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SECOND ORDER LINEAR DIFFERENTIAL SYSTEMS

BY F. NEUMAN

I. – Introduction

We shall deal with second order linear differential systems

$$\text{(Q)} \qquad y'' = Q(t)\, y,$$

where n by n real symmetric continuous matrices $Q : \mathbf{R} \to \mathbf{R}^{n^2}$ satisfy

$$Q(t+\pi) = PQ(t)P^{-1}$$

for a constant orthogonal matrix P. We shall derive a sufficient condition under which all solutions of (Q) comply with

$$(1) \qquad y(t+\pi) = P\, y(t),$$

and we shall construct some (Q) of the property (1). If $P = \pm I$ (I denoting the unit matrix), all solutions of (Q) are periodic or half-periodic. For the case we shall construct an example of two-dimensional system (Q) having only half-periodic solutions so that Q is not diagonalizable, i.e., it is not of the form

$$C^{-1} \operatorname{diag}(q_1, \ldots, q_n)\, C,$$

C being a real constant regular n by n matrix, and q_i are scalar functions such that all solutions of

$$y'' = q_i(t)\, y$$

are half-periodic. For constructing such q_i (*see* [5], pp. 573-589).

Systems (Q) with solutions satisfying (1) are in close connection with investigations in differential geometry, especially with Blaschke's conjecture *see* [1], pp. 225-230.

The problem considered here was proposed by Professor M. Berger.

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II. – Notations and basic properties

For an integer $m \geqq 0$, let $C^m(J, \mathbf{R}^{n^2})$ denote the set of all matrices $T: J \to \mathbf{R}^{n^2}$, $J \subset \mathbf{R}$, having continuous derivatives up to and including the m-th order. T^* means the tranpose of T, ′ denotes d/dt. Throughout this paper the matrix Q in (Q) is supposed to be continuous on $\mathbf{R}: Q \in C^0(\mathbf{R}, \mathbf{R}^{n^2})$.

If Y_1 and Y_2 are two matrix-solutions of (Q) on $\mathbf{R}$ such that the $2n$ by $2n$ matrix $\begin{pmatrix} Y_1 & Y_2 \\ Y_1' & Y_2' \end{pmatrix}$ is regular at least at some t_0 (then it is regular on $\mathbf{R}$), then $Y_1(t)C_1 + Y_2(t)C_2$ is a general matrix-solution of (Q), C_1 and C_2 being arbitrary constant n by n matrices.

For each solution Y of (Q) with symmetric Q, $Q^* = Q$, the expression $Y^*(t)Y'(t) - Y^{*\prime}(t)Y(t)$ is a constant matrix, say C. If $C = 0$ (the null matrix), then Y is called isotropic. For each isotropic solution Y of (Q) such that Y is regular on an interval J, the matrix

$$Y(t)\int_d^t Y^{-1}(s)Y^{*-1}(s)\,ds, \qquad d \in J,$$

is a solution of (Q) on J, *see* e.g. [2] or [3].

LEMMA 1. – *Let* Y *be a solution of* (Q) *satisfying* $Y(a) = 0$, $Y'(a)$ *being regular. Then there exists a neighbourhood* V *of* a *such that* $Y(t)$ *is regular on* $V - \{a\}$.

Remark 1. – We need not suppose the symmetry of Q for the Lemma. However, if $Q^* = Q$, then the Y in Lemma 1 is isotropic.

Proof. – If such a V does not exist, there is a sequence $\{t_i\}_{i=1}^{\infty}$, $t_i \neq a$, $t_i \to a$ as $i \to \infty$, such that $\det Y(t_i) = 0$. Because of the continuity of det as a function of n^2 variables, we have

$$\det Y'(a) = \det\{\lim_{i\to\infty} [Y(t_i) - Y(a)].[t_i - a]^{-1}\}$$

$$= \lim_{i\to\infty} \det\{[Y(t_i) - Y(a)].[t_i - a]^{-1}\}$$

$$= \lim_{i\to\infty} (t_i - a)^{-n} \det Y(t_i) = 0,$$

that contradicts the regularity of $Y'(a)$. ■

LEMMA 2. – *Suppose* $Q^* = Q$. *Let a solution* Y_1 *of* (Q) *satisfy:* $Y_1(a) = 0$, $Y_1'(a)$ *is regular. Let* Y_1 *be regular on* (a, b). *For*

$$Y_2(t) := Y_1(t)\int_d^t Y_1^{-1}(s)Y_1^{*-1}(s)\,ds, \qquad d \in (a, b),$$

the expression $Y_1(t)C_1 + Y_2(t)C_2$ *is a general solution of* (Q) *on* (a, b).

