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INDUCED REPRESENTATIONS OF REDUCTIVE 𝔭-ADIC GROUPS II. ON IRREDUCIBLE REPRESENTATIONS OF GL (n)

By A. V. ZELEVINSKY

Introduction

This paper is a continuation of that of I. N. Bernstein and the author [1], whose notation and terminology we will freely use. In [1] we have dealt mainly with non-degenerate representations of the groups $G_n = GL(n, F)$; the technique developped there is applied here to the investigation of all irreducible representations of these groups. Our main method is the same as in [1] and is based on studying the restriction of representations of G_n to the subgroup $P_n \subset G_n$ consisting of matrices with last row $(0, 0, \ldots, 0, 1)$. This restriction is described in terms of the derivatives of ω (*see* [1], § 3, 4).

The results of this paper were announced in [14], [15]. The main ones are these:

(1) The classification of all irreducible representations of G_n modulo that of cuspidal representations (Thm. 6.1).

(2) The generalization of results of I. M. Gelfand and D. A. Kazhdan concerning the Kirillov model and the Whittaker model (Thm. 8.1, Cor. 8.2 and 8.3).

(3) The description of non-degenerate irreducible representations of G_n in terms of square-integrable ones (Thm. 9.7).

We give now a more detailed account of the contents. In paragraph 1 general results on induced representations from [1] are applied to the groups G_n. Let $\mathscr{R}(G_n)$ be the Grothendieck group of the category of algebraic G_n-modules of finite length and $\mathscr{R} = \oplus \mathscr{R}(G_n)(n=0, 1, 2, \ldots)$. It turns out that the functors $i_{G, M}$ and $r_{M, G}$ of inducing and localisation (*see* [1], 2.3) give rise to the structure of a bialgebra on $\mathscr{R}$ (Prop. 1.7). The significance of this structure is not yet well understood, but we think it must be of importance. Two other results of paragraph 1: Theorem 1.9 asserts that the ring $\mathscr{R}$ is commutative and Proposition 1.11 gives the complete description of the product of two irreducible cuspidal representations.

Section 2 is devoted to an investigation of the product $\pi = \rho_1 \times \ldots \times \rho_r$ in the case when all ρ_i are irreducible, cuspidal, and distinct. In this "regular" case π is multiplicity-free and we give the complete description of its composition factors and lattice of submodules (Prop. 2.1, Thms. 2.2 and 2.8). Results of paragraphs 1, 2 were obtained by the author together with I. N. Bernstein; results of paragraph 2 were announced in [3].

In paragraph 3 we introduce and study certain irreducible representations $\langle \Delta \rangle$; one of our main results is that all irreducible representations of G_n may be expressed in terms of products of such representations (§ 6).

Let $\mathscr{C}$ be the set of equivalence classes of irreducible cuspidal representations of the groups G_n $(n=1, 2, \ldots)$. Call a segment in $\mathscr{C}$ any subset of $\mathscr{C}$ of the form $\Delta = \{\rho, \nu\rho, \nu^2\rho, \ldots, \nu^k\rho = \rho'\}$ where ν is the character $\nu(g) = |\det g|$ and k is an integer $\geqq 0$ (we use the notation $\Delta = [\rho, \rho']$). To each segment $\Delta = [\rho, \rho']$ in $\mathscr{C}$ we associate the irreducible representation $\langle \Delta \rangle = \langle [\rho, \rho'] \rangle$ (3.1); it may be defined as the (unique) irreducible submodule of $\rho \times \nu\rho \times \ldots \times \rho'$. Note that both cuspidal and one-dimensional irreducible representations have such a form; cuspidal representations correspond to one-element segments, one-dimensional ones correspond to segments consisting of characters of $G_1 = F^*$ (3.1, 3.2). The representations $\langle \Delta \rangle$ remain irreducible when restricted to the subgroup $P_n \subset G_n$ (3.5, 3.6); in fact they may be characterized by this property (7.9).

In paragraph 4 we establish a criterion of irreducibility of the product $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ where the Δ_i are segments in $\mathscr{C}$ (Thm. 4.2). It generalizes Theorem 4.2 from [1]. Furthermore, for the product $\langle \Delta \rangle \times \langle \Delta' \rangle$ the complete description of its lattice of submodules is given (Prop. 4.6).

In paragraph 5 we discuss the property of homogeneity of representations of P_n and G_n. The representation ω of P_n or G_n is called homogeneous if it has no non-zero P_n-submodules more degenerate than ω itself (for the precise definition *see* 5.1). This property is closely connected with the generalization of the Kirillov model (5.2); it is used in paragraph 6.

In paragraph 6 we prove the central result of this paper, Theorem 6.1. Roughly speaking it asserts that irreducible representations of groups G_n are parametrized by families of segments. More precisely let $\mathcal{O}$ be the set consisting of all finite multisets (1) $a = \{\Delta_1, \ldots, \Delta_r\}$, where each Δ_i is a segment in $\mathscr{C}$. Theorem 6.1 says that there exists a natural bijection $a \mapsto \langle a \rangle = \langle \Delta_1, \ldots, \Delta_r \rangle$ between $\mathcal{O}$ and the set of equivalence classes of all irreducible representations of the groups G_n $(n=0, 1, 2, \ldots)$.

To construct $\langle a \rangle$ for $a = \{\Delta_1, \ldots, \Delta_r\} \in \mathcal{O}$ one must order $\Delta_1, \ldots, \Delta_r$ in a certain way [precisely described in 6.1.(*a*)] and consider the representation

$$\pi(a) = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle.$$

Theorem 6.1 (*a*) claims that $\pi(a)$ has a unique irreducible submodule; this is just $\langle a \rangle$. Conversely, the multiset $a \in \mathcal{O}$ may be directly reconstructed from the irreducible representation $\omega = \langle a \rangle$ in terms of the functor $r_{M,G}$ (6.9).

The proof of Theorem 6.1 is based on results of paragraphs 4, 5 and Theorem 6.2, which is of independent interest. It gives sufficient conditions for the product

(1) The term "multiset" means that elements of a may be repeated, i. e. each element occurs in a with some finite multiplicity. This notion will be used throughout the whole paper. The rigorous definitions and necessary terminology on multisets are collected together after the introduction.

$\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ to be homogeneous. Both 6.1 and 6.2 imply that every irreducible representation of G_n is homogeneous (Cor. 6.8). Note that Corollary 6.8 and Theorem 6.2 generalize Theorems 4.9 and 4.11 from [1].

To make our classification more explicit one has to express the basic operations on representations in terms of the set $\mathcal{O}$. Some results of this kind are collected in paragraphs 7, 8.

In paragraph 7 we compute composition factors of the product $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$, where the Δ_i are segments in $\mathscr{C}$ (Thm. 7.1). To formulate Theorem 7.1 one needs some definitions. For $a=(\Delta_1, \ldots, \Delta_r) \in \mathcal{O}$ set $\pi(a)=\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$; let $m(b; a)$ be the multiplicity of $\langle b \rangle$ in the Jordan-Hölder series of $\pi(a)$ (this means that $\pi(a)=\sum_{b \in \mathcal{O}} m(b; a) . \langle b \rangle$ in $\mathscr{R}$). Segments Δ and Δ' in $\mathscr{C}$ are called linked if $\Delta \not\subset \Delta'$, $\Delta' \not\subset \Delta$ and $\Delta \cup \Delta'$ is a segment (4.1). Call an elementary operation on the multiset $a \in \mathcal{O}$ replacement of a pair (Δ, Δ') of linked segments by the pair $(\Delta \cup \Delta', \Delta \cap \Delta')$ (note that $\Delta \cap \Delta'$ is either $\emptyset$ or a segment in $\mathscr{C}$; in the first case $\Delta \cap \Delta'$ is dropped). We write $b < a$ if b may be obtained from a by a chain of elementary operations. One obtains the structure of a partially ordered set on $\mathcal{O}$ (7.1). Theorem 7.1 claims that $m(b; a) \neq 0$ if and only if $b \leqq a$; moreover $m(a; a)=1$ for any $a \in \mathcal{O}$. In particular $\pi(a)$ is irreducible if and only if any two of segments of a are not linked (this is precisely Theorem 4.2, so Theorem 7.1 is its generalization).

Using 7.1 one can express the contragredient of any irreducible representation of G_n in terms of $\mathcal{O}$ (Thm. 7.10). For any representation ω denote its contragredient by $\tilde{\omega}$. Clearly for any segment Δ in $\mathscr{C}$ the set $\tilde{\Delta}=\{\tilde{\rho}/\rho \in \Delta\}$ is also a segment in $\mathscr{C}$. For any $a=(\Delta_1, \ldots, \Delta_r) \in \mathcal{O}$ set $\tilde{a}=(\tilde{\Delta}_1, \ldots, \tilde{\Delta}_r) \in \mathcal{O}$. Theorem 7.10 says that $\widetilde{\langle a \rangle} = \langle \tilde{a} \rangle$ for any $a \in \mathcal{O}$.

The other consequence of Theorem 7.1 is that monomials $\pi(a)$ $(a \in \mathcal{O})$ form a $\mathbb{Z}$-basis of $\mathscr{R}$; it means that $\mathscr{R}$ is a polynomial ring in the indeterminates $\langle \Delta \rangle$, where Δ runs over all segments in $\mathscr{C}$ (Cor. 7.5). This result allows one to describe completely the bialgebra $\mathscr{R}$ in a purely algebraic way (7.6). But realizing $\mathscr{R}$ as a polynomial ring, one may ask: which elements of $\mathscr{R}$ are represented by ordinary representations of G_n (i. e. not only by virtual ones). This important problem is yet unsolved. Clearly it is equivalent to the problem of explicit evaluation of coefficients $m(b; a)$. Some partial results in this direction are collected in paragraph 11.

In paragraph 8 we compute the highest derivative of any irreducible representation of G_n (Thm. 8.1). For any segment $\Delta=[\rho, \rho']$ in $\mathscr{C}$ set $\Delta^- = \Delta \setminus \{\rho'\}$; for any $a=(\Delta_1, \ldots, \Delta_r) \in \mathcal{O}$ set $a^- = (\Delta_1^-, \ldots, \Delta_r^-) \in \mathcal{O}$ (here terms $\Delta_i^- = \emptyset$ are dropped). Theorem 8.1 states that for any $a \in \mathcal{O}$ the highest derivative of the irreducible representation $\langle a \rangle$ is isomorphic to $\langle a^- \rangle$; in particular it is irreducible. This generalizes the well-known result of I. M. Gelfand and D. A. Kazhdan ([8], p. 97, Thm. C or [2], 5.16).

Combining Theorem 8.1 with Corollary 6.8, one obtains a generalization of the Kirillov model (Cor. 8.2). In other words, every irreducible representation ω of G_n may be realized on some space of vector-valued functions on P_n (for precise definitions *see* 5.2).

The another application of Theorem 8.1 is the generalization of the Whittaker model (Cor. 8.3). For any irreducible representation ω of G_n we construct a character θ of the subgroup $U_n \subset G_n$ of unipotent upper triangular matrices such that ω may in a unique way be embedded into the representation of G_n, induced by θ (for non-degenerate ω this was done in [8], p. 97, Thm. D; *see* also [2], 5.17).

In the remainder of paragraph 8 we derive from Theorem 8.1 some consequences about products $\langle a_1 \rangle \times \ldots \times \langle a_p \rangle$ of general irreducible representations (Prop. 8.4-8.6). In particular Proposition 8.5 gives a sufficient condition for irreducibility of such a product. Let $a=(\Delta_1, \ldots, \Delta_r)$, $a'=(\Delta'_1, \ldots, \Delta'_s) \in \mathcal{O}$ be such that Δ_i and Δ'_j are not linked for all i, j. Proposition 8.5 says that $\langle a \rangle \times \langle a' \rangle$ is irreducible and moreover equals $\langle \Delta_1, \ldots, \Delta_r, \Delta'_1, \ldots, \Delta'_s \rangle$.

In paragraph 9 we associate to each segment $\Delta=[\rho, \rho']$ in $\mathscr{C}$ the irreducible representation $\langle \Delta \rangle^t$. It may be defined as the (unique) irreducible quotient module of $\rho \times \nu\rho \times \ldots \times \rho'$. An example of such a representation is the Steinberg representation of G_n (9.2).

The representations $\langle \Delta \rangle^t$ may be characterized as irreducible quasi-square-integrable representations of G_n (Thm. 9.3) ($\omega \in \operatorname{Alg} G_n$ is called quasi-square-integrable if it becomes square-integrable after multiplying by a suitable character of G_n). This result is due to I. N. Bernstein. Note that half of it, namely that representations $\langle \Delta \rangle^t$ are quasi-square-integrable follows directly from the criterion for square-integrability obtained by W. Casselman ([6], Thm. 6.5.1); the converse is based on a refinement of results of his.

It is known that all irreducible representations of semisimple real groups may be obtained by inducing from square-integrable ones ([12], [11]). It is thus natural to try to classify irreducible representations of G_n in terms of products $\langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_r \rangle^t$. The first result in this direction is Theorem 9.7, classifying non-degenerate irreducible representations of G_n. It consists of two parts:

(*a*) Each product $\langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_r \rangle^t$ is non-degenerate. It is irreducible if and only if any two of segments $\Delta_1, \ldots, \Delta_r$ are not linked.

(*b*) Any irreducible non-degenerate representation ω of G_n decomposes into the product $\omega = \langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_r \rangle^t$ where any two of segments $\Delta_1, \ldots, \Delta_r$ are not linked. Moreover the multiset $a=(\Delta_1, \ldots, \Delta_r) \in \mathcal{O}$ is uniquely determined by ω.

Further results are concerned with the remarkable duality between representations $\langle \Delta \rangle$ and $\langle \Delta \rangle^t$. Since $\mathscr{R}$ is a polynomial ring in indeterminates $\langle \Delta \rangle$, the mapping $\langle \Delta \rangle \mapsto \langle \Delta \rangle^t$ is uniquely extended to the endomorphism $\omega \mapsto \omega^t$ of the ring $\mathscr{R}$. Proposition 9.12 claims that this endomorphism is an involutive automorphism of $\mathscr{R}$. In fact it may be defined in a purely algebraic way in terms of the bialgebra structure on $\mathscr{R}$ (Prop. 9.16). We conjecture (9.17) that the automorphism $\omega \mapsto \omega^t$ carries irreducible representations into irreducible ones. There is much evidence for this (e.g. Thms. 4.2 and 9.7); we also mention the interesting analogies with groups over finite fields (*see* [15]).

In paragraph 10 we discuss relationships between our results and the (hypothetical) reciprocity law of Langlands. Roughly speaking our results allow one to extend the reciprocity law from cuspidal to all irreducible representations of G_n.

The final section 11 contains some partial results about coefficients $m(b; a)$. We mention here Proposition 11.4, which describes irreducible components of $\rho \times \nu\rho \times \nu\rho \times \nu^2 \rho$ for

$\rho \in \mathscr{C}$. It provides counter-examples for many possible general conjectures about coefficients $m(b; a)$, derivatives of irreducible representations, and so on. I. N. Bernstein communicated to me that an analogous example (playing a similar "destructive" role) exists in the theory of Verma modules (*see* [4], p. 9, Remark).

I am glad to express my deep gratitude to I. N. Bernstein for invaluable discussions and help and for allowing me to include here some of his results. I would like to thank V. G. Drinfeld, who called my attention to relationships between the present results and the reciprocity law. I am also indebted to the referee for some useful remarks and suggestions.

Notation related to multisets

Fix a set Ω. A multiset on Ω is by definition a function $\chi : \Omega \to \mathbb{Z}_+$ (the set of non-negative integers). Since subsets of Ω may be represented by their characteristic functions the notion of a multiset on Ω generalizes that of a subset of Ω. It is often convenient to extend the set theoretic language to multisets. So we write down the multiset $\chi : \Omega \to \mathbb{Z}_+$ as $a = \{ \ldots, x, \ldots, x, y, \ldots, y, \ldots \}$ where each element $x \in \Omega$ is repeated $\chi(x)$ times. The function χ is called the characteristic function of a and is denoted by χ_a; the value $\chi_a(x)$ is called a multiplicity of x in a. We write $x \in a$ if $\chi_a(x) > 0$, $a \subset b$ if $\chi_a(x) \leqq \chi_b(x)$ for all $x \in \Omega$; the empty multiset $a = \emptyset$ corresponds to $\chi \equiv 0$. A multiset a is called finite if χ_a has finite support; in this paper we need only finite multisets.

The sum $a+b$ of two multisets is defined by $\chi_{a+b} = \chi_a + \chi_b$. Note that the sum of two subsets of Ω is itself a subset iff these subsets have an empty intersection; in this case the sum coincides with the (disjoint) union.

If the multiset a occurs in expressions such as $\sum_{x \in a} f(x)$ one must take into account the multiplicities, i. e. one has

$$\sum_{x \in a} f(x) = \sum_{x \in \Omega} \chi_a(x) . f(x)$$

In particular for any finite multiset a put

$$|a| = \sum_{x \in a} 1 = \sum_{x \in \Omega} \chi_a(x).$$

Let a be a finite multiset and $|a| = \mathrm{N}$. Define an ordering of a as a map $\lambda : [1, \mathrm{N}] \to \Omega$ such that $|\lambda^{-1}(x)| = \chi_a(x)$ for $x \in \Omega$ (here $[1, \mathrm{N}] = \{ 1, 2, \ldots, \mathrm{N} \}$). We will sometimes write down orderings as sequences $(\lambda(1), \lambda(2), \ldots, \lambda(\mathrm{N}))$.

Example. — Let $\pi \in \mathrm{Alg}\, \mathrm{G}$ be a representation of finite length. Then its composition series $\mathscr{J}\mathrm{H}^0(\pi)$ is a finite multiset on $\overline{\mathrm{Irr}}\ \mathrm{G}$. If π is glued together from $\pi_1, \ldots, \pi_k$ then $\mathscr{J}\mathrm{H}^0(\pi) = \mathscr{J}\mathrm{H}^0(\pi_1) + \ldots + \mathscr{J}\mathrm{H}^0(\pi_k)$ (*see* [1], 1.11).

1. The functors $i_{\alpha, \beta}$ and $r_{\beta, \alpha}$

In this section the results of [1], paragraph 2 are applied to the case $\mathrm{G} = \mathrm{GL}(n)$.

1.1. Fix from now on a local nonarchimedean field F and set $G_n = GL(n, F)$ for $n > 0$, $G_0 = \{e\}$. For each ordered partition $\alpha = (n_1, \ldots, n_r)$ of n let G_α be the subgroup $G_{n_1} \times \ldots \times G_{n_r}$ of G_n, embedded in G_n as the subgroup of block-diagonal matrices. By blocks of α we mean the sets of indices

$$I_1 = \{1, 2, \ldots, n_1\}, I_2 = \{n_1+1, \ldots, n_1+n_2\}, \ldots, I_r = \{n_1 + \ldots + n_{r-1}+1, \ldots, n\}.$$

The partition β is called a subpartition of α if every block of β is contained in some block of α (notation: $\beta \leqq \alpha$). If $\beta \leqq \alpha$ then G_β is a standard subgroup of G_α (*see* [1], 2.1). Thus there are defined functors

$$i_{G_\alpha, G_\beta} : \quad \operatorname{Alg} G_\beta \to \operatorname{Alg} G_\alpha \qquad \text{and} \qquad r_{G_\beta, G_\alpha} : \quad \operatorname{Alg} G_\alpha \to \operatorname{Alg} G_\beta$$

(*see* [1], 2.3). We will write $i_{\alpha, \beta}$ and $r_{\beta, \alpha}$ instead of i_{G_α, G_β} and r_{G_β, G_α}. Let us state in these notations Proposition 2.3 of [1].

PROPOSITION. — (*a*) *The functors* $i_{\alpha, \beta}$ *and* $r_{\beta, \alpha}$ *are exact.*

(*b*) *The functor* $r_{\beta, \alpha}$ *is left adjoint to* $i_{\alpha, \beta}$.

(*c*) *If* $\gamma \leqq \beta \leqq \alpha$ *then*

$$i_{\alpha, \beta} \circ i_{\beta, \gamma} = i_{\alpha, \gamma}, \qquad r_{\gamma, \beta} \circ r_{\beta, \alpha} = r_{\gamma, \alpha}.$$

(*d*) $\widetilde{i_{\alpha, \beta}(\rho)} = i_{\alpha, \beta}(\tilde{\rho})$ *for any* $\rho \in \operatorname{Alg} G_\beta$ (*here* $\tilde{\rho}$ *is the contragredient representation to* ρ).

1.2. Let $\beta, \gamma \leqq (n)$. We will compute the composition of functors

$$i_{(n), \beta} : \quad \operatorname{Alg} G_\beta \to \operatorname{Alg} G_n \qquad \text{and} \qquad r_{\gamma, (n)} : \quad \operatorname{Alg} G_n \to \operatorname{Alg} G_\gamma.$$

Consider the group $W = W_n$ of all permutations of the set $[1, n]$. We will identify an element $w \in W$ with the matrix $w = (\delta_{i, w(j)}) \in G_n$. So W is a subgroup of $G = G_n$; it is called the Weyl group of G. For any $w \in W$ denote by the same symbol w the corresponding inner automorphism of G, i.e. $w(g) = wgw^{-1}$.

Let $I_1, \ldots, I_r$ and $\mathcal{J}_1, \ldots, \mathcal{J}_s$ be blocks of β and γ respectively. Set

$$\begin{aligned} W^{\beta, \gamma} = \{ w \in W \mid\ & w(k) < w(l) \text{ if } k < l \\ & \text{and both } k \text{ and } l \text{ belong to the same } I_i; \\ & w^{-1}(k) < w^{-1}(l) \text{ if } k < l \text{ and both } k \text{ and } l \text{ belong to the same } \mathcal{J}_j \}. \end{aligned}$$

Let $w \in W^{\beta, \gamma}$. It is clear that all sets $w(I_i) \cap \mathcal{J}_j$ are blocks. They induce the subpartition $\gamma' \leqq \gamma$ (notation: $\gamma' = \gamma \cap w(\beta)$). Similarly the sets $I_i \cap w^{-1}(\mathcal{J}_j)$ are the blocks of the partition $\beta' = \beta \cap w^{-1}(\gamma) \leqq \beta$. It is clear that $w(G_{\beta'}) = G_{\gamma'}$. Define the functor $F_w : \operatorname{Alg} G_\beta \to \operatorname{Alg} G_\gamma$, by

$$F_w = i_{\gamma, \gamma'} \circ w \circ r_{\beta', \beta}$$

(here w is considered as a functor $\operatorname{Alg} G_{\beta'} \to \operatorname{Alg} G_{\gamma'}$, *see* [1], 1.7).

THEOREM. — *The functor* $F = r_{\gamma, (n)} \circ i_{(n), \beta} : \operatorname{Alg} G_\beta \to \operatorname{Alg} G_\gamma$ *is glued together from the* F_w, *where* $w \in W^{\beta, \gamma}$.

This is a particular case of Lemma 2.12 *from* [1].

