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ON THE POINT SPECTRUM OF SCHRÖDINGER OPERATORS

BY ANNE BERTHIER

1. Introduction

This paper is an extension of a work [2] on the spectral analysis of partial differential operators of Schrödinger type. The problem was the following: Let $A$ be a compact subset of $\mathbb{R}^n$, $\Sigma$ a finite interval in $\mathbb{R}$ and $H$ a self-adjoint elliptic differential operator in the complex Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. We define $F(\Sigma)$ to be the spectral projection of $H$ associated with the interval $\Sigma$ and $E(A)$ the multiplication operator by the characteristic function $\chi_A$ of $A$. Do there exist vectors in $L^2(\mathbb{R}^n)$ which are contained both in the range $E(A)\mathcal{H}$ of $E(A)$ and in $F(\Sigma)\mathcal{H}$?

It turns out that the closed subspace $\mathcal{H}_p(H)$ generated by the set of eigenvectors of $H$ plays a different role from the subspace $\mathcal{H}_c(H) = \mathcal{H}_p(H)^\perp$ associated with the continuous spectrum of $H$. Notice that it is shown in [2], under regularity and integrability conditions on the coefficients of the differential operator, that there do not exist vectors of $\mathcal{H}_c(H)$ which belong both to $E(A)\mathcal{H}$ and to $F(\Sigma)\mathcal{H}$. On the other hand, to prove the non-existence of vectors in $\mathcal{H}_p(H)$ belonging to $E(A)\mathcal{H}$ and $F(\Sigma)\mathcal{H}$, we used an unique continuation theorem for solutions of the differential equation associated with $H$. Now, if for example $H = -\Delta + V$, where $V$ is the multiplication operator by a real function $v(x)$, the known results on unique continuation require a condition $L^\infty(\mathbb{R}^n\setminus N)$ on $v$, where $N$ is a closed set of measure zero such that $\mathbb{R}^n\setminus N$ is connected ([3], [5]).

In the present paper, we propose to show that:

(1) $\mathcal{H}_p(H) \cap E(A)\mathcal{H} \cap F(\Sigma)\mathcal{H} = \{0\}$,

by imposing only an integrability condition on the function $v$. More precisely, we will prove (1) under the hypothesis that $v \in L^s_{\text{loc}}(\mathbb{R}^n)$ with $s = 2$ if $n = 1, 2, 3$ and $s > n - 2$ if $n \geq 4$.

This result shows that, under the above conditions on $v$, the operator $-\Delta + v$ has no eigenvector with compact support. This is essentially the content of our Theorem 1 in paragraph 2. (In the case $n = 1$, one obtains ordinary differential operators for which results of this type have been known for a long time [9]).
This result is also interesting from the point of view of "non-existence of positive eigenvalues of the operator $H$". In the literature (for example [2], [12]) the non-existence of positive eigenvalues is obtained in two steps:

(i) under suitable decay conditions at infinity on the function $v$, it is shown that all eigenfunctions $f$ associated with a strictly positive eigenvalue of $H$ have compact support;

(ii) then one imposes suitable local conditions on $v$ (e.g. $v \in L^\infty_\text{loc} (\mathbb{R}^n \setminus N)$ in order to apply the unique continuation theorem, which then leads to $f \equiv 0$. It turns out that the non-existence of positive eigenvalues is also obtained by assuming in (ii) a local condition that $v \in L^s_\text{loc} (\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$ and $s>n-2$ if $n \geq 4$ (Thm. 2).

Finally our method implies also the spectral continuity of a class of Schrödinger operators with periodic potentials $v(x)$.

The organization of the paper is as follows: first we give the principal results and deduce Theorems 1 and 2 from Theorem 3 in section 2, and we introduce a direct integral representation of Schrödinger operators in section 3. This representation will be used in section 4 for proving Theorem 3. The principal estimate of the proof is the subject of the last section 5.

2. Statements of the results

Let $v : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We always suppose that:

(2) $v \in L^s_\text{loc} (\mathbb{R}^n)$ with $s=2$ if $n=1, 2, 3$; $s>n-2$ if $n \geq 4$.

Notice that $s>n-2$ in all cases.

The function $v$ will be called periodic if there exist $n$ linearly independent vectors $\vec{a}_1, \ldots, \vec{a}_n \in \mathbb{R}^n$ such that $v(x+\vec{a}_i)=v(x)$ for all $x \in \mathbb{R}^n$. A periodic function will be called ortho-periodic if:

(3) $\vec{a}_j, \vec{a}_k = L^2 \delta_{jk},$

with $L>0$, i.e. if the vectors of the form $\sum_{i=1}^n \alpha_i \vec{a}_i$, $0 \leq \alpha_i < 1$, define a cube $C^n$ with side $L$.

We denote by $\hat{H}$ the symmetric operator:

(4) $\hat{H} = -\Delta + v(x),$

with domain $D(\hat{H})=C_0^\infty (\mathbb{R}^n)$ and by $H_0$ the unique self-adjoint extension of $\hat{H}_0 = -\Delta$, $D(H_0)=C_0^\infty (\mathbb{R}^n)$. Let $H$ a self-adjoint extension of $\hat{H}$. We have the following lemma:

**Lemma 1.** Assume that (2) and one of the following conditions are satisfied:

(i) $v$ is periodic;

(ii) $v \in L^\infty (\mathbb{R}^n)$ where $B_R = \{ x \in \mathbb{R}^n | |x| \leq R \}$ and $B_R$ denotes the complement of $B_R$.

Then:

(a) $v$ is $H_\infty$-bounded with $H_\infty$-bound $0$;

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(b) $\hat{H}$ is essentially self-adjoint;
(c) $D(H) = D(H_0)$, where $H$ is the unique self-adjoint extension of $H$.

**Proof.** — (b) and (c) follow from (a) by using the Kato-Rellich Theorem ([7], Chapt. 5.4.1). Under hypothesis (i), (a) follows from Theorem XIII.96 of [11], whereas under the assumption (ii), (a) can be proved by the method used in the proof of Lemma 3 in [10]. Both cases are treated in [4].

We now state our principal results. In Theorem 2 we choose as conditions on the potential $v$ at infinity those used in [4].