Proof. — It is sufficient to show that

$$\begin{pmatrix} Y_1(t); & Y_1(t)\int_d^t Y_1^{-1}(s)Y_1^{*-1}(s)\,ds \\ Y_1'(t); & Y_1'(t)\int_d^t Y_1^{-1}(s)Y_1^{*-1}(s)\,ds+Y_1^{*-1}(t) \end{pmatrix}$$

is regular at least at some $t_0\in(a, b)$. For $t_0=d$ we get

$$\begin{pmatrix} Y_1(d); & 0 \\ Y_1'(d); & Y_1^{*-1}(d) \end{pmatrix},$$

whose determinant is $\det Y_1(d)$. $\det Y^{*-1}(d)=1$. ■

III. — Sufficient condition for $y(t+\pi)=Py(t)$

Suppose that a matrix-solution Y_1 of (Q), $Q^*=Q$,

$$Q(t+\pi)=PQ(t)P^{-1}, \tag{2}$$

P being a real constant orthogonal matrix, satisfies:

$$Y_1(a)=0, \qquad Y_1'(a) \text{ is regular,}$$
$$Y_1(t) \text{ is regular on } (a, a+\pi),$$
$$Y_1(t+\pi)=PY_1(t).$$

Evidently $Y_1\in C^2(\mathbf{R}, \mathbf{R}^{n^2})$, and $a+\pi$ is the first conjugate point to a, [2]. The matrix

$$Y_2:\ t\mapsto Y_1(t)\int_d^t Y_1^{-1}(s)Y_1^{*-1}(s)\,ds, \qquad d\in(a, a+\pi),$$

is also a solution of (Q) on $(a, a+\pi)$. Let $\overline{Y}_2\in C^2(\mathbf{R}, \mathbf{R}^{n^2})$ denote the (unique) continuation of Y_2. Due to Lemma 2 every solution y of (Q) satisfies (1) if and only if

$$\overline{Y}_2(t+\pi)=P\overline{Y}_2(t) \quad \text{on } \mathbf{R}. \tag{3}$$

Because of the uniqueness of solutions, the relation (3) holds if and only if

$$\overline{Y}_2(a+\pi)=P\overline{Y}(a) \qquad \text{and} \qquad \overline{Y}_2'(a+\pi)=P\overline{Y}_2'(a).$$

Since $\overline{Y}_2(t)=Y_2(t)$ on $(a,a+\pi)$, and $\overline{Y}_2\in C^2(\mathbf{R},\mathbf{R}^{n^2})$, there exist

$$\lim_{t\to a_+} Y_2(t)=\overline{Y}_2(a), \qquad \lim_{t\to a+\pi_-} Y_2(t)=\overline{Y}_2(a+\pi),$$
$$\lim_{t\to a_+} Y_2'(t)=\overline{Y}_2'(a), \qquad \lim_{t\to a+\pi_-} Y_2'(t)=\overline{Y}_2'(a+\pi).$$

Hence (3) holds iff both

(4) $$\lim_{t\to a+\pi_-} Y_2(t)=P\lim_{t\to a_+} Y_2(t),$$

(5) $$\lim_{t\to a+\pi_-} Y_2'(t)=P\lim_{t\to a_+} Y_2(t).$$

Define

$$A(t):=Y_1(t).\sin^{-1}(t-a)\qquad\text{for}\quad t\in(a+k\pi,\ a+\overline{k+1}\,\pi),$$
$$A(t):=(-P)^k Y_1'(a)\qquad\text{for}\quad t=a+k\pi,\quad k=0,\ \pm 1,\ \ldots;$$