1.3. Let us give a more detailed formulation of Theorem 1.2. It claims that there exists an ordering $w_1, \ldots, w_k$ of elements of $W^{\beta, \gamma}$ satisfying the following condition: for any $\rho \in \operatorname{Alg} G_\beta$ the representation $F(\rho)$ has a filtration $0=\tau_0 \subset \tau_1 \subset \ldots \subset \tau_k = F(\rho)$ such that τ_i/τ_{i-1} is isomorphic to $F_{w_i}(\rho)$.

Denote by P_β the parabolic subgroup of G_n corresponding to β, i.e. P_β is a subgroup of upper block-triangular matrices. By Theorem 5.2 from [1], to choose a sequence $w_1, \ldots, w_k$ as above it is sufficient to require all sets $(P_\beta w_1^{-1} P_\gamma \cup P_\beta w_2^{-1} P_\gamma \cup \ldots \cup P_\beta w_i^{-1} P_\gamma)$ to be open in G_n $(i=1, 2, \ldots, k)$. In particular one can set $w_k = e$ and we derive.

COROLLARY. — *For any* $\rho \in \operatorname{Alg} G_\beta$ *the representation* $r_{\gamma, (n)} \circ i_{(n), \beta}(\rho)$ *has a quotient module isomorphic to* $i_{\gamma, \gamma \cap \beta} \circ r_{\gamma \cap \beta, \beta}(\rho)$.

1.4. PROPOSITION. — *The functors* $i_{\alpha, \beta}$ *and* $r_{\beta, \alpha}$ *take representations of finite length into ones of finite length.*

Proof. — Let $\gamma \leqq \beta \leqq \alpha$ be partitions and $\omega \in \operatorname{Alg} G_\beta$ be of finite length. We must prove that $r_{\gamma, \beta}(\omega)$ and $i_{\alpha, \beta}(\omega)$ are of finite length. Since $r_{\gamma, \beta}$ and $i_{\alpha, \beta}$ are exact it suffices to consider the case when ω is irreducible. By [1], 2.5 one can embed ω into some representation $i_{\beta, \delta}(\rho)$ where ρ is irreducible and cuspidal. So it suffices to prove that

$$r_{\gamma, \beta} \circ i_{\beta, \delta}(\rho) \qquad \text{and} \qquad i_{\alpha, \beta} \circ i_{\beta, \delta}(\rho)$$

are of finite length. By 1.1 (*c*) $i_{\alpha, \beta} \circ i_{\beta, \delta}(\rho) = i_{\alpha, \delta}(\rho)$; this representation has finite length by Theorem 2.8 from [1]. The statement that $r_{\gamma, \beta} \circ i_{\beta, \delta}(\rho)$ has finite length follows directly from [1], 2.8 and 2.12.

1.5. Let $\alpha = (n_1, \ldots, n_r)$ be an ordered partition. Consider the functor

$$\otimes: \quad \operatorname{Alg} G_{n_1} \times \ldots \times \operatorname{Alg} G_{n_r} \to \operatorname{Alg} G_\alpha ((\rho_1, \ldots, \rho_r) \mapsto \rho_1 \otimes \ldots \otimes \rho_r)$$

of tensor product. Since any irreducible representation of G_α is admissible (*see* [1], 2.5), this functor induces a bijection $\otimes : \overline{\operatorname{Irr}}\, G_{n_1} \times \ldots \times \overline{\operatorname{Irr}}\, G_{n_r} \cong \overline{\operatorname{Irr}}\, G_\alpha$ (*see* [1], 1.2, 1.6). Let us write functors $i_{\alpha, \beta}$ and $r_{\beta, \alpha}$ in this "coordinate form". Let $\beta \leqq \alpha$; denote by $\beta_1, \ldots, \beta_r$ the partitions induced by β on blocks of α. It is clear that $G_\beta = G_{\beta_1} \times \ldots \times G_{\beta_r}$.

PROPOSITION. — (*a*) *Let* $\pi_i \in \operatorname{Alg} G_{n_i}$, $i=1, \ldots, r$. *Then*

$$r_{\beta, \alpha}(\pi_1 \otimes \ldots \otimes \pi_r) = r_{\beta_1, (n_1)}(\pi_1) \otimes \ldots \otimes r_{\beta_r, (n_r)}(\pi_r);$$

(*b*) *Let* $\rho_i \in \operatorname{Alg} G_{\beta_i}$, $i=1, \ldots, r$. *Then*

$$i_{\alpha, \beta}(\rho_1 \otimes \ldots \otimes \rho_r) = i_{(n_1), \beta_1}(\rho_1) \otimes \ldots \otimes i_{(n_r), \beta_r}(\rho_r).$$

This follows immediately from [1], 1.9 (*c*), (*g*).

1.6. We give now the "coordinate form" of Theorem 1.2. One can use it to compute composition factors of the representation $r_{\gamma, (n)} \circ i_{(n), \beta}(\rho)$ where $\rho \in \operatorname{Alg} G_\beta$ is of finite length.

Let $\beta=(n_1, \ldots, n_r)$, $\gamma=(m_1, \ldots, m_s)$, let $I_1, \ldots, I_r$ and $\mathscr{J}_1, \ldots, \mathscr{J}_s$ be blocks of β and γ respectively, so $n_i=|I_i|$, $m_j=|\mathscr{J}_j|$. To each $w\in W^{\beta,\gamma}$ there corresponds the rectangular $r\times s$-matrix $B(w)=(b_{ij})$, where $b_{ij}=|I_i\cap w^{-1}(\mathscr{J}_j)|$. It is clear that the correspondence $w\mapsto B(w)$ is the bijection of $W^{\beta,\gamma}$ with the set $M^{\beta,\gamma}$ of matrices $B=(b_{ij})$ such that:

(1) All b_{ij} are integers $\geqq 0$.

(2) $\sum_j b_{ij}=n_i$ for any $i=1, \ldots, r$; $\sum_i b_{ij}=m_j$ for any $j=1, \ldots, s$.

Let $\rho_i\in \operatorname{Irr} G_{n_i}$, $\rho=\rho_1\otimes\ldots\otimes\rho_r\in\operatorname{Irr} G_\beta$, $w\in W^{\beta,\gamma}$. We will compute composition factors of $F_w(\rho)$ (*see* 1.2) in terms of the matrix $B(w)=(b_{ij})\in M^{\beta,\gamma}$. Denote by β_i the partition $(b_{i1}, \ldots, b_{is})$ of n_i and by γ_j the partition $(b_{1j}, \ldots, b_{rj})$ of m_j. By 1.4 the representations $r_{\beta_i,(n_i)}(\rho_i)$ have finite length. Let $\mathscr{J}H^0(r_{\beta_i,(n_i)}(\rho_i)=\{\sigma_i^{(1)}, \sigma_i^{(2)}, \ldots\}$ (*see* [1], 1.2), where

$$\sigma_i^{(k)}=\sigma_{i1}^{(k)}\otimes\ldots\otimes\sigma_{is}^{(k)}, \qquad \sigma_{ij}^{(k)}\in\operatorname{Irr} G_{b_{ij}}.$$

For each $k_1, \ldots, k_r$ put

$$\sigma_j=\sigma_{1j}^{(k_1)}\otimes\sigma_{2j}^{(k_2)}\otimes\ldots\otimes\sigma_{rj}^{(k_r)}\in\operatorname{Irr} G_{\gamma_j}$$

and

$$\sigma(k_1, \ldots, k_r)=i_{(m_1),\gamma_1}(\sigma_1)\otimes i_{(m_2),\gamma_2}(\sigma_2)\otimes\ldots\otimes i_{(m_s),\gamma_s}(\sigma_s)\in\operatorname{Alg} G_\gamma.$$

PROPOSITION. — $F_w(\rho)$ *is glued together from the representations* $\sigma(k_1, \ldots, k_r)$.

This follows immediately from definition of F_w, the exactness of $r_{\beta,\alpha}$ and $i_{\alpha,\beta}$ and 1.5.

1.7. Denote by $\mathscr{R}(G_\alpha)$ the Grothendieck group of the category of algebraic G_α-modules of finite length. By definition $\mathscr{R}(G_\alpha)$ is a free abelian group with basis $\overline{\operatorname{Irr}}\, G_\alpha$. We will denote by the same symbol the representation $\pi\in\operatorname{Alg} G_\alpha$ of finite length and its image in $\mathscr{R}(G_\alpha)$, i. e. in $\mathscr{R}(G_\alpha)$ one has

$$\pi=\sum\omega, \qquad \omega\in\mathscr{J}H^0(\pi).$$

Let $\alpha=(n_1, \ldots, n_r)$. Using tensor products one can identify $\overline{\operatorname{Irr}}\, G_\alpha$ with $\overline{\operatorname{Irr}}\, G_{n_1}\times\ldots\times\overline{\operatorname{Irr}}\, G_{n_r}$; hence $\mathscr{R}(G_\alpha)$ may be identified with $\mathscr{R}(G_{n_1})\otimes\ldots\otimes\mathscr{R}(G_{n_r})$. By 1.1 (*a*) and 1.4 the functors $i_{\alpha,\beta}$ and $r_{\beta,\alpha}$ induce homomorphisms

$$i_{\alpha,\beta}:\ \mathscr{R}(G_\beta)\to\mathscr{R}(G_\alpha) \qquad\text{and}\qquad r_{\beta,\alpha}:\ \mathscr{R}(G_\alpha)\to\mathscr{R}(G_\beta).$$

By 1.1 (*c*) and 1.5 to compute all such homomorphisms one needs only to know maps

$$i_{(n),(k,l)}:\ \mathscr{R}(G_k)\otimes\mathscr{R}(G_l)\to\mathscr{R}(G_n);$$
$$r_{(k,l),(n)}:\ \mathscr{R}(G_n)\to\mathscr{R}(G_k)\otimes\mathscr{R}(G_l).$$

To give a more condensed form of this put $\mathscr{R}=\oplus\mathscr{R}(G_n)$ $(n=0, 1, 2, \ldots)$ and define the multiplication $m:\mathscr{R}\otimes\mathscr{R}\to\mathscr{R}$ and comultiplication $c:\mathscr{R}\to\mathscr{R}\otimes\mathscr{R}$ by

$$m(\rho_1\otimes\rho_2)=i_{(n),(k,l)}(\rho_1\otimes\rho_2),\qquad \rho_1\in\mathscr{R}(G_k),\quad \rho_2\in\mathscr{R}(G_l),\quad n=k+l,$$

$$c(\pi)=\sum_{0\leq k\leq n} r_{(k,n-k),(n)}(\pi),\qquad \pi\in\mathscr{R}(G_n).$$

Using m, one obtains the structure of a $\mathbb{Z}$-algebra on $\mathscr{R}$ [put $\rho_1\times\rho_2=m(\rho_1\otimes\rho_2)$]; similarly c determines the structure of coalgebra on $\mathscr{R}$.

PROPOSITION. — *By means of m and c, the group $\mathscr{R}$ becomes a graded bialgebra over $\mathbb{Z}$. It is called the representation bialgebra of groups G_n (for definitions see* [5], *Chapt. III, § 11; nevertheless they are explained in the proof below).*

Proof. — We must prove the following properties.

(*a*) Associativity. This means that both diagrams

$$\begin{array}{ccc} \mathscr{R}\otimes\mathscr{R}\otimes\mathscr{R} & \xrightarrow{m\otimes\mathrm{id}} & \mathscr{R}\otimes\mathscr{R} \\ \big\downarrow{\scriptstyle \mathrm{id}\otimes m} & & \big\downarrow{\scriptstyle m} \\ \mathscr{R}\otimes\mathscr{R} & \xrightarrow{m} & \mathscr{R} \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{R} & \xrightarrow{c} & \mathscr{R}\otimes\mathscr{R} \\ \big\downarrow{\scriptstyle c} & & \big\downarrow{\scriptstyle \mathrm{id}\otimes c} \\ \mathscr{R}\otimes\mathscr{R} & \xrightarrow{c\otimes\mathrm{id}} & \mathscr{R}\otimes\mathscr{R}\otimes\mathscr{R} \end{array}$$

are commutative. Let $\rho_1\in\mathscr{R}(G_k)$, $\rho_2\in\mathscr{R}(G_l)$, $\rho_3\in\mathscr{R}(G_m)$, $n=k+l+m$. One can easily derive from 1.1 (*c*) and 1.5 that

$$m\circ(m\otimes\mathrm{id})(\rho_1\otimes\rho_2\otimes\rho_3)=m\circ(\mathrm{id}\otimes m)(\rho_1\otimes\rho_2\otimes\rho_3)=i_{(n),(k,l,m)}(\rho_1\otimes\rho_2\otimes\rho_3)$$

[in other words

$$(\rho_1\times\rho_2)\times\rho_3=\rho_1\times(\rho_2\times\rho_3)=i_{(n),(k,l,m)}(\rho_1\otimes\rho_2\otimes\rho_3)]$$

For the second diagram the arguments are similar.

(*b*) The map $c:\mathscr{R}\to\mathscr{R}\otimes\mathscr{R}$ is a homomorphism of rings (the multiplication in $\mathscr{R}\otimes\mathscr{R}$ is defined as usual:

$$(\rho_1\otimes\rho_2)\times(\rho_1'\otimes\rho_2')=(\rho_1\times\rho_1')\otimes(\rho_2\times\rho_2')).$$

This follows directly from 1.6. Note that this statement is a condensed form of Theorem 1.2.

(*c*) Other properties are of less importance. There exist a unit $1\in\mathscr{R}(G_0)$ (it is represented by a trivial representation of $G_0=\{e\}$) and counit $\gamma:\mathscr{R}\to\mathbb{Z}$ [$\gamma=0$ on $\bigoplus_{n>0}\mathscr{R}(G_n)$, $\gamma(1)=1$] with the usual properties (*see* [5], Chapt. III, § 11, No. 2,4). In conclusion, $\mathscr{R}$ and $\mathscr{R}\otimes\mathscr{R}$ are graded by

$$\mathscr{R}_n=\mathscr{R}(G_n),\qquad (\mathscr{R}\otimes\mathscr{R})_n=\bigoplus_{k+l=n}(\mathscr{R}_k\otimes\mathscr{R}_l)$$

and m and c are homogeneous of degree 0.

1.8. *Remark.* — If $\rho_i \in \operatorname{Alg} G_{n_i}$ not be required to have finite length ($i=1, \ldots, r$), we will still use the notation

$$\rho_1 \times \ldots \times \rho_r = i_{(n),(n_1, \ldots, n_r)}(\rho_1 \otimes \ldots \otimes \rho_r) \in \operatorname{Alg} G_n (n = n_1 + \ldots + n_r)$$

and call this representation the product of $\rho_1, \ldots, \rho_r$.

1.9. THEOREM. — *The ring $\mathcal{R}$ is commutative. In other words if $\pi_1 \in \operatorname{Alg} G_k$, $\pi_2 \in \operatorname{Alg} G_l$ are of finite length then $\pi_1 \times \pi_2$ and $\pi_2 \times \pi_1$ have the same composition factors* [*i.e.* $\mathcal{J}H^0(\pi_1 \times \pi_2) = \mathcal{J}H^0(\pi_2 \times \pi_1)$].

Proof. — For each n define the matrix $s_n \in G_n$ and the automorphism $s: G_n \to G_n$ by

$$(s_n)_{ij} = (-1)^i \delta_{i, n+1-j}, \qquad s(g) = s_n . g'^{-1} . s_n^{-1}$$

(here g' is the matrix transpose to g). One obtains thus the functor

$$s: \quad \operatorname{Alg} G_n \to \operatorname{Alg} G_n.$$

LEMMA. — (*a*) $s(\pi_1 \times \pi_2) \simeq s(\pi_2) \times s(\pi_1)$ *for each* $\pi_1 \in \operatorname{Alg} G_k$, $\pi_2 \in \operatorname{Alg} G_l$.
(*b*) *If $\pi \in \operatorname{Alg} G_n$ is of finite length then so is $s(\pi)$ and*

$$\mathcal{J}H^0(s(\pi)) = \mathcal{J}H^0(\tilde{\pi}).$$

Part (*a*) follows immediately from definitions. To prove (*b*) it suffices to consider the case when π is irreducible. In this case our statement is due to Gelfand and Kazhdan (*see* [8] or [2], 7.3).

Now one has

$$\mathcal{J}H^0(\pi_1 \times \pi_2) = \mathcal{J}H^0(\widetilde{s(\pi_1 \times \pi_2)})$$
$$= \mathcal{J}H^0(\widetilde{s(\pi_2) \times s(\pi_1)}) = \mathcal{J}H^0(\widetilde{s(\pi_2)} \times \widetilde{s(\pi_1)}) = \mathcal{J}H^0(\pi_2 \times \pi_1)$$

[use in consecutive order part (*b*), (*a*), 1.1 (*d*) and again part (*b*)]. The Theorem is done.

1.10. The representation $\rho \in \operatorname{Alg} G_\alpha$ is called quasicuspidal if $r_{\beta, \alpha}(\rho) = 0$ for all $\beta < \alpha$, $\beta \neq \alpha$. Admissible quasicuspidal representations are called cuspidal. If $\alpha = (n_1, \ldots, n_r)$, $\rho_i \in \operatorname{Alg} G_{n_i}$, $0 \neq \rho = \rho_1 \otimes \ldots \otimes \rho_r \in \operatorname{Alg} G_\alpha$ then by 1.5 ρ is quasicuspidal iff all ρ_i are quasicuspidal. Note that the representation $\rho \in \operatorname{Alg} G_n$ of finite length is cuspidal iff $c(\rho) = 1 \otimes \rho + \rho \otimes 1$ (*see* 1.7); it means that ρ is a primitive element of the coalgebra $\mathcal{R}$ (*see* [5], Chapt. III, § 11, No. 8, Remark 2). Denote by $\mathcal{C}$ the set of equivalence classes of irreducible cuspidal representations of groups G_n ($n = 1, 2, \ldots$).

PROPOSITION. — *Let $\omega \in \operatorname{Irr} G_n$. There exists a partition $(n_1, \ldots, n_r)$ of n and cuspidal representations $\rho_i \in \operatorname{Irr} G_{n_i}$ such that $\omega \in \mathcal{J}H(\rho_1 \times \ldots \times \rho_r)$. The multiset $\{\rho_1, \ldots, \rho_r\}$ on $\mathcal{C}$* (cf. *list of notations*) *is determined by ω; it is called the support of ω* (*notation* $\operatorname{supp} \omega = \{\rho_i, \ldots, \rho_r\}$). *One can choose an ordering $(\rho_{i_1}, \ldots, \rho_{i_r})$ of $\operatorname{supp} \omega$ such that ω can be embedded into $\rho_{i_1} \times \ldots \times \rho_{i_r}$* (see [1], 2.5, 2.9).

One can define $\operatorname{supp}\omega$ *in an another way. Choose* $\beta=(n_1, \ldots, n_r)$ *such that* $\rho=r_{\beta,(n)}(\omega)\in \operatorname{Alg} G_\beta$ *is a non-zero cuspidal representation. Let* $\rho_1\otimes\ldots\otimes\rho_r$ *be a composition factor of* ρ. *Then* ω *can be embedded into* $\rho_1\times\ldots\times\rho_r$ *hence* $\operatorname{supp}\omega=\{\rho_1, \ldots, \rho_r\}$ [*this follows from* 1.1 (*b*) *and* [1], 2.4 (*b*)].

1.11. In conclusion we describe the product of two irreducible cuspidal representations. Denote by ν the character of G_n defined by $\nu(g)=|\det g|$ where $|\ |$ is the standard norm of F.

PROPOSITION. — *Let* $\rho\in \operatorname{Irr} G_k$, $\rho'\in\operatorname{Irr} G_l$ *be cuspidal:*

(*a*) *If* $\rho'\not\simeq\nu\rho$ *and* $\rho\not\simeq\nu\rho'$ (*in particular if* $k\neq l$) *then* $\rho\times\rho'$ *is irreducible.*

(*b*) *Suppose that* $k=l$ *and either* $\rho'\simeq\nu\rho$ *or* $\rho\simeq\nu\rho'$. *Then the representation* $\rho\times\rho'$ *has length* 2. *It has a unique proper submodule* ω; *the quotient* $\omega'=(\rho\times\rho')/\omega$ *is irreducible and one has*

$$r_{(k,k),(2k)}(\omega)=\rho\otimes\rho', \qquad r_{(k,k),(2k)}(\omega')=\rho'\otimes\rho.$$

Proof. — It follows immediately from [1], 2.16 (1) that any product $\rho\times\rho'$ is either irreducible or satisfies the conclusions of part (*b*). Part (*a*) is a particular case of [1]. Theorem 4.2. It remains only to prove that the representation $\rho\times\nu\rho$ is reducible. This statement was announced in [1], Remark 4.2 (2). Here we will only sketch the proof.

Suppose that $\rho\times\nu\rho$ is irreducible; by 1.9 so is $\nu\rho\times\rho$. Multiplying by the appropriate power of ν, one may assume $\rho=\nu^{-1/2}\rho_0$ where ρ_0 is unitary (*see* [9], prop. 5.1). Set

$$\pi_s=\nu^{-s}\rho_0\times\nu^s\rho_0 \qquad (s\in\mathbb{R}).$$

According to Part (*a*) and our supposition π_s is irreducible for each $s\in\mathbb{R}$. Using the restriction of π_s to the maximal compact subgroup of G_{2k}, one can realize all π_s on the same space E. It is easy to see that for each $s\in\mathbb{R}$ there exists an Hermitian π_s-invariant form B_s on E. Moreover B_s is unique up to a scalar multiple and one can choose B_s analytically depending on s (*see* e. g. [13], §4).

The representation π_0 is unitary (*see* [9], p. 22) so one can assume B_0 to be positive and non-degenerate. It follows from irreducibility of π_s that all B_s are non-degenerate. One can derive from these facts that all B_s are positive hence all π_s are unitary.

On the other hand the matrix coefficients of π_s may be computed directly and one obtains that for large $|s|$ there exists a non-bounded matrix coefficient of π_s. Therefore such π_s cannot be unitary and we obtain a contradiction.

1.12. *Remarks.* — (*a*) Stated appropriately, Theorem 1.9 may be generalized to all reductive groups (*see* e. g. [1], Remark 2.10). One may prove it by a direct computation of characters but one meets with some technical difficulties. The present proof for GL(n) is perhaps the simplest possible. For cuspidal representations the statement is proved e. g. in [1], 2.9.

(*b*) We will explicitly compute the bialgebra $\mathcal{R}$ in paragraph 7 (*see* 7.6). Already now one may see that it is not cocommutative: for ω defined in 1.11 (*b*) one has

$$c(\omega)=1\otimes\omega+\rho\otimes\rho'+\omega\otimes 1$$

so $c(\omega)$ is not stable under the transposition

$$\sigma : \quad \mathcal{R} \otimes \mathcal{R} \to \mathcal{R} \otimes \mathcal{R} \qquad (\sigma(x \otimes y) = y \otimes x).$$

2. Representations of multiplicity-free support

Fix until the end of this section a finite subset $R = \{ \rho_1, \ldots, \rho_r \} \subset \mathcal{C}$ (*see* 1.10). We will describe irreducible representations whose support is R, with multiplicities equal to 1, i. e. composition factors of the product $\rho_1 \times \ldots \times \rho_r$ (*see* 1.10). Moreover we will give the complete description of the lattice of submodules of $\rho_1 \times \ldots \times \rho_r$.