**Theorem 1.** — Let $v \in L^{1,\infty}(\mathbb{R}^n)$ with $s$ satisfying (2) and let $H$ be a self-adjoint extension of $H$:
(a) suppose that $f \in L^{2}(\mathbb{R}^n)$ satisfies $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $E(A)f = f$ for some compact subset $A$ of $\mathbb{R}^n$. (i.e. $f$ is an eigenvector of $H$ with compact support in $\mathbb{R}^n$). Then $f = 0$;
(b) for each compact subset $A$ of $\mathbb{R}^n$ and each bounded interval $\Sigma$, one has:
$$\mathcal{H}_P(H) \cap E(A) \cap F(\Sigma) = \{0\}.$$

**Theorem 2.** — Suppose that:
(i) $v \in L^s(\mathbb{R}_R)$ with $s$ satisfying (2) for some $R < \infty$;
(ii) $v = v_1 + v_2$ such that:
(α) $v_1, v_2 \in L^{\infty}(\mathbb{R}_R),$
(β) $|v(x)|v_1(x) \to 0$ as $|x| \to \infty,$
(γ) $v_2(x) \to 0$ as $|x| \to \infty,$
(δ) $r \mapsto v_2(r, \cdot)$ is differentiable as a function from $(R, \infty)$ to $L^\infty(S^{s-1}),$ and
$$\limsup_{r \to \infty} v_2(r, \cdot) \leq 0.$$  
($S^{s-1}$ denotes the unit sphere in $\mathbb{R}^n.$)

Then $H = H_0 + V$ has no eigenvalues in $(0, \infty)$.

**Theorem 3.** — Let $v$ be ortho-periodic and $v \in L_1^{1,\infty}(\mathbb{R}^n)$ with $s$ satisfying (2). Then the spectrum of $H = H_0 + V$ is purely continuous.

**Remark 1.** — By following the proof of Theorem XIII.100 in [11], it is possible to show that the operator $H$ in Theorem 3 is absolutely continuous. Other comments on Theorem 3 will be made at the end of this paper.

**Remark 2.** — Contrarily to [2], where the operator $\hat{H}$ was defined by:
$$\hat{H} = \sum_{j,k=1}^n a_{jk} \left( -i \frac{\partial}{\partial x_j} + b_j(\tilde{x}) \right) \left( -i \frac{\partial}{\partial x_k} + b_k(\tilde{x}) \right) + V(\tilde{x}),$$
we assume here that the vector potential $\tilde{b} = \{b_k\}$ is equal to zero. It is possible to generalize Theorem 1 to the case where $\tilde{b} \neq 0$.

Theorem 2 follows from results of [11] and [6], and from Theorem 1 as indicated in the introduction. (If $Hf = \lambda f$ with $\lambda > 0$, then $f$ has compact support by Theorem XIII.58 of
[11], and consequently \( f = 0 \) by our Theorem 1.) Theorem 1 (a) is deduced from Theorem 3: By the proof of Proposition 4 of [2], the vector \( f \) belongs to \( D(H_0) \cap D(V) \) and \( H f = H_0 f + V E(A) f \). Let \( w \) be an ortho-periodic function such that \( w \in L^2_{\text{loc}}(\mathbb{R}^n) \) and \( w(\tilde{x}) = v(\tilde{x}) \) for \( \tilde{x} \in A \). If \( H_1 \) denotes the periodic Schrödinger operator \( H_1 = H_0 + W \) then \( H_1 f = H f = \lambda f \). Therefore we deduce from Theorem 3 that \( f = 0 \).

To show Theorem 1(b), let \( S = E(A) \cap F(S) \) (the orthogonal projection with range \( E(A) \cap F(S) \)) and suppose that \( f \in \mathcal{H}_p(H) \) satisfies \( S f = f \). \( f \) is a linear combination of eigenvectors of \( H \), i.e. \( f = \sum \alpha_k g_k \), where \( H g_k = \lambda_k g_k \) with \( \lambda_k \in \Sigma \). It follows that:

\[
S f = f = \sum \alpha_k S g_k.
\]

Now, by Proposition 2 of [2], \( S \) commutes with \( H \); in particular \( H S g_k = S H g_k = \lambda_k S g_k \). This implies that each \( S g_k \) is an eigenvector of \( H \) of compact support in \( A \), hence \( S g_k = 0 \) by the part (a) of Theorem 1. We deduce from this that \( f = \sum \alpha_k S g_k = 0 \). The condition "\( \Sigma \) bounded" is fundamental: we can choose a potential \( V \) such that \( \mathcal{H}_p(H) = \mathcal{H} \), i.e. such that the eigenvectors of \( H \) generate \( \mathcal{H} \). In this case, we have:

\[
\mathcal{H}_p(H) \cap E(A) \mathcal{H} = E(A)  \mathcal{H} \neq \{0\}.
\]

### 3. Reduction of the translation group of the lattice

In this part, let \( v \) be an ortho-periodic potential. In a natural way, this implies a decomposition of the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n) \) and of the operators \( H \) and \( H_0 \) into direct integrals. This decomposition will be used in the next part for the proof of Theorem 3.

The potential \( v \) satisfies \( v(\tilde{x} + \tilde{a}_i) = v(\tilde{x}) \) where \( \tilde{a}_1, \ldots, \tilde{a}_n \) are as in (3). The points of the form \( \tilde{z} = \sum_{i=1}^n q_i \tilde{a}_i, \tilde{q} = \{q_i\} \in \mathbb{Z}^n \), form a cubic lattice in \( \mathbb{R}^n \) which is invariant under the translations:

\[
\tilde{z} \mapsto \tilde{z} + \sum_i q_i \tilde{a}_i, \quad \tilde{q} \in \mathbb{Z}^n.
\]

In \( L^2(\mathbb{R}^n) \), we consider the unitary representation \( U(\tilde{q}) \) of the additive group \( \mathbb{Z}^n \) given by:

\[
[U(\tilde{q}) f](\tilde{x}) = f(\tilde{x} - \sum_i q_i \tilde{a}_i) = f(\tilde{x} - L \tilde{q}),
\]

where we have written \( \sum_i q_i \tilde{a}_i = L \tilde{q} \), assuming that the directions of the \( \tilde{a}_i \) coincide with Cartesian coordinate system.