$\sin^{-k}s$ denoting $(\sin s)^{-k}$ throughout this paper. We have

$$\lim_{t\to a+k\pi} A(t)=(-P)^k Y_1'(a),\qquad \lim_{t\to a+k\pi} A'(t)=0,$$
$$\lim_{t\to a+k\pi} A''(t)=\frac{1}{3}(-P)^k(Q(a)+I)Y_1'(a).$$

Hence $A\in C^2(\mathbf{R},\mathbf{R}^{n^2})$, $A(t+\pi)=-PA(t)$, A being regular on the whole $\mathbf{R}$. Using l'Hospital rule we get

$$\lim_{t\to a_+} Y_2(t)=\lim_{t\to a_+} A(t)\frac{\int_d^t (A^*(s)A(s))^{-1}\sin^{-2}(s-a)\,ds}{\sin^{-1}(t-a)}$$
$$=A(a)\lim_{t\to a_+}\frac{(A^*(t)A(t))^{-1}}{-\cos(t-a)}=-A^{*-1}(a),$$

and

$$\lim_{t\to a+\pi_-} Y_2(t)=\lim_{a\to a+\pi_-} A(a+\pi)\frac{(A^*(t)A(t))^{-1}}{-\cos(t-a)}=-PA^{*-1}(a).$$

Thus the condition (4) gives no further restriction on A. For (5) we have:

$$\lim_{t\to a_+} Y_2'(t)=\lim_{t\to a_+}\Bigg\{(A(t)\sin(t-a))'\int_d^t\frac{(A^*(s)A(s))^{-1}-(A^*(a)A(s))^{-1}}{\sin^2(s-a)}ds$$
$$+(A(t)\sin(t-a))'(A^*(a)A(a))^{-1}[\operatorname{ctg}(d-a)-\operatorname{ctg}(t-a)]+A^{*-1}(t)\sin^{-1}(t-a)\Bigg\}$$
$$=A(a)\int_d^a\frac{(A^*(s)A(s))^{-1}-(A^*(a)A(a))^{-1}}{\sin^2(s-a)}ds+A^{*-1}(a)\operatorname{ctg}(d-a),$$

because of

$$\lim_{t\to a_+}[-(A(t)\sin(t-a))'(A^*(a)A(a))^{-1}\operatorname{ctg}(t-a)+A^{*-1}(t)\sin^{-1}(t-a)]=0.$$

Analogously

$$\lim_{t \to a+\pi_-} Y_2'(t) = PA(a) \int_d^{a+\pi} \frac{(A^*(s) A(s))^{-1} - (A^*(a) A(a))^{-1}}{\sin^2(s-a)} \times ds + PA^{*-1}(a) \operatorname{ctg}(d-a).$$

Due to our conditions on A the expression

$$\frac{(A^*(s) A(s))^{-1} - (A^*(a) A(a))^{-1}}{\sin^2(s-a)}$$

has limits both for $t \to a$ and for $t \to a+\pi$, hence the above definite integrals are well defined and we may equivalently rewrite the condition (5) as

$$(6) \qquad \int_a^{a+\pi} \frac{(A^*(t) A(t))^{-1} - (A^*(a) A(a))^{-1}}{\sin^2(t-a)} dt = 0.$$

Let us summarize our considerations in:

THEOREM. — *Let* $Q^* = Q$, $a \in R$, Y_1 *be a matrix-solution of* (Q) *such that* $Y_1(a) = 0$, $Y_1'(a)$ *is regular*, $Y_1(t+\pi) = PY_1(t)$ *for an orthogonal constant matrix* P, Y_1 *being regular on* $(a, a+\pi)$ (*or equivalently*, $a+\pi$ *being the* 1st *conjugate point to* a).

Then

$$Y_1(t) = A(t) \sin(t-a),$$

where

$$A \in C^2(\mathbf{R}, \mathbf{R}^{n^2}), \quad A \textit{ is regular on } \mathbf{R},$$

$$(7) \qquad A(t+\pi) = -PA(t), \qquad A(a) = Y_1'(a), \qquad A'(a) = 0,$$

and

$$(8) \qquad Q(t) = A''(t) A^{-1}(t) + 2 A'(t) A^{-1}(t) \operatorname{ctg}(t-a) - I.$$

Moreover, every solution y *of* (Q) *satisfies* (1) *if and only if* (6) *holds.*

Remark 2. $A'(t) A^{-1}(t) \operatorname{ctg}(t-a)$ in (8) is continuous by defining its value at $a+k\pi$ as $P^k A''(a) A^{-1}(a) P^{-k}$.