2.1. If $\pi \in \operatorname{Alg} G_n$, put $n(\pi) = n$; fix now $n = \sum_{\rho \in R} n(\rho)$. Denote by S the set of orderings of R i. e. bijections $\lambda : [1, r] \to R$ (*see* the summary of notation). Assign to each $\lambda \in S$ the partition $\beta(\lambda) = (n \circ \lambda(1), \ldots, n \circ \lambda(r))$ and representations

$$\rho(\lambda) = \lambda(1) \otimes \lambda(2) \otimes \ldots \otimes \lambda(r) \in \overline{\operatorname{Irr}}\, G_{\beta(\lambda)},$$
$$\pi(\lambda) = \lambda(1) \times \ldots \times \lambda(r) \in \operatorname{Alg} G_n.$$

Denote by P the set of partitions of the form $\beta(\lambda)$, $\lambda \in S$; e. g. if all $n(\rho)$, $\rho \in R$ are the same then P consists of one element. Denote by Ω the set $\{ \rho(\lambda) \mid \lambda \in S \} \subset \bigcup_{\beta \in P} \overline{\operatorname{Irr}}\, G_\beta$; clearly the map $\lambda \mapsto \rho(\lambda)$ is a bijection $S \simeq \Omega$. For each $\pi \in \operatorname{Alg} G_n$ define the multiset $\Omega(\pi)$ on $\bigcup_{\beta \in P} \overline{\operatorname{Irr}}\, G_\beta$ to be $\sum_{\beta \in P} \mathcal{J} H^0(r_{\beta, (n)}(\pi))$. It follows from the exactness of $r_{\beta, (n)}$ that the map $\pi \mapsto \Omega(\pi)$ is additive i. e. if π is glued together from $\pi_1, \ldots, \pi_k$ then

$$\Omega(\pi) = \Omega(\pi_1) + \ldots + \Omega(\pi_k).$$

By Theorem 1.9 (or [1], 2.9) the multiset $\mathcal{J} H^0(\pi(\lambda))$ on $\overline{\operatorname{Irr}}\, G_n$ does not depend on λ; denote it by $\mathcal{J}$.

PROPOSITION. — (*a*) *For each* $\lambda \in S$ *the multiset* $\Omega(\pi(\lambda))$ *equals* Ω (*in particular it is a set, i. e. all multiplicities are equal to* 1).

(*b*) *For each* $\omega \in \mathcal{J}$ *one has* $\Omega(\omega) \neq \emptyset$ *and* $\sum_{\omega \in \mathcal{J}} \Omega(\omega) = \Omega$ (see *list of notations*).

(*c*) *The representation* $\pi(\lambda)$ *is multiplicity-free, i. e. each composition factor of* $\pi(\lambda)$ *occurs in* $\mathcal{J}$ *with multiplicity* 1.

Proof. — (*a*) follows immediately from [1], 2.13 (*c*) or from 1.6; (*b*) from 1.10, (*a*) and additivity of $(\pi \mapsto \Omega(\pi))$; (*c*) from (*a*) and (*b*).

2.2. By 2.1 any $\omega \in \mathcal{J}$ is uniquely determined by the subset $\Omega(\omega)$ of Ω. So to compute $\mathcal{J}$ one has only to describe the partition:

$$\Omega = \bigcup_{\omega \in \mathcal{J}} \Omega(\omega).$$

Define the (non-oriented) graph Γ in a following way. The set of vertices of Γ is $\Gamma^0=\{\rho_1, \ldots, \rho_r\}$. Two vertices ρ and ρ' define an edge if either $\rho' \simeq \nu\rho$ or $\rho \simeq \nu\rho'$ (*see* 1.11). Call an orientation of Γ the choice of a direction on each edge of Γ. To each $\lambda \in S$ there corresponds an orientation $\vec{\Gamma}(\lambda)$ of Γ: the edge $\{\rho, \rho'\}$ is oriented from ρ to ρ' if $\lambda^{-1}(\rho) < \lambda^{-1}(\rho')$.

Evidently any orientation $\vec{\Gamma}$ of Γ has the form $\vec{\Gamma}=\vec{\Gamma}(\lambda)$ for some $\lambda \in S$.

THEOREM. — *There exists a bijection $\vec{\Gamma} \mapsto \omega(\vec{\Gamma})$ between the set of all orientations of Γ and the set $\mathcal{J}$ such that*

$$\Omega(\omega(\vec{\Gamma}))=\{\rho(\lambda) \mid \vec{\Gamma}(\lambda)=\vec{\Gamma}\}.$$

2.3. COROLLARY. — *The length of $\pi(\lambda)$ is 2^k where k is the number of pairs of the form* $\{\rho, \nu\rho\} \subset R$.

2.4. We will simultaneously prove Theorem 2.2 and describe the lattice of submodules of $\pi(\lambda)$. We begin with a general result about multiplicity-free modules.

PROPOSITION. — *Let π be a multiplicity-free module and $\mathcal{J}=\mathcal{J}\,H(\pi)$. Then the map*

$$\pi' \mapsto \mathcal{J}(\pi')=\mathcal{J}\,H(\pi') \subset \mathcal{J}$$

is the embedding of the lattice of submodules of π into the lattice of subsets of $\mathcal{J}$ (this means that

$$\mathcal{J}(\pi_1 \cap \pi_2)=\mathcal{J}(\pi_1) \cap \mathcal{J}(\pi_2),\ \mathcal{J}(\pi_1+\pi_2)=\mathcal{J}(\pi_1) \cup \mathcal{J}(\pi_2)).$$

Proof. — (1) Let $\pi_1, \pi_2 \subset \pi$ and $\mathcal{J}(\pi_1) \subset \mathcal{J}(\pi_2)$. Since π is multiplicity-free, one has $\mathrm{Hom}(\pi_1, \pi/\pi_2)=0$ so $\pi_1 \subset \pi_2$. Therefore $\mathcal{J}(\pi_1)=\mathcal{J}(\pi_2) \Rightarrow \pi_1=\pi_2$, so $\pi_0 \mapsto \mathcal{J}(\pi_0)$ is an embedding.

(2) The submodule $\pi_1+\pi_2$ is glued together from π_1 and

$$(\pi_1+\pi_2)/\pi_1 \simeq \pi_2/(\pi_1 \cap \pi_2).$$

Hence $\mathcal{J}(\pi_1+\pi_2) \subset \mathcal{J}(\pi_1) \cup \mathcal{J}(\pi_2)$. The inverse inequality $\mathcal{J}(\pi_1+\pi_2) \supset \mathcal{J}(\pi_1) \cup \mathcal{J}(\pi_2)$ is trivial. So

$$\mathcal{J}(\pi_1+\pi_2)=\mathcal{J}(\pi_1) \cup \mathcal{J}(\pi_2)$$

(for this statement the fact that π is multiplicity-free is not needed).

(3) Evidently

$$\mathcal{J}(\pi_1 \cap \pi_2) \subset \mathcal{J}(\pi_1) \cap \mathcal{J}(\pi_2).$$

Suppose there exists $\omega \in \mathcal{J}(\pi_1) \cap \mathcal{J}(\pi_2)$, $\omega \notin \mathcal{J}(\pi_1 \cap \pi_2)$. Then $\omega \in \mathcal{J}(\pi_1)$ and $\omega \in \mathcal{J}(\pi_2/(\pi_1 \cap \pi_2))$ hence by (2) ω occurs in $\mathcal{J}\,H^0(\pi_1+\pi_2)$ with multiplicity at least 2 and we obtain a contradiction.

2.5. Return to the notation of 2.1-2.2. From 2.1 and 2.4 one derives:

LEMMA. — *For each* $\lambda \in S$ *the map* $\pi' \mapsto \Omega(\pi')$ *is an isomorphism of the lattice of submodules of* $\pi(\lambda)$ *with the sublattice of the lattice of all subsets of* Ω.

2.6. BASIC LEMMA. — *Let* $\lambda, \mu \in S$ *and* π' *be a submodule of* $\pi(\lambda)$ *such that* $\rho(\mu) \in \Omega(\pi')$. *Let* $\rho = \mu(j)$, $\rho' = \mu(j+1)$ *be two neighbour elements of the ordering* μ, *satisfying one of the two following conditions:*

(1) $\{\rho, \rho'\}$ *is not the edge of* Γ, *i.e.* $\rho \neq \nu\rho'$ *and* $\rho' \neq \nu\rho$.

(2) $\lambda^{-1}(\rho') < \lambda^{-1}(\rho)$.

Then $\Omega(\pi') \ni \rho(\mu')$, *where* $\mu' = \mu \circ (j, j+1) \in S$ *is obtained from* μ *by the transposition of* ρ *and* ρ'.

Proof. — (1) Let $\mu_0 = \mu$ if $\lambda^{-1}(\rho) < \lambda^{-1}(\rho')$ and $\mu_0 = \mu'$ if $\lambda^{-1}(\rho') < \lambda^{-1}(\rho)$. Let $\beta = \beta(\lambda)$ and γ be obtained from $\beta(\mu_0)$ by the union of its j-th and $(j+1)$-th blocks into a single block. Put $\sigma = i_{\gamma, \beta(\mu_0)}(\rho(\mu_0)) \in \operatorname{Alg} G_\gamma$. We shall prove that σ is a subquotient of $r_{\gamma, (n)}(\pi(\lambda))$.

We note that $r_{\gamma, (n)}(\pi(\lambda)) = r_{\gamma, (n)} \circ i_{(n), \beta}(\rho(\lambda))$ and apply Theorem 1.2. Consider the permutation $w \in W_n$ which transfers i-th block of β to $\mu_0^{-1} \circ \lambda(i)$-th block of $\beta(\mu_0)$. Clearly $w \in W^{\beta, \gamma}$ and $\sigma = F_w(\rho(\lambda))$ (*see* 1.2). So our statement follows directly from Theorem 1.2.

(2) Now we shall prove that $r_{\gamma, (n)}(\pi')$ has a non-zero subquotient which is a submodule of σ.

By step (1) $r_{\gamma, (n)}(\pi(\lambda))$ is glued together from σ and some other modules $\sigma_1, \sigma_2, \ldots, \sigma_k$. Using 1.1 (*a*) and (*c*), one obtains that

$$r_{\beta(\mu), (n)}(\pi(\lambda)) = r_{\beta(\mu), \gamma} \circ r_{\gamma, (n)}(\pi(\lambda))$$

is glued together from $r_{\beta(\mu), \gamma}(\sigma)$ and $r_{\beta(\mu), \gamma}(\sigma_i)$ $(i = 1, \ldots, k)$. Evaluating $r_{\beta(\mu), \gamma}(\sigma)$ as in 1.6, we obtain that $\rho(\mu) \in \mathscr{J} H^0(r_{\beta(\mu), \gamma}(\sigma))$. Since $\Omega(\pi(\lambda))$ is multiplicity-free [*see* 2.1 (*a*)] it follows that $\rho(\mu) \notin \mathscr{J} H^0(r_{\beta(\mu), \gamma}(\sigma_i))$ for $i = 1, \ldots, k$.

The exactness of $r_{\gamma, (n)}$ implies that $r_{\gamma, (n)}(\pi')$ is a submodule of $r_{\gamma, (n)}(\pi(\lambda))$. So $r_{\gamma, (n)}(\pi')$ is glued from some submodules $\sigma' \subset \sigma$ and $\sigma_i' \subset \sigma_i$ $(i = 1, \ldots, k)$. It remains only to prove that $\sigma' \neq 0$. By assumption $\rho(\mu) \in \Omega(\pi')$. Since $\rho(\mu) \notin \mathscr{J} H^0(r_{\beta(\mu), \gamma}(\sigma_i'))$ it follows that $\rho(\mu) \in \mathscr{J} H^0(r_{\beta(\mu), \gamma}(\sigma'))$ so $\sigma' \neq 0$ and our statement is proved.

(3) Now take into account conditions (1) and (2). By 1.11 if one of them holds then either σ is irreducible or it has the unique proper submodule σ' and $r_{\beta(\mu'), \gamma}(\sigma') = \rho(\mu')$. In each case $\rho(\mu') \in \mathscr{J} H^0(r_{\beta(\mu), \gamma}(\sigma'))$ for any non-zero submodule σ' of σ. By step (2) $\rho(\mu') \in \Omega(\pi')$ and our Lemma is proved.

2.7. Call an elementary transposition of an ordering $\lambda \in S$ the transposition of two neighbours in λ which don't belong to the same edge of Γ.

LEMMA. — (*a*) *Let* $\lambda, \mu \in S$. *Then the following conditions are equivalent:*

(1) $\vec{\Gamma}(\lambda) = \vec{\Gamma}(\mu)$ (see 2.2).

(2) μ *may be obtained from* λ *by a chain of elementary transpositions.*

(*b*) *For each orientation* $\vec{\Gamma}$ *of* Γ *define the subset* $\Omega(\vec{\Gamma}) \subset \Omega$ *by*

$$\Omega(\vec{\Gamma}) = \{ \rho(\lambda) \mid \vec{\Gamma}(\lambda) = \vec{\Gamma} \}.$$

Then for each $\omega \in \mathcal{J}$ *the set* $\Omega(\omega)$ *is the union of such subsets.*

(*c*) *If* $\vec{\Gamma}(\lambda)=\vec{\Gamma}(\mu)$ *then* $\pi(\lambda) \simeq \pi(\mu)$.

Proof. — The easy combinatorial proof of (*a*) is omitted. Part (*b*) follows immediately from (*a*) and 2.6 since ω may be embedded into some $\pi(\lambda)$ (*see* 1.10).

(*c*) By (*a*) it suffices to consider the case when λ is obtained from μ by an elementary transposition. In this case our statement follows immediately from associativity and commutativity of product (*see* 1.7, 1.9) and 1.11 (*a*).

2.8. Denote by $\mathcal{O}(\Gamma)$ the set of all orientations of Γ. By 2.7 (*b*) for each submodule π' of some $\pi(\lambda)$ one has

$$\Omega(\pi')=\Omega(\vec{\Gamma}_1) \cup \ldots \cup \Omega(\vec{\Gamma}_k) \qquad (\vec{\Gamma}_1, \ldots, \vec{\Gamma}_k \in \mathcal{O}(\Gamma)).$$

Denote the set $\{\vec{\Gamma}_1, \ldots, \vec{\Gamma}_k\}$ by $\mathcal{O}(\pi')$. By 2.5 the map $\pi' \mapsto \mathcal{O}(\pi')$ is an isomorphism of the lattice of submodules of $\pi(\lambda)$ with some sublattice $\mathcal{B}(\lambda)$ of the lattice of all subsets of $\mathcal{O}(\Gamma)$.

Denote by $\overline{\mathcal{B}}(\lambda)$ the lattice consisting of all subsets $\mathcal{O} \subset \mathcal{O}(\Gamma)$ satisfying the following condition:

($\star$) If $\vec{\Gamma} \in \mathcal{O}$ and the edge γ of Γ has different orientations in $\vec{\Gamma}$ and $\vec{\Gamma}(\lambda)$ then the orientation $\vec{\Gamma}'$, obtained from $\vec{\Gamma}$ by the inversion of the direction on γ, also belongs to $\mathcal{O}$.

For each edge γ of Γ and $\lambda \in S$ consider the subset $\mathcal{O}(\lambda, \gamma)=\{\vec{\Gamma} \mid \gamma$ has the same directions in $\vec{\Gamma}$ and $\vec{\Gamma}(\lambda)\} \subset \mathcal{O}(\Gamma)$. Denote by $\underline{\mathcal{B}}(\lambda)$ the lattice of subsets of $\mathcal{O}(\Gamma)$ generated by the $\mathcal{O}(\lambda, \gamma)$, where γ ranges over all edges of Γ.

THEOREM. — $\underline{\mathcal{B}}(\lambda)=\mathcal{B}(\lambda)=\overline{\mathcal{B}}(\lambda)$.

Proof. — (1) The equality $\underline{\mathcal{B}}(\lambda)=\overline{\mathcal{B}}(\lambda)$ is of combinatorial nature and it holds for each finite graph Γ. Obviously each $\mathcal{O}(\lambda, \gamma)$ satisfies ($\star$) so $\underline{\mathcal{B}}(\lambda) \subset \overline{\mathcal{B}}(\lambda)$. Let $\mathcal{O} \in \overline{\mathcal{B}}(\lambda)$. Consider the orientation $\vec{\Gamma}^t$ obtained from $\vec{\Gamma}(\lambda)$ by inversion of the direction on all edges of Γ. If $\vec{\Gamma}^t \in \mathcal{O}$ then by ($\star$) $\mathcal{O}=\mathcal{O}(\Gamma)$. If $\vec{\Gamma}^t \notin \mathcal{O}$ then $\mathcal{O} \subset \bigcup_\gamma \mathcal{O}(\lambda, \gamma)$ so $\mathcal{O}=\cup \mathcal{O}_\gamma$ where $\mathcal{O}_\gamma=\mathcal{O} \cap \mathcal{O}(\lambda, \gamma)$. Using obvious induction on the number of edges of Γ, one may assume each $\mathcal{O}_\gamma$ to belong to $\underline{\mathcal{B}}(\lambda)$. It follows that $\mathcal{O} \in \underline{\mathcal{B}}(\lambda)$.

(2) For each $\vec{\Gamma} \in \mathcal{O}(\Gamma)$ and $\rho, \rho' \in R$ there exists $\lambda \in S$ such that $\vec{\Gamma}(\lambda)=\vec{\Gamma}$ and ρ and ρ' are neighbours in λ. The simple combinatorial proof of this statement is omitted.

(3) $\mathcal{B}(\lambda) \subset \overline{\mathcal{B}}(\lambda)$: it follows immediately from (2) and 2.6 (2).

(4) To prove $\underline{\mathcal{B}}(\lambda) \subset \mathcal{B}(\lambda)$ one has only to construct the submodule $\pi(\lambda, \gamma)$ of $\pi(\lambda)$ such that $\mathcal{O}(\pi(\lambda, \gamma))=\mathcal{O}(\lambda, \gamma)$. Let $\gamma=\{\rho, \rho'\}$. By (2) and 2.7 (*c*) one may assume that ρ and ρ' are neighbours in λ so

$$\pi(\lambda)=\lambda(1) \times \ldots \times \lambda(i-1) \times \rho \times \rho' \times \lambda(i+2) \times \ldots \times \lambda(r).$$

By 1.11 the representation $\rho \times \rho'$ has the unique proper submodule ω and we set

$$\pi(\lambda, \gamma)=\lambda(1) \times \ldots \times \lambda(i-1) \times \omega \times \lambda(i+2) \times \ldots \times \lambda(r).$$

Using 1.6 and 1.11 one can easily compute $\Omega(\pi(\lambda, \gamma))$ and obtain the desired $\mathcal{O}(\pi(\lambda, \gamma))=\mathcal{O}(\lambda, \gamma)$. The Theorem is done.

2.9. Corollary. — *The lattice of submodules of* $\pi(\lambda)$ *is generated by submodules* $\pi(\lambda, \gamma)$ *where* γ *ranges over all edges of* Γ.

2.10. Proposition. — *The representation* $\pi(\lambda)$ *has the unique irreducible submodule* $\omega(\lambda)$ *and the unique irreducible factor-module* $\omega(\lambda)^t$. *Moreover*

$$\Omega(\omega(\lambda))=\Omega(\vec{\Gamma}(\lambda)), \qquad \Omega(\omega(\lambda)^t)=\Omega(\vec{\Gamma}^t) \quad \text{(see 2.7)}$$

where $\vec{\Gamma}^t$ *is obtained from* $\vec{\Gamma}(\lambda)$ *by the inversion of the direction on each edge of* Γ.

Proof. — By 2.9 $\pi(\lambda)$ has the unique minimal and maximal proper submodules namely $\bigcap_\gamma \pi(\lambda, \gamma)$ and $\sum_\gamma \pi(\lambda, \gamma)$ respectively. By 2.8 (4) one has

$$\mathcal{O}(\bigcap_\gamma \pi(\lambda, \gamma))=\bigcap_\gamma \mathcal{O}(\lambda, \gamma)=\{\vec{\Gamma}(\lambda)\},$$

$$\mathcal{O}(\sum_\gamma \pi(\lambda, \gamma))=\bigcup_\gamma \mathcal{O}(\lambda, \gamma)=\mathcal{O}(\Gamma)\diagdown\{\vec{\Gamma}^t\}.$$

So $\omega(\lambda)=\bigcap_\gamma \pi(\lambda, \gamma)$, $\omega(\lambda)^t=\pi(\lambda)/\sum_\gamma \pi(\lambda, \gamma)$ and the Proposition is proven.

Note that Theorem 2.2 follows immediately from this proposition.

2.11. Corollary. — *The following conditions are equivalent:*

(1) $\vec{\Gamma}(\lambda)=\vec{\Gamma}(\mu)$.

(2) $\pi(\lambda)\simeq\pi(\mu)$.

(3) $\omega(\lambda)\simeq\omega(\mu)$.

This follows immediately from 2.7 (*c*), 2.10 *and* 2.2.

3. Segments and corresponding representations

In this section we introduce and study a class of irreducible representations of the groups G_n which plays the main role in our classification of irreducible representations.

3.1. Recall that we denote by $\mathcal{C}$ the set of equivalence classes of irreducible cuspidal representations of groups G_n $(n=1, 2, \ldots)$. Call a segment in $\mathcal{C}$ a subset $\Delta\subset\mathcal{C}$ of the form $\Delta=\{\rho, \nu\rho, \ldots, \nu^k\rho=\rho'\}$ (k is an integer $\geqq 0$); we write $\Delta=[\rho, \rho']$. The element ρ is called the beginning of Δ and ρ' the end of Δ.

Let $\Delta=[\rho, \rho']$ be a segment in $\mathcal{C}$. Denote by $\langle\Delta\rangle$ the irreducible representation with support $\{\rho, \nu\rho, \ldots, \rho'\}$, which corresponds to the orientation

$$\rho\to\nu\rho\to\nu^2\rho\to\ldots\to\rho' \quad (\textit{see } 2.2).$$

Let $\rho\in\operatorname{Irr} G_m$, $\rho'=\nu^{k-1}\rho$ so $\langle\Delta\rangle\in\operatorname{Irr} G_n$, where $n=km$. Let $\beta=(m, m, \ldots, m)$ be a partition of n. Then by definition

$$r_{\beta, (n)}(\langle\Delta\rangle)=\rho\otimes\nu\rho\otimes\ldots\otimes\rho'$$

and $\langle \Delta \rangle$ may be defined by this property. Another way to define $\langle \Delta \rangle$ is to say that $\langle \Delta \rangle$ is the unique irreducible submodule of $\rho \times \nu\rho \times \ldots \times \rho'$ (or the unique irreducible quotient module of $\rho' \times \nu^{-1} \rho' \times \ldots \times \rho$), *see* 2.10.