We also introduce the **reciprocal lattice** which is the set of points of the following form:

\[
\tilde{z} = \sum_{i=1}^n q_i \tilde{e}_i, \quad \tilde{q} \in \mathbb{Z}^n,
\]

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where the vectors $\tilde{e}_1, \ldots, \tilde{e}_n$ are defined by:

\begin{equation}
\tilde{e}_j, \tilde{a}_k = 2\pi \delta_{jk}.
\end{equation}

We may write $\tilde{z} = E \tilde{q}$, with $E = 2\pi L^{-1}$. Let again:

\begin{equation}
\Gamma^n = \left\{ k \in \mathbb{R}^n \mid k = \sum_{i=1}^n \lambda_i e_i, \ 0 \leq \lambda_i < 1 \right\}.
\end{equation}

Consider the Hilbert space $\mathcal{H}$ of square-integrable functions $f : \Gamma^n \to l^2_n \equiv l^2(\mathbb{Z}^n)$:

\begin{equation}
\mathcal{H} = L^2(\Gamma^n; l^2_n).
\end{equation}

We write $f(\tilde{k})$ for the component $\tilde{q}(\tilde{q} \in \mathbb{Z}^n)$ of $f$ at the point $\tilde{k} \in \mathbb{Z}^n$. Thus, we have:

\begin{equation}
\|f\|^2 = \int_{\Gamma^n} \sum_{\tilde{k} \in \mathbb{Z}^n} |f(\tilde{k})|^2.
\end{equation}

Now, let $\mathcal{U} : \mathcal{H} \to \mathcal{H}$ be the operator defined by:

\begin{equation}
(\mathcal{U} f)(\tilde{k}) = f(\tilde{k} + E \tilde{q}),
\end{equation}

where $\tilde{f}$ is the Fourier transform of the function $f$:

\begin{equation}
\tilde{f}(\tilde{\xi}) = (2\pi)^{-n/2} \lim_{\Gamma^n} \int_{\mathbb{R}^n} dx \exp(-i\tilde{x} \cdot \tilde{\xi}) f(\tilde{x}).
\end{equation}

It follows from Plancherel's Theorem that the operator $\mathcal{U}$ is unitary, and its inverse is given by:

\begin{equation}
\mathcal{U}^{-1} \{ f(\cdot) \}(\tilde{\xi}) = f(\tilde{k}),
\end{equation}

where $\tilde{q} \in \mathbb{Z}^n$ and $\tilde{k} \in \Gamma^n$ are determined by $\tilde{k} + E \tilde{q} = \tilde{\xi}$. If $m \in \mathbb{Z}^n$, one has:

\begin{equation}
[\mathcal{U} U(m)](\tilde{k}) = \exp(-i\tilde{k} \cdot \tilde{m})(\mathcal{U} f)(\tilde{k}),
\end{equation}

i.e. $\mathcal{U} U(m) \mathcal{U}^{-1}$ is diagonalizable in $\mathcal{H}$ (i.e. a multiplication operator by a function of $\tilde{k}$). As the functions $\{ \exp(i\tilde{k} \cdot \tilde{m}) \}_{m \in \mathbb{Z}^n}$ form a basis of $L^2(\Gamma^n)$, each bounded diagonalizable operator is a function of $\{ \mathcal{U} U(m) \mathcal{U}^{-1} \}$. As $H_0, V$ and $H$ commute with every $U(m)$, these operators commute with each diagonalizable operator, i.e. $\mathcal{U} H_0 \mathcal{U}^{-1}, \mathcal{U} V \mathcal{U}^{-1}$ and $\mathcal{U} H \mathcal{U}^{-1}$ are decomposable in $L^2(\Gamma^n; l^2_n)$. Therefore there exist in $l^2_n$ measurable families of self-adjoint operators $H_0(\tilde{k}), V(\tilde{k})$ and $H(\tilde{k})(\tilde{k} \in \Gamma^n)$ such that, for $f \in D(H_0)$:

\begin{equation}
\begin{cases}
(\mathcal{U} H_0 f)(\tilde{k}) = H_0(\tilde{k}) f(\tilde{k}), \\
(\mathcal{U} V f)(\tilde{k}) = V(\tilde{k}) f(\tilde{k}), \\
(\mathcal{U} H f)(\tilde{k}) = H(\tilde{k}) f(\tilde{k}).
\end{cases}
\end{equation}

Now let us give the explicit form and the properties of these three families of operators.
LEMMA 2. — (i) $H_0(k)$ is the self-adjoint multiplication operator in $l^2_n$ by $\varphi_\xi(\xi) = (k + E \xi)^2$: If $g = \left\{ g_\xi \right\} \in l^2_n$, then:

$$ (H_0(k)g) = (k + E \xi)^2 g_\xi $$

(ii) the domain of $D(H_0(k))$ is independent of $k$ and is given by:

$$ D(H_0(k)) = \left\{ g \in l^2_n : \sum_{\xi \in \mathbb{Z}^n} \left| q^2 g_\xi \right|^2 < \infty \right\} $$

(iii) the resolvent $(H_0(k) - \mu)^{-1}$ of $H_0(k)$ is a compact operator for all $\mu \not\in \sigma(H_0(k))$, where $\sigma(H_0(k))$ is the spectrum of $H_0(k)$.

Proof. — (i) and (ii) are obvious, since:

$$ (H_0 f)(\xi) = \xi^2 f(\xi) $$

(iii) The resolvent $(H_0(k) - \mu)^{-1}$ is the multiplication operator by:

$$ \psi(\xi) = \left( (k + E \xi)^2 - \mu \right)^{-1} $$

Let $\chi_M$ be the characteristic function of the set $\left\{ \xi \in \mathbb{Z}^n : q^2 \leq M \right\}$ and $D_M$ the multiplication operator by $\psi(\xi) \chi_M(\xi)$. $D_\mu$ is a compact (even nuclear) operator, and:

$$ \| (H_0(k) - \mu)^{-1} - D_M \| = \sup_{\xi^2 \geq M} \left| \left( (k + E \xi)^2 - \mu \right)^{-1} \right| \to 0, $$

as $M \to \infty$. Thus $(H_0(k) - \mu)^{-1}$ is compact as the uniform limit of the sequence $\left\{ D_M \right\}$ of compact operators.