Remark 3. — We may always take Y_1 normalized by $Y_1'(a) = I$ that gives $A(a) = I$ and

$$(9) \qquad \int_a^{a+\pi} \frac{(A^*(t) A(t))^{-1} - I}{\sin^2(t-a)} dt = 0$$

instead of (6).

IV. — Constructions

In the first part of the paragraph we shall use the condition (9) for constructing some differential systems (Q) with all solutions satisfying (1).

In the second part we shall construct a two-dimensional differential system (Q) with all solutions satisfying

$$y(t+\pi)=-y(t),$$

[i.e. $P=-I$ in (1)], the system (Q) being non diagonalizable, i.e., Q being not of the form $C^{-1}\operatorname{diag}(q_1, \ldots, q_n)C$ for a regular constant matrix C.

For both the parts relation (8) with a suitable A satisfying (7) and (9) will be considered. If such an A is taken, the only one requirement we need to guarantee is the symmetry of Q. In can easily be checked that for

$$S(t):=A'(t)A^{-1}(t)$$

the relation (8) reads

$$Q(t)=S'(t)+S^2(t)+2S(t)\operatorname{ctg}(t-a)-I. \tag{10}$$

Compare with formulae in [5].

We shall prove:

LEMMA 3. *$Q=Q^*$ if and only if $S=S^*$.*

Proof. ($\Leftarrow$) If $S=S^*$ then (10) gives $Q=Q^*$.

($\Rightarrow$) For $Q=Q^*$, the solution $Y(t):=A(t)\sin(t-a)$ [hence $Y(a)=Y^*(a)=0$] is isotropic:

$$Y^*Y'-Y^{*\prime}Y=0,$$

or

$$(A^*A'-A^{*\prime}A)\sin^2(t-a)=0.$$

Because of continuity of A' we get $A^*A'-A^{*\prime}A=0$, or $A'A^{-1}=A^{*-1}A^{*\prime}=(A'A^{-1})^*$. ■

As a sufficient condition for Q being not diagonalizable we shall use the following two Lemmas:

LEMMA 4. — *Let $Q=Q^*$ and Q be diagonalizable, i.e. $Q(t)=C^{-1}D(t)C$, where $D(t)=\operatorname{diag}(d_1(t), \ldots, d_n(t))$. Then for $R(t):=(A^*(t)A(t))^{-1}$ the matrix $R'R^{-1}R''$ is symmetric.*

Proof. — Let Z be a solution of

$$Z''=\operatorname{diag}(d_1(t), \ldots, d_n(t)).Z$$

determined by $Z(a)=0$, $Z'(a)=I$. Then

$$Z(t)=\operatorname{diag}(z_1(t), \ldots, z_n(t)),$$

where

$$z_i''(t)=d_i(t)z_i(t),$$
$$z_i(a)=0, \qquad z_i'(a)=1.$$

Put $Y(t) := C^{-1} Z(t) C$.

Then

$$Y(a)=0, \qquad Y'(a)=I,$$

and

$$Y'' = C^{-1} D(t) ZC = C^{-1} D(t) CY = Q(t) Y.$$

For $Y(t) = A(t) \sin(t-a)$ we have $A(t) = C^{-1} \delta(t) C$, where C is a regular constant matrix and δ is a diagonal matrix.

According to Lemma 3 it holds $A^* A' = A^{*\prime} A$. Hence

$$\begin{aligned} R' R^{-1} &= -(A^* A)^{-1} (A^* A)' = -A^{-1} A^{*-1} (A^{*\prime} A + A^* A') \\ &= -2 A^{-1} A^{*-1} (A^* A') = -2 A^{-1} A' = -2 C^{-1} \delta^{-1} \delta' C, \end{aligned}$$

i.e. $R' R^{-1}$ is diagonalizable.