In particular any $\rho \in \mathscr{C}$ is expressed as $\langle [\rho, \rho] \rangle$; it is convenient to denote by $\langle \emptyset \rangle$ the identity representation of the group $G_0 = \{e\}$.

3.2. *Example*. — Let $\rho \in \operatorname{Irr} G_1$. Since $G_1 = F^*$ is abelian ρ is one-dimensional, i. e. it is a multiplicative character of F. If $\rho' = \nu^k \rho$ and $\Delta = [\rho, \rho']$ then $\omega = \langle \Delta \rangle \in \operatorname{Irr} G_{k+1}$ is one-dimensional:

$$\omega(g) = \nu^{k/2}(g) . \rho(\det g), \qquad g \in G_{k+1}.$$

The easiest way to check it is to compute $r_{(1, 1, \ldots, 1), (k+1)}(\omega)$ directly by definition (*see* [1], 1.8).

3.3. For each segment $\Delta = [\rho, \rho']$ in $\mathscr{C}$ set $\tilde{\Delta} = \{ \tilde{\rho} \mid \rho \in \Delta \}$ ($\tilde{\rho}$ is contragredient to ρ). It is clear that $\tilde{\Delta}$ is a segment in $\mathscr{C}$ with the beginning $\tilde{\rho}'$ and the end $\tilde{\rho}$.

PROPOSITION. — $\widetilde{\langle \Delta \rangle} \doteq \langle \tilde{\Delta} \rangle$.

Proof. — Since $\langle \Delta \rangle$ is a submodule of $\rho \times \nu\rho \times \ldots \times \rho'$ one has $\widetilde{\langle \Delta \rangle}$ to be a quotient of $(\widetilde{\rho \times \ldots \times \rho'}) = \tilde{\rho} \times \tilde{\nu\rho} \times \ldots \times \tilde{\rho}'$. So $\widetilde{\langle \Delta \rangle} = \langle \tilde{\Delta} \rangle$, *see* 3.1.

3.4. PROPOSITION. — *Let* $\Delta = [\rho, \rho']$ *be a segment in* $\mathscr{C}$, $\rho \in \operatorname{Irr} G_m$, $\rho' = \nu^{k-1} \rho$ *so* $\langle \Delta \rangle \in \operatorname{Irr} G_n$, $n = km$. *If* l *is not divisible by* m *then* $r_{(l, n-l), (n)}(\langle \Delta \rangle) = 0$. *If* $l = mp$ *then*

$$r_{(l, n-l), (n)}(\langle \Delta \rangle) = \langle [\rho, \nu^{p-1} \rho] \rangle \otimes \langle [\nu^p \rho, \rho'] \rangle.$$

In other words

$$c(\langle \Delta \rangle) = \langle \emptyset \rangle \otimes \langle \Delta \rangle + \rho \otimes \langle [\nu\rho, \rho'] \rangle + \langle [\rho, \nu\rho] \rangle \otimes \langle [\nu^2 \rho, \rho'] \rangle + \ldots + \langle [\rho, \nu^{-1} \rho'] \rangle \otimes \rho' + \langle \Delta \rangle \otimes \langle \emptyset \rangle \quad \text{(see 1.7)}.$$

Proof. — Let $\pi = \rho \times \nu\rho \times \ldots \times \rho'$, $\beta = (m, \ldots, m) < (n)$. By [1], 2.13 (*a*) (or 1.6) $r_{\gamma, (n)}(\pi) = 0$ if $\beta \not\leq \gamma$. In particular $r_{(l, n-l), (n)}(\pi) = 0$ if l is not divisible by m so $r_{(l, n-l), (n)}(\langle \Delta \rangle) = 0$.

Let now $l = mp$ and $\gamma = (l, n-l) < n$. It follows easily from 1.10 that $r_{\beta, \gamma}(\sigma) \neq 0$ for each composition factor σ of $r_{\gamma, (n)}(\langle \Delta \rangle)$. Since

$$r_{\beta, \gamma} \circ r_{\gamma, (n)}(\langle \Delta \rangle) = r_{\beta, (n)}(\langle \Delta \rangle) = \rho \otimes \nu\rho \otimes \ldots \otimes \rho' \quad [\textit{see } 1.1\ (c),\ 3.1]$$

is irreducible, one obtains that $r_{\gamma, (n)}(\langle \Delta \rangle)$ is irreducible so

$$r_{\gamma, (n)}(\langle \Delta \rangle) = \sigma_1 \otimes \sigma_2, \qquad \sigma_1 \in \operatorname{Irr} G_l, \quad \sigma_2 \in \operatorname{Irr} G_{n-l}.$$

Moreover

$$r_{\beta, \gamma}(\sigma_1 \otimes \sigma_2) = r_{(m, \ldots, m), (l)}(\sigma_1) \otimes r_{(m, \ldots, m), (n-l)}(\sigma_2) \quad [\textit{see } 1.5\ (a)]$$

hence

$$r_{(m, \ldots, m), (l)}(\sigma_1) = \otimes \nu\rho \otimes \ldots \otimes \nu^{p-1}\rho,$$
$$r_{(m, \ldots, m), (n-l)}(\sigma_2) = \nu^p \rho \otimes \ldots \otimes \rho'.$$

By 3.1 $\sigma_1 = \langle [\rho, \nu^{p-1}\rho] \rangle$, $\sigma_2 = \langle [\nu^p \rho, \rho'] \rangle$ and the Proposition is proven.

3.5. Now we compute the derivatives of the representation $\langle \Delta \rangle \in \operatorname{Irr} G_n$, i.e. its restriction to the subgroup $P_n \subset G_n$ (*see* [1], 4.3). For each segment $\Delta = [\rho, \rho']$ in $\mathscr{C}$ set $\Delta^- = \Delta \setminus \{\rho'\}$ (in particular if $\Delta = \{\rho\}$ then $\Delta^- = \emptyset$).

THEOREM. — *Let Δ be a segment in $\mathscr{C}$. Then exactly one of derivatives $\langle \Delta \rangle^{(k)}$ for $k > 0$ is non-zero and this derivative equals $\langle \Delta^- \rangle$. In other words*

$$\mathscr{D}(\langle \Delta \rangle) = \langle \Delta \rangle + \langle \Delta^- \rangle \quad \text{(see [1], 4.5).}$$

3.6. *Remark.* — Theorem 3.5 means that the restriction of $\langle \Delta \rangle$ to the subgroup P_n remains irreducible and is isomorphic to $(\Phi^+)^{m-1} \circ \Psi^+(\langle \Delta^- \rangle)$ (where elements of Δ belong to $\operatorname{Irr} G_m$), *see* [1], 3.5. In fact representations of the form $\langle \Delta \rangle$ may be characterized as those irreducible representations of G_n that remain irreducible, when restricted to P_n (this will be proven in paragraph 7).

3.7. The rest of this section is devoted to the proof of Theorem 3.5. We begin with connections between derivatives and functors $r_{\beta, (n)}$.

Let k, m be integer $\geqq 0$. We will define the functor

$$\partial : \quad \operatorname{Alg} G_{(k, m)} \to \operatorname{Alg} G_k \quad \text{(partial derivative).}$$

Let U be the subgroup of unipotent upper triangular matrices in

$$G_m = \{e\} \times G_m \subset G_k \times G_m = G_{(k, m)}.$$

Define the character θ of U by

$$\theta((u_{ij})) = \left(\sum_{1 \leqq i \leqq m-1} u_{k+i, k+i+1} \right).$$

(here Ψ is the additive character of the field F, which was used in definition of derivatives, *see* [1], § 3). Set

$$\partial = r_{U, \theta} : \quad \operatorname{Alg} G_{(k, m)} \to \operatorname{Alg} G_k \quad (\textit{see } [1], 1.8).$$

PROPOSITION. — (*a*) *If $\sigma \in \operatorname{Alg} G_k$, $\tau \in \operatorname{Alg} G_m$ then*

$$\partial(\sigma \otimes \tau) = \sigma \otimes \tau^{(m)} \in \operatorname{Alg}(G_k \times G_0) = \operatorname{Alg} G_k.$$

(*b*) *If $n = k + m$ then the functor of m-th derivative $\operatorname{Alg} G_n \to \operatorname{Alg} G_k$ is isomorphic to the composition $\partial \circ r_{(k, m), (n)}$.*

The Proposition follows immediately from the definition of m-th derivative and [1], 1.9 (*c*), (*g*).

3.8. Let us reformulate Proposition 3.7 in terms of the representation bialgebra $\mathscr{R}$ (*see* 1.7). Consider the $\mathbb{Z}$-linear form

$$\delta : \mathscr{R} \to \mathscr{R}(G_0) = \mathbb{Z},$$

defined by $\delta(\pi) = \pi^{(n)}$ for $\pi \in \operatorname{Alg} G_n$. Lemma 4.5 from [1] implies that δ is a ring homomorphism. Note that

(!) $$\delta(\rho) = 1 \quad \text{for any} \quad \rho \in \mathscr{C} \quad (\textit{see } [1], 4.4).$$

This deep result of Gelfand-Kazhdan is a corner-stone of our theory.

PROPOSITION. — *The homomorphism* $\mathscr{D} : \mathscr{R} \to \mathscr{R}$ (see [1], 4.5) *equals the composition*

$$\mathscr{R} \xrightarrow{c} \mathscr{R} \otimes \mathscr{R} \xrightarrow{\text{id} \otimes \delta} \mathscr{R} \otimes \mathbb{Z} \simeq \mathscr{R} \quad (\textit{see } 1.7).$$

This follows immediately from the definitions and 3.7.

3.9. LEMMA. — *Let* $\Delta = [\rho, \rho']$ *be a segment in* $\mathscr{C}$ *and* $\rho \neq \rho'$. *Then the representation* $\langle \Delta \rangle \in \operatorname{Irr} G_n$ *is degenerate (this means* $\langle \Delta \rangle^{(n)} = 0$ *or* $\delta(\langle \Delta \rangle) = 0$).

Proof. — (1) Consider the case $\rho' = \nu\rho$. Let $\rho \in \operatorname{Irr} G_m$ hence $n = 2m$. By 1.11 (*b*) $\mathscr{J}\, H^0(\rho \times \rho') = \{\omega, \omega'\}$ $(\omega = \langle \Delta \rangle)$ where

$$r_{(m, m), (n)}(\omega) = \rho \otimes \rho', \qquad r_{(m, m), (n)}(\omega') = \rho' \otimes \rho.$$

By [1], 2.13 (*a*) $r_{(n-l, l), (n)}(\omega) = r_{(n-l, l), (n)}(\omega') = 0$ if $l \neq 0, m, n$. Applying 3.7 and 3.8 one obtains

$$\omega^{(l)} = \omega'^{(l)} = 0 \qquad \text{for} \quad l \neq 0, m, n; \qquad \omega^{(m)} = \rho, \qquad \omega'^{(m)} = \rho'.$$

Moreover $\delta(\omega) + \delta(\omega') = \delta(\rho \times \rho') = \delta(\rho).\delta(\rho') = 1$ so exactly one of ω and ω' is non-degenerate. Suppose ω is non-degenerate. Then ω' is degenerate so its highest derivative equals ρ'. But this contradicts [1], Lemma 4.7 (*b*). Therefore $\omega = \langle \Delta \rangle$ is degenerate.

(2) General case. Set $\Delta' = [\rho, \nu\rho]$, $\Delta'' = \Delta \setminus \Delta'$. Then

$$\langle \Delta' \rangle \subset \rho \times \nu\rho, \qquad \langle \Delta'' \rangle \subset \nu^2 \rho \times \ldots \times \rho'$$

hence $\langle \Delta' \rangle \times \langle \Delta'' \rangle \subset \rho \times \nu\rho \times \ldots \times \rho'$. Since $\langle \Delta \rangle$ is the unique irreducible submodule of $\rho \times \ldots \times \rho'$, $\langle \Delta \rangle \subset \langle \Delta' \rangle \times \langle \Delta'' \rangle$ (another way to prove it is to use 1.1 and 3.4). It is clear that

$$0 \leqq \delta(\langle \Delta \rangle) \leqq \delta(\langle \Delta' \rangle \times \langle \Delta'' \rangle).$$

But by step (1):

$$\delta(\langle \Delta' \rangle \times \langle \Delta'' \rangle) = \delta(\langle \Delta' \rangle).\delta(\langle \Delta'' \rangle) = 0$$

and our Lemma is proven.

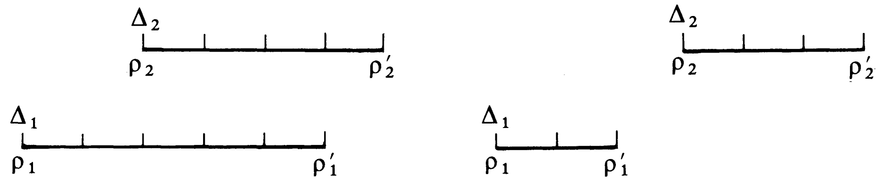
3.10. Theorem 3.5 follows immediately from 3.8, 3.4 and 3.9.

4. Criterion for irreducibility of the product

$$\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle.$$

In this section we establish a criterion for irreducibility of the product $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$, where the Δ_i are segments in $\mathscr{C}$ (Thm. 4.2). Furthermore we give a complete description of the lattice of submodules of the product $\langle \Delta \rangle \times \langle \Delta' \rangle$ (Prop. 4.6).

4.1. Let $\Delta_1 = [\rho_1, \rho'_1]$, $\Delta_2 = [\rho_2, \rho'_2]$ be segments in $\mathscr{C}$. We say that Δ_1 and Δ_2 are linked if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$ and $\Delta_1 \cup \Delta_2$ is also a segment. If Δ_1 and Δ_2 are linked and $\Delta_1 \cap \Delta_2 = \emptyset$ then we say that Δ_1 and Δ_2 are juxtaposed (this means that either $\rho_2 = \nu \rho'_1$ or $\rho_1 = \nu \rho'_2$). If Δ_1 and Δ_2 are linked and $\rho_2 = \nu^k \rho_1$ where $k > 0$ then we say that Δ_1 precedes Δ_2:



(*a*) (*b*)

[in figure (*a*) Δ_1 and Δ_2 are linked, in (*b*) Δ_1 and Δ_2 are linked and juxtaposed, in both figures Δ_1 precedes Δ_2].

We mention one trivial but useful property of these notions. If $\rho_1 \neq \rho'_1$, $\rho_2 \neq \rho'_2$ then the following conditions are equivalent:

(1) The segments Δ_1^- and Δ_2^- are linked (*see* 3.5).

(2) The segments Δ_1 and Δ_2 are linked but not juxtaposed.

4.2. Theorem. — *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathscr{C}$. The following conditions are equivalent:*

(1) *The representation $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ is irreducible.*

(2) *For each $i, j = 1, \ldots, r$ the segments Δ_i and Δ_j are not linked.*

4.3. Let $\pi \in \operatorname{Alg} G_n$ or $\pi \in \operatorname{Alg} P_n$. The maximal number k such that $\pi^{(k)} \neq 0$ is called the level (of non-degeneracy) of π [notation $k = \lambda(\pi)$]. In other words $\pi^{(\lambda(\pi))}$ is the highest derivative of π (*see* [1], 4.3).

Proposition. — *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathscr{C}$ and $\pi = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$. Suppose that Δ_i and Δ_j are not juxtaposed for any $i, j = 1, \ldots, r$. Then for each $\omega \in \mathscr{J}\, H(\pi)$ one has $\lambda(\omega) = \lambda(\pi)$.*

First we derive the implication (2) $\Rightarrow$ (1) in 4.2 from this proposition. In condition 4.2 (2) let $\pi = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle \in \operatorname{Alg} G_n$. By 3.5 and [1], 4.6 the highest derivative of π equals

$$\pi^{(k)} = \langle \Delta_1^- \rangle \times \ldots \times \langle \Delta_r^- \rangle.$$

By 4.1 no Δ_i^- and Δ_j^- are linked. Using induction on n one may assume that $\pi^{(k)}$ is irreducible. So exactly one element of $\mathscr{J}\, H^0(\pi)$ has the same level as π. On the other hand our proposition implies that each element of $\mathcal{J}\,H(\pi)$ has the same level as π. So $\mathcal{J}\,H^0(\pi)$ contains only one element, i. e. π is irreducible.

4.4. For each $\omega\in \operatorname{Irr} G_n$ denote by $\varphi_\omega : \mathcal{C}\to \mathbb{Z}_+$ the characteristic function of supp ω (*see* 1.10 and the summary of notation). The correspondence $\omega\mapsto\varphi_\omega$ has the following properties:

(*a*) $\operatorname{supp}\omega'\subset\operatorname{supp}\omega \Leftrightarrow \varphi_\omega-\varphi_{\omega'}\geqq 0$.

(*b*) If $\omega\in\mathcal{J}\,H(\omega_1\times\ldots\times\omega_r)$ then $\varphi_\omega=\varphi_{\omega_1}+\ldots+\varphi_{\omega_r}$.

(*c*) If Δ is a segment in $\mathcal{C}$ then $\varphi_{\langle\Delta\rangle}=\chi_\Delta$ is the characteristic function of the subset $\Delta\subset\mathcal{C}$.

(*d*) If $\omega\in\operatorname{Irr} G_n$, $k=\lambda(\omega)$, and σ is an irreducible submodule of $\omega^{(k)}$ then $\nu\varphi_\omega-\varphi_\sigma\geqq 0$ where $\nu\varphi(\rho)=\varphi(\nu\rho)$ (*see* [1], 4.7 (*b*)).

4.5. *Proof of Proposition* 4.3. — By 3.5 and [1], 4.6 all derivatives of π are glued together from representations of the form $\langle\Delta'_1\rangle\times\ldots\times\langle\Delta'_r\rangle$ where each Δ'_i is either Δ_i or Δ_i^-. Let $\omega\in\mathcal{J}\,H(\pi)$, $k=\lambda(\omega)$, and σ be an irreducible submodule of $\omega^{(k)}$. Then $\sigma\in\mathcal{J}\,H(\langle\Delta'_1\rangle\times\ldots\times\langle\Delta'_r\rangle)$ for some $\Delta'_1,\ldots,\Delta'_r$. One has

$$\varphi_\omega=\chi_{\Delta_1}+\ldots+\chi_{\Delta_r},\qquad \varphi_\sigma=\chi_{\Delta'_1}+\ldots+\chi_{\Delta'_r}\quad [\textit{see } 4.4\ (b), (c)].$$

By 4.4 (*d*) $\nu\varphi_\omega-\varphi_\sigma\geqq 0$. Let $\Delta_i=[\rho_i,\rho'_i]$. It is clear that

$$\nu\varphi_\omega-\varphi_\sigma=\sum_{1\leqq i\leqq r}\chi_{\{\nu^{-1}\rho_i\}}-\sum_j\chi_{\{\rho'_j\}},$$

where j ranges over such indices that $\Delta'_j=\Delta_j$. Therefore if $\Delta'_j=\Delta_j$ for some j then ρ'_j equals one of the representations $\nu^{-1}\rho_i$. This means that Δ_i and Δ_j are juxtaposed, which contradicts the condition of 4.3. So $\Delta'_i=\Delta_i^-$ for $i=1,\ldots,r$ and our proposition follows.

4.6. To prove the implication (1) $\Rightarrow$ (2) in 4.2 it suffices to check the reducibility of $\langle\Delta\rangle\times\langle\Delta'\rangle$ where Δ and Δ' are linked segments. We get the more precise information about this representation. Set

$$\Delta^\cup=\Delta\cup\Delta',\qquad \Delta^\cap=\Delta\cap\Delta'.$$

By definition $\Delta^\cup$ is a segment in $\mathcal{C}$; $\Delta^\cap$ is a segment if Δ and Δ' are not juxtaposed otherwise $\Delta^\cap=\emptyset$. Set $\omega=\langle\Delta^\cup\rangle\times\langle\Delta^\cap\rangle$. By the implication (2) $\Rightarrow$ (1) in 4.2 which is already proved, ω is irreducible.

PROPOSITION. — *Suppose* Δ' *precedes* Δ (see 4.1) *and set* $\pi=\langle\Delta\rangle\times\langle\Delta'\rangle$. *Then* π *has the unique irreducible submodule* ω_0. *Moreover* $\omega_0\not\simeq\omega$ *and* $\pi/\omega_0\simeq\omega$.

Proof. — (1) Suppose Δ and Δ' are juxtaposed. Let

$$\Delta'=[\rho,\nu^{p-1}\rho],\qquad \Delta=[\nu^p\rho,\nu^{r-1}\rho]\qquad \text{where}\quad 0<p<r$$

and $\rho\in\operatorname{Irr} G_m$. Apply the results of paragraph 2 to the case $\rho_i=\nu^{i-1}\rho$ ($i=1,\ldots,r$). Using 1.6 and 3.1 one may easily compute the set

$$\Omega(\pi)=\mathcal{J}\,H^0(r_{(m,\ldots,m),(rm)}\circ i_{(rm),((r-p)m,pm)}(\langle\Delta\rangle\otimes\langle\Delta'\rangle)).$$

One obtains

$$\mathscr{J}H^0(\pi)=\{\omega(\vec{\Gamma}_1),\ \omega(\vec{\Gamma}_2)\}$$

where $\vec{\Gamma}_1$ and $\vec{\Gamma}_2$ are orientations

$$\rho_1\to\rho_2\to\ldots\to\rho_r \qquad\text{and}\qquad \rho_1\to\rho_2\to\ldots\to\rho_p\leftarrow\rho_{p+1}\to\ldots\to\rho_r$$

(*see* 2.2). By 2.10 π has the unique irreducible submodule ω_0 and $\omega_0\simeq\omega(\vec{\Gamma}_2)$. By definition $\omega(\vec{\Gamma}_1)=\langle\Delta^{\cup}\rangle=\omega$ so in this case the Proposition is proven.

(2) Suppose now that Δ and Δ' are linked but not juxtaposed. Let us prove that π has a quotient isomorphic to ω i.e. $\operatorname{Hom}(\pi,\omega)\neq 0$.

