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ON THE POINT SPECTRUM OF SCHRÖDINGER OPERATORS

BY ANNE BERTHIER

1. Introduction

This paper is an extension of a work [2] on the spectral analysis of partial differential operators of Schrödinger type. The problem was the following: Let A be a compact subset of $\mathbb{R}^n$, Σ a finite interval in $\mathbb{R}$ and H a self-adjoint elliptic differential operator in the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. We define $F(\Sigma)$ to be the spectral projection of H associated with the interval Σ and $E(A)$ the multiplication operator by the characteristic function χ_A of A . Do there exist vectors in $L^2(\mathbb{R}^n)$ which are contained both in the range $E(A)\mathcal{H}$ of $E(A)$ and in $F(\Sigma)\mathcal{H}$?

It turns out that the closed subspace $\mathcal{H}_p(H)$ generated by the set of eigenvectors of H plays a different role from the subspace $\mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$ associated with the continuous spectrum of H . Notice that it is shown in [2], under regularity and integrability conditions on the coefficients of the differential operator, that there do not exist vectors of $\mathcal{H}_c(H)$ which belong both to $E(A)\mathcal{H}$ and to $F(\Sigma)\mathcal{H}$. On the other hand, to prove the non-existence of vectors in $\mathcal{H}_p(H)$ belonging to $E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H}$, we used an unique continuation theorem for solutions of the differential equation associated with H . Now, if for example $H = -\Delta + V$, where V is the multiplication operator by a real function $v(\vec{x})$, the known results on unique continuation require a condition $L^\infty(\mathbb{R}^n \setminus N)$ on v , where N is a closed set of measure zero such that $\mathbb{R}^n \setminus N$ is connected ([3], [5]).

In the present paper, we propose to show that:

$$(1) \quad \mathcal{H}_p(H) \cap E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H} = \{0\},$$

by imposing only an integrability condition on the function v . More precisely, we will prove (1) under the hypothesis that $v \in L_{loc}^s(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s > n-2$ if $n \geq 4$.

This result shows that, under the above conditions on v , the operator $-\Delta + v$ has no eigenvector with compact support. This is essentially the content of our Theorem 1 in paragraph 2. (In the case $n=1$, one obtains *ordinary* differential operators for which results of this type have been known for a long time [9]).

This result is also interesting from the point of view of "non-existence of positive eigenvalues of the operator H ". In the literature (for example [2], [12]) the non-existence of positive eigenvalues is obtained in two steps:

- (i) under suitable decay conditions at infinity on the function v , it is shown that all eigenfunctions f associated with a strictly positive eigenvalue of H have compact support;
- (ii) then one imposes suitable local conditions on v (e.g. $v \in L_{\text{Loc}}^{\infty}(\mathbb{R}^n \setminus N)$) in order to apply the unique continuation theorem, which then leads to $f \equiv 0$. It turns out that the non-existence of positive eigenvalues is also obtained by assuming in (ii) as a local condition that $v \in L_{\text{Loc}}^s(\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s>n-2$ if $n \geq 4$ (Thm. 2).

Finally our method implies also the spectral continuity of a class of Schrödinger operators with periodic potentials $v(\vec{x})$.

The organization of the paper is as follows: first we give the principal results and deduce Theorems 1 and 2 from Theorem 3 in section 2, and we introduce a direct integral representation of Schrödinger operators in section 3. This representation will be used in section 4 for proving Theorem 3. The principal estimate of the proof is the subject of the last section 5.

2. Statements of the results

Let $v: \mathbb{R}^n \rightarrow \mathbb{R}$ be a measurable function. We always suppose that:

$$(2) \quad v \in L_{\text{Loc}}^s(\mathbb{R}^n) \quad \text{with } s=2 \quad \text{if } n=1, 2, 3; \quad s>n-2 \quad \text{if } n \geq 4.$$

Notice that $s>n-2$ in all cases.

The function v will be called *periodic* if there exist n linearly independent vectors $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^n$ such that $v(\vec{x} + \vec{a}_i) = v(\vec{x})$ for all $\vec{x} \in \mathbb{R}^n$. A periodic function will be called *ortho-periodic* if:

$$(3) \quad \vec{a}_j \cdot \vec{a}_k = L^2 \delta_{jk},$$

with $L>0$, i.e. if the vectors of the form $\sum_{i=1}^n \alpha_i \vec{a}_i$, $0 \leq \alpha_i < 1$, define a cube C^n with side L .

We denote by $\hat{H}$ the symmetric operator:

$$(4) \quad \hat{H} = -\Delta + v(\vec{x}),$$

with domain $D(\hat{H}) = C_0^{\infty}(\mathbb{R}^n)$ and by H_0 the unique self-adjoint extension of $\hat{H}_0 = -\Delta$, $D(\hat{H}_0) = C_0^{\infty}(\mathbb{R}^n)$. Let H a self-adjoint extension of $\hat{H}$. We have the following lemma:

LEMMA 1. — Assume that (2) and one of the following conditions are satisfied:

- (i) v is periodic;
- (ii) $v \in L^{\infty}(\complement B_R)$ where $B_R = \{\vec{x} \in \mathbb{R}^n \mid |\vec{x}| \leq R\}$ and $\complement B_R$ denotes the complement of B_R .

Then:

- (a) v is H_0 -bounded with H_0 -bound 0;

- (b) $\hat{H}$ is essentially self-adjoint;
(c) $D(H) = D(H_0)$, where H is the unique self-adjoint extension of H .

Proof. — (b) and (c) follow from (a) by using the Kato-Rellich Theorem ([7], Chapt. 5.4.1). Under hypothesis (i), (a) follows from Theorem XIII.96 of [11], whereas under the assumption (ii), (a) can be proved by the method used in the proof of Lemma 3 in [10]. Both cases are treated in [4].

We now state our principal results. In Theorem 2 we choose as conditions on the potential v at infinity those used in [4].