Thus it commutes with its derivative

$$(R' R^{-1})(R' R^{-1})' = (R' R^{-1})' (R' R^{-1}),$$

or

$$R' R^{-1} (R'' R^{-1} - (R' R^{-1})^2) = (R'' R^{-1} - (R' R^{-1})^2)(R' R^{-1}).$$

We get $R' R^{-1} R'' = R'' R^{-1} R'$. Because of symmetricity of $R = (A^* A)^{-1}$,

$$R' R^{-1} R'' = (R' R^{-1} R'')^*. \quad \blacksquare$$

LEMMA 5. — *Let* $R(t) = \begin{pmatrix} u_1(t) & u_2(t) \\ u_2(t) & u_3(t) \end{pmatrix}$ *be a* 2 *by* 2 *regular real symmetric matrix of the class* $C^2(J, \mathbf{R}^{2^2})$. *Then* $R' R^{-1} R''$ *is symmetric on* J *if and only if*

$$\det \begin{pmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ u_1''(t) & u_2''(t) & u_3''(t) \end{pmatrix} = W(u_1, u_2, u_3) = 0 \quad \textit{on } J.$$

Proof. — Let $\Delta := \det R$. Then

$$R^{-1} = \Delta^{-1} . \begin{pmatrix} u_3 & -u_2 \\ -u_2 & u_1 \end{pmatrix},$$

$$R' R^{-1} R'' = \Delta^{-1} . \begin{pmatrix} u_1' u_3 - u_2 u_2' & -u_1' u_2 + u_1 u_2' \\ u_2' u_3 - u_2 u_3' & -u_2 u_2' + u_1 u_3' \end{pmatrix} \begin{pmatrix} u_1'' & u_2'' \\ u_2'' & u_3'' \end{pmatrix}$$

and $R' R^{-1} R''$ is symmetric if and only if

$$u_1' u_2'' u_3 - u_2 u_2' u_2'' - u_1' u_2 u_3'' + u_1 u_2' u_3'' = u_1'' u_2' u_3 - u_1'' u_2 u_3' - u_2 u_2' u_2'' + u_1 u_2'' u_3',$$

or

$$u_1(u_2' u_3'' - u_2'' u_3') - u_2(u_1' u_3'' - u_1'' u_3') + u_3(u_1' u_2'' - u_1'' u_2') = 0,$$

or $W(u_1, u_2, u_3) = 0$. ■

PART I. — We are going to construct a system (Q) with all solutions satisfying (1) for an orthogonal constant matrix P.

Let a symmetric matrix $M \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$ be periodic,

$$M(t+\pi) = M(t), \qquad \text{and} \qquad \int_0^\pi M(t)\,dt = 0.$$

Moreover, let the eigenvalues of M be greater than -1. Then the matrix $M(t)\sin^2 t + I$ has only positive eigenvalues. Let $N(t)$ denote the symmetric square root with only positive eigenvalues of the symmetric matrix $(I + M(t)\sin^2 t)^{-1}$. Then $N \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$, $\det N(t)$ is always positive,

$$N(t+\pi) = N(t), \qquad N^*(t) = N(t), \qquad N(0) = I, \qquad N'(0) = 0,$$

and

$$\int_0^\pi \frac{N^{-2}(t) - I}{\sin^2 t}\,dt = \int_0^\pi M(t)\,dt \doteq 0.$$

We put $A(t) := B(t)N(t)$, where $B \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$ is an orthogonal matrix. With respect to Lemma 3 we are looking for such a B, that $S := A'A^{-1}$ is symmetric. Hence we need

$$0 = S - S^* = (BN)'(BN)^{-1} - (BA)^{*-1}(BA)^{*\prime} = 2B'B^{-1} + B(N'N^{-1} - (N'N^{-1})^*)B^{-1},$$

because of orthogonality of B and skew-symmetricity of $B'B^{-1}$, *see* e. g. [4]. We get

$$B' = B.\frac{1}{2}(N'N^{-1} - (N'N^{-1})^*). \tag{10}$$

Since $1/2(N'N^{-1} - (N'N^{-1})^*) \in C^1(\mathbf{R}, \mathbf{R}^{n^2})$ is skew-symmetric, B is orthogonal for every t if it is orthogonal at some t_0.