Let

$$\Delta'=[\rho,\ \nu^{p-1}\rho],\qquad \Delta=[\nu^q\rho,\ \nu^{r-1}\rho]\qquad (0<q<p<r)$$

and $\rho\in\operatorname{Irr}G_m$. Then $\pi,\ \omega\in\operatorname{Alg}G_n$ where $n=(p+r-q)m$. Let

$$\beta=((r-q)m,\ pm),\qquad \gamma=(rm,\ (p-q)m),$$

be partitions of n. By definition

$$\pi=i_{(n),\beta}(\langle\Delta\rangle\otimes\langle\Delta'\rangle),\qquad \omega=i_{(n),\gamma}(\langle\Delta^{\cup}\rangle\otimes\langle\Delta^{\cap}\rangle).$$

So by 1.1 (*b*):

$$\operatorname{Hom}(\pi,\omega)=\operatorname{Hom}(\sigma,\ \langle\Delta^{\cup}\rangle\otimes\langle\Delta^{\cap}\rangle)$$

where

$$\sigma=r_{\gamma,(n)}\circ i_{(n),\beta}(\langle\Delta\rangle\otimes\langle\Delta'\rangle).$$

One must prove that σ has a quotient isomorphic to $\langle\Delta^{\cup}\rangle\otimes\langle\Delta^{\cap}\rangle$. Now use Corollary 1.3. One obtains that σ has the quotient $\sigma'=i_{\gamma,\gamma\cap\beta}\circ r_{\gamma\cap\beta,\beta}(\langle\Delta\rangle\otimes\langle\Delta'\rangle)$. Using 3.4 and 1.5 one can show that

$$\sigma'=(\langle\Delta\rangle\times\langle\Delta'\diagdown\Delta^{\cap}\rangle)\otimes\langle\Delta^{\cap}\rangle.$$

Segments Δ and $\Delta'\diagdown\Delta^{\cap}$ are juxtaposed and $\Delta'\diagdown\Delta^{\cap}$ precedes Δ; so by step (1) $\langle\Delta\rangle\times\langle\Delta'\diagdown\Delta^{\cap}\rangle$ has a quotient isomorphic to $\langle\Delta\cup(\Delta'\diagdown\Delta^{\cap})\rangle=\langle\Delta^{\cup}\rangle$. It follows that σ' (and so σ) has a quotient isomorphic to $\langle\Delta^{\cup}\rangle\otimes\langle\Delta^{\cap}\rangle$, hence $\operatorname{Hom}(\pi,\omega)\neq 0$.

(3) In the situation of (2) consider the highest derivative $\pi^{(k)}$ of π. By 3.5 and [1], 4.5 one has $\pi^{(k)}=\langle\Delta^{-}\rangle\times\langle\Delta'^{-}\rangle$. The segments Δ^{-} and Δ'^{-} are linked (*see* 4.1); induction on n allows us to assume that the statements of our Proposition hold for them. Since Δ and Δ' are not juxtaposed, Proposition 4.3 implies that the map $\tau\mapsto\tau^{(k)}$ induces the embedding of the lattice of submodules of π into the lattice of submodules of $\pi^{(k)}$. Therefore to prove our Proposition it remains only to check that π is reducible. By step (2) it suffices to prove that $\pi\not\simeq\omega$. One has $\omega^{(k)}=\langle(\Delta^{\cup})^{-}\rangle\times\langle(\Delta^{\cap})^{-}\rangle$. Using induction on n, one may assume $\pi^{(k)}\not\simeq\omega^{(k)}$. Consequently $\pi\not\simeq\omega$ and our Proposition is done.

The remaining part of Theorem 4.2 follows immediately from this Proposition.

5. Homogeneous and strongly indecomposable representations of P_n

In this section we discuss some properties of representations of P_n concerning the degenerate Kirillov model.

5.1. Let $P=P_n$ (*see* [1], 3.1). A representation $\tau\in \operatorname{Alg} P$ is called homogeneous if $\lambda(\sigma)=\lambda(\tau)$ for each non-zero submodule $\sigma\subset\tau$ (*see* 4.3).

PROPOSITION. — *Let* $\tau\in\operatorname{Alg}P$, $\lambda(\tau)=k$. *The following conditions are equivalent:*

(1) τ *is homogeneous.*

(2) *Any non-zero submodule of* τ *has non-zero intersection with the submodule* $\tau_k=(\Phi^+)^{k-1}\circ\Psi^+(\tau^{(k)})\subset\tau$ (see [1], 3.5).

(3) τ *may be embedded into* $(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\tau^{(k)})$.

Proof. — (1)$\Leftrightarrow$(2). Let $\sigma\subset\tau, \sigma\neq 0$. Let $\sigma_k=\sigma\cap\tau_k$. It follows easily from [1], 3.5 that $(\sigma/\sigma_k)^{(k)}=0$ so $\sigma^{(k)}=\sigma_k^{(k)}$. Furthermore by [1], 3.3 (*a*) $\sigma_k^{(k)}=0\Leftrightarrow\sigma_k=0$. Hence $\sigma^{(k)}=0\Leftrightarrow\sigma_k=0$.

(1)$\Leftrightarrow$(3). By [1], 3.2 (*b*) and (*c*):

$$(\star)\qquad \operatorname{Hom}(\sigma, (\hat{\Phi}^+)^{k-1}\circ\Psi^+(\tau^{(k)}))=\operatorname{Hom}(\sigma^{(k)}, \tau^{(k)})$$

for any $\sigma\in\operatorname{Alg}P$. If (3) holds and $0\neq\sigma\subset\tau$ then the left part of ($\star$) is non-zero hence $\sigma^{(k)}\neq 0$; this proves (1).

Now consider the morphism $A:\tau\to(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\tau^{(k)})$ corresponding by ($\star$) to the identical morphism $\tau^{(k)}\to\tau^{(k)}$. It follows from [1], 3.2 (*f*) that the functor $\tau\mapsto\tau^{(k)}$ carries A into the identical morphism $\tau^{(k)}\to\tau^{(k)}$. Therefore $(\operatorname{Ker} A)^{(k)}=0$. If (1) holds it follows that $\operatorname{Ker} A=0$ i.e. A is an embedding; this proves (3).

5.2. A representation $\pi\in\operatorname{Alg}G_n$ is called homogeneous if so is its restriction to P. Suppose that:

(1) π is homogeneous.

(2) The highest derivative $\pi^{(k)}$ of π is irreducible. Then 5.1 ($\star$) and Schur's Lemma imply that $\operatorname{Hom}(\pi|P, (\hat{\Phi}^+)^{k-1}\circ\Psi^+(\pi^{(k)}))$ is one-dimensional. So by 5.1 (3) $\pi|P$ may be embedded into $(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\pi^{(k)})$ in a unique (up to a scalar multiple) way. Let L be the space of the representation $(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\pi^{(k)})$; it is a space of certain vector-valued functions on P (*see* [1], 1.8). We have proved that there exists a unique realization of π on the subspace $V_\pi\subset L$ such that P acts by right translations. Call this realization the degenerate Kirillov model of π (*cf.* [2], 5.19).

5.3. PROPOSITION. — *Let* $\tau\in\operatorname{Alg}P_n$ *be homogeneous and* $\rho\in\operatorname{Alg}G_m$. *Then the representation* $\rho\times\tau\in\operatorname{Alg}P_{n+m}$ *is homogeneous* (*for the definition of* $\rho\times\tau$ see [1], 4.12).

Proof. — Let $\rho\neq 0$. By [1], 4.14 (*a*) $(\rho\times\tau)^{(i)}=\rho\times\tau^{(i)}$ for $i\geqq 1$ so $\lambda(\rho\times\tau)=\lambda(\tau)$ (*see* 4.3). Let $\lambda(\rho\times\tau)=\lambda(\tau)=k$. By 5.1 (3) τ may be embedded into $(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\tau^{(k)})$ hence

$$\rho\times\tau\hookrightarrow\rho\times(\hat{\Phi}^+)^{k-1}\circ\Psi^+(\tau^{(k)}).$$

Note now that

$$\rho \times \hat{\Phi}^{+}(\sigma) \subsetneq \hat{\Phi}^{+}(\rho \times \sigma)$$

for each $\sigma \in \operatorname{Alg} P_N$ (the proof of this is straightforward). It follows that

$$\rho \times (\hat{\Phi}^{+})^{k-1} \circ \Psi^{+}(\tau^{(k)}) \subsetneq (\hat{\Phi}^{+})^{k-1} \circ \Psi^{+}(\rho \times \tau^{(k)}) = (\hat{\Phi}^{+})^{k-1} \circ \Psi^{+}((\rho \times \tau)^{(k)}).$$

Therefore $\rho \times \tau$ may be embedded into $(\hat{\Phi}^{+})^{k-1} \circ \Psi^{+}((\rho \times \tau)^{(k)})$ and it remains only to use 5.1.

5.4. Let G be an *l*-group (*see* [1], 1.1). Call $\pi \in \operatorname{Alg} G$ strongly indecomposable if each two non-zero submodules of π have non-zero intersection. Clearly if π has finite length then it is strongly indecomposable iff it has a unique non-zero irreducible submodule.

PROPOSITION. — *Let $\tau \in \operatorname{Alg} P_n$ and $k = \lambda(\tau)$ be the level of τ. The following are equivalent:*

(1) *τ is strongly indecomposable.*

(2) *τ is homogeneous and $\tau^{(k)}$ is strongly indecomposable.*

Proof. — Consider the submodule $\tau_k = (\Phi^{+})^{k-1} \circ \Psi^{+}(\tau^{(k)}) \subset \tau$. By [1], 3.3 (*a*) the lattices of submodules of τ_k and $\tau^{(k)}$ are isomorphic. Hence $\tau^{(k)}$ is strongly indecomposable iff τ_k is. Therefore (2) means that τ_k is strongly indecomposable and has the non-zero intersection with each non-zero submodule of τ [*see* 5.1 (2)]. Equivalence of that and (1) is trivial.

6. Classification of irreducible non-cuspidal representations of G_n

This section contains the first main result of the paper.

6.1. THEOREM. — (*a*) *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathscr{C}$. Suppose for each pair of indices i, j such that $i<j$, Δ_i does not precede Δ_j* (see 4.1). *Then the representation $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ has a unique irreducible submodule; denote it by $\langle \Delta_1, \ldots, \Delta_r \rangle$.*

(*b*) *The representations $\langle \Delta_1, \ldots, \Delta_r \rangle$ and $\langle \Delta'_1, \ldots, \Delta'_s \rangle$ are isomorphic iff the sequences $(\Delta_1, \ldots, \Delta_r)$ and $(\Delta'_1, \ldots, \Delta'_s)$ are equal up to a rearrangement.*

(*c*) *Any irreducible representation of G_n is isomorphic to some representation of the form $\langle \Delta_1, \ldots, \Delta_r \rangle$.*

6.2. THEOREM. — *Let $\Delta_i = [\rho_i, \rho'_i]$ be segments in $\mathscr{C}$ $(i=1, \ldots, r)$. Suppose for each pair of indices i, j such that $i<j$, $\rho_j \not\simeq \nu\rho'_i$ (in other words if $i<j$ and Δ_i and Δ_j are juxtaposed then Δ_j precedes Δ_i). Then the representation $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ is homogeneous* (see § 5).

We first derive Part (*a*) of 6.1 from this Theorem. Let $\Delta_1, \ldots, \Delta_r$ satisfy the conditions of 6.1 (*a*) (thus also the conditions of 6.2); let $\pi = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle \in \operatorname{Alg} G_n$. By [1], 4.6 and 3.5 the highest derivative of π equals $\langle \Delta_1^{-} \rangle \times \ldots \times \langle \Delta_r^{-} \rangle$. Using 6.2, 5.4 and proceeding by induction on n, one sees that the restriction of π to P_n is strongly indecomposable. This proves a result which is stronger than 6.1 (*a*): π has a unique irreducible P_n-submodule.

6.3. Theorem 6.2 generalizes Theorem 4.11 of [1]. One can prove it by arguments similar to those in [1], 4.15. Let us indicate necessary modifications:

(1) Everywhere in [1], 4.15 one has to replace ρ_i by $\langle \Delta_i \rangle$ and references to [1], 4.4 by references to 3.5.

(2) Reasoning as in [1], 4.15, Case 1, one concludes now that $\langle \Delta_1^- \rangle \times \pi^0 | P$ is not homogeneous ($\pi^0 = \langle \Delta_2 \rangle \times \ldots \times \langle \Delta_r \rangle$). By 5.3 π^0 is not homogeneous, which as in [1], 4.15 contradicts the inductive assumption.

(3) In Case 2 of [1], 4.15 one has to use the reasoning of 4.5 which leads to the equality $\rho_i \simeq \nu\rho_1'$. This contradicts the assumptions of 6.2.

Theorem 6.2 and the claim 6.1 (*a*) are proven.

6.4. We now prove half of 6.1 (*b*).

PROPOSITION. — *Let* $(\Delta_1, \ldots, \Delta_r)$ *and* $(\Delta_1', \ldots, \Delta_r')$ *be ordered sequences of segments in* $\mathscr{C}$. *Suppose one of the following conditions holds:*

(1) $(\Delta_1, \ldots, \Delta_r)$ *differs from* $(\Delta_1', \ldots, \Delta_r')$ *only by a transposition of two neighbours which are not linked.*

(2) *Both* $(\Delta_1, \ldots, \Delta_r)$ *and* $(\Delta_1', \ldots, \Delta_r')$ *satisfy the condition of* 6.1 (*a*) *and are equal up to a rearrangement.*

Then $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle \simeq \langle \Delta_1' \rangle \times \ldots \times \langle \Delta_r' \rangle$. *Therefore in condition* (2) $\langle \Delta_1, \ldots, \Delta_r \rangle \simeq \langle \Delta_1', \ldots, \Delta_r' \rangle$.

Proof. — Part (1) follows immediately from associativity and commutativity of the multiplication (*see* 1.7, 1.9) and 4.2. Note now that ordered sequences $(\Delta_1, \ldots, \Delta_r)$ and $(\Delta_1', \ldots, \Delta_r')$ satisfying (2) may be obtained each from other by a chain of transpositions as in (1) (the easy combinatorial proof of this fact is omitted). So (2) follows from (1).

6.5. Introduce some useful notations. Denote by $\mathscr{S}$ the set of all segments in $\mathscr{C}$ and by $\mathcal{O}$ the set of all finite multisets on $\mathscr{S}$ (see the summary notation). For each $a \in \mathcal{O}, a \neq \emptyset$ one can choose an ordering $(\Delta_1, \ldots, \Delta_r)$ of a, satisfying 6.1(*a*). By 6.4 the representations $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ and $\langle \Delta_1, \ldots, \Delta_r \rangle$ depend only on a; denote them by $\pi(a)$ and $\langle a \rangle$ respectively. For the empty multiset $\emptyset \in \mathcal{O}$ define $\pi(\emptyset) = \langle \emptyset \rangle$ to be the identity representation of the group $G_0 = \{ e \}$.

Using this notation, one may state the remaining part of Theorem 6.1 as follows:

THEOREM. — *The* map $a \mapsto \langle a \rangle$ *is a bijection between* $\mathcal{O}$ *and the set of equivalence classes of irreducible representations of all the* G_n $(n = 0, 1, 2, \ldots)$.

6.6. For each $a = \{ \Delta_1, \ldots, \Delta_r \} \in \mathcal{O}$ denote by $a^- \in \mathcal{O}$ the multiset $\{ \Delta_1^-, \ldots, \Delta_r^- \}$ (*see* 3.5; here the empty parts Δ_i^- are dropped).

LEMMA. — *The highest derivative of* $\pi(a)$ *equals* $\pi(a^-)$. *The highest derivative of* $\langle a \rangle$ *has a unique irreducible submodule, which is isomorphic to* $\langle a^- \rangle$.

The first statement follows immediately from 3.5, 4.1 and [1], 4.6; the second part follows from the first one and 6.1 (*a*), 6.2.

Now derive from this Lemma the remaining part of 6.1 (*b*) (or in other words, the injectivity of $a \mapsto \langle a \rangle$). Let $a = \{ \Delta_1, \ldots, \Delta_r \} \in \mathcal{O}$, $\omega = \langle a \rangle \in \operatorname{Irr} G_n$. We want to reconstruct a from ω. Use induction on n. Without loss of generality one may assume that Δ_i contains more than one element when $1 \leqq i \leqq s$ and exactly one element ρ_i when

$s<i\leqq r$. By our Lemma and the inductive assumption one can reconstruct a^- from ω so the multiset $\{\Delta_1, \ldots, \Delta_s\}$ may be reconstructed from ω. One has

$$\varphi_\omega = \sum_{1\leqq i\leqq r} \chi_{\Delta_i} \quad [\textit{see } 4.4\ (b), (c)].$$

Therefore $\sum_{s<i\leqq r} \chi_{\{\rho_i\}} = \varphi_\omega - \chi_{\Delta_1} - \ldots - \chi_{\Delta_s}$ so representations $\rho_s, \rho_{s+1}, \ldots, \rho_r$ also are determined by ω.

6.7. Now prove 6.1 (*c*) (or, in other words, the surjectivity of $a \mapsto \langle a \rangle$). Let $\omega \in \operatorname{Irr} G_n$. Consider the set $\vec{\mathcal{O}} = \vec{\mathcal{O}}(\omega)$ consisting of ordered sequences $\vec{a} = (\Delta_1, \ldots, \Delta_r)$ of segments such that ω may be embedded into $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$. By 1.10 $\vec{\mathcal{O}}$ is non-empty; furthermore if $\vec{a} = (\Delta_1, \ldots, \Delta_r) \in \vec{\mathcal{O}}$ then $\chi_{\Delta_1} + \ldots + \chi_{\Delta_r} = \varphi_\omega$ (*see* 4.4) so $\vec{\mathcal{O}}$ is finite. Call an inversion of $\vec{a} = (\Delta_1, \ldots, \Delta_r) \in \vec{\mathcal{O}}$ a pair of indices (i, j) such that $i<j$ and Δ_i precedes Δ_j. We must prove that there exists an element $\vec{a} \in \vec{\mathcal{O}}$ which has no inversions.

Choose the element $\vec{a} = (\Delta_1, \ldots, \Delta_r) \in \vec{\mathcal{O}}$ with the least number of inversions. Suppose $\vec{a}$ has some inversions. Applying to $\vec{a}$ some transpositions as in 6.4 (1) one can obtain the element with the inversion of the form $(i, i+1)$. Clearly such transpositions don't change the number of inversions and carry elements of $\vec{\mathcal{O}}$ into ones of $\vec{\mathcal{O}}$ (*see* 6.4). So one may assume that $\vec{a}$ has the inversion $(i, i+1)$, i.e. Δ_i precedes Δ_{i+1}. Set $\Delta_i = \Delta'$, $\Delta_{i+1} = \Delta$.

By 4.6, 1.9 and 6.1 (*a*) $\langle \Delta' \rangle \times \langle \Delta \rangle$ is glued together from $\langle \Delta, \Delta' \rangle$ and $\langle \Delta^\cup, \Delta^\cap \rangle = \langle \Delta^\cup \rangle \times \langle \Delta^\cap \rangle$. Therefore $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ is glued together from

$$\sigma_1 = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_{i-1} \rangle \times \langle \Delta, \Delta' \rangle \times \langle \Delta_{i+2} \rangle \times \ldots \times \langle \Delta_r \rangle$$

and

$$\sigma_2 = \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_{i-1} \rangle \times \langle \Delta^\cup \rangle \times \langle \Delta^\cap \rangle \times \langle \Delta_{i+2} \rangle \times \ldots \times \langle \Delta_r \rangle.$$

So one may embed ω in either σ_1 or σ_2.

Case 1. — $\omega \subset \sigma_1$. Since $\langle \Delta, \Delta' \rangle \subset \langle \Delta \rangle \times \langle \Delta' \rangle$ [*see* 6.1 (*a*)], one has

$$\omega \subset \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_{i-1} \rangle \times \langle \Delta \rangle \times \langle \Delta' \rangle \times \langle \Delta_{i+2} \rangle \times \ldots \times \langle \Delta_r \rangle.$$

This means that the sequence $\vec{b} = (\Delta_1, \ldots, \Delta_{i-1}, \Delta, \Delta', \Delta_{i+2}, \ldots, \Delta_r)$ obtained from $\vec{a}$ by the transposition of Δ and Δ' also belongs to $\vec{\mathcal{O}}$. Evidently $\vec{b}$ has fewer inversions than $\vec{a}$. This contradicts the choice of $\vec{a}$.

Case 2. — $\omega \subset \sigma_2$. This means that $\vec{c} = (\Delta_1, \ldots, \Delta_{i-1}, \Delta^\cup, \Delta^\cap, \Delta_{i+2}, \ldots, \Delta_r)$ belongs to $\vec{\mathcal{O}}$. Apply the following trivial.

LEMMA. — *Let Δ and Δ' be linked segments, $\Delta^\cup = \Delta \cup \Delta'$, $\Delta^\cap = \Delta \cap \Delta'$. If a segment Δ^0 precedes one of $\Delta^\cup$, $\Delta^\cap$ then Δ^0 precedes one of Δ, Δ'; if Δ^0 precedes both of $\Delta^\cup$, $\Delta^\cap$ then Δ^0 precedes both of Δ, Δ'. Similarly if one of $\Delta^\cup$, $\Delta^\cap$ precedes Δ^0 then one of Δ, Δ' precedes Δ^0; if both $\Delta^\cup$, $\Delta^\cap$ precede Δ^0 then both Δ, Δ' precede Δ^0.*

By this Lemma $\vec{c}$ has fewer inversions than $\vec{a}$ and one again obtains a contradiction.

Theorem 6.1 is completely proven.

6.8. COROLLARY. — *Any irreducible representation of* G_n *is homogeneous.*

This follows immediately from 6.1 and 6.2.

6.9. Now we show that the element $a \in \mathcal{O}$ may be explicitly reconstructed from the irreducible representation $\langle a \rangle$ in terms of functors $r_{\beta,(n)}$. Consider a certain structure of a totally ordered set on $\mathcal{C}$, satisfying the following condition: for any $\rho \in \mathcal{C}$ elements ρ and $\nu\rho$ are neighbour in $\mathcal{C}$ and $\rho < \nu\rho$. Let us order finite sequences $(\rho_1, \ldots, \rho_r)$ of elements of $\mathcal{C}$ lexicographically.

For any $\pi \in \operatorname{Alg} G_n$ consider the set $\Omega(\pi)$, consisting of all sequences $(\rho_1, \ldots, \rho_r)$ of elements of $\mathcal{C}$ such that $\rho_1 \otimes \ldots \otimes \rho_r \in \mathcal{J} H(r_{\beta,(n)}(\pi))$ $(\beta = (n_1, \ldots, n_r)$ where $\rho_i \in \operatorname{Irr} G_{n_i})$. By 1.10 if $\omega \in \operatorname{Irr} G_n$ then $\{\rho_1, \ldots, \rho_r\} = \operatorname{supp} \omega$ for any $(\rho_1, \ldots, \rho_r) \in \Omega(\omega)$ so elements of $\Omega(\omega)$ differ from one another only order.