Let us denote by $\left\{ \tilde{v}_\xi \right\}_{\xi \in \mathbb{Z}^n}$ the Fourier coefficients of the periodic function $v$:

$$ \tilde{v}_\xi = \mathcal{L}^{-n/2} \int_{\mathbb{C}^n} dx \exp(-iE \xi \cdot \bar{x}) v(x). $$

Notice that $v \in L^p(\mathbb{C}^n)$ for all $p \in [1, s]$. To establish the relation between the Fourier coefficients of $v$ and the operator $V(k)$ we need the following result:

LEMMA 3. — Given $\varphi, \psi : \mathbb{Z}^n \to \mathbb{C}$, we define an operator $A_\varphi : l^2_n \to l^2_n$ as follows:

$$ A_\varphi g = \sum_{m \in \mathbb{Z}^n} \varphi(\xi) \psi(\xi - m) g_{\xi - m}. $$

Assume that $2 \leq p < \infty$, $\psi \in l^p(\mathbb{Z}^n)$ and let $\left\{ \varphi(\xi) \right\}$ be the Fourier coefficients of a function $\Phi$ belonging to $L^p(\mathbb{C}^n)$. Then $A_\varphi$ is a compact operator and one has:

$$ \| A_\varphi \| \leq \mathcal{L}^{-\left( n/2 \right) - \left( n/p \right)} \| \Phi \|_{L^p(\mathbb{C}^n)} \| \psi \|_{l^p(\mathbb{Z}^n)}. $$

Proof. — For $g = \left\{ g_\xi \right\} \in l^2_n$, define $g = \left\{ \psi(\xi) g_\xi \right\}$. By the Hölder inequality, $g \in l^r_n$ with $r^{-1} = (1/2) + p^{-1}$, i.e. $1 \leq r < 2$, and:

$$ \| g \|_r \leq \| \psi \|_p \| g \|_2. $$
Let:

\[ \gamma(x) = L^{-n/2} \sum_{q \in \mathbb{Z}_n} \exp(i \xi \cdot \tilde{x}) \psi(q) g_{\tilde{q}}, \quad x \in \mathbb{C}^n. \]

By the Hausdorff-Young inequality [8], \( \gamma \in L^r(\mathbb{C}^n) \) with \( r^{-1} = 1 - r^{-1} = 1/2 - p^{-1} \) and:

\begin{equation}
\| \gamma \|_r \leq L^{(n/r) - (n/2)} \| \psi \|_r \leq L^{(n/r) - (n/2)} \| \psi \|_p \| g \|_2.
\end{equation}

Since \( 1/2 = p^{-1} + (r')^{-1} \) and \( \Phi \in L^p(\mathbb{C}^n) \), the Hölder inequality implies that \( \Phi \gamma \in L^2(\mathbb{C}^n) \) and:

\begin{equation}
\| \Phi \gamma \|_2 \leq \| \Phi \|_p \| \gamma \|_r \leq L^{(n/r) - (n/2)} \| \Phi \|_p \| \psi \|_p \| g \|_2.
\end{equation}

Now:

\[ (A_{\psi\psi} g)_{\tilde{q}} = \int_{\mathbb{C}^n} dx \exp(-i \xi \cdot \tilde{x}) \Phi(\tilde{x}) \gamma(\tilde{x}), \]

and by Plancherel’s theorem we have:

\begin{equation}
\| A_{\psi\psi} g \|_2 = L^{n/2} \| \Phi \gamma \|_2 \leq L^{n/r} \| \Phi \|_p \| \psi \|_p \| g \|_2.
\end{equation}

This shows that \( A_{\psi\psi} \) is defined everywhere with the bound (12):

(b) Let \( D_M \) be the multiplication operator by \( \psi_M(\tilde{q}) = \psi(q) \chi_M(q) \) (see the proof of Lemma 2). By (a), \( A_{\psi\psi} \) is bounded, and \( A_{\psi\psi} \) is non-zero only on a subspace of finite dimension. Therefore \( A_{\psi\psi} \) is nuclear. By using (12) we obtain:

\begin{equation}
\| A_{\psi\psi} - A_{\psi\psi} \| \leq L^{(n/2) - (n/p)} \| \Phi \|_p \| (1 - \chi_M) \Phi \|_p.
\end{equation}

Since \( \psi \in L^p \), \( \| (1 - \chi_M) \psi \|_p \to 0 \) as \( M \to \infty \). This proves the compactness of \( A_{\psi\psi} \).

**Lemma 4.** — Let \( Y \) be the operator in \( l^2 \) defined by:

\begin{equation}
(Y g)_{\tilde{q}} = L^{-n/2} \sum_{\tilde{m} \in \mathbb{Z}_n} \tilde{v}_{\tilde{m}} g_{\tilde{q} \cdot \tilde{m}}.
\end{equation}

Then:

(i) \( D_0 \subseteq D(Y) \) and \( Y \) is symmetric on \( D_0 \);
(ii) \( Y \) is relatively compact with respect to \( H_0(\tilde{k}) \);
(iii) \( V(\tilde{k}) = Y \) on \( D_0 \), for all \( \tilde{k} \in \Gamma_n \) (in particular \( V(\tilde{k}) \) is independent of \( \tilde{k} \));
(iv) \( H(\tilde{k}) = H_0(\tilde{k}) + Y \) and \( D(H(\tilde{k})) = D_0 \).

**Proof.** — (i) If \( g \in D_0 \), then \( g = [H(\tilde{0}) + 1]^{-1} \) for some \( h \in l^2 \). (15) shows that \( \| Y g \|_2 < \infty \), therefore \( D_0 \subseteq D(Y) \). By using \( \tilde{v}_{\tilde{q} \cdot \tilde{m}} = \tilde{v}_{\tilde{m}} \), one obtains easily that \( (f, Y g) = (Y f, g) \) for \( f, g \in D_0 \);
(ii) \( Y(H_0(\tilde{k}) + 1)^{-1} \) is of the form \( A_{\psi\psi} \), with \( \Phi(x) = L^{-n/2} \psi(\tilde{x}) \) and \( \psi(q) = [(k + E \tilde{q})^2 + 1]^{-1} \). Notice that \( \psi \in L^p \) for each \( p > n/2 \). As \( v \in L^p(\mathbb{C}^n) \) for \( s = 2 \) if \( n = 2, 3 \) and \( s > n/2 \) if \( n \geq 4 \), Lemma 3 implies that \( Y(H_0(\tilde{k}) + 1)^{-1} \) is compact.
(iii) this can be verified by calculating the Fourier transform of $Vf$.

(iv) by (i) and (ii), $H_0(\tilde{k})$ is self-adjoint. $H(\tilde{k}) = H_0(\tilde{k}) + \gamma$ follows from (iii) and Lemmas 1 and 2.