By taking $B(0) = I$ we have $B \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$ and orthogonal for every t. Then $S = S^*$ and also $Q = Q^*$ due to lemma 3. For $A = B.N$ we get

$$\int_0^\pi \frac{(A^*(t)A(t))^{-1} - I}{\sin^2 t}\,dt = \int_0^\pi \frac{N^{-2}(t) - I}{\sin^2 t}\,dt = 0.$$

Evidently $A \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$, $A(0) = N(0) = I$, $A'(0) = B'(0) + N'(0) = 0$, and A is regular on $\mathbf{R}$. Moreover, since N is periodic, the system (10) is also periodic and due to Floquet Theory, there exists a regular real constant matrix C such that $B(t+\pi) = CB(t)$ for all t. Because of orthogonality of B, C is also orthogonal. Hence

$$A(t+\pi) = B(t+\pi)N(t+\pi) = CB(t)N(t) = CA(t).$$

For $P := -C$ we have

$$A(t+\pi) = -PA(t) \quad \text{for all } t.$$

Let us summarize our construction. $M \in C^2(\mathbf{R}, \mathbf{R}^{n^2})$ is symmetric, periodic with all eigenvalues >-1, and $\int_0^\pi M(t)\,dt=0$. N is the symmetric square root of $(I+M(t)\sin^2 t)^{-1}$ with only positive eigenvalues. B is a solution of (10) with $B(0)=I$. Thus (9) is satisfied for $A:=BN$, $a=0$, and Q defined by (8) is symmetric. Also $P:=-B(t+\pi)B^{-1}(t)$ is a constant real orthogonal matrix and $A(t+\pi)=-PA(t)$.

Due to Theorem 1, all solutions of the system (Q) with Q given by (8) satisfy (1).

PART II. — Now we are going to specify the matrix P in (1), namely we take $P=-I$. The aim of this part is to construct a two-dimensional system (Q) with non-diagonalizable Q having only half-periodic solutions, $y(t+\pi)=-y(t)$.

Again we use Theorem 1 and relation (8) for constructing Q. We are looking for A of the form

$$A(t)=H(t)D(t)G(t),$$

where periodic H, D, $G \in C^2(\mathbf{R}, \mathbf{R}^{2^2})$,

$$D(t)=\begin{pmatrix} d_1(t) & 0 \\ 0 & d_2(t) \end{pmatrix}$$

is diagonal,

$$G(t)=\begin{pmatrix} \cos\alpha(t) & \sin\alpha(t) \\ -\sin\alpha(t) & \cos\alpha(t) \end{pmatrix}, \qquad H(t)=\begin{pmatrix} \cos\beta(t) & \sin\beta(t) \\ -\sin\beta(t) & \cos\beta(t) \end{pmatrix}$$

are orthogonal 2 by 2 matrices such that

$$H(0)=I, \quad H'(0)=0; \qquad D(0)=I; \qquad D'(0)=0;$$
$$G(0)=I, \qquad G'(0)=0;$$

that is satisfied by

$$\begin{gathered} \alpha,\ \beta,\ d_i \in C^2(\mathbf{R}, \mathbf{R}), \\ \alpha(0)=0, \quad \alpha'(0)=0, \quad \beta(0)=0, \quad \beta'(0)=0, \quad d_i(0)=1, \\ d_i'(0)=0; \qquad i=1, 2. \end{gathered} \tag{11}$$

With respect to Lemma 3 we need $A^{*\prime}A=A^{*\prime}A$, or

$$D(H^*H'-H^{*\prime}H)D=GG^{*\prime}D^2-D^2G'G^*,$$

or

$$2\beta'(t)\begin{pmatrix} 0 & d_1 d_2 \\ -d_1 d_2 & 0 \end{pmatrix}=\begin{pmatrix} 0 & -d_1^2-d_2^2 \\ d_1^2+d_2^2 & 0 \end{pmatrix}\alpha'(t),$$

or equivalently

(12) $$2\beta' d_1 d_2+\alpha'(d_1^2+d_2^2)=0 \quad \text{on } \mathbf{R}.$$

Consider now (9) for $a=0$:

$$\int_0^\pi \frac{(\mathrm{A}^*(t)\,\mathrm{A}(t))^{-1}-\mathrm{I}}{\sin^2 t}\,dt=\int_0^\pi (\mathrm{G}^*\,\mathrm{D}^2\,\mathrm{G}-\mathrm{I})\sin^{-2} t\,dt$$

$$=\int_0^\pi \begin{bmatrix} (d_1^{-2}-1)\cos^2\alpha+(d_2^{-2}-1)\sin^2\alpha & \frac{1}{2}(d_1^{-2}-d_2^{-2})\sin^2\alpha \\ \frac{1}{2}(d_1^{-2}-d_2^{-2})\sin 2\alpha & (d_1^{-2}-1)\sin^2\alpha+(d_2^{-2}-1)\cos^2\alpha \end{bmatrix} \sin^{-2} t\,dt.$$