Proposition. — *Let $\omega \in \operatorname{Irr} G_n$ and $(\rho_1, \ldots, \rho_r)$ be the (lexicographically) highest term of $\Omega(\omega)$. Divide the sequence $(\rho_1, \ldots, \rho_r)$ into segments $\Delta_1, \ldots, \Delta_k$ (with the least possible number k) so that elements of each segment $\Delta_i = [\rho_{i1}, \rho_{i2}]$ follow in a natural order $\rho_{i1}, \nu\rho_{i1}, \ldots, \rho_{i2}$; in other words $(\rho_1, \ldots, \rho_r) = (\rho_{11}, \nu\rho_{11}, \ldots, \rho_{12}, \rho_{21}, \ldots, \rho_{k2})$ and $\rho_{i+1,1} \neq \nu\rho_{i,2}$. Then segments $\Delta_1, \ldots, \Delta_k$ satisfy the conditions of* 6.1 (*a*) *and $\omega \simeq \langle \Delta_1, \ldots, \Delta_k \rangle$.*

Proof. — (1) According to Theorem 6.1 $\omega = \langle \Delta'_1, \ldots, \Delta'_l \rangle$ for some segments $\Delta'_i = [\rho'_{i1}, \rho'_{i2}]$ in $\mathcal{C}$. Choose the ordering $(\Delta'_1, \ldots, \Delta'_l)$ such that the sequence $\lambda = (\rho'_{11}, \nu\rho'_{11}, \ldots, \rho'_{12}, \rho'_{21}, \ldots, \rho'_{l2})$ is the lexicographically highest of all possible ones. Evidently this ordering $(\Delta'_1, \ldots, \Delta'_l)$ satisfies the conditions of 6.1 (*a*) and is obtained from λ in the way described above. It remains to prove that λ is the highest term in $\Omega(\omega)$.

(2) By 6.1 (*a*) ω embeds into $\langle \Delta'_1 \rangle \times \ldots \times \langle \Delta'_l \rangle$. Therefore by 1.1 (*b*) $\langle \Delta'_1 \rangle \otimes \ldots \otimes \langle \Delta'_l \rangle \in \mathcal{J} H(r_{\gamma,(n)}(\omega))$ (for the appropriate γ). Applying 1.1 (*c*), 1.5 (*a*) and 3.1, one concludes that $\lambda \in \Omega(\omega)$.

(3) Using 1.6 and 3.4 one may easily find the set $\Omega(\langle \Delta'_1 \rangle \times \ldots \times \langle \Delta'_l \rangle)$. It consists of all sequences, obtained from λ by "shuffling" permutations, i.e. preserving the order of elements on each Δ'_i. Easy combinatorial reasoning shows that λ is the highest term of $\Omega(\langle \Delta'_1 \rangle \times \ldots \times \langle \Delta'_l \rangle)$. For all the more reason, λ is the highest term of $\Omega(\omega)$ and the Proposition is proven.

6.10. Compare Theorem 6.1 with results of paragraph 2. Let $\Gamma^0 = \{\rho_1, \ldots, \rho_r\} \subset \mathcal{C}$ (all ρ_i are different). By 6.1 any irreducible representation with support Γ^0 is of the form $\omega = \langle \Delta_1, \ldots, \Delta_k \rangle$ where Γ^0 is a disjoint union

$$\Gamma^0 = \Delta_1 \cup \ldots \cup \Delta_k.$$

On the other hand $\omega = \omega(\vec{\Gamma})$ for some orientation $\vec{\Gamma}$ of the graph Γ constructed in 2.2 (*see* Thm. 2.2).

Proposition. — *The edge $\{\rho, \nu\rho\}$ of Γ is oriented in $\vec{\Gamma}$ from ρ to $\nu\rho$ iff ρ and $\nu\rho$ belong to the same segment Δ_i.*

This follows immediately from definitions and 2.10.

7. Decomposition of the product $\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ and some applications

In this section we compute $\mathcal{J}\mathrm{H}(\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle)$ (Thm. 7.1) and obtain some corollaries.

7.1. For each $a, b \in \mathcal{O}$ denote by $m(b; a)$ the multiplicity, with which $\langle b \rangle$ occurs in $\mathcal{J}\mathrm{H}^0(\pi(a))$ (*see* 6.5).

Another definition of $m(b; a)$ may be given in terms of the representation bialgebra $\mathcal{R}$ (*see* 1.7). By 6.5 the set $\{\langle b \rangle \mid b \in \mathcal{O}\}$ is the basis of $\mathcal{R}$; multiplicities $m(b; a)$ are the coefficients of the decomposition of $\pi(a)$ with respect to this basis so in $\mathcal{R}$ one has

$$\pi(a) = \sum m(b; a) . \langle b \rangle, \qquad b \in \mathcal{O}.$$

Let $a = \{\Delta_1, \ldots, \Delta_r\} \in \mathcal{O}$. Call an elementary operation on a the replacement in it of the pair $\{\Delta, \Delta'\}$ of linked segments by the pair $\{\Delta^{\cup} = \Delta \cup \Delta', \Delta^{\cap} = \Delta \cap \Delta'\}$. We write $b < a$ if b may be obtained from a by a chain of elementary operations. By Lemma 6.7 if $b < a$ then the number of pairs of linked segments of the multiset b is less than that of a. It follows that the relation "$b < a$" defines the structure of partially ordered set on $\mathcal{O}$.

THEOREM. — *The coefficient $m(b; a)$ is non-zero iff $b \leqq a$. Moreover $m(a; a) = 1$ for any $a \in \mathcal{O}$.*

7.2. First of all prove the equality $m(a; a) = 1$. The inequality $m(a; a) > 0$ follows immediately from 6.1 (*a*). By 6.6 and the exactness of derivatives one has

$$m(a; a) \leqq m(a^-; a^-).$$

So to prove $m(a; a) \leqq 1$ it suffices to use the obvious induction.

7.3. PROPOSITION. — *If $a, b, c \in \mathcal{O}$ and $b \leqq a$ then $m(c; b) \leqq m(c; a)$. In particular one obtains*

$$1 = m(b; b) \leqq m(b; a).$$

Proof. — It suffices to consider the case when b may be obtained from a by an elementary operation $(\Delta, \Delta') \mapsto (\Delta^{\cup}, \Delta^{\cap})$. In this case our statement follows from the fact that $\langle \Delta^{\cup} \rangle \times \langle \Delta^{\cap} \rangle$ is a composition factor of $\langle \Delta \rangle \times \langle \Delta' \rangle$ (*see* 4.6).

7.4. To finish the proof of Theorem 7.1 it remains only to check the implication

$$m(b; a) \neq 0 \quad \Rightarrow \quad b \leqq a.$$

Let $a, a' \in \mathcal{O}$. The multiset a' is called subordinate to a if it may be obtained from a by the replacement of some segments Δ by Δ^- (notation $a' \dashv a$). In particular $a^- \dashv a$.

Let $\langle a \rangle, \langle b \rangle \in \mathrm{Irr}\, G_n$ and $m(b; a) \neq 0$. Prove that $b \leqq a$ using induction on n. By 6.6 the highest derivative of $\langle b \rangle$ has a submodule isomorphic to $\langle b^- \rangle$. Reasoning as in 4.5 one obtains $m(b^-; a') \neq 0$ for some $a' \dashv a$. By the inductive assumption $b^- \leqq a'$.

For any $a=\{\Delta_1, \ldots, \Delta_r\}\in\mathcal{O}$ set

$$\varphi_a=\varphi_{\langle a\rangle}=\chi_{\Delta_1}+\ldots+\chi_{\Delta_r} \quad (\textit{see } 4.4).$$

Clearly if $m(b; a)\neq 0$ then $\varphi_b=\varphi_a$.

Summarizing the above arguments one concludes that it suffices to prove the following:

($\star$) If $a, b\in\mathcal{O}$, $\varphi_a=\varphi_b$ and $b^-\leqq a'$ for some $a'\dashv a$ then $b\leqq a$.

Divide this statement into two parts:

(a) If $a, a', c'\in\mathcal{O}$, $a'\dashv a$ and $c'\leqq a'$ then there exists $c\in\mathcal{O}$ such that $c'\dashv c$ and $c\leqq a$:

$$\begin{matrix} ? & \leqq & a \\ \top & & \top \\ c' & \leqq & a' \end{matrix}$$

(b) If $b, c\in\mathcal{O}$, $\varphi_b=\varphi_c$ and $b^-\dashv c$ then $b\leqq c$.

The easy, purely combinatorial, proof of these facts is omitted. Theorem 7.1 follows.

7.5. Corollary. — *The ring $\mathscr{R}$ is a polynomial ring in indeterminates $\langle\Delta\rangle$ over $\mathbb{Z}$ ($\Delta\in\mathscr{S}$*; see 6.5).

Proof. — One has to prove that monomials $\pi(a)$, $a\in\mathcal{O}$ form a $\mathbb{Z}$-basis of $\mathscr{R}$. More precisely, for any function $\varphi:\mathscr{C}\to\mathbb{Z}_+$ with finite support set

$$\mathcal{O}(\varphi)=\{a\in\mathcal{O}\mid\varphi_a=\varphi\}, \qquad \mathscr{R}(\varphi)=\bigoplus_{a\in\mathcal{O}(\varphi)}\mathbb{Z}.\langle a\rangle.$$

Evidently, $\mathscr{R}=\bigoplus_{\varphi}\mathscr{R}(\varphi)$ and the relation $b\leqq a$ implies that a and b belong to the same $\mathcal{O}(\varphi)$. We prove that monomials $\pi(a)$, $a\in\mathcal{O}(\varphi)$ form a $\mathbb{Z}$-basis of $\mathscr{R}(\varphi)$.

Choose an ordering $(b_1, \ldots, b_k)$ of $\mathcal{O}(\varphi)$ such that

$$b_i\leqq b_j \Rightarrow i\leqq j.$$

By 7.1 the matrix $(m(b_i; b_j))$ $(i, j=1, \ldots, k)$ is triangular and unipotent. Hence it is invertible and the inverse matrix is also integer. Our statement follows.

7.6. *Remark.* — Corollary 7.5 and Proposition 3.4 give an explicit description of the bialgebra $\mathscr{R}$. It may be useful to translate these results into the language of algebraic groups [2]. Let $\mathscr{A}$ be the formal power series algebra generated by (non-commuting) indeterminates X_ρ $(\rho\in\mathscr{C})$, with the relations $X_\rho.X_{\rho'}=0$ if $\rho'\neq\nu\rho$. For $\Delta=[\rho, \rho']\in\mathscr{S}$ put $X_\Delta=X_\rho.X_{\nu\rho}.\ldots.X_{\rho'}\in\mathscr{A}$; clearly elements of $\mathscr{A}$ are the linear combinations $m_0+\sum_{\Delta\in\mathscr{S}} m_\Delta X_\Delta$ $(m_0, m_\Delta\in\mathbb{Z})$. It is easy to check that the set G of elements of the form $1+\sum_{\Delta\in\mathscr{S}} m_\Delta X_\Delta$ is a multiplicative subgroup of $\mathscr{A}$. By 7.5 and 3.4 $\mathscr{R}$ is the bialgebra corresponding to the algebraic group G. This means that $\mathscr{R}$ is identified with the affine ring of G (to $\langle\Delta\rangle$ corresponds the function m_Δ on G) and the comultiplication on $\mathscr{R}$ is induced by the group law on G.

(2) The following elegant construction was suggested by the referee.

Note that the linear form $\delta : \mathscr{R} \to \mathbb{Z}$ and the homomorphism $\mathscr{D} : \mathscr{R} \to \mathscr{R}$ also have a nice interpretation in terms of G. Identifying $\mathscr{R}$ with $\mathbb{Z}[G]$ and reformulating 3.8, 3.9, one obtains that $\delta(f)=f(g)$ and $\mathscr{D}f(x)=f(xg)$ where $g=1+\sum_{\rho\in\mathscr{C}} X_\rho \in G$.

7.7. PROPOSITION. — *Let* $b', a\in\mathcal{O}$. *Then the following are equivalent:*

(1) $\langle b' \rangle$ *is a composition factor of some derivative of the representation* $\pi(a)$.

(2) $b' \leqq a'$ *for some* a' *subordinate to* a.

(3) b' *is subordinate to some* $b \leqq a$.

Proof. — The equivalence (1)⇔(2) follows immediately from 7.1, 3.5 and [1], 4.6; implication (2)⇒(3) is the statement (*a*) from 7.4. Although surely one can prove (3)⇒(2) in a purely combinatorial way, we give another proof of the implication (3)⇒(1).

Apply the already proved implication (2)⇒(1) to b' and b. One obtains that $\langle b' \rangle$ is a composition factor of some derivative of some representation $\langle c \rangle$ such that $m(c; b)\neq 0$. By 7.1 $c \leqq b$. Since $b \leqq a$ one has $c \leqq a$. Therefore $m(c; a)\neq 0$ and (1) follows.

7.8. An important unsolved problem is to find necessary and sufficient conditions on $a, a'\in\mathcal{O}$ in order that $\langle a' \rangle$ be a composition factor of some derivative of $\langle a \rangle$. The following sufficient condition follows immediately from 7.7.

COROLLARY. — *If* $a, a'\in\mathcal{O}$, $a' \dashv a$ *and* a' *is not subordinate to any* $b<a$ *then* $\langle a' \rangle$ *is a composition factor of some derivative of* $\langle a \rangle$.

7.9. COROLLARY. — *If the representation* $\omega\in \operatorname{Irr} G_n$ *is irreducible as* P_n*-module then* $\omega=\langle \Delta \rangle$ *for some segment* Δ *in* $\mathscr{C}$.

Proof. — By 6.1, 7.8 and [1], 3.5 it suffices to prove the following statement: if the multiset $a\in\mathcal{O}$ contains more than one segment then there exist at least two elements $a'\neq a$ satisfying the condition of 7.8. By 7.4 (*b*) one such element is a^-.

Suppose

$$a=\{\Delta_1, \ldots, \Delta_r\}, \qquad r\geqq 2$$

and

$$\nu.(\text{end of } \Delta_r)\notin \operatorname{supp}(\langle a \rangle).$$

One can easily check that the element $a'=\{\Delta_1^-, \ldots, \Delta_{r-1}^-, \Delta_r\}$ also satisfies 7.8.

7.10. For any $a=\{\Delta_1, \ldots, \Delta_r\}\in\mathcal{O}$ set $\tilde{a}=\{\tilde{\Delta}_1, \ldots, \tilde{\Delta}_r\}\in\mathcal{O}$ (*see* 3.3).

THEOREM. — *The contragredient representation to* $\langle a \rangle$ *equals* $\langle \tilde{a} \rangle$.

Proof. — Since $\widetilde{\langle a \rangle}$ is irreducible, one has $\widetilde{\langle a \rangle}=\langle \hat{a} \rangle$ for some $\hat{a}\in\mathcal{O}$. The mappings $a\mapsto\tilde{a}$ and $a\mapsto\hat{a}$ have the following properties:

(1) $\hat{\hat{a}}=a$, $\tilde{\tilde{a}}=a$.

(2) If $b<a$ then $\tilde{b}<\tilde{a}$.

(3) $\hat{a}\leqq\tilde{a}$ for any $a\in\mathcal{O}$.

Properties (1) and (2) follow immediately from definitions. To prove (3): Let $a=\{\Delta_1, \ldots, \Delta_r\}$ and $\pi(a)=\langle \Delta_1 \rangle \times \ldots \times \langle \Delta_r \rangle$ (*see* 6.5). Since $\langle a \rangle \in \mathscr{J}\,\mathrm{H}(\pi(a))$ one has

$$\langle \hat{a} \rangle = \widetilde{\langle a \rangle} \in \mathscr{J}\,\mathrm{H}(\widetilde{\pi(a)}) = \mathscr{J}\,\mathrm{H}(\langle \tilde{\Delta}_1 \rangle \times \ldots \times \langle \tilde{\Delta}_r \rangle)$$

[*see* 1.1 (*d*) and 3.3]. By 1.9 $\langle \hat{a} \rangle \in \mathscr{J}\,\mathrm{H}(\pi(\tilde{a}))$. Therefore by 7.1 $\hat{a} \leqq \tilde{a}$.

We derive from (1), (2), (3) that $\hat{a}=\tilde{a}$ for any $a \in \mathcal{O}$. Suppose $\hat{a} \neq \tilde{a}$. By (3) $\hat{a} < \tilde{a}$. Using (1), (3), (2) and again (1) one obtains

$$a = \hat{\hat{a}} \leqq \tilde{\hat{a}} < \tilde{\tilde{a}} = a.$$

This contradiction proves that $\hat{a}=\tilde{a}$. The Theorem is proven.

8. Evaluation of the highest derivative and some applications

In this section we evaluate the highest derivative of any irreducible representation of G_n (Thm. 8.1). This allows one to generalize the models of Kirillov and Whittaker (Cor. 8.2, 8.3). Furthermore we obtain some results about products of irreducible representations (Prop. 8.4-8.6).

8.1. Theorem. — *The highest derivative of any irreducible representation of* G_n *is also irreducible. Moreover for any* $a \in \mathcal{O}$ *the highest derivative of* $\langle a \rangle$ *equals* $\langle a^- \rangle$ (see 6.5, 6.6).

Proof. — Let $a=\{\Delta_1, \ldots, \Delta_r\} \in \mathcal{O}$, $\omega = \langle a \rangle$. Set $\sigma = \nu\tilde{\omega}$. By 7.10 $\sigma = \langle \nu\tilde{a} \rangle$, where $\nu\tilde{a} = \{\nu\tilde{\Delta}_1, \ldots, \nu\tilde{\Delta}_r\} \in \mathcal{O}$. Let ω' and σ' be highest derivatives of ω and σ respectively. Reasoning as in [1], 4.8, one obtains that there exists a 1-pairing of ω' with σ', non-degenerate w.r.t. ω' so ω' may be embedded into $\tilde{\sigma}'$. Similarly σ' may be embedded into $\tilde{\omega}'$. Since ω' and σ' have finite length one obtains

$$\omega' \simeq \tilde{\sigma}'.$$

By 6.6. σ' has a unique irreducible submodule and it is isomorphic to $\langle (\nu\tilde{a})^- \rangle$; by definitions and 7.10 one has $\langle (\nu\tilde{a})^- \rangle = \langle \widetilde{a^-} \rangle = \widetilde{\langle a^- \rangle}$. It follows that $\omega' \simeq \tilde{\sigma}'$ has a unique irreducible quotient and it is isomorphic to $\langle a^- \rangle$. On the other hand applying again 6.6 one obtains that ω' has a unique irreducible submodule which is also isomorphic to $\langle a^- \rangle$. It follows that either ω' equals $\langle a^- \rangle$ or $\langle a^- \rangle$ occurs in $\mathscr{J}\,\mathrm{H}^0(\omega')$ with multiplicity $\geqq 2$. The latter possibility contradicts Theorem 7.1 so our Theorem is done.

8.2. Corollary. — *Any irreducible representation of* G_n *has a degenerate Kirillov model* (see 5.2, 6.8).

8.3. Let $\omega \in \mathrm{Irr}\,\mathrm{G}_n$. Define representations $\omega_0, \omega_1, \omega_2, \ldots$, and numbers $\lambda_1, \lambda_2, \ldots$, inductively by $\omega_0 = \omega$, $\lambda_i = \lambda(\omega_{i-1})$ (*see* 4.3), $\omega_i = \omega_{i-1}^{(\lambda_i)}$ for $i \geqq 1$. Let k be the least number such that $\omega_k \in \mathrm{Alg}\,\mathrm{G}_0$, i.e. $\lambda_1 + \ldots + \lambda_k = n$. Let $\mathrm{U} \subset \mathrm{G}_n$ be the subgroup of all unipotent upper triangular matrices. Define the character θ of U by $\theta((u_{ij}))=\Psi(\sum u_{i,\,i+1})$ where i runs over all indices 1, 2, ..., $n-1$ except

$$n-\lambda_1,\quad n-\lambda_1-\lambda_2,\ \ldots,\ n-\lambda_1-\lambda_2-\ldots-\lambda_{k-1}.$$

Set $\tau(\theta)=I_{U,\theta}(G_n, \{e\}, 1)$ (*see* [1], 1.8); this means that $\tau(\theta)\in \mathrm{Alg}\, G_n$ is induced by the character θ of U_n.

COROLLARY. — *ω may be in a unique way realized as a submodule of $\tau(\theta)$.*
We call this realization a (degenerate) Whittaker model of ω (cf. [8], p. 97, Thm. $\mathscr{D}$, *or* [2], 5.17).

Proof. — By 8.1 all ω_i are irreducible. In particular $\omega_k\in \mathrm{Irr}\, G_0$, i. e. $\omega_k=\langle \emptyset \rangle$. Using the definition of derivatives ([1], 4.3) and [1], 1.9 (*c*), one obtains that $\omega_k=r_{U,\theta}(\omega)$.

$$\mathrm{Hom}(\omega, \tau(\theta))=\mathrm{Hom}(\omega, I_{U,\theta}(\langle \emptyset \rangle))=\mathrm{Hom}(r_{U,\theta}(\omega), \langle \emptyset \rangle)$$

is one-dimensional [*see* [1], 1.9 (*b*)]. Since ω is irreducible each non-zero element of $\mathrm{Hom}(\omega, \tau(\theta))$ is an embedding.

Q.E.D.

8.4. PROPOSITION. — *For each $a_1, a_2, \ldots, a_p\in\mathcal{O}$ the representation $\langle a_1+a_2+\ldots+a_p\rangle$ occurs in $\mathscr{J}H^0(\langle a_1\rangle\times\langle a_2\rangle\times\ldots\times\langle a_p\rangle)$ with multiplicity* 1 (see *the summary of notation*).

Proof. — By 8.1 and [1], 4.6 the highest derivatives of $\langle a_1\rangle\times\ldots\times\langle a_p\rangle$ and $\langle a_1+\ldots+a_p\rangle$ equal respectively $\langle a_1^-\rangle\times\ldots\times\langle a_p^-\rangle$ and $\langle a_1^-+\ldots+a_p^-\rangle$. Note now that each $b\in\mathcal{O}$ is uniquely determined by φ_b and b^- (*see* 6.6). So Theorem 8.1 implies that the multiplicity of $\langle a_1+\ldots+a_p\rangle$ in $\mathscr{J}H^0(\langle a_1\rangle\times\ldots\times\langle a_p\rangle)$ is equal to that of $\langle a_1^-+\ldots+a_p^-\rangle$ in $\mathscr{J}H^0(\langle a_1^-\rangle\times\ldots\times\langle a_p^-\rangle$. It remains to use the obvious induction.

8.5. PROPOSITION. — *Let $a_1, \ldots, a_p\in\mathcal{O}$. Suppose for $i\neq j$ each segments $\Delta\in a_i$, $\Delta'\in a_j$ are not linked. Then*

$$\langle a_1\rangle\times\ldots\times\langle a_p\rangle=\langle a_1+\ldots+a_p\rangle.$$

Proof. — It suffices to consider the case $p=2$. The hypotheses imply that

$$\pi(a_1)\times\pi(a_2)=\pi(a_1+a_2)\quad (\textit{see } 6.5).$$

By 6.1 (*a*) for each $a\in\mathcal{O}$ the representation $\pi(a)$ has a unique irreducible submodule, which is isomorphic to $\langle a\rangle$. Applying this to $a=a_1, a_2$ and a_1+a_2 one obtains that $\langle a_1\rangle\times\langle a_2\rangle$ has a unique irreducible submodule, which is isomorphic to $\langle a_1+a_2\rangle$. Apply now this statement to $\tilde{a}_1$ and $\tilde{a}_2$ instead of a_1 and a_2 and use that $\langle a_1\rangle\times\langle a_2\rangle$ is contragredient to $\langle\tilde{a}_1\rangle\times\langle\tilde{a}_2\rangle$ [*see* 1.1 (*d*) and 7.10]. One obtains that $\langle a_1\rangle\times\langle a_2\rangle$ has a unique irreducible quotient, which is also isomorphic to $\langle a_1+a_2\rangle$. It follows that either $\langle a_1\rangle\times\langle a_2\rangle=\langle a_1+a_2\rangle$ or $\langle a_1+a_2\rangle$ occurs in $\mathscr{J}H^0(\langle a_1\rangle\times\langle a_2\rangle)$ with multiplicity $\geqq 2$. The latter possibility contradicts 8.4.