### 4. Proof of Theorem 3

Let $f$ be an eigenvector of $H$, i.e. $Hf = \lambda f$ for some $\lambda \in \mathbb{R}$. By defining $v'(x) = v(x) - \lambda$ and $H' = H_0 + V'$, we have $H'f = 0$. Since $V'$ satisfies also the hypothesis (2), it is possible to assume without loss of generality that $\lambda = 0$.

Let $\Gamma_0 = \{ \tilde{k} \in \Gamma \mid (\mathcal{U}f)(\tilde{k}) \neq 0 \text{ in } l_\mathbb{R}^n \}$. $\Gamma_0$ is measurable. Since $H(\tilde{k})(\mathcal{U}f)(\tilde{k}) = 0$, $H(\tilde{k})$ must have the eigenvalue 0 for almost all $\tilde{k} \in \Gamma_0$. We will show that, for all $p \in (k_1, \ldots, k_{n-1}, 0) \in \mathbb{R}^{n-1}$ the set $\theta(p)$ of the points $k_n \in (0, E)$ such that $0 \in \sigma(H(p + k_n E^{-1} \tilde{e}_n))$ is a set of measure zero. Thus the measure of $\Gamma_0$ is zero, i.e. $\sigma((\mathcal{U}f)(\tilde{k})) = 0$ a.e., i.e. $\lambda = 0$. Therefore $H'$ cannot have any eigenvalues.

Fix $\tilde{p} = (\tilde{k}_1, \ldots, \tilde{k}_{n-1})$. To show that the measure of $\theta(\tilde{p})$ is zero, we shall use the Fredholm theory of holomorphic families of operators of type (A), [7]. Let $\Omega$ be the following complex domain:

\begin{equation}
\Omega = \{ \mathcal{X} + ir \mid \mathcal{X} \in (0, 1), r \in \mathbb{R} \}.
\end{equation}

For $z \in \Omega$, we define $H_0(\tilde{p}, z\tilde{e}_n)$ to be the multiplication operator in $l_\mathbb{R}^n$ by $(\tilde{p} + z\tilde{e}_n + E \tilde{q})^2$ and:

\begin{equation}
H(\tilde{p}, z\tilde{e}_n) = H_0(\tilde{p}, z\tilde{e}_n) + \gamma.
\end{equation}

We shall see that:

1. $\{H(\tilde{p}, z\tilde{e}_n)\}$ is a holomorphic family of type (A) with respect to $z$. (See the terminology in [7]);

2. the resolvent of $H(\tilde{p}, z\tilde{e}_n)$ is compact;

3. the resolvent set of $H(\tilde{p}, z\tilde{e}_n)$ is not empty.

Under these conditions, Theorem VII.1.10 of [7] says that we have the following alternative:

- either $0 \in \sigma(H(\tilde{p}, z\tilde{e}_n))$ for each $z \in \Omega$;

- or every compact $\Omega_0$ in $\Omega$ contains only a finite number of points $z$ such that $0 \in \sigma(H(\tilde{p}, z\tilde{e}_n))$.

We shall show that:

- $0 \in \sigma(H(\tilde{p}, z\tilde{e}_n))$ for $\text{Im} z$ sufficiently large. Hence the first alternative is excluded, so that the measure of $\theta(\tilde{p})$ is zero.

The remainder of the paper is devoted to the verification of the properties I to IV of $H(\tilde{p}, z\tilde{e}_n)$. To simplify the notations we write $H(\tilde{p}, \tilde{z})$ for $H(\tilde{p}, z\tilde{e}_n)$.

**Lemma 5.** (i) $H_0(\tilde{p}, z)$ is a self-adjoint holomorphic family of type (A) in $\Omega$ with domain $D(H_0(\tilde{p}, z)) = D_0$;
(ii) \( \forall z \in \Omega \), the resolvent of \( H_0(\vec{p}, z) \) is compact;

(iii) \( 0 \) belongs to the resolvent set \( \sigma(H_0(\vec{p}, z)) \) of \( H_0(\vec{p}, z) \) for all \( z \) with \( \text{Im} \, z \neq 0 \).

**Proof.** — (i) Let \( P_j (j = 1, \ldots, n) \) be the following operator in \( L^2 \):

\[
P_j \phi = g_j \phi.
\]

One has:

\[
H_0(\vec{p}, z) = (\vec{p} + E \vec{q} + z \vec{e}_n)^2 = (\vec{p} + E \vec{q})^2 + E^2 z^2 + 2 E^2 z \vec{p}.
\]

and the result is immediate:

(ii) the proof is the same as in Lemma 2 (iv).

(iii) for \( z = \Re + ir \), we have:

\[
\text{Im}(\vec{p} + E \vec{q} + z \vec{e}_n)^2 = 2 E^2 r (\Re + q_n),
\]

which is different from zero if \( r \neq 0 \). Since \( q_n \in \mathbb{Z} \) and \( \Re \in (0, 1) \) it follows that:

\[
||[H_0(\vec{p}, z)]^{-1}|| = \sup_{q \in \mathbb{Z}} |(\vec{p} + E \vec{q} + z \vec{e}_n)^2|^{-1} < \infty,
\]

i.e. \( 0 \in \sigma(H_0(\vec{p}, z)) \). \( \Box \)

**Lemma 6.** — (i) \( H(\vec{p}, z) \) is a self-adjoint holomorphic family of type (A) in \( \Omega \) with domain \( D_0 \);

(ii) \( \forall z \in \Omega \) the resolvent of \( H(\vec{p}, z) \) is compact;

(iii) for all \( \vec{p} \in \Gamma^{*} \) and \( z \in \Omega \), \( \sigma(H(\vec{p}, z)) \) is not empty.