Let

(13) $$\begin{cases} f_i\in \mathrm{C}^2(\mathbf{R},\mathbf{R}), \qquad i=1,2, \\ \text{and} \\ f_i(t+\pi)=f_i(t), \\ f_i(\pi/2+t)=-f_i(\pi/2-t), \quad \text{or} \quad f_i(t)=-f_i(\pi-t), \\ |f_i(t)|<1, \\ f_i(0)=0, \qquad f'_i(0)=0. \end{cases}$$

Then $d_i:=(1+f_i(t))^{-1/2}$ satisfy

(14) $$\begin{cases} d_i\in \mathrm{C}^2(\mathbf{R},\mathbf{R}), \\ d_i(t)>0, \qquad d_i(0)=1, \qquad d'_i(0)=0, \\ d_i(t+\pi)=d_i(t), \\ d_i^{-2}(t)-1=-(d_i^{-2}(\pi-t)-1), \qquad i=1,2. \end{cases}$$

Hence

$$\int_0^\pi (d_i^{-2}(t)-1)\frac{\cos^2\alpha(t)}{\sin^2 t}\,dt$$

$$=\int_0^{\pi^2} (d_i^{-2}(t)-1)\frac{\cos^2\alpha(t)}{\sin^2 t}\,dt+\int_0^{\pi^2} (d_i^{-2}(\pi-t)-1)\frac{\cos^2\alpha(\pi-t)}{\sin^2(\pi-t)}\,dt=0$$

if

(15) $$\alpha(t)=\alpha(\pi-t).$$

Similarly

$$\int_0^\pi (d_i^{-2}(t)-1)\frac{\sin^2\alpha(t)}{\sin^2 t}dt=0$$

and

$$\int_0^\pi (d_1^{-2}(t)-d_2^{-2}(t)\frac{\sin 2\alpha(t)}{\sin^2 t}dt = \int_0^\pi [(d_1^{-2}(t)-1)-(d_2^{-2}(t)-1)]\frac{\sin 2\alpha(t)}{\sin^2 t}dt=0,$$

because of $\sin 2\alpha(\pi-t)=\sin 2\alpha(t)$.

Let us see for conditions on α and β. If

$$(16)\qquad \begin{cases} \alpha\in C^2(\mathbf{R}, \mathbf{R}), \quad \alpha(t+\pi)=\alpha(t), \\ \alpha(t)=\alpha(\pi-t) \qquad [see (15)], \\ \alpha(0)=0,\ \alpha'(0)=0 \qquad [see (11)], \end{cases}$$

then $G\in C^2(\mathbf{R}, \mathbf{R}^{2^2})$ is periodic, $G(0)=I$, $G'(0)=0$. The same remains true for G if instead of α the function $k\alpha$ is taken, k being a constant.

Due to (12):

$$\beta(t)=-\int_0^t \frac{\alpha'(s)}{2}\left(\frac{d_1(s)}{d_2(s)}+\frac{d_2(s)}{d_1(s)}\right)ds,$$

and hence

$$\beta\in C^2(\mathbf{R}, \mathbf{R}),$$
$$\beta(0)=\beta'(0)=0,$$

and because of periodicity of α, d_1, d_2 also

$$\beta(t+\pi)=\beta(t)-k_0,$$

where

$$k_0=\int_0^\pi \frac{\alpha'}{2}\left(\frac{d_1}{d_2}+\frac{d_2}{d_1}\right)ds.$$

If $k_0=0$, then $H\in C^2(\mathbf{R}, \mathbf{R}^{2^2})$ is periodic, and that is what we need.