THEOREM 1. — *Let $v \in L^s_{loc}(\mathbb{R}^n)$ with s satisfying (2) and let H be a self-adjoint extension of H :
(a) suppose that $f \in L^2(\mathbb{R}^n)$ satisfies $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $E(A)f = f$ for some compact subset A of $\mathbb{R}^n$. (i.e. f is an eigenvector of H with compact support in $\mathbb{R}^n$). Then $f = 0$;
(b) for each compact subset A of $\mathbb{R}^n$ and each bounded interval Σ , one has:*

$$\mathcal{H}_p(H) \cap E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\}.$$

THEOREM 2. — *Suppose that:*

- (i) $v \in L^s(B_R)$ with s satisfying (2) for some $R < \infty$;
(ii) $v = v_1 + v_2$ such that:
 (α) $v_1, v_2 \in L^\infty(\mathbb{R}^n)$,
 (β) $|\vec{x}| v_1(\vec{x}) \rightarrow 0$ as $|\vec{x}| \rightarrow \infty$,
 (γ) $v_2(\vec{x}) \rightarrow 0$ as $|\vec{x}| \rightarrow \infty$,
 (δ) $r \mapsto v_2(r, \cdot)$

is differentiable as a function from (R, ∞) to $L^\infty(S^{n-1})$, and $\limsup_{r \rightarrow \infty} \partial v_2 / \partial r \leq 0$. (S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.)

Then $H = H_0 + V$ has no eigenvalues in $(0, \infty)$.

THEOREM 3. — *Let v be ortho-periodic and $v \in L^s_{loc}(\mathbb{R}^n)$ with s satisfying (2). Then the spectrum of $H = H_0 + V$ is purely continuous.*

Remark 1. — By following the proof of Theorem XIII.100 in [11], it is possible to show that the operator H in Theorem 3 is absolutely continuous. Other comments on Theorem 3 will be made at the end of this paper.

Remark 2. — Contrarily to [2], where the operator $\hat{H}$ was defined by:

$$\hat{H} = \sum_{j,k=1}^n a_{jk} \left(-i \frac{\partial}{\partial x_j} + b_j(\vec{x}) \right) \left(-i \frac{\partial}{\partial x_k} + b_k(\vec{x}) \right) + V(\vec{x}),$$

we assume here that the vector potential $\vec{b} = \{b_k\}$ is equal to zero. It is possible to generalize Theorem 1 to the case where $\vec{b} \neq 0$.

Theorem 2 follows from results of [11] and [6], and from Theorem 1 as indicated in the introduction. (If $Hf = \lambda f$ with $\lambda > 0$, then f has compact support by Theorem XIII.58 of

[11], and consequently $f=0$ by our Theorem 1.) Theorem 1 (a) is deduced from Theorem 3: By the proof of Proposition 4 of [2], the vector f belongs to $D(H_0) \cap D(V)$ and $Hf = H_0 f + VE(A)f$. Let w be an ortho-periodic function such that $w \in L^s_{\text{loc}}(\mathbb{R}^n)$ and $w(\vec{x}) = v(\vec{x})$ for $\vec{x} \in A$. If H_1 denotes the periodic Schrödinger operator $H_1 = H_0 + W$ then $H_1 f = Hf = \lambda f$. Therefore we deduce from Theorem 3 that $f=0$.

To show Theorem 1 (b), let $S = E(A) \cap F(\Sigma)$ (the orthogonal projection with range $E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H}$) and suppose that $f \in \mathcal{H}_p(H)$ satisfies $Sf = f$. f is a linear combination of eigenvectors of H , i. e. $f = \sum_k \alpha_k \cdot g_k$, where $Hg_k = \lambda_k g_k$ with $\lambda_k \in \Sigma$. It follows that:

$$Sf = f = \sum_k \alpha_k Sg_k.$$

Now, by Proposition 2 of [2], S commutes with H ; in particular $HSg_k = SHg_k = \lambda_k Sg_k$. This implies that each Sg_k is an eigenvector of H of compact support in A , hence $Sg_k = 0$ by the part (a) of Theorem 1. We deduce from this that $f = \sum_k \alpha_k Sg_k = 0$. The condition “ Σ bounded” is fundamental: we can choose a potential V such that $\mathcal{H}_p(H) = \mathcal{H}$, i. e. such that the eigenvectors of $\mathcal{H}$ generate $\mathcal{H}$. In this case, we have:

$$\mathcal{H}_p(H) \cap E(A) \mathcal{H} = E(A) \mathcal{H} \neq \{0\}.$$

3. Reduction of the translation group of the lattice

In this part, let v be an ortho-periodic potential. In a natural way, this implies a decomposition of the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$ and of the operators H and H_0 into direct integrals. This decomposition will be used in the next part for the proof of Theorem 3.

The potential v satisfies $v(\vec{x} + \vec{a}_i) = v(\vec{x})$ where $\vec{a}_1, \dots, \vec{a}_n$ are as in (3). The points of the form $\vec{z} = \sum_{i=1}^n q_i \vec{a}_i$, $\vec{q} = \{q_i\} \in \mathbb{Z}^n$, form a cubic lattice in $\mathbb{R}^n$ which is invariant under the translations:

$$\vec{z} \mapsto \vec{z} + \sum_i q'_i \vec{a}_i, \quad \vec{q}' \in \mathbb{Z}^n.$$

In $L^2(\mathbb{R}^n)$, we consider the unitary representation $U(\vec{q})$ of the additive group $\mathbb{Z}^n$ given by:

$$(5) \quad [U(\vec{q})f](\vec{x}) = f(\vec{x} - \sum_i q_i \vec{a}_i) = f(\vec{x} - L\vec{q}),$$

where we have written $\sum_i q_i \vec{a}_i = L\vec{q}$, assuming that the directions of the $\vec{a}_i$ coincide with Cartesian coordinate system.