8.6. Consider the subsets $\Pi\subset\mathscr{C}$ of the form $\Pi=\{\nu^k\rho\,|\,\rho\in\mathscr{C}$ is fixed, k ranges over $\mathbb{Z}\}$. Call such subsets straight lines in $\mathscr{C}$. Set

$$\mathcal{O}(\Pi)=\bigcup_{\mathrm{supp}\,\varphi\subset\Pi}\mathcal{O}(\varphi);\qquad \mathscr{R}(\Pi)=\bigoplus_{\mathrm{supp}\,\varphi\subset\Pi}\mathscr{R}(\varphi)=\bigoplus_{a\in\mathcal{O}(\Pi)}\mathbb{Z}.\langle a\rangle\quad (\textit{see } 7.5).$$

Evidently each $a \in \mathcal{O}$ decomposes uniquely up to a permutation as

$$a = a_1 + a_2 + \ldots + a_p$$

where the a_i belong to different $\mathcal{O}(\Pi_i)$.

PROPOSITION. — *If* $\Pi_1, \ldots, \Pi_p$ *are different straight lines in* $\mathcal{C}$, $a_i \in \mathcal{O}(\Pi_i)$ *and* $a = a_1 + \ldots + a_p$, *then*

$$\langle a \rangle = \langle a_1 \rangle \times \ldots \times \langle a_p \rangle.$$

In other words any irreducible representation of G_n decomposes (uniquely up to a permutation) as the product of irreducible representations with supports belonging to different straight lines in $\mathcal{C}$.

This follows immediately from 8.5.

8.7. *Remark.* — It is easy to see that each $\mathcal{R}(\Pi)$ is a sub-bialgebra of $\mathcal{R}$, stable under $\mathcal{D}$, and that $\mathcal{R}$ decomposes as the tensor product of $\mathcal{R}(\Pi)$, where Π ranges over all straight lines in $\mathcal{C}$. Proposition 8.6 reduces the problem of the computation of all coefficients $m(b; a)$ (*see* 7.1) to the case when a and b belong to the same $\mathcal{O}(\Pi)$.

All our computations (*see* e.g. § 11 below) lead to the conjecture that coefficients $m(b; a)$ when a and b belong to the same $\mathcal{O}(\Pi)$, depend only on mutual relationships between segments of a and b and don't depend on Π.

9. Representations $\langle \Delta \rangle^t$ and duality

In this section we introduce and study another important class of irreducible representations parametrized by segments in $\mathcal{C}$.

9.1. Let $\Delta = [\rho, \rho']$ be a segment in $\mathcal{C}$. Denote by $\langle \Delta \rangle^t$ the irreducible representation with support $\{\rho, \nu\rho, \ldots, \rho'\}$ which corresponds to the orientation

$$\rho \leftarrow \nu\rho \leftarrow \ldots \leftarrow \nu^{-1}\rho' \leftarrow \rho' \quad (\textit{see } 2.2).$$

Using notations of 3.1, one may define $\langle \Delta \rangle^t$ by the property

$$(\star) \qquad r_{\beta, (n)}(\langle \Delta \rangle^t) = \rho' \otimes \nu^{-1}\rho' \otimes \ldots \otimes \rho.$$

Another way to define $\langle \Delta \rangle^t$ is to say that $\langle \Delta \rangle^t$ is the unique irreducible submodule of $\rho' \times \nu^{-1}\rho' \times \ldots \times \rho$ (or the unique irreducible quotient of $\rho \times \nu\rho \times \ldots \times \rho'$), *see* 2.10.

Applying 6.10 [or, more directly, 6.1 (*a*)] one obtains

$$\langle \Delta \rangle^t = \langle \{\Delta\}^t \rangle,$$

where $\{\Delta\}^t \in \mathcal{O}$ is the family of one-element segments $\{\{\rho\}, \{\nu\rho\}, \ldots, \{\rho'\}\}$.

9.2. *Example.* — Let $\Delta=[\rho, \rho']$ be the segment in $\mathscr{C}$ such that $\langle \Delta \rangle$ is the identity representation of G_n (by 3.2 $\rho=\nu^{-(n-1)/2}$, $\rho'=\nu^{(n-1)/2}\in \operatorname{Irr} G_1$). Then $\langle \Delta \rangle$ is the Steinberg representation (*see* [6], § 8).

9.3. Representations $\langle \Delta \rangle^t$ may be characterized in terms of the asymptotic behavior of matrix coefficients. Call the representation $\pi \in \operatorname{Alg} G_n$ quasi-square-integrable if its matrix coefficients become square-integrable modulo the centre Z_n of G_n after multiplying by a suitable character of G_n.

THEOREM (I. N. Bernstein). — *The representation $\omega \in \operatorname{Irr} G_n$ is quasi-square-integrable iff it is isomorphic to $\langle \Delta \rangle^t$ for some segment Δ in $\mathscr{C}$.*

The claim that representations $\langle \Delta \rangle^t$ are quasi-square-integrable, follows directly from the criterion for square-integrability, given by W. Casselman (*see* [6], Theorem 6.5.1). One has only to reformulate this criterion in terms of functors $r_{\beta, (n)}$ (i.e. to take into account multiples of the form $\operatorname{mod}_U^{1/2}(g)$, which are included in the definition of $r_{\beta, (n)}$) and then apply 9.1 ($\star$).

The converse is due to Bernstein and is based on a refinement of results of Casselman. The proof is omitted.

9.4. PROPOSITION. — $\widetilde{\langle \Delta \rangle^t}=\langle \tilde{\Delta} \rangle^t$ (see 3.3).

The proof is the same as that of 3.3.

9.5. PROPOSITION. — *Under the hypotheses of* 3.4, *if l is not divisible by m then* $r_{(n-l, l), (n)}(\langle \Delta \rangle^t)=0$. *If $l=mp$ then*

$$r_{(n-l, l), (n)}(\langle \Delta \rangle^t)=\langle [\nu^p \rho, \rho'] \rangle^t \otimes \langle [\rho, \nu^{p-1} \rho] \rangle^t.$$

In other words,

$$c(\langle \Delta \rangle^t)=\langle \emptyset \rangle \otimes \langle \Delta \rangle^t+\rho' \otimes \langle [\rho, \nu^{-1} \rho'] \rangle^t+\langle [\nu^{-1} \rho', \rho'] \rangle^t$$

$$\otimes \langle [\rho, \nu^{-2} \rho'] \rangle^t+\ldots+\langle [\nu\rho, \rho'] \rangle^t \otimes \rho+\langle \Delta \rangle^t \otimes$$

The proof is the same as that of 3.4.

9.6. PROPOSITION. — *Under the hypothèses of* 3.4, *if l is not divisible by m then the l-th derivative of $\langle \Delta \rangle^t$ is* 0. *If $l=mp$ then*

$$(\langle \Delta \rangle^t)^{(l)}=\langle [\nu^p \rho, \rho'] \rangle^t \qquad (p=0, 1, \ldots, k-1);$$
$$(\langle \Delta \rangle^t)^{(n)}=\langle \emptyset \rangle.$$

In other words

$$\mathscr{D}(\langle \Delta \rangle^t)=\langle \Delta \rangle^t+\langle [\nu\rho, \rho'] \rangle^t+\ldots+\rho'+1.$$

Proof. — By 8.1 $(\langle \Delta \rangle^t)^{(n)}=\langle \emptyset \rangle$ so $\delta(\langle \Delta \rangle^t)=1$ (*see* 3.8). Apply 3.8 and 9.5.

9.7. Now we classify all the irreducible non-degenerate representations of G_n in terms of the representations $\langle \Delta \rangle^t$.

THEOREM. — *(a) For each $a=\{\Delta_1, \ldots, \Delta_r\}\in\mathcal{O}$ the representation $\pi=\langle\Delta_1\rangle^t\times\ldots\times\langle\Delta_r\rangle^t$ is non-degenerate. It is irreducible if and only if no two of segments $\Delta_1, \ldots, \Delta_r$ are linked.*

(b) Any non-degenerate $\omega\in\operatorname{Irr}G_n$ decomposes as the product $\omega=\langle\Delta_1\rangle^t\times\ldots\times\langle\Delta_r\rangle^t$, where $\Delta_1, \ldots, \Delta_r$ are segments in $\mathcal{C}$ no two of which are linked. Moreover the multiset $a=\{\Delta_1, \ldots, \Delta_r\}\in\mathcal{O}$ is uniquely determined by ω.

Part (*a*) is proved in 9.8-9.9, part (*b*) in 9.10.

9.8. By 9.6 and [1], 4.6 $\delta(\pi)=1$ (*see* 3.8). In particular π is non-degenerate. Now suppose that no two of segments $\Delta_1, \ldots, \Delta_r$ are linked. We must prove that π is irreducible.

Since $\delta(\pi)=1$, exactly one element of $\mathcal{J}H^0(\pi)$ is non-degenerate. So it suffices to prove that π has no degenerate composition factors. We introduce some notation. Let $\Delta=[\rho, \rho']$ be a segment in $\mathcal{C}$. Write $\Delta'\leftarrow\!\dashv\Delta$ if either $\Delta'=\emptyset$ or Δ' is a segment such that $\rho'\in\Delta'\subset\Delta$; write $\Delta''\mapsto\Delta$ if either $\Delta''=\emptyset$ or Δ'' is a segment such that $\rho\in\Delta''\subset\Delta$. Using these notations one may rewrite Proposition 9.6 as follows:

$$\mathcal{D}(\langle\Delta\rangle^t)=\sum_{\Delta'\leftarrow\!\dashv\Delta}\langle\Delta'\rangle^t.$$

Suppose there exists a degenerate $\omega\in\mathcal{J}H(\pi)$. Let σ be the highest derivative of ω; by 8.1 σ is irreducible. By 9.6 and [1], 4.6 one has

$$\mathcal{D}(\pi)=\sum\langle\Delta_1'\rangle^t\times\ldots\times\langle\Delta_r'\rangle^t, \qquad \Delta_i'\leftarrow\!\dashv\Delta_i.$$

Therefore $\sigma\in\mathcal{J}H(\langle\Delta_1'\rangle^t\times\ldots\times\langle\Delta_r'\rangle^t)$ for some $\Delta_i'\leftarrow\!\dashv\Delta_i$. It follows that

$$\varphi_\sigma=\chi_{\Delta_1'}+\ldots+\chi_{\Delta_r'} \quad (\textit{see } 4.4).$$

Note that some of Δ_i' are non-empty since ω is degenerate.

By 1.1 (*d*) and 9.4, the contragredient representation to $\langle\Delta_1\rangle^t\times\ldots\times\langle\Delta_r\rangle^t$ is $\langle\tilde{\Delta}_1\rangle^t\times\ldots\times\langle\tilde{\Delta}_r\rangle^t$. Therefore $\tilde{\omega}\in\mathcal{J}H(\langle\tilde{\Delta}_1\rangle^t\times\ldots\times\langle\tilde{\Delta}_r\rangle^t)$. By 8.1 the highest derivative of $\tilde{\omega}$ is $\nu^{-1}\tilde{\sigma}$. Reasoning as above, one obtains

$$\varphi_{\nu^{-1}\tilde{\sigma}}=\chi_{\bar{\Delta}_1}+\ldots+\chi_{\bar{\Delta}_r} \qquad \text{for some } \bar{\Delta}_i\leftarrow\!\dashv\tilde{\Delta}_i.$$

It follows that

$$\varphi_{\nu\sigma}=\chi_{\Delta_1''}+\ldots+\chi_{\Delta_r''},$$

where $\Delta_i''=\tilde{\bar{\Delta}}_i$; clearly $\Delta_i''\mapsto\Delta_i$. Since $\operatorname{supp}\varphi_\sigma=\bigcup\Delta_i'$, $\operatorname{supp}\varphi_{\nu\sigma}=\bigcup\Delta_i''$ one has:

$$\bigcup\Delta_i''=\bigcup(\nu.\Delta_i').$$

It remains to use the following combinatorial:

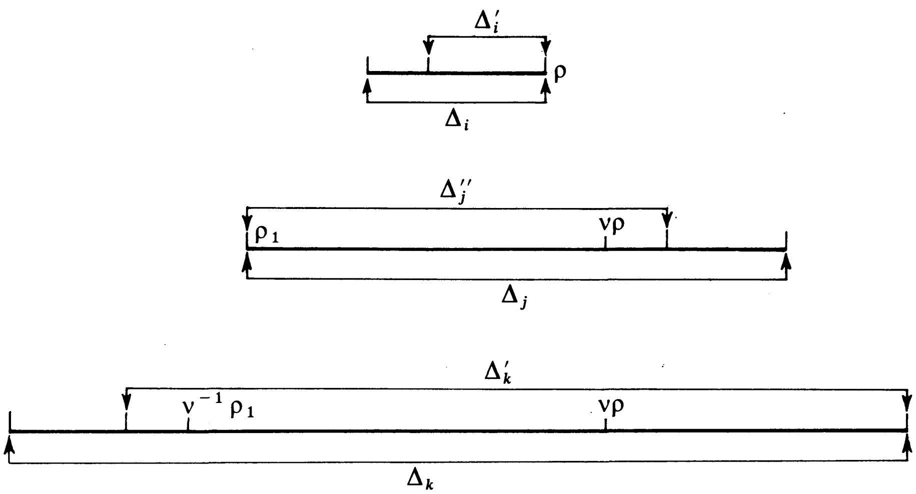
LEMMA. — *Let $\Delta_1, \ldots, \Delta_r$ be segments in $\mathcal{C}$ and $\Delta_i'\leftarrow\!\dashv\Delta_i$, $\Delta_i''\mapsto\Delta_i$ be chosen such that*

(★) $$\bigcup\Delta_i''=\bigcup(\nu\Delta_i')\neq\emptyset.$$

Then there exist two linked segments among $\Delta_1, \ldots, \Delta_r$.

Proof. — Suppose no two of $\Delta_1, \ldots, \Delta_r$ are linked. Choose $\rho \in \mathscr{C}$ such that $\rho \in \bigcup \Delta_i'$, $\nu\rho \notin \bigcup \Delta_i'$. Let $\rho \in \Delta_i'$. Then ρ is the end of Δ_i', hence the end of Δ_i. By ($\star$) $\nu\rho \in \bigcup \Delta_j''$. Let $\nu\rho \in \Delta_j'' \subset \Delta_j$; clearly $\Delta_j \neq \Delta_i$. Since Δ_i and Δ_j are not linked, one obtains that $\Delta_i \subset \Delta_j$. Let ρ_1 be the beginning of Δ_j; since $\Delta_j'' \mapsto \Delta_j$ it is also the beginning of Δ_j''. By ($\star$) $\nu^{-1}\rho_1 \in \bigcup \Delta_k'$; let $\nu^{-1}\rho_1 \in \Delta_k' \subset \Delta_k$. Clearly $\Delta_k \neq \Delta_j$; the condition that Δ_k and Δ_j not be linked, implies $\Delta_j \subset \Delta_k$. In particular $\nu\rho \in \Delta_k$. Since $\Delta_k' \leftarrow\!\dashv \Delta_k$, one obtains that $\nu\rho \in \Delta_k'$. This contradicts the choice of ρ.

Illustrate this proof by the following figure.



9.9. Prove now the statement converse to 9.8: if there exist two linked segments among $\Delta_1, \ldots, \Delta_r$, then $\langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_r \rangle^t$ is reducible. It suffices to prove the reducibility of $\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t$ where Δ_1 and Δ_2 are linked. Set $\Delta^\cup = \Delta_1 \cup \Delta_2$, $\Delta^\cap = \Delta_1 \cap \Delta_2$.

By 8.4 and 9.1 one has

$$\langle \{\Delta_1\}^t + \{\Delta_2\}^t \rangle \in \mathscr{J}\mathrm{H}(\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t).$$

Clearly by definition $\{\Delta_1\}^t + \{\Delta_2\}^t = \{\Delta^\cup\}^t + \{\Delta^\cap\}^t$. Applying 8.4, 9.1, and the already proven irreducibility of $\langle \Delta^\cup \rangle^t \times \langle \Delta^\cap \rangle^t$, one obtains

$$\langle \Delta^\cup \rangle^t \times \langle \Delta^\cap \rangle^t \in \mathscr{J}\mathrm{H}(\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t).$$

So it suffices to prove that $\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t \not\leq \langle \Delta^\cup \rangle^t \times \langle \Delta^\cap \rangle^t$. One has

$$\mathscr{D}(\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t) = \sum \langle \Delta_1' \rangle^t \times \langle \Delta_2' \rangle^t; \qquad \Delta_1' \leftarrow\!\dashv \Delta_1, \quad \Delta_2' \leftarrow\!\dashv \Delta_2,$$

$$\mathscr{D}(\langle \Delta^\cup \rangle^t \times \langle \Delta^\cap \rangle^t) = \sum \langle \Delta^{\cup'} \rangle^t \times \langle \Delta^{\cap'} \rangle^t;$$

$$\Delta^{\cup'} \leftarrow\!\dashv \Delta^\cup, \qquad \Delta^{\cap'} \leftarrow\!\dashv \Delta^\cap \quad (\textit{see } 9.8).$$

Suppose Δ_1 to precede Δ_2, and consider the segment $\Delta'_1 = \Delta_1 \cap \nu^{-1}\Delta_2$. Clearly $\Delta'_1 \leftarrow\!\!\!| \Delta_1$. It is easy to see that $\chi_{\Delta'_1}$ may not be represented in the form $\chi_{\Delta^{\cup'}} + \chi_{\Delta^{\cap'}}$ where $\Delta^{\cup'} \leftarrow\!\!\!| \Delta^{\cup}$, $\Delta^{\cap'} \leftarrow \Delta^{\cap}$. It follows that $\langle \Delta'_1 \rangle^t$ occurs in $\mathcal{D}(\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t)$ and does not occur in $\mathcal{D}(\langle \Delta^{\cup} \rangle^t \times \langle \Delta^{\cap} \rangle^t)$. So $\langle \Delta_1 \rangle^t \times \langle \Delta_2 \rangle^t$ and $\langle \Delta^{\cup} \rangle^t \times \langle \Delta^{\cap} \rangle^t$ are not isomorphic.

9.10. By 8.1 the representation $\langle a \rangle$ is non-degenerate iff the multiset $a \in \mathcal{O}$ consists only of one-element segments. Clearly any set $\mathcal{O}(\varphi)$ (*see* 7.5) contains exactly one such element, say $a(\varphi)$ (it is the maximal element of $\mathcal{O}(\varphi)$, *see* 7.1). Let $b = \{\Delta_1, \ldots, \Delta_r\}$ be a minimal element of $\mathcal{O}(\varphi)$, i.e. no two of $\Delta_1, \ldots, \Delta_r$ are linked. Then by 9.8 and 8.4 one has

$$\langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_r \rangle^t = \langle \{\Delta_1\}^t + \ldots + \{\Delta_r\}^t \rangle = \langle a(\varphi) \rangle.$$

The uniqueness of such a decomposition follows from the uniqueness of the minimal element in $\mathcal{O}(\varphi)$. The following lemma establishes this uniqueness; moreover it gives the algorithm which allows one to construct this minimal element.

LEMMA. — *Let φ be any function on $\mathcal{C}$ with non-negative integral values and finite support. Define the sequence $(\Delta_0, \Delta_1, \ldots)$ by induction in the following way: $\Delta_0 = \emptyset$ and for $i \geq 1$, Δ_i is a maximal segment, containing in* $\operatorname{supp}(\varphi - \chi_{\Delta_0} - \chi_{\Delta_1} - \ldots - \chi_{\Delta_{i-1}})$. *Set $b = \{\Delta_1, \Delta_2, \ldots\}$. Then $b \leq a$ for each $a \in \mathcal{O}(\varphi)$.*

The easy combinatorial proof is omitted. The proof of Theorem 9.7 is concluded.

9.11. Corollary to 9.3 and 9.7. — *Any irreducible non-degenerate representation of* G_n *decomposes into the product of irreducible quasi-square-integrable representations.*

9.12. The comparison between 9.1-9.7 and the results of paragraphs 3, 4 shows that there exists a certain duality between representations $\langle \Delta \rangle$ and $\langle \Delta \rangle^t$. To formalize this consider the mapping $\langle \Delta \rangle \mapsto \langle \Delta \rangle^t$. According to 7.5 it may be extended uniquely to an endomorphism $\omega \mapsto \omega^t$ of the ring $\mathcal{R}$.

PROPOSITION. — *The endomorphism $\omega \mapsto \omega^t$ is an involutive automorphism of $\mathcal{R}$, i.e. $(\omega^t)^t = \omega$ for any $\omega \in \mathcal{R}$.*

It suffices to check that $(\langle \Delta \rangle^t)^t = \langle \Delta \rangle$ for any segment Δ in $\mathcal{C}$. We give two proofs of this fact.

9.13. PROPOSITION. — *Let $a \in \mathcal{O}$ be such that any two of its segments have an empty intersection. Then in $\mathcal{R}$ one has*

$$\pi(a) = \sum_{b \leq a} \langle b \rangle, \qquad \langle a \rangle = \sum_{b \leq a} (-1)^{|a| - |b|} . \pi(b)$$

(*for the meaning of $|a|$ see the summary of notation*).

Proof. — By hypothesis the function $\varphi = \varphi_a$ is the characteristic function of some subset $\Gamma^0 \subset \mathcal{C}$ (*see* 7.4). If $a \in \mathcal{O}(\varphi)$ (*see* 7.5) then $\langle a \rangle$ has multiplicity-free support Γ^0. It follows from 2.1 (*c*) that the representation $\pi(a)$ is multiplicity-free, so $m(b; a) \leq 1$ for any $b \in \mathcal{O}$ (*see* 7.1). Combining this with Theorem 7.1, one obtains

$$\pi(a) = \sum_{b \leq a} \langle b \rangle .$$

Now consider the Möbius function $\mu(b, a)$ of the partially ordered set $\mathcal{O}(\varphi)$ (*see* [10], 2.2). By Theorem 2.2.1 of [10] one has

$$\langle a \rangle = \sum_{b \leqq a} \mu(b, a) . \pi(b).$$

Using 6.9, one may identify $\mathcal{O}(\varphi)$ with the set $\mathcal{O}(\Gamma)$ of all orientations of the graph Γ constructed in 2.2. Furthermore consider the set Γ^1 of all edges of Γ. There is a natural bijection between $\mathcal{O}(\Gamma)$ and the set $\mathcal{B}$ of all subsets of Γ^1 (the subset corresponding to the orientation $\vec{\Gamma}$ consists of all edges $\{\rho, \nu\rho\}$ which are oriented in $\vec{\Gamma}$ from $\nu\rho$ to ρ). One obtains the bijection $i : \mathcal{O}(\varphi) \xrightarrow{\sim} \mathcal{B}$. It is easy to see that i is an isomorphism of partially ordered sets ($\mathcal{B}$ is naturally ordered by inclusions). The Möbius function of $\mathcal{B}$ is well-known (*see* e.g. [10], (2.2.10)). Applying this one obtains

$$\mu(b, a) = (-1)^{|a| - |b|}.$$

The proof is done.