**Proof.** — (i) this follows from Lemmas 5 (i) and 4 (ii);

(iii) it suffices to show:

\[
\lim_{\lambda \to +\infty} ||Y[H_0(\vec{p}, z) - i \lambda]^{-1}|| = 0,
\]

since then the Neumann series for \( [H(\vec{p}, z) - i \lambda]^{-1} \), i.e.:

\[
[H(\vec{p}, z) - i \lambda]^{-1} = [H_0(\vec{p}, z) - i \lambda]^{-1} \sum_{n=0}^{\infty} \{ -Y[H_0(\vec{p}, z) - i \lambda]^{-1} \}_{n},
\]

is convergent if \( \lambda \) is sufficiently large. Now, by (12):

\[
||Y[H_0(\vec{p}, z) - i \lambda]^{-1}|| \leq L^{-n} \|v\| \sum_{q \in \mathbb{Z}} |(\vec{p} + E \vec{q} + z \vec{e}_n)^2 - i \lambda|^{-1},
\]

We have with the notations \( z = \Re + ir, \vec{k} = (\vec{p}, \Re \vec{e}_n) \in \Gamma^{*} \):

\[
|(\vec{p} + E \vec{q} + z \vec{e}_n)^2 - i \lambda|^2 \leq \left\{ \left[ (\vec{k} + E \vec{q})^2 - E^2 r^2 \right]^2 + 4 E^4 r^2 [\Re^2 + q_n - \lambda (2 E^2 r)^{-1}]^{-1} \right\}^{-1} \leq \left[ (\vec{k} + E \vec{q})^2 - E^2 r^2 \right]^{-2}.
\]
This shows that each term of the sum in (26) converges to zero as \( \lambda \to +\infty \), and that the series in (26) is uniformly majorized in \( \lambda \) by a convergent series (since \( s > n/2 \)). Therefore (23) is proven.

(If \( z \) is such that \((k+\bar{q})^2-E^2 \overset{2}{=}-0\) for certain \( \bar{q} \in \mathbb{Z}^n \), then there exist \( c > 0 \) and \( \lambda_0 < \infty \) such that \( 4 \overset{2}{E}^2 \overset{2}{[\bar{x} + q_n - \lambda(2 \overset{2}{E}^2 \overset{2}{r})] \overset{2}{\geq} c \) for all these \( \bar{q} \) and for each \( \lambda \geq \lambda_0 \). For these values of \( \bar{q} \) we can take as majorization in (26) the number \( c^{-1} \).

(ii) Now we use the first and the second resolvent equation:

\[
\begin{align*}
(27) \quad [\mathcal{H}(\bar{p}, z) - \xi]^{-1} &= [\mathcal{H}(\bar{p}, z) - \mu]^{-1} + (\xi - \mu)[\mathcal{H}(\bar{p}, z) - \xi]^{-1} [\mathcal{H}(\bar{p}, z) - \mu]^{-1}, \\
(28) \quad [\mathcal{H}(\bar{p}, z) - \mu]^{-1} &= [\mathcal{H}_0(\bar{p}, z) - \mu]^{-1} - [\mathcal{H}(\bar{p}, z) - \mu]^{-1} \mathcal{Y}[\mathcal{H}_0(\bar{p}, z) - \mu]^{-1}.
\end{align*}
\]

(27) shows that if \([\mathcal{H}(\bar{p}, z) - \mu]^{-1}\) is compact for \( \mu \in \rho(\mathcal{H}(\bar{p}, z)) \) then \([\mathcal{H}(\bar{p}, z) - \xi]^{-1}\) is compact for each \( \xi \in \rho(\mathcal{H}(\bar{p}, z)) \). Since \([\mathcal{H}_0(\bar{p}, z) - \mu]^{-1}\) and \(\mathcal{Y}[\mathcal{H}_0(\bar{p}, z) - \mu]^{-1}\) are compact if \( \mu \in \rho(\mathcal{H}_0(\bar{p}, z)) \), by (28) it suffices to show that:

\[\rho(\mathcal{H}_0(\bar{p}, z)) \cap \rho(\mathcal{H}(\bar{p}, z)) \neq \emptyset.\]

We know from (iii) that there exists a point \( \mu_0 \in \rho(\mathcal{H}(\bar{p}, z)) \). If \( \mu_0 \notin \rho(\mathcal{H}_0(\bar{p}, z)) \), there exists a point close to \( \mu \in \rho(\mathcal{H}_0(\bar{p}, z)) \cap \rho(\mathcal{H}(\bar{p}, z)) \), since:

(1) \( \rho(\mathcal{H}(\bar{p}, z)) \) is open;

(2) \( \sigma(\mathcal{H}_0(\bar{p}, z)) \) consists of isolated eigenvalues only, because the resolvent of \( \mathcal{H}_0(\bar{p}, z) \) is compact (I.3, Thm. III 6.29).

By Lemma 6 we have verified the properties (I) to (III) of the family \( \{\mathcal{H}(\bar{p}, z)\} \). It now remains to prove (IV) i.e. \( 0 \in \rho(\mathcal{H}(\bar{p}, z)) \) for some \( z = \bar{x} + ir \) in \( \Omega \). We have seen that \( 0 \in \rho(\mathcal{H}_0(\bar{p}, z)) \) if \( r \neq 0 \). We shall show that:

\[\lim_{r \to \infty} \| \mathcal{Y}[\mathcal{H}_0(\bar{p}, \bar{x} + ir)]^{-1} \| = 0.\]

By using the Neumann series (24) with \( \lambda = 0 \) and \( r \) sufficiently large, (29) implies \( 0 \in \rho(\mathcal{H}(\bar{p}, z)) \) if \( r = \text{Im} \ z \) is sufficiently large.

To obtain (29), we use the inequality (25). By virtue of the first inequality in (26), it suffices to show that:

\[
\begin{align*}
\lim_{r \to \infty} \sum_{q \in \mathbb{Z}^n} \left\{ \left( \frac{q + \frac{k}{E}}{r^2} \right)^2 + 4 r^2 \left| q_n + \bar{x} \right|^2 \right\}^{-s/2} &= 0,
\end{align*}
\]

which will be done in the next section.

5. Estimation of the series (30)

We now show that (30) holds if \( s = 2 \) for \( n = 2, 3, s > n - 2 \) for \( n \geq 4 \) and \( \bar{x} \in (0, 1) \). We use the following notations:

\[
\begin{align*}
a &= 2 r \left| q_n + \bar{x} \right|, \quad b = (q_n + \bar{x})^2 - r^2.
\end{align*}
\]
We set \( \tilde{p} = E^{-1}(k_1, \ldots, k_{n-1}) \in \Gamma_{-1}^{n-1} \), where \( \Gamma_{-1}^{n-1} = \{ \tilde{p} \in \mathbb{R}^{n-1} \mid 0 \leq p_j < 1 \} \), and:

(32) \[ S(q_m, r) = \sum_{m \in \mathbb{Z}} \frac{[|m+p|^2 + b^2]^2 + a^2}{r^s}. \]

(30) is then equivalent to:

(33) \[ \lim_{r \to \infty} \sum_{q_n \in \mathbb{Z}} S(q_m, r) = 0. \]

To prove (33), we first give a preliminary estimate in Lemma 7.