We also introduce the *reciprocal lattice* which is the set of points of the following form:

$$\vec{z} = \sum_{i=1}^n q_i \vec{e}_i, \quad \vec{q} \in \mathbb{Z}^n,$$

where the vectors $\vec{e}_1, \dots, \vec{e}_n$ are defined by:

$$(6) \quad \vec{e}_i \cdot \vec{a}_k = 2\pi \delta_{ik}.$$

We may write $\vec{z} = E \vec{q}$, with $E = 2\pi L^{-1}$. Let again:

$$\Gamma^n = \left\{ k \in \mathbb{R}^n \mid k = \sum_{i=1}^n \lambda_i e_i, 0 \leq \lambda_i < 1 \right\}.$$

Consider the Hilbert space $\mathcal{G}$ of square-integrable functions $f : \Gamma^n \rightarrow l_n^2 \equiv l^2(\mathbb{Z}^n)$:

$$\mathcal{G} = L^2(\Gamma^n; l_n^2).$$

We write $f(\vec{k})_{\vec{q}}$ for the component $\vec{q}(\vec{q} \in \mathbb{Z}^n)$ of f at the point $\vec{k} \in \mathbb{Z}^n$. Thus, we have:

$$\|f\|_{\mathcal{G}}^2 = \int_{\Gamma^n} dk \sum_{\vec{q} \in \mathbb{Z}^n} |f(\vec{k})_{\vec{q}}|^2.$$

Now, let $\mathcal{U} : \mathcal{H} \rightarrow \mathcal{G}$ be the operator defined by:

$$(7) \quad (\mathcal{U} f)(\vec{k})_{\vec{q}} = \hat{f}(\vec{k} + E \vec{q}),$$

where $\hat{f}$ is the Fourier transform of the function f :

$$\hat{f}(\xi) = (2\pi)^{-n/2} \lim \int_{\mathbb{R}^n} dx \exp(-i\vec{x} \cdot \vec{\xi}) f(\vec{x}).$$

It follows from Plancherel's Theorem that the operator $\mathcal{U}$ is unitary, and its inverse is given by:

$$\mathcal{F}[\mathcal{U}^{-1}\{f(\cdot)\}](\xi) = f(\vec{k})_{\vec{q}},$$

where $\vec{q} \in \mathbb{Z}^n$ and $\vec{k} \in \Gamma^n$ are determined by $\vec{k} + E \vec{q} = \vec{\xi}$. If $\vec{m} \in \mathbb{Z}^n$, one has:

$$(8) \quad [\mathcal{U} U(\vec{m}) f](\vec{k})_{\vec{q}} = \exp(-i L \vec{k} \cdot \vec{m}) (\mathcal{U} f)(\vec{k})_{\vec{q}},$$

i.e. $\mathcal{U} U(m) \mathcal{U}^{-1}$ is diagonalizable in $\mathcal{G}$ (i.e. a multiplication operator by a function of $\vec{k}$). As the functions $\{\exp(i L \vec{k} \cdot \vec{m})\}_{\vec{m} \in \mathbb{Z}^n}$ form a basis of $L^2(\Gamma^n)$, each bounded diagonalizable operator is a function of $\{\mathcal{U} U(m) \mathcal{U}^{-1}\}$. As H_0, V and H commute with every $U(\vec{m})$, these operators commute with each diagonalizable operator, i.e. $\mathcal{U} H_0 \mathcal{U}^{-1}$, $\mathcal{U} V \mathcal{U}^{-1}$ and $\mathcal{U} H \mathcal{U}^{-1}$ are decomposable in $L^2(\Gamma^n; l_n^2)$. Therefore there exist in l_n^2 measurable families of self-adjoint operators $H_0(\vec{k})$, $V(\vec{k})$ and $H(\vec{k})$ ($\vec{k} \in \Gamma^n$) such that, for $f \in D(H_0)$:

$$(9) \quad \begin{cases} (\mathcal{U} H_0 f)(\vec{k}) = H_0(\vec{k}) f(\vec{k}), \\ (\mathcal{U} V f)(\vec{k}) = V(\vec{k}) f(\vec{k}), \\ (\mathcal{U} H f)(\vec{k}) = H(\vec{k}) f(\vec{k}). \end{cases}$$

Now let us give the explicit form and the properties of these three families of operators.

LEMMA 2. — (i) $H_0(\vec{k})$ is the self-adjoint multiplication operator in l_n^2 by $\varphi_{\vec{k}}(\vec{q}) = (\vec{k} + E\vec{q})^2$. If $g = \{g_{\vec{q}}\} \in l_n^2$, then:

$$(H_0(\vec{k})g)_{\vec{q}} = (\vec{k} + E\vec{q})^2 g_{\vec{q}}.$$

(ii) the domain of $D(H_0(\vec{k}))$ is independent of $\vec{k}$ and is given by:

$$D(H_0(\vec{k})) = D_0 = \left\{ g \in l_n^2 \mid \sum_{\vec{q} \in \mathbb{Z}^n} |\vec{q}^2 g_{\vec{q}}|^2 < \infty \right\};$$

(iii) the resolvent $(H_0(\vec{k}) - \mu)^{-1}$ of $H_0(\vec{k})$ is a compact operator for all $\mu \notin \sigma(H_0(\vec{k}))$, where $\sigma(H_0(\vec{k}))$ is the spectrum of $H_0(\vec{k})$.

Proof. — (i) and (ii) are obvious, since:

$$(H_0 f)(\vec{\xi}) = \vec{\xi}^2 \hat{f}(\vec{\xi}).$$

(iii) The resolvent $(H_0(\vec{k}) - \mu)^{-1}$ is the multiplication operator by:

$$\psi(\vec{q}) = [(\vec{k} + E\vec{q})^2 - \mu]^{-1}.$$

Let χ_M be the characteristic function of the set $\{\vec{q} \in \mathbb{Z}^n \mid \vec{q}^2 \leq M\}$ and D_M the multiplication operator by $\psi(\vec{q})\chi_M(\vec{q})$. D_M is a compact (even nuclear) operator, and:

$$(10) \quad \|(H_0(\vec{k}) - \mu)^{-1} - D_M\| = \sup_{\vec{q} > M} [(\vec{k} + E\vec{q})^2 - \mu]^{-1} \rightarrow 0,$$

as $M \rightarrow \infty$. Thus $(H_0(\vec{k}) - \mu)^{-1}$ is compact as the uniform limit of the sequence $\{D_M\}$ of compact operators. ■

Let us denote by $\{\vec{v}_{\vec{q}}\}_{\vec{q} \in \mathbb{Z}^n}$ the Fourier coefficients of the periodic function v :

$$(11) \quad \vec{v}_{\vec{q}} = L^{-n/2} \int_{C^n} dx \exp(-i E \cdot \vec{q} \cdot \vec{x}) v(\vec{x}).$$

Notice that $v \in L^p(C^n)$ for all $p \in [1, s]$. To establish the relation between the Fourier coefficients of v and the operator $V(\vec{k})$ we need the following result:

LEMMA 3. — Given $\varphi, \psi : \mathbb{Z}^n \rightarrow \mathbb{C}$, we define an operator $A_{\varphi\psi} : l_n^2 \rightarrow l_n^2$ as follows:

$$(A_{\varphi\psi}g)_{\vec{q}} = \sum_{\vec{m} \in \mathbb{Z}^n} \varphi(\vec{m}) \psi(\vec{q} - \vec{m}) g_{\vec{q} - \vec{m}}.$$

Assume that $2 \leq p < \infty$, $\psi \in l^p(\mathbb{Z}^n)$ and let $\{\varphi(\vec{q})\}$ be the Fourier coefficients of a function Φ belonging to $L^p(C^n)$. Then $A_{\varphi\psi}$ is a compact operator and one has:

$$(12) \quad \|A_{\varphi\psi}\| \leq L^{-(n/2) - (n/p)} \|\Phi\|_{L^p(C^n)} \|\psi\|_{l^p(\mathbb{Z}^n)}.$$

Proof. — For $g = \{g_{\vec{q}}\} \in l_n^2$, define $\psi g = \{\psi(\vec{q})g_{\vec{q}}\}$. By the Hölder inequality, $\psi g \in l_n^r$ with $r^{-1} = (1/2) + p^{-1}$, i.e. $1 \leq r < 2$, and:

$$\|\psi g\|_r \leq \|\psi\|_p \|g\|_2.$$

Let:

$$\gamma(x) = L^{-n/2} \sum_{q \in \mathbb{Z}^n} \exp(i E \vec{q} \cdot \vec{x}) \psi(\vec{q}) g_{\vec{q}}, \quad x \in \mathbb{C}^n.$$

By the Hausdorff-Young inequality [8], $\gamma \in L^{r'}(\mathbb{C}^n)$ with $(r')^{-1} = 1 - r^{-1} = 1/2 - p^{-1}$ and:

$$(13) \quad \|\gamma\|_{r'} \leq L^{(n/r') - (n/2)} \|\psi g\|_r \leq L^{(n/r') - (n/2)} \|\psi\|_p \|g\|_2.$$

Since $1/2 = p^{-1} + (r')^{-1}$ and $\Phi \in L^p(\mathbb{C}^n)$, the Hölder inequality implies that $\Phi \gamma \in L^2(\mathbb{C}^n)$ and:

$$(14) \quad \|\Phi \gamma\|_2 \leq \|\Phi\|_p \|\gamma\|_{r'} \leq L^{(n/r') - (n/2)} \|\Phi\|_p \|\psi\|_p \|g\|_2.$$

Now:

$$(A_{\Phi\psi} g)_{\vec{q}} = \int_{\mathbb{C}^n} dx \exp(-i E \cdot \vec{q} \cdot \vec{x}) \Phi(\vec{x}) \gamma(\vec{x}),$$

and by Plancherel's theorem we have:

$$(15) \quad \|A_{\Phi\psi} g\|_2 = L^{n/2} \|\Phi \gamma\|_2 \leq L^{n/r'} \|\Phi\|_p \|\psi\|_p \|g\|_2.$$

This shows that $A_{\Phi\psi}$ is defined everywhere with the bound (12) :

(b) Let D_M be the multiplication operator by $\psi_M(\vec{q}) = \psi(\vec{q}) \chi_M(\vec{q})$ (see the proof of Lemma 2). By (a), $A_{\Phi\psi_M}$ is bounded, and $A_{\Phi\psi_M}$ is non-zero only on a subspace of finite dimension. Therefore $A_{\Phi\psi_M}$ is nuclear. By using (12) we obtain:

$$(16) \quad \|A_{\Phi\psi} - A_{\Phi\psi_M}\| \leq L^{(n/2) - (n/p)} \|\Phi\|_p \|(1 - \chi_M)\psi\|_p.$$

Since $\psi \in l_n^p$, $\|(1 - \chi_M)\psi\|_p \rightarrow 0$ as $M \rightarrow \infty$. This proves the compactness of $A_{\Phi\psi}$.

LEMMA 4. — Let Y be the operator in l_n^2 defined by:

$$(17) \quad (Yg)_{\vec{q}} = L^{-n/2} \sum_{\vec{m} \in \mathbb{Z}^n} \hat{v}_{\vec{m}} g_{\vec{q} - \vec{m}}.$$

Then:

- (i) $D_0 \subseteq D(Y)$ and Y is symmetric on D_0 ;
- (ii) Y is relatively compact with respect to $H_0(\vec{k})$;
- (iii) $V(\vec{k}) = Y$ on D_0 , for all $\vec{k} \in \Gamma_n$ (in particular $V(\vec{k})$ is independent of $\vec{k}$);
- (iv) $H(\vec{k}) = H_0(\vec{k}) + Y$ and $D(H(\vec{k})) = D_0$.

Proof. — (i) If $g \in D_0$, then $g = [H(\vec{0}) + 1]^{-1}$ for some $h \in l_n^2$. (15) shows that $\|Yg\|_2 < \infty$, therefore $D_0 \subseteq D(Y)$. By using $\bar{v}_{-\vec{q}} = v_{\vec{q}}$, one obtains easily that $(f, Yg) = (Yf, g)$ for $f, g \in D_0$;

(ii) $Y(H_0(\vec{k}) + 1)^{-1}$ is of the form $A_{\Phi\psi}$, with $\Phi(x) = L^{-n/2} v(\vec{x})$ and $\psi(\vec{q}) = [(k + E \vec{q})^2 + 1]^{-1}$. Notice that $\psi \in l_n^p$ for each $p > n/2$. As $v \in L^s(\mathbb{C}^n)$ for $s = 2$ if $n = 2, 3$ and $s > n/2$ if $n \geq 4$, Lemma 3 implies that $Y(H_0(\vec{k}) + 1)^{-1}$ is compact;

- (iii) this can be verified by calculating the Fourier transform of Vf ;
 (iv) by (i) and (ii), $H_0(\vec{k})$ is self-adjoint. $H(\vec{k})=H_0(\vec{k})+Y$ follows from (iii) and Lemmas 1 and 2.