If $k_0\neq 0$, then take $(2\pi/k_0)\,\alpha(t)$ instead of $\alpha(t)$.

Then $\beta(t+\pi)=\beta(t)-2\pi$, and $H\in C^2(\mathbf{R}, \mathbf{R}^{2^2})$ is periodic.

Since again $\beta(0)=\beta'(0)=0$, we have $H(0)=I$, $H'(0)=0$.

It remains to look for conditions of non-diagonalization of Q. According to Lemma 5 it would be sufficient to have

$$R=(A^* A)^{-1}=\begin{pmatrix} u_1 & u_2 \\ u_2 & u_3 \end{pmatrix},$$

such that W (u_1, u_2, u_3), Wronskian of u_1, u_2, u_3, be different from zero. Since

$$\mathrm{R}=\mathrm{G}^*\mathrm{D}^{-2}\mathrm{G} = \begin{bmatrix} d_1^{-2}\cos^2\alpha+d_2^{-2}\sin^2\alpha & \frac{1}{2}(d_1^{-2}-d_2^{-2})\sin 2\alpha \\ \frac{1}{2}(d_1^{-2}-d_2^{-2})\sin 2\alpha & d_1^{-2}\sin^2\alpha+d_2^{-2}\cos^2\alpha \end{bmatrix},$$

if

(17) d_1 and d_2 have different positive constant values on some subinterval (c, d) of $(\pi/4, \pi/3)$,

then the Wronskian of

$$d_1^{-2}\cos^2\alpha+d_2^{-2}\sin^2\alpha, \qquad \frac{1}{2}(d_1^{-2}-d_2^{-2})\sin 2\alpha,$$
$$d_1^{-2}\sin^2\alpha+d_2^{-2}\cos^2\alpha$$

on the interval (c, d) has the value $(\alpha'(t))^3$. W (y_1, y_2, y_3), where

$$y_1(t)=d_1^{-2}\cos^2 t+d_2^{-2}\sin^2 t,$$
$$y_2(t)=\frac{1}{2}(d_1^{-2}-d_2^{-2})\sin 2t,$$
$$y_3(t)=d_1^{-2}\sin^2 t+d_2^{-2}\cos^2 t,$$

$d_1^{-2}\neq d_2^{-2}$ being constants, are three linearly independent solutions of $y'''+4y'=0$, having $c_1+c_2\sin 2t+c_3\cos 2t$ as its general solution. Hence $\mathrm{W}(y_1, y_2, y_3)\neq 0$ and if α besides of above restrictions complies with

$$\alpha'(t)\neq 0 \quad \text{on } (c, d), \tag{18}$$

then our Q is not diagonalizable.

We summarize our considerations. Let f_i satisfy (13), f_1 and f_2 being different constants on $(c, d)\subset(\pi/4, \pi/3)$, then $d_i(t):=(1+f_i(t))^{-1/2}$ comply with (14), and (17). Take α satisfying (16) and (18). If

$$k_0=\int_0^\pi \frac{\alpha'}{2}\left(\frac{d_1}{d_2}+\frac{d_2}{d_1}\right)ds\neq 0,$$

take $(2\pi/k_0)\,\alpha(t)$ instead of the $\alpha(t)$. Define

$$\beta(t):=-\int_0^t \frac{\alpha'(s)}{2}\left(\frac{d_1(s)}{d_2(s)}+\frac{d_2(s)}{d_1(s)}\right)ds.$$

Using α, β, and d_i we get periodic matrices G. H, and D. For $A := HDG$ we define Q by means of (8). This Q is symmetric [Lemma 3 and relation (12)], non-diagonalizable [Lemma 5 and conditions (17) and (18)]. Our A complies with Theorem 1 for $P = -I$ [i. e. $A(t+\pi) = A(t)$] and satisfies relation (9) with $a=0$. Hence all solutions of (Q) satisfy $y(t+\pi) = -y(t)$.

Remark 4. — Having a two-dimensional second order non-diagonalizable system (Q) with all solutions satisfying $y(t+\pi) = -y(t)$, we may construct a non-diagonalizable system of the same property for any dimension n ($n>2$) simply by extending the second order system (Q) by adding $n-2$ equations $y_i'' = -y_i$, $i=3, \ldots, n$.

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