9.14. We now derive the equality $(\langle \Delta \rangle^t)^t = \langle \Delta \rangle$ from 9.13. By 9.13 one has

$$\langle \Delta \rangle^t = \sum_{b \leqq \{\Delta\}^t} (-1)^{|\{\Delta\}^t| - |b|} . \pi(b)$$

so

$$(\langle \Delta \rangle^t)^t = \sum_{b \leqq \{\Delta\}^t} (-1)^{|\{\Delta\}^t| - |b|} . \pi(b)^t.$$

Elements $b \leqq \{\Delta\}^t$ are of the form $b = \{\Delta_1, \ldots, \Delta_k\}$, where Δ_i precedes Δ_{i+1} and Δ is the disjoint union of Δ_i. One has $\pi(b)^t = \langle \Delta_1 \rangle^t \times \ldots \times \langle \Delta_k \rangle^t$. Again apply 9.13:

$$(\langle \Delta \rangle^t)^t = \sum_{\Delta_1, \ldots, \Delta_k} (-1)^{|\{\Delta\}^t| - k} \times \sum_{b_i \leqq \{\Delta_i\}^t} (-1)^{\sum(|\{\Delta_i\}^t| - |b_i|)} . \pi(b_1 + \ldots + b_k) = \sum_{b \leqq \{\Delta\}^t} \pi(b) . \sum_{b_1, \ldots, b_k} (-1)^{|b| - k}$$

(here $b_1 + \ldots + b_k = b$ and $\varphi_{b_i} = \chi_{\Delta_i}$ where the Δ_i are as above). Clearly if $b = \{\Delta\}$, i.e. $|b| = 1$, then the inner sum equals 1. Easy combinatorial arguments, similar to those of 9.13, show that the inner sum is 0 when $|b| > 1$. Therefore $(\langle \Delta \rangle^t)^t = \langle \Delta \rangle$.

Q.E.D.

9.15. The equality $(\langle \Delta \rangle^t)^t = \langle \Delta \rangle$ is a particular case of the following.

PROPOSITION. — *Let* $\Gamma^0 \subset \mathcal{C}$ *and let* $\omega = \omega(\vec{\Gamma})$ *be the irreducible representation with multiplicity-free support* Γ^0 *corresponding to the orientation* $\vec{\Gamma}$ (see 2.2). *Then* $\omega^t = \omega(\vec{\Gamma}^t)$, *where* $\vec{\Gamma}^t$ *is the opposite orientation to* $\vec{\Gamma}$ (see 2.7).

Proof. — Let $\varphi = \chi_{\Gamma^0}$ so $\omega \in \mathcal{R}(\varphi)$ (*see* 7.5). Consider the free abelian group $\mathcal{M}$ generated by the set Ω, and extend the mapping $\pi \mapsto \Omega(\pi)$ to the $\mathbb{Z}$-linear operator $s : \mathcal{R}(\varphi) \to \mathcal{M}$ (*see* 2.1). Clearly elements $s(\omega(\vec{\Gamma}))$ are linearly independent so s is an embedding. Define the involutive automorphism $t: \mathcal{M} \to \mathcal{M}$ by $t(\rho(\lambda)) = \rho(\mu)$ where μ is an ordering opposite to λ (*see* 2.1). By definition $(t \circ s)(\omega(\vec{\Gamma})) = s(\omega(\vec{\Gamma}^t))$. Therefore our proposition follows from the commutativity of the diagram

$$\begin{array}{ccc} \mathcal{R}(\varphi) & \xrightarrow{t} & \mathcal{R}(\varphi) \\ s\downarrow & & s\downarrow \\ \mathcal{M} & \xrightarrow{t} & \mathcal{M} \end{array}$$

It suffices to check that $t(s(\pi(a))) = s(\pi(a)^t)$ for $a \in \mathcal{O}(\varphi)$ (*see* 7.5) Using 1.6, 3.1 and 9.1, one may directly compute the sets $\Omega(\pi(a))$ and $\Omega(\pi(a)^t)$. One obtains that $\Omega(\pi(a))$ [respectively $\Omega(\pi(a)^t)$] consists of all $\rho(\lambda)$ such that for any ρ and $\nu\rho$ belonging to the same segment of a $\lambda^{-1}(\rho) < \lambda^{-1}(\nu\rho)$ [respectively $\lambda^{-1}(\nu\rho) < \lambda^{-1}(\rho)$]. It follows that $t(s(\pi(a))) = s(\pi(a)^t)$ so the Proposition is proven.

9.16. The automorphism $\omega \mapsto \omega^t$ may be described in terms of the structure of a bialgebra on $\mathcal{R}$ (*see* 1.7). For each function φ on $\mathcal{C}$ set

$$|\varphi| = \sum_{\rho \in \mathcal{C}} \varphi(\rho).$$

Define the $\mathbb{Z}$-linear map $i: \mathcal{R} \to \mathcal{R}$ by

$$i(\pi) = (-1)^{|\varphi|} . \pi^t \qquad \text{for} \quad \pi \in \mathcal{R}(\varphi) \quad (\textit{see } 7.5).$$

As $\pi \mapsto \pi^t$ is an involutive automorphism, so is i.

PROPOSITION. — *The map i is an inversion of the bialgebra $\mathcal{R}$ (for the definition of an inversion* see [5], chapt. III, § 11, ex. 4; *we recall it below*).

Proof. — We must check that each of the compositions

$$\mathcal{R} \xrightarrow{c} \mathcal{R} \otimes \mathcal{R} \underset{i \otimes \mathrm{id}}{\overset{\mathrm{id} \otimes i}{\rightrightarrows}} \mathcal{R} \otimes \mathcal{R} \xrightarrow{m} \mathcal{R}$$

is the identity on $\mathcal{R}_0$ and 0 on $\bigoplus_{n>0} \mathcal{R}_n$ (*see* 1.7). Clearly

$$m \circ (\mathrm{id} \otimes i) \circ c(\langle \emptyset \rangle) = m \circ (i \otimes \mathrm{id}) \circ c(\langle \emptyset \rangle) = \langle \emptyset \rangle.$$

By 7.5 it remains only to prove

$$m \circ (\mathrm{id} \otimes i) \circ c(\langle \Delta \rangle) = m \circ (i \otimes \mathrm{id}) \circ c(\langle \Delta \rangle) = 0$$

for any segment Δ in $\mathcal{C}$. One has

$$(\star) \qquad m \circ (\mathrm{id} \otimes i) \circ c(\langle \Delta \rangle) = \langle \Delta \rangle + \sum (-1)^{|\chi_{\Delta'}|} \langle \Delta_1 \rangle \times \langle \Delta' \rangle^t + (-1)^{|\chi_\Delta|} . \langle \Delta \rangle^t$$

(here the sum is over all pairs of segments (Δ_1, Δ') such that $\Delta_1 \cap \Delta' = \emptyset$, $\Delta_1 \cup \Delta' = \Delta$ and Δ_1 precedes Δ'). By 9.13

$$\langle \Delta' \rangle^t = \sum (-1)^{|\{\Delta'\}^t| - (k-1)} . \langle \Delta_2 \rangle \times \ldots \times \langle \Delta_k \rangle$$

(the sum is over all families of segments $(\Delta_2, \ldots, \Delta_k)$ such that Δ_i precedes Δ_{i+1} and Δ' is the disjoint union of Δ_i). Substitute this into ($\star$) and use that $|\chi_{\Delta'}| = |\{\Delta'\}^t|$. One obtains

$$\langle \Delta \rangle + \sum_{\Delta_1, \Delta'} (-1)^{|\chi_{\Delta'}|} \langle \Delta_1 \rangle \times \langle \Delta' \rangle^t$$
$$= \langle \Delta \rangle + \sum_{\Delta_1, \ldots, \Delta_k} (-1)^{k-1} . \langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \times \ldots \times \langle \Delta_k \rangle$$
$$= (-1)^{|\{\Delta\}^t|-1} . \sum_{b \leqq \{\Delta\}^t} (-1)^{|\{\Delta\}^t| - |b|} . \pi(b) = (-1)^{|\{\Delta\}^t|-1} . \langle \Delta \rangle^t.$$

Therefore the right side of ($\star$) is 0. The equality $m \circ (i \otimes \mathrm{Id}) \circ c(\langle \Delta \rangle) = 0$ is proved similarly.

9.17. Duality conjecture (³). The automorphism $\omega \mapsto \omega^t$ carries irreducible representations into irreducible ones. In other words for any $a \in \mathcal{O}$ there exists $a^t \in \mathcal{O}$ such that $\langle a \rangle^t = \langle a^t \rangle$.

This conjecture is confirmed by 9.15 and both 4.2 and 9.7. More evidence is provided by the analogy with groups over finite fields (*see* [15]).

10. The relationships with the Langlands reciprocity law

In this section we show that our results are in good accordance with the (hypothetical) reciprocity law of Langlands.

10.1. Let W be the Weil group of the field F (*see* [7], 3.1.1); supplied with the usual topology, it becomes an l-group countable at infinity.

By local class field theory there exists a natural bijection between the characters of W and those of $F^* = G_1$ (we normalize this bijection as in [7], 3.1.1). The reciprocity law generalizes this statement: it connects irreducible representations of G_n with n-dimensional representations of W. More precisely, let ν' be the character of W corresponding to the norm character of F^*. Denote by $\mathscr{W}$ the category of pairs (σ, N) where σ is a finite-dimensional algebraic completely reducible representation of W on the space V and $N : V \to V$ belongs to $\mathrm{Hom}(\nu' \sigma, \sigma)$; morphisms in $\mathscr{W}$ are defined naturally (*see* [7], 3.1.1).

The reciprocity law claims that for any $n = 1, 2, \ldots$ there exists a natural bijection $\omega \mapsto (\sigma(\omega), N(\omega))$ between $\overline{\mathrm{Irr}}\, G_n$ and the set of isomorphism classes of objects $(\sigma, N) \in \mathscr{W}$ such that $\dim \sigma = n$. This bijection is supposed to satisfy certain conditions (*see* [7], 3.2.3); moreover one expects that these conditions determine the bijection uniquely. The case $n = 2$ is considered in [7]. Using analogies one may expect that in the general case the mapping $\omega \mapsto (\sigma(\omega), N(\omega))$ has following properties:

(1) To $\nu\omega$ there corresponds the pair $(\nu' \sigma(\omega), N(\omega))$.

(2) ω is cuspidal if and only if $\sigma(\omega)$ is irreducible (clearly in this case $N(\omega) = 0$).

(3) If $\mathrm{supp}\, \omega = \{\rho_1, \ldots, \rho_r\}$ then $\sigma(\omega) \simeq \sigma(\rho_1) \oplus \ldots \oplus \sigma(\rho_r)$.

(³) *Added in proof:* this conjecture has recently been proved by I. N. Bernstein by means of homological methods; the idea to apply such methods is due to V. G. Drinfeld.

10.2. Now we obtain the classification of objects of $\mathscr{W}$; it is quite similar to that of irreducible representations of G_n. To indicate this it is useful to introduce some terminology. Denote by $\mathscr{C}'$ the set of equivalence classes of irreducible finite dimensional representations of W. Call a segment in $\mathscr{C}'$ any subset $\Delta'=[\sigma, \sigma']\subset\mathscr{C}'$ of the form $\Delta'=\{\sigma, \nu'\sigma, \nu'^2\sigma, \ldots, \nu'^k\sigma=\sigma'\}$ $(k\in\mathbb{Z}_+)$; denote by $\mathscr{S}'$ the set consisting of all segments in $\mathscr{C}'$.

Assign to each segment $\Delta'=[\sigma, \sigma']\in\mathscr{S}'$ the object

$$\tau(\Delta')=(\sigma(\Delta'), N(\Delta'))\in\mathscr{W}.$$

Put $\sigma(\Delta')=\sigma\oplus\nu'\sigma\oplus\ldots\oplus\sigma'$. Let V_i be the space of the representation $\nu'^i\sigma$ $(i=0, 1, \ldots, k)$; clearly all V_i may be identified with the same space V. Let $N(\Delta')|_{V_0}=0$ and $N(\Delta'): V_i\to V_{i-1}$ be the obvious (identity) isomorphism for $i=1, \ldots, k$.

PROPOSITION. — *(a) The objects $\tau(\Delta')$ $(\Delta'\in\mathscr{S})$ are indecomposable and mutually non-isomorphic, and each indecomposable object of $\mathscr{W}$ is of this form.*

(b) Each object of $\mathscr{W}$ decomposes into the direct sum $\tau(\Delta'_1)\oplus\ldots\oplus\tau(\Delta'_r)$. This decomposition is unique up to permutation.

Proof. — Claim (*a*) is proved in [7], 3.1.3 (ii). Claim (*b*) follows from (*a*) and the theorem of Krull-Remak-Schmidt: in any abelian category where objects are of finite length, the decomposition into the sum of indecomposable objects is unique up to automorphisms (*see* e. g. I. Bucur and A. Deleanu, "Introduction to the theory of categories and functors"); it is also easy to prove directly.

Let us reformulate this Proposition to look like 6.5. Denote by $\mathcal{O}'$ the set consisting of all finite multisets on $\mathscr{S}'$ (*see* list of notations). To each $a'\in\mathcal{O}'$ assign the object

$$\tau(a')=\sum_{\Delta'\in a'}\tau(\Delta')\in\mathscr{W}.$$

Our Proposition means that the map $a'\mapsto\tau(a')$ is a bijection between $\mathcal{O}'$ and the set of isomorphism classes of objects of $\mathscr{W}$.

10.3. We suppose now the reciprocity law to be established for cuspidal representations of G_n, i. e. that the natural bijection $\mathscr{C}\to\mathscr{C}'$ has been constructed [*see* 10.1 (2)]. By 10.1 (1) this bijection maps segments in $\mathscr{C}$ into segments in $\mathscr{C}'$ so it induces the bijections $\mathscr{S}\to\mathscr{S}'$ and $\mathcal{O}\to\mathcal{O}'$.

We want to extend the reciprocity law to all irreducible representations of G_n. One must take into account the restrictive condition 10.1 (3). For this define for each $\tau=(\sigma, N)\in\mathscr{W}$ its support $\operatorname{supp}\tau$ as $\mathscr{J}\,H^0(\sigma)$; it is a finite multiset on $\mathscr{C}'$. Then 10.1 (3) means that the irreducible representations ω of G_n with support $\varphi:\mathscr{C}\to\mathbb{Z}_+$ must correspond to objects $\tau\in\mathscr{W}$ with support $\varphi':\mathscr{C}'\to\mathbb{Z}_+$ corresponding to φ under the bijection $\mathscr{C}\to\mathscr{C}'$. By our classification irreducible representations with support φ are of the form $\langle a\rangle$ where $a\in\mathcal{O}(\varphi)$ (*see* 7.5). On the other hand put $\mathcal{O}'(\varphi')=\{a'\in\mathcal{O}'/\sum_{\Delta'\in a'}\chi_{\Delta'}=\varphi'\}$; then by 10.2 objects of $\mathscr{W}$ with support φ' are just $\tau(a')$, $a'\in\mathcal{O}'(\varphi')$. Since $\mathcal{O}(\varphi)$ and $\mathcal{O}'(\varphi')$ have the same number of elements the desired bijection exists. In this sense our classification is compatible with the reciprocity law.

To make the Langlands correspondence more explicit one needs some additional conditions. By analogy with the case $n=2$ (*see* [7], 3.2.3 (B)) the indecomposable objects of $\mathscr{W}$ must correspond to quasi-square-integrable irreducible representations of G_n. Combining 9.3 and 10.2 (*a*) one concludes that the Langlands correspondence turns $\langle \Delta \rangle^t$ into $\tau(\Delta')$ where $\Delta \in \mathscr{S}$ and $\Delta' \in \mathscr{S}'$ correspond to each other under the bijection $\mathscr{S} \to \mathscr{S}'$. This makes natural the following:

CONJECTURE. — *If $a \in \mathscr{O}$ and $a' \in \mathscr{O}'$ correspond to each other under the bijection $\mathscr{O} \to \mathscr{O}'$ then the object $\tau(a') \in \mathscr{W}$ corresponds to the representation $\langle a \rangle^t$ (it is irreducible by Conjecture* 9.17).

11. Examples

In this section we collect some partial results about coefficients $m(b;\ a)$ and derivatives of irreducible representations of G_n.

11.1. By 7.1 $m(b;\ a) \neq 0$ iff $b \leqq a$; moreover $m(a;\ a)=1$. So the only interesting is the case when $b<a$. Clearly in this case representations $\langle a \rangle$ and $\langle b \rangle$ have the same support; in other words a and b belong to the same $\mathscr{O}(\varphi)$ (*see* 7.5). By 9.13 $m(b;\ a)=1$ if $b<a$ and $\operatorname{supp}\langle a \rangle$ is multiplicity-free. Consider now the first non-trivial case, when the support is not multiplicity-free.

11.2. *Example.* — Representations with support $\{\rho, \nu\rho, \nu\rho\}$. In other words, consider $\mathscr{O}(\varphi)$, where $\operatorname{supp}\varphi=\{\rho, \nu\rho\}$, $\varphi(\rho)=1$, $\varphi(\nu\rho)=2$. The set $\mathscr{O}(\varphi)$ consists of two elements a_0 and a_1 where $a_0=\{\{\rho\}, \{\nu\rho\}, \{\nu\rho\}\}$, $a_1=\{[\rho, \nu\rho], \nu\rho\}$. Clearly $a_1<a_0$. In $\mathscr{R}$ one has

$$\pi(a_0)=\nu\rho \times \nu\rho \times \rho=\nu\rho \times (\langle \Delta \rangle + \langle \Delta \rangle^t)=\nu\rho \times \langle \Delta \rangle + \nu\rho \times \langle \Delta \rangle^t$$

where $\Delta=[\rho, \nu\rho]$. By 4.2 and 9.7

$$\nu\rho \times \langle \Delta \rangle = \langle a_1 \rangle, \qquad \nu\rho \times \langle \Delta \rangle^t = \langle a_0 \rangle,$$

so $m(a_1;\ a_0)=1$. Give the interpretation of this fact in terms of representations. Consider the module $\pi=\nu\rho \times \rho \times \nu\rho$. By definitions $\langle \Delta \rangle$ and $\langle \Delta \rangle^t$ are submodules of $\rho \times \nu\rho$ and $\nu\rho \times \rho$ respectively. It follows that π has submodules $\pi_0=\langle \Delta \rangle^t \times \nu\rho \simeq \langle a_0 \rangle$ and $\pi_1=\nu\rho \times \langle \Delta \rangle \simeq \langle a_1 \rangle$.

By commutativity of $\mathscr{R}$, $\mathscr{J}H^0(\pi)=\{\langle a_0 \rangle, \langle a_1 \rangle\}$ and we conclude that π is a direct sum of submodules π_0 and π_1. This phenomenon shows the difference between our case and the case of multiplicity-free support (*see* §2).

11.3. *Example.* — Representations with support $\{\rho, \rho, \ldots, \rho, \nu\rho, \ldots, \nu\rho\}$ (multiplicities of ρ and $\nu\rho$ equal k_0 and k_1). Let $k_1 \geqq k_0$ (the case $k_0>k_1$ may be reduced to this one by passing to contragredient representations). The set $\mathscr{O}(\varphi)$ consists of elements a_0, $a_1, \ldots, a_{k_0}$ where

$$a_i=\{\underbrace{\Delta, \Delta, \ldots, \Delta}_{i \text{ times}}, \underbrace{\{\rho\}, \ldots, \{\rho\}}_{(k_0-i) \text{ times}}, \underbrace{\{\nu\rho\}, \ldots, \{\nu\rho\}}_{(k_1-i) \text{ times}}\} \ (\Delta=[\rho, \nu\rho]).$$

Clearly $a_0>a_1>\ldots>a_{k_0}$.

PROPOSITION. — (*a*):

$$\langle a_i \rangle = \underbrace{\langle \Delta \rangle \times \ldots \times \langle \Delta \rangle}_{i \text{ times}} \times \underbrace{\langle \Delta \rangle^t \times \ldots \times \langle \Delta \rangle^t}_{(k_0 - i) \text{ times}} \times \underbrace{\nu\rho \times \ldots \times \nu\rho}_{(k_1 - k_0) \text{ times}}.$$

(*b*) *In* $\mathscr{R}$ *one has*

$$\pi(a_i) = \sum_{i \leqq j \leqq k_0} C_{k_0 - i}^{k_0 - j} . \langle a_j \rangle,$$

so $m(a_j;\, a_i) = C_{k_0 - i}^{k_0 - j}$ *for* $j \geqq i$.

(*c*) $\langle a_i \rangle^t = \langle a_{k_0 - i} \rangle$ $(i = 0, 1, \ldots, k_0)$.

Proof. — (*a*) Denote the right side of (*a*) by ω. By 8.4 $\langle a_i \rangle$ occurs in $\mathscr{J}\,H^0(\omega)$ with multiplicity 1. One has

$$(\star) \quad \mathscr{D}(\omega) = \mathscr{D}(\langle \Delta \rangle)^i\, \mathscr{D}(\langle \Delta \rangle^t)^{k_0 - i}\, \mathscr{D}(\nu\rho)^{k_1 - k_0}$$
$$= (\rho + \langle \Delta \rangle)^i . (1 + \nu\rho + \langle \Delta \rangle^t)^{k_0 - i} . (1 + \nu\rho)^{k_1 - k_0}$$

(*see* 3.5, 9.6).

In particular the highest derivative of ω equals $\rho \times \rho \times \ldots \times \rho$ (*i* times). By 8.1 and 8.5 the highest derivative of $\langle a_i \rangle$ is the same. Suppose $\omega' \in \mathscr{J}\,H(\omega)$, $\omega' \not\simeq \langle a_i \rangle$. Then $\lambda(\omega') < \lambda(\omega)$ (*see* 4.3). The highest derivative σ of ω' is the composition factor of some derivative of ω. Using ($\star$) one obtains that $\nu\rho \in \operatorname{supp} \sigma$. It follows from 4.4 (*d*) that $\operatorname{supp} \omega' \ni \nu^2 \rho$. This yields a contradiction, so $\omega \simeq \langle a_i \rangle$.

(*b*) *One has*

$$\pi(a_i) = \langle \Delta \rangle^i \times \rho^{k_0 - i} \times \nu\rho^{k_1 - i} = \langle \Delta \rangle^i \