4. Proof of Theorem 3

Let f be an eigenvector of H , i. e. $Hf=\lambda f$ for some $\lambda \in \mathbb{R}$. By defining $v'(x)=v(x)-\lambda$ and $H'=H_0+V'$, we have $H'f=0$. Since V' satisfies also the hypothesis (2), it is possible to assume without loss of generality that $\lambda=0$.

Let $\Gamma_0=\{\vec{k} \in \Gamma \mid (\mathcal{U}f)(\vec{k}) \neq 0 \text{ in } l_n^2\}$. Γ_0 is measurable. Since $H(\vec{k})(\mathcal{U}f)(\vec{k})=0$, $H(\vec{k})$ must have the eigenvalue 0 for almost all the $\vec{k} \in \Gamma_0$. We will show that, for all $p \in (k_1, \dots, k_{n-1}, 0) \in \mathbb{R}^{n-1}$ the set $\theta(\vec{p})$ of the points $k_n \in (0, E)$ such that $0 \in \sigma(H(\vec{p} + k_n E^{-1} \vec{e}_n))$ is a set of measure zero. Thus the measure of Γ_0 is zero, i. e. $(\mathcal{U}f)(\vec{k})=0$ a. e., i. e. $f=0$. Therefore H cannot have any eigenvalues.

Fix $\vec{p}=(\vec{k}_1, \dots, \vec{k}_{n-1})$. To show that the measure of $\theta(\vec{p})$ is zero, we shall use the Fredholm theory of holomorphic families of operators of type (A), [7]. Let Ω be the following complex domain:

$$(18) \quad \Omega = \{\mathcal{X} + ir \mid \mathcal{X} \in (0, 1), r \in \mathbb{R}\}.$$

For $z \in \Omega$, we define $H_0(\vec{p}, z\vec{e}_n)$ to be the multiplication operator in l_n^2 by $(\vec{p} + z\vec{e}_n + E\vec{q})^2$ and:

$$(19) \quad H(\vec{p}, z\vec{e}_n) = H_0(\vec{p}, z\vec{e}_n) + Y.$$

We shall see that:

(I) $\{H(\vec{p}, z\vec{e}_n)\}$ is a holomorphic family of type (A) with respect to z . (See the terminology in [7]);

(II) the resolvent of $H(\vec{p}, z\vec{e}_n)$ is compact;

(III) the resolvent set of $H(\vec{p}, z\vec{e}_n)$ is not empty.

Under these conditions, Theorem VII.1.10 of [7] says that we have the following alternative:

- either $0 \in \sigma(H(\vec{p}, z\vec{e}_n))$ for each $z \in \Omega$;
- or every compact Ω_0 in Ω contains only a finite number of points z such that $0 \in \sigma(H(\vec{p}, z\vec{e}_n))$.

We shall show that:

(IV) 0 belongs to the resolvent set of $H(\vec{p}, z\vec{e}_n)$ for $\text{Im } z$ sufficiently large. Hence the first alternative is excluded, so that the measure of $\theta(\vec{p})$ is zero.

The remainder of the paper is devoted to the verification of the properties I to IV of $H(\vec{p}, z\vec{e}_n)$. To simplify the notations we write $H(\vec{p}, z)$ for $H(\vec{p}, z\vec{e}_n)$.

LEMMA 5. – (i) $H_0(\vec{p}, z)$ is a self-adjoint holomorphic family of type (A) in Ω with domain $D(H_0(\vec{p}, z))=D_0$;

- (ii) $\forall z \in \Omega$, the resolvent of $H_0(\vec{p}, z)$ is compact;
- (iii) 0 belongs to the resolvent set $\rho(H_0(\vec{p}, z))$ of $H_0(\vec{p}, z)$ for all z with $\text{Im } z \neq 0$.

Proof. — (i) Let $P_j (j=1, \dots, n)$ be the following operator in l_n^2 :

$$(20) \quad P_j g_{\vec{q}} = g_j g_{\vec{q}}.$$

One has:

$$(21) \quad H_0(\vec{p}, z) = (\vec{p} + E\vec{P} + z\vec{e}_n)^2 = (\vec{p} + E\vec{P})^2 + E^2 z^2 + 2E^2 z P_n,$$

and the result is immediate:

- (ii) the proof is the same as in Lemma 2 (iv).
- (iii) for $z = \mathcal{X} + ir$, we have:

$$(22) \quad \text{Im}(\vec{p} + E\vec{q} + z\vec{e}_n)^2 = 2E^2 r(\mathcal{X} + q_n),$$

which is different from zero if $r \neq 0$. Since $q_n \in \mathbb{Z}$ and $\mathcal{X} \in (0, 1)$ it follows that:

$$\| [H_0(\vec{p}, z)]^{-1} \| = \sup_{\vec{q} \in \mathbb{Z}^n} |(\vec{p} + E\vec{q} + z\vec{e}_n)^2|^{-1} < \infty,$$

i. e. $0 \in \rho(H_0(\vec{p}, z))$. ■

LEMMA 6. — (i) $H(\vec{p}, z)$ is a self-adjoint holomorphic family of type (A) in Ω with domain D_0 ;

- (ii) $\forall z \in \Omega$ the resolvent of $H(\vec{p}, z)$ is compact;
- (iii) for all $\vec{p} \in \Gamma^{n-1}$ and $z \in \Omega$, $\rho(H(\vec{p}, z))$ is not empty.

Proof. — (i) this follows from Lemmas 5 (i) and 4 (ii);

(iii) it suffices to show:

$$(23) \quad \lim_{\lambda \rightarrow +\infty} \| Y[H_0(\vec{p}, z) - i\lambda]^{-1} \| = 0,$$

since then the Neumann series for $[H(\vec{p}, z) - i\lambda]^{-1}$, i. e.:

$$(24) \quad [H(\vec{p}, z) - i\lambda]^{-1} = [H_0(\vec{p}, z) - i\lambda]^{-1} \sum_{n=0}^{\infty} \{ -Y[H_0(\vec{p}, z) - i\lambda]^{-1} \}^n,$$

is convergent if λ is sufficiently large. Now, by (12):

$$(25) \quad \| Y[H_0(\vec{p}, z) - i\lambda]^{-1} \| \leq L^{-n/s} \| v \|_s \left\{ \sum_{\vec{q} \in \mathbb{Z}^n} |(\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda|^{-s} \right\}^{1/s}.$$

We have with the notations $z = \mathcal{X} + ir$, $\vec{k} = (\vec{p}, \mathcal{X}\vec{e}_n) \in \Gamma^n$:

$$(26) \quad |(\vec{p} + E\vec{q} + z\vec{e}_n)^2 - i\lambda|^{-2} \leq \{[(\vec{k} + E\vec{q})^2 - E^2 r^2]^2 + 4E^4 r^2 [\mathcal{X} + q_n - \lambda(2E^2 r)^{-1}]^2\}^{-1} \leq [(\vec{k} + E\vec{q})^2 - E^2 r^2]^{-2}.$$

This shows that each term of the sum in (26) converges to zero as $\lambda \rightarrow +\infty$, and that the series in (26) is uniformly majorized in λ by a convergent series (since $s > n/2$). Therefore (23) is proven.