**Lemma 7.** Let \( \delta > 0, \, c > 0 \) and \( R > 0 \). Then:

(34) \[ \epsilon \equiv \inf_{r \geq R} \inf_{a \geq \delta} \inf_{b \geq -\delta} \left( \frac{(z^2 + b)^2 + a^2}{(t^2 + b)^2 + a^2} \right) > 0. \]

Proof. Setting \( \alpha = a/r, \beta = b/r^2, \sigma = z/r, \tau = t/r \) and \( \Omega_r = \{ (\alpha, \beta, \sigma, \tau) \mid \alpha \geq \delta, \beta \geq -1, \sigma \geq 0, \tau \geq 0, |\sigma - \tau| \leq cr^{-1} \} \), we see that (34) is equivalent to:

(35) \[ \epsilon = \inf_{r \geq R} \inf_{\Omega_r} \left( \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \right) > 0. \]

The quotient on the r.h.s. of (35) is \( \geq 1 \) if \( |\tau^2 + \beta| \leq |\sigma^2 + \beta| \). Hence the infimum is obtained by taking \( |\tau^2 + \beta| \geq |\sigma^2 + \beta| \). Under this restriction we have:

(36) \[ \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \geq \max \left[ \frac{(\sigma^2 + \beta)^2}{(\tau^2 + \beta)^2}, \frac{(\sigma^2 + \beta)^2 + (\alpha/r)^2}{(\tau^2 + \beta)^2 + (\alpha/r)^2} \right]. \]

Also notice the following inequalities, valid on each \( \Omega_r \) with \( r \geq R \):

(37) \[ \tau^2 + \beta = [(|\tau - \sigma| + \sigma)]^2 + \beta \leq 2(\tau - \sigma)^2 + 2\sigma^2 + \beta \]
\[ = 2(\sigma^2 + \beta) - \beta + 2(\tau - \sigma)^2 \leq 2(\sigma^2 + \beta) + 1 + 2c^2 R^{-2}. \]

(38) \[ |(\sigma^2 + \beta) - (\tau^2 + \beta)| \leq (\sigma + \tau) |\sigma - \tau| \leq (\sigma + \tau) cr^{-1}. \]

(38) implies that:

(39) \[ (\tau^2 + \beta)^2 \leq 2(\sigma^2 + \beta)^2 + 2(\sigma + \tau)^2 c^2 r^{-2}. \]

We denote by \( \epsilon_+ \) and \( \epsilon_- \) the infimum in (35) under the restriction \( \sigma^2 + \beta \geq 1 \) and \( \sigma^2 + \beta \in [-1, +1] \) respectively. It suffices to show that \( \epsilon_+ > 0 \) and \( \epsilon_- > 0 \). In the first case (i.e. for \( \sigma^2 + \beta \geq 1 \)), we use the first expression on the r.h.s. of (36) and the inequality (37). Setting \( x = \sigma^2 + \beta \), we see that:

(40) \[ \epsilon_+ = \inf_{x \geq 1} \frac{x^2}{(2x + 1 + 2c^2 R^{-2})^2} > 0. \]
In the second case (i.e. for \( \sigma^2 + \beta \in [-1, +1] \)), we have \( \sigma^2 \leq 2 \), hence \( \sigma + \tau \leq 2 \sqrt{2} + c R^{-2} \equiv \eta. \) After inserting this into (39) and using the second expression on the r.h.s. of (36), one obtains by setting \( y = (\sigma^2 + \beta)^2 \):

\[
(41) \quad \epsilon_- = \inf_{r \geq R} \inf_{0 \leq y \leq 1} \frac{y + (\alpha/r)^2}{2y + 2 \eta^2 c^2 r^{-2} + 2(\alpha/r)^2} = \inf_{r \geq R} \inf_{x \leq \delta} \frac{(\alpha/r)^2}{2 \eta^2 c^2 r^{-2} + 2(\alpha/r)^2} = \frac{\delta^2}{2 \eta^2 c^2 + 2 \delta^2} > 0.
\]

**Proof of (33).** — Let \( \vec{m} \in \mathbb{Z}^{n-1} \) and \( \Gamma(\vec{m}) \) be the cube:

\[
\Gamma(\vec{m}) = \{ \vec{x} \in \mathbb{R}^{n-1} \mid \vec{x} = \vec{p} + \vec{m} + \vec{y}, \vec{y} \in \Gamma_1^{-1} \}.
\]

We have \( \Gamma(\vec{m}) \cap \Gamma(\vec{m}') = \emptyset \) if \( \vec{m} \neq \vec{m}' \) and:

\[
\mathbb{R}^{n-1} = \bigcup_{\vec{m} \in \mathbb{Z}^{n-1}} \Gamma(\vec{m}).
\]

Let \( c = \sqrt{n-1} \). Then for each \( \vec{x} \in \Gamma(\vec{m}) \) and each \( \vec{m} \in \mathbb{Z}^{n-1} \):

\[
||\vec{m} + \vec{p}|| - ||\vec{x}|| \leq c.
\]

Let \( \delta = 1/2 \min (\vec{a}, 1 - \vec{a}) \). By assumption \( \delta > 0 \); since \( a \geq \delta r \) and \( b \geq -r^2 \), Lemma 7 implies the existence of a number \( \epsilon > 0 \) such that, for each \( \vec{m} \in \mathbb{Z}^{n-1} \), each \( \vec{x} \in \Gamma(\vec{m}) \), each \( a \geq \delta r \) and \( b \geq -r^2 \) and all \( r \geq R \):

\[
(42) \quad [(\vec{m} + \vec{p})^2 + b^2] + a^2 \geq \epsilon [(\vec{x}^2 + b)^2 + a^2].
\]

Thus:

\[
(43) \quad S(q_n, r) = \sum_{\vec{m} \in \mathbb{Z}^{n-1}} [(\vec{m} + \vec{p})^2 + b^2] + a^2 - \frac{s}{2} = \frac{1}{2} \epsilon^{-1} w_{n-1} \int_0^\infty y^{(n-3)/2} [(y + b)^2 + a^2]^{-s/2} dy,
\]

where we have introduced spherical polar coordinates, \( y = ||\vec{x}||^2 \) and \( w_{n-1} \) denotes the area of the unit sphere in \( \mathbb{R}^{n-1} \).
To estimate the integral in (43), we distinguish the two cases $b \geq 0$ and $b < 0$. For $b \geq 0$, we have $(y + b)^2 + a^2 \leq y^2 + a^2 + b^2$, and (43) leads to:

$$S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} (a^2 + b^2)^{-s/2} \int_{0}^{\infty} z^{(n-3)/2} (z^2 + 1)^{-s/2} dz.$$  

Notice that the integral in this expression is convergent since $s > n/2$. By observing that:

$$a^2 + b^2 = [(q + \mathcal{X})^2 + r^2],$$  

we obtain:

$$\sum_{|q_n + \mathcal{X}| \geq r} S(q_n, r) \leq C \sum_{|q_n + \mathcal{X}| \geq r} |q_n + \mathcal{X}|^{-2s+n-1}.$$  

The hypothesis $s > n/2$ implies that the last series is convergent so that this term tends to zero as $r \to \infty$.

