ldots, \rho(x', z_{i+1}) \geq \rho(x_{j_0+1}, z_{j_0+1}) \geq \rho(x_{j_0+1}, z_{j_0+1}).
\]
It follows that:

\[ \rho(x', y') = \rho(x', z_{j_{b+1}}, y') + \rho(z_{j_{b+1}}, x) \geq \rho(x_{j_{b+1}}, z_{j_{b+1}}) + \rho(z_{j_{b+1}}, y') \geq \rho(x_{j_{b+1}}, y'). \]

Therefore we have:

\[ \delta(c) \geq \rho(x', y') \geq \rho(x, y) = d(H \circ \psi(M'_c)) \]

which contradicts the first assumption.

Q.E.D.

By Proposition 3.4 it is possible to take a point \( p_0 \) which belongs to \( M'_c - H \circ \psi(M'_c) \). Coosing:

\[ p_1 \in \pi_{i_2}^{-1}(p_0) \cap M'_{i_2}, \quad p_2 \in \pi_{i_1}^{-1}(p_1) \cap M'_{i_1}, \ldots, p_k \in \pi_{i_{k-1}}^{-1}(p_{k-1}) \cap M'_{i_{k-1}} \]

and joining \( p_0 \) to \( p_1, p_2, \ldots, p_{k-1} \) to \( p_k \) in this order by minimizing geodesics we obtain a broken geodesic \( \sigma \) from \( p_0 \) to \( p_k \) which does not intersect \( \psi(M'_c) \). It is easy to construct a continuous extension \( \sigma : \mathbb{R} \to M \) of \( \sigma \) such that \( \sigma(\mathbb{R}) \cap \psi(M'_c) = \emptyset \) and \( f \circ \sigma(\mathbb{R}) = (m, \infty) \). Since \( M \) is topologically a product of a level set and \( \mathbb{R} \), it turns out that \( f \circ \psi^{-1} \circ \sigma(\mathbb{R}) = (m, \infty) \). This contradicts the fact that \( \sigma(\mathbb{R}) \cap \psi(M'_c) = \emptyset \).

The rest of the proof of Theorem B is a direct consequence of the following:

**Corollary C.** — Under the same hypothesis as in Theorem B, every isometry of \( M \) fixes each of the two ends of \( M \).

**Proof.** — If some \( \psi \in I(M) \) permutes the ends, then there is a compact set \( K \) of \( M \) such that \( \psi \) maps one component \( U_1 \) of \( M - K \) into the other component \( U_2 \) and maps \( U_2 \) into \( U_1 \). It turns out that \( \psi \) maps a low level set to a much higher level. This is impossible.

**Proof of Theorem B in the case \( \psi(M'_c(f)) \subset M' \).** — We note that since \( f \circ \psi^{-1} \) is strictly convex, it follows from Theorem A in [4] that every level set of \( f \circ \psi^{-1} \) is connected. Let \( A \) be the closure of the component of \( M - \psi(M'_c(f)) \) which does not contain \( M'_c(f) \), then we get that \( M'((f \circ \psi^{-1}) = A \) or \( M'((f \circ \psi^{-1}) = M - A \). If \( M'((f \circ \psi^{-1}) = (\psi(M'(f))) = M - A \), it contradicts Corollary C. Hence \( M'((f \circ \psi^{-1}) = A \). We set \( \alpha = \max \{ f(x); x \in \psi(M'_c(f)) \} \) and \( d = \max \{ f \circ \psi^{-1}(x); x \in M'_c(f) \} \). Notice that \( \delta(c) < \delta \) and \( M'_c(f) \subset M'_d(f \circ \psi^{-1}) \). Now we can use the same argument as in the case \( \psi(M'_c(f)) \subset M' \) with \( f \circ \psi^{-1} \) in place of \( f \) and define a projection from \( M'_d(f \circ \psi^{-1}) \) onto \( M'((f \circ \psi^{-1}) \) as before. Then projecting \( M'_c(f) \) to \( M'((f \circ \psi^{-1}) \) derives a contradiction. This completes the proof of Theorem B.

Q.E.D.

In general, in the situation of Theorem B a level set is not invariant under the isometries. It is not difficult to exhibit the examples.
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