(If z is such that $(\vec{k} + E\vec{q})^2 - E^2 r^2 = 0$ for certain $\vec{q} \in \mathbb{Z}^n$, then there exist $c > 0$ and $\lambda_0 < \infty$ such that $4E^4 r^2 [\mathcal{X} + q_n - \lambda(2E^2 r)^{-1}]^2 \geq c$ for all these $\vec{q}$ and for each $\lambda \geq \lambda_0$. For these values of $\vec{q}$ we can take as majorization in (26) the number c^{-1}).

(ii) Now we use the first and the second resolvent equation:

$$(27) \quad [H(\vec{p}, z) - \xi]^{-1} = [H(\vec{p}, z) - \mu]^{-1} + (\xi - \mu)[H(\vec{p}, z) - \xi]^{-1}[H(\vec{p}, z) - \mu]^{-1}.$$

$$(28) \quad [H(\vec{p}, z) - \mu]^{-1} = [H_0(\vec{p}, z) - \mu]^{-1} - [H(\vec{p}, z) - \mu]^{-1}Y[H_0(\vec{p}, z) - \mu]^{-1}.$$

(27) shows that if $[H(\vec{p}, z) - \mu]^{-1}$ is compact for $\mu \in \rho(H(\vec{p}, z))$ then $[H(\vec{p}, z) - \xi]^{-1}$ is compact for each $\xi \in \rho(H(\vec{p}, z))$. Since $[H_0(\vec{p}, z) - \mu]^{-1}$ and $Y[H_0(\vec{p}, z) - \mu]^{-1}$ are compact if $\mu \in \rho(H_0(\vec{p}, z))$, by (28) it suffices to show that:

$$\rho(H_0(\vec{p}, z)) \cap \rho(H(\vec{p}, z)) \neq \emptyset.$$

We know from (iii) that there exists a point $\mu_0 \in \rho(H(\vec{p}, z))$. If $\mu_0 \notin \rho(H_0(\vec{p}, z))$, there exists a point close to $\mu \in \rho(H_0(\vec{p}, z)) \cap \rho(H(\vec{p}, z))$, since:

(α) $\rho(H(\vec{p}, z))$ is open;

(β) $\sigma(H_0(\vec{p}, z))$ consists of isolated eigenvalues only, because the resolvent of $H_0(\vec{p}, z)$ is compact ([7], Thm. III 6.29).

By Lemma 6 we have verified the properties (I) to (III) of the family $\{H(\vec{p}, z)\}$. It now remains to prove (IV) i.e. $0 \in \rho(H(\vec{p}, z))$ for some $z = \mathcal{X} + ir$ in Ω . We have seen that $0 \in \rho(H_0(\vec{p}, z))$ if $r \neq 0$. We shall show that:

$$(29) \quad \lim_{r \rightarrow \infty} \|Y[H_0(\vec{p}, \mathcal{X} + ir)]^{-1}\| = 0.$$

By using the Neumann series (24) with $\lambda = 0$ and r sufficiently large, (29) implies $0 \in \rho(H(\vec{p}, z))$ if $r = \text{Im } z$ is sufficiently large.

To obtain (29), we use the inequality (25). By virtue of the first inequality in (26), it suffices to show that:

$$(30) \quad \lim_{r \rightarrow \infty} \sum_{\vec{q} \in \mathbb{Z}^n} \left\{ \left[\left(\vec{q} + \frac{\vec{k}}{E} \right)^2 - r^2 \right]^2 + 4r^2 |q_n + \mathcal{X}|^2 \right\}^{-s/2} = 0,$$

which will be done in the next section.

5. Estimation of the series (30)

We now show that (30) holds if $s = 2$ for $n = 2, 3$, $s > n - 2$ for $n \geq 4$ and $\mathcal{X} \in (0, 1)$. We use the following notations:

$$(31) \quad a = 2r |q_n + \mathcal{X}|, \quad b = (q_n + \mathcal{X})^2 - r^2.$$

We set $\vec{p} = E^{-1}(k_1, \dots, k_{n-1}) \in \Gamma_1^{n-1}$, where $\Gamma_1^{n-1} = \{\vec{p} \in \mathbb{R}^{n-1} \mid 0 \leq p_j < 1\}$, and:

$$(32) \quad S(q_n, r) = \sum_{\vec{m} \in \mathbb{Z}^{n-1}} \{[(\vec{m} + \vec{p})^2 + b]^2 + a^2\}^{-s/2}.$$

(30) is then equivalent to:

$$(33) \quad \lim_{r \rightarrow \infty} \sum_{q_n \in \mathbb{Z}} S(q_n, r) = 0.$$

To prove (33), we first give a preliminary estimate in Lemma 7.