We now turn to the case $b < 0$. We set $z = (y + b)/a$. (43) then gives:

$$S(q_n, r) \leq \frac{1}{2} e^{-1} w_{n-1} a^{-s+1} \int_{b/a}^{+\infty} (az - b)^{(n-3)/2} (1 + z^2)^{-s/2} dz.$$  

If $n \geq 3$, this leads to:

$$S(q_n, r) \leq c_2 a^{-s+1} \int_{b/a}^{+\infty} (az - b)^{(n-3)/2} (1 + z^2)^{-s/2} dz$$

$$\leq c_3 a^{-s+1} (a^2 + b^2)^{(n-3)/4}.$$  

Using (47), (44) and (31), we obtain in this case that:

$$\sum_{|q_n + \mathcal{X}| < r} S(q_n, r) \leq c_4 r^{-s+1} \sum_{|q_n + \mathcal{X}| < r} |q_n + \mathcal{X}|^{-s+1} = C (r^{-s+n-2} \log r),$$

since $s \geq 2$. Under the hypothesis $s > n - 2$, this converges to zero as $r \to \infty$.

Finally, if $n = 2$, one may bound the integral in (46) by a constant which is independent of $a$ and $b$ on the set $\{a \geq a_0 > 0, b < 0 \}$; this is easily achieved by splitting the domain of integration into $\{z \mid az - b \leq 1\} \cup \{z \mid az - b > 1\}$. Thus:

$$S(q_n, r) \leq c_5 r^{-s+1} |q_n + \mathcal{X}|^{-s+1}, \quad \forall q_n, \forall r \geq r_0.$$  

For any $s > 3/2$, this implies that:

$$\lim_{r \to \infty} \sum_{|q_n + \mathcal{X}| < r} S(q_n, r) = 0.$$  

Remark 3. — One sees from the preceding proof that, for $n = 3$, the limit in (33) is zero under the weaker hypothesis that $s > 3/2$. By using a modified resolvent equation, one obtains the result of Theorem 1 for $s > 3/2$. The case $s = 2, n = 3$ was first treated by Thomas.
Remark 4. — Theorem 3 remains true if the condition of ortho-periodicity of \( v \) is replaced by the weaker condition of periodicity. Indeed, the estimation of the series given in section 5, may be applied if, instead of \( \hat{a}_i, \hat{a}_j = \delta_{ij} \), one requires only that \( \hat{a}_i, \hat{a}_n = \delta_{in} \) (i.e. the vector \( \hat{a}_n \) is orthogonal to the hyperplane spanned by \( \hat{a}_1, \ldots, \hat{a}_{n-1} \)). Clearly the direction \( \hat{a}_n \) is distinguished in our estimation. A similar result for an arbitrary periodic lattice is given in Theorem XIII.100 of [11], under a more restrictive assumption on the local behaviour of the function \( v(\vec{x}) \) than that of Theorem 3.

Remark 5. — We also have the following result which generalizes Theorem 1:

**Theorem 5'.** — Let \( v \in L^\infty_{\text{loc}}(\mathbb{R}^n \setminus N) \), where \( s \) satisfies \( s = 2 \) if \( n = 1, 2, 3 \) and \( s > n - 2 \) if \( n \geq 4 \), and where \( N \) is a closed set of measure zero. Let \( H \) be a self-adjoint extension of \( \hat{H} \), \( D(\hat{H}) = C_0^\infty(\mathbb{R}^n \setminus N) \). Suppose that \( f \in L^2(\mathbb{R}^n) \) satisfies \( Hf = \lambda f \) for some \( \lambda \in \mathbb{R} \) and \( E(\Lambda)f = f \) for some compact subset of \( \mathbb{R}^n \setminus N \) (i.e. \( f \) is an eigenvector of \( H \) having compact support in \( \mathbb{R}^n \setminus N \)). Then \( f = 0 \).

**Proof.** — One has \( \chi_A(\cdot)v(\cdot) \in L^s(\mathbb{R}^n) \). Let \( C \) be a cube in \( \mathbb{R}^n \) such that \( A \subseteq C \). Define \( w \) by:

\[
w(x + \sum q_i \hat{a}_j) = \chi_A(\vec{x})v(\vec{x}), \quad \vec{x} \in C,
\]

\( w \) is ortho-periodic and in \( L^s_{\text{loc}}(\mathbb{R}^n) \). Since \( (H_0 + w)f = \lambda f \), one has \( f = 0 \) by Theorem 3.

Remark 6. — The hypothesis "\( \Sigma \) bounded" in Theorem 1(b) is essential. Assume for example that \( v \) is such that \( H_0 + v \) has pure point spectrum (e.g. \( v(\vec{x}) \to +\infty \) as \( |\vec{x}| \to \infty \)). Take \( \Sigma = \mathbb{R} \). Then:

\[
F(\Sigma) \mathcal{H} = \mathcal{H} \quad \text{and} \quad E(\Lambda) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = E(\Lambda) \mathcal{H}.
\]

Since \( E(\Lambda) \mathcal{H} \neq \{0\} \) if \( A \) has positive measure, it is clear that one cannot have \( E(\Lambda) \mathcal{H} \cap F(\Sigma) \mathcal{H} \cap \mathcal{H}_p(H) = \{0\} \) in this case.

Remark 7. — By combining our Theorem 1 with Proposition 4 of [2], one may also prove that \( E(\Lambda) \mathcal{H} \cap F(\Sigma) \mathcal{H} = \{0\} \) under assumptions of Theorem 1(b).

**References**


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