LEMMA 7. — Let $\delta > 0$, $c > 0$ and $R > 0$. Then:

$$(34) \quad \varepsilon \equiv \inf_{r \geq R} \inf_{\substack{a \geq \delta r \\ b \geq -r^2 \\ t, z \geq 0 \\ |t-z| \leq c}} \frac{(z^2 + b)^2 + a^2}{(t^2 + b)^2 + a^2} > 0.$$

Proof. — Setting $\alpha = a/r$, $\beta = br^{-2}$, $\sigma = z/r$, $\tau = t/r$ and $\Omega_r = \{(\alpha, \beta, \sigma, \tau) \mid \alpha \geq \delta, \beta \geq -1, \sigma \geq 0, \tau \geq 0, |\sigma - \tau| \leq cr^{-1}\}$, we see that (34) is equivalent to:

$$(35) \quad \varepsilon = \inf_{r \geq R} \inf_{\Omega_r} \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} > 0.$$

The quotient on the r. h. s. of (35) is ≥ 1 if $|\tau^2 + \beta| \leq |\sigma^2 + \beta|$. Hence the infimum is obtained by taking $|\tau^2 + \beta| \geq |\sigma^2 + \beta|$. Under this restriction we have:

$$(36) \quad \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \geq \max \left[\frac{(\sigma^2 + \beta)^2}{(\tau^2 + \beta)^2}, \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + 2(\alpha/r)^2} \right].$$

Also notice the following inequalities, valid on each Ω_r with $r \geq R$:

$$(37) \quad \begin{aligned} \tau^2 + \beta &= [(\tau - \sigma) + \sigma]^2 + \beta \leq 2(\tau - \sigma)^2 + 2\sigma^2 + \beta \\ &= 2(\sigma^2 + \beta) - \beta + 2(\tau - \sigma)^2 \leq 2(\sigma^2 + \beta) + 1 + 2c^2 R^{-2}. \end{aligned}$$

$$(38) \quad |(\sigma^2 + \beta) - (\tau^2 + \beta)| \leq (\sigma + \tau) |\sigma - \tau| \leq (\sigma + \tau) cr^{-1}.$$

(38) implies that:

$$(39) \quad (\tau^2 + \beta)^2 \leq 2(\sigma^2 + \beta)^2 + 2(\sigma + \tau)^2 c^2 r^{-2}.$$

We denote by ε_+ and ε_- the infimum in (35) under the restriction $\sigma^2 + \beta \geq 1$ and $\sigma^2 + \beta \in [-1, +1]$ respectively. It suffices to show that $\varepsilon_+ > 0$ and $\varepsilon_- > 0$. In the first case (i. e. for $\sigma^2 + \beta \geq 1$), we use the first expression on the r. h. s. of (36) and the inequality (37). Setting $x = \sigma^2 + \beta$, we see that:

$$(40) \quad \varepsilon_+ = \inf_{x \geq 1} \frac{x^2}{(2x + 1 + 2c^2 R^{-2})^2} > 0.$$

In the second case (i.e. for $\sigma^2 + \beta \in [-1, +1]$), we have $\sigma^2 \leq 2$, hence $\sigma + \tau \leq 2\sqrt{2} + cR^{-2} \equiv \eta$. After inserting this into (39) and using the second expression on the r.h.s. of (36), one obtains by setting $y = (\sigma^2 + \beta)^2$:

$$(41) \quad \varepsilon_- = \inf_{r \geq R} \inf_{\substack{0 \leq y \leq 1 \\ \alpha \geq \delta}} \frac{y + (\alpha/r)^2}{2y + 2\eta^2 c^2 r^{-2} + 2(\alpha/r)^2} \\ = \inf_{r \geq R} \inf_{\alpha \geq \delta} \frac{(\alpha/r)^2}{2\eta^2 c^2 r^{-2} + 2(\alpha/r)^2} = \frac{\delta^2}{2\eta^2 c^2 + 2\delta^2} > 0. \quad \blacksquare$$

Proof of (33). — Let $\vec{m} \in \mathbb{Z}^{n-1}$ and $\Gamma(\vec{m})$ be the cube:

$$\Gamma(\vec{m}) = \{\vec{x} \in \mathbb{R}^{n-1} \mid \vec{x} = \vec{p} + \vec{m} + \vec{y}, \vec{y} \in \Gamma_1^{n-1}\}.$$

We have $\Gamma(\vec{m}) \cap \Gamma(\vec{m}') = \emptyset$ if $\vec{m} \neq \vec{m}'$ and:

$$\mathbb{R}^{n-1} = \bigcup_{\vec{m} \in \mathbb{Z}^{n-1}} \Gamma(\vec{m}).$$

Let $c = \sqrt{n-1}$. Then for each $\vec{x} \in \Gamma(\vec{m})$ and each $\vec{m} \in \mathbb{Z}^{n-1}$:

$$||\vec{m} + \vec{p}| - |\vec{x}|| \leq c.$$

Let $\delta = 1/2 \min(\mathcal{X}, 1 - \mathcal{X})$. By assumption $\delta > 0$; since $a \geq \delta r$ and $b \geq -r^2$, Lemma 7 implies the existence of a number $\varepsilon > 0$ such that, for each $\vec{m} \in \mathbb{Z}^{n-1}$, each $x \in \Gamma(\vec{m})$, each $a \geq \delta r$ and $b \geq -r^2$ and all $r \geq R$:

$$(42) \quad [(\vec{m} + \vec{p})^2 + b^2] + a^2 \geq \varepsilon[(\vec{x}^2 + b)^2 + a^2].$$

Thus:

$$(43) \quad S(q_n, r) = \sum_{\vec{m} \in \mathbb{Z}^{n-1}} \{[(\vec{m} + \vec{p})^2 + b^2] + a^2\}^{-s/2} \\ = \sum_{\vec{m} \in \mathbb{Z}^{n-1}} \int_{\Gamma(\vec{m})} dx \{[(\vec{m} + \vec{p})^2 + b^2] + a^2\}^{-s/2} \\ \leq \varepsilon^{-1} \sum_{\vec{m}} \int_{\Gamma(\vec{m})} dx \{(\vec{x}^2 + b)^2 + a^2\}^{-s/2} \\ = \varepsilon^{-1} \int_{\mathbb{R}^{n-1}} dx \{(\vec{x}^2 + b)^2 + a^2\}^{-s/2} \\ = \frac{1}{2} \varepsilon^{-1} w_{n-1} \int_0^\infty y^{(n-3)/2} \{(y+b)^2 + a^2\}^{-s/2} dy,$$

where we have introduced spherical polar coordinates, $y = |\vec{x}|^2$ and w_{n-1} denotes the area of the unit sphere in $\mathbb{R}^{n-1}$.

To estimate the integral in (43), we distinguish the two cases $b \geq 0$ and $b < 0$. For $b \geq 0$, we have $\{(y+b)^2 + a^2\}^{-s/2} \leq \{y^2 + a^2 + b^2\}^{-s/2}$, and (43) leads to:

$$S(q_n, r) \leq \frac{1}{2} \varepsilon^{-1} w_{n-1} (a^2 + b^2)^{-s/2 + (n-1)/4} \int_0^\infty z^{(n-3)/2} (z^2 + 1)^{-s/2} dz.$$

Notice that the integral in this expression is convergent since $s > n/2$. By observing that:

$$(44) \quad a^2 + b^2 = [(q + \mathcal{X})^2 + r^2]^2.$$

we obtain:

$$(45) \quad \sum_{|q_n + \mathcal{X}| \geq r} S(q_n, r) \leq Cte \sum_{|q_n + \mathcal{X}| \geq r} |q_n + \mathcal{X}|^{-2s+n-1}.$$

The hypothesis $s > n/2$ implies that the last series is convergent so that this term tends to zero as $r \rightarrow \infty$.

We now turn to the case $b < 0$. We set $z = (y+b)/a$. (43) then gives:

$$(46) \quad S(q_n, r) \leq \frac{1}{2} \varepsilon^{-1} w_{n-1} a^{-s+1} \int_{b/a}^{+\infty} (az-b)^{(n-3)/2} \{1+z^2\}^{-s/2} dz.$$

If $n \geq 3$, this leads to:

$$(47) \quad S(q_n, r) \leq c_1 a^{-s+1} \int_{-r}^{+\infty} [|az|^{(n-3)/2} + |b|^{(n-3)/2}] \{1+z^2\}^{-s/2} dz \\ \leq c_2 a^{-s+1} [|a|^{(n-3)/2} + |b|^{(n-3)/2}] \leq c_3 a^{-s+1} (a^2 + b^2)^{(n-3)/4}.$$

Using (47), (44) and (31), we obtain in this case that:

$$\sum_{|q_n + \mathcal{X}| < r} S(q_n, r) \leq c_4 r^{-s+1} r^{n-3} \sum_{|q_n + \mathcal{X}| < r} |q_n + \mathcal{X}|^{-s+1} = \mathcal{O}(r^{-s+n-2} \log r),$$

since $s \geq 2$. Under the hypothesis $s > n-2$, this converges to zero as $r \rightarrow \infty$.

Finally, if $n=2$, one may bound the integral in (46) by a constant which is independent of a and b on the set $\{a \geq a_0 > 0, b < 0\}$; this is easily achieved by splitting the domain of integration into $\{z \mid az-b \leq 1\} \cup \{z \mid az-b > 1\}$. Thus:

$$S(q_n, r) \leq c_5 r^{-s+1} |q_n + \mathcal{X}|^{-s+1}, \quad \forall q_n, \quad \forall r \geq r_0.$$

For any $s > 3/2$, this implies that:

$$\lim_{r \rightarrow \infty} \sum_{|q_n + \mathcal{X}| < r} S(q_n, r) = 0. \quad \blacksquare$$

Remark 3. — One sees from the preceding proof that, for $n=3$, the limit in (33) is zero under the weaker hypothesis that $s > 3/2$. By using a modified resolvent equation, one obtains the result of Theorem 1 for $s > 3/2$. The case $s=2, n=3$ was first treated by Thomas

in [12]. Similarly, for $n=2$, a more careful estimate of the integral in (46) shows that it suffices to require $s > 1$.

Remark 4. — Theorem 3 remains true if the condition of ortho-periodicity of v is replaced by the weaker condition of periodicity. Indeed, the estimation of the series given in section 5, may be applied if, instead of $\vec{a}_i \cdot \vec{a}_j = \delta_{ij}$, one requires only that $\vec{a}_i \cdot \vec{a}_n = \delta_{in}$ (i.e. the vector $\vec{a}_n$ is orthogonal to the hyperplane spanned by $\vec{a}_1, \dots, \vec{a}_{n-1}$). Clearly the direction $\vec{a}_n$ is distinguished in our estimation. A similar result for an arbitrary periodic lattice is given in Theorem XIII.100 of [11], under a more restrictive assumption on the local behaviour of the function $v(\vec{x})$ than that of Theorem 3.

Remark 5. — We also have the following result which generalizes Theorem 1:

THEOREM 1'. — Let $v \in L^s_{\text{loc}}(\mathbb{R}^n \setminus N)$, where s satisfies $s=2$ if $n=1, 2, 3$ and $s > n-2$ if $n \geq 4$, and where N is a closed set of measure zero. Let H be a self-adjoint extension of $\hat{H}$, $D(\hat{H}) = C_0^\infty(\mathbb{R}^n \setminus N)$. Suppose that $f \in L^2(\mathbb{R}^n)$ satisfies $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $E(A)f = f$ for some compact subset of $\mathbb{R}^n \setminus N$ (i.e. f is an eigenvector of H having compact support in $\mathbb{R}^n \setminus N$). Then $f=0$.

Proof. — One has $\chi_A(\cdot) v(\cdot) \in L^s(\mathbb{R}^n)$. Let C be a cube in $\mathbb{R}^n$ such that $A \subseteq C$. Define w by:

$$w(\vec{x} + \sum q_i \vec{a}_i) = \chi_A(\vec{x}) v(\vec{x}), \quad \vec{x} \in C,$$

w is ortho-periodic and in $L^s_{\text{loc}}(\mathbb{R}^n)$. Since $(H_0 + w)f = \lambda f$, one has $f=0$ by Theorem 3. ■

Remark 6. — The hypothesis “ Σ bounded” in Theorem 1(b) is essential. Assume for example that v is such that $H_0 + v$ has pure point spectrum (e.g. $v(\vec{x}) \rightarrow +\infty$ as $|\vec{x}| \rightarrow \infty$). Take $\Sigma = \mathbb{R}$. Then:

$$F(\Sigma) \mathcal{H} = \mathcal{H} \quad \text{and} \quad E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = E(A) \mathcal{H}.$$

Since $E(A) \mathcal{H} \neq \{0\}$ if A has positive measure, it is clear that one cannot have $E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = \{0\}$ in this case.

Remark 7. — By combining our Theorem 1 with Proposition 4 of [2], one may also prove that $E(A) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\}$ under assumptions of Theorem 1(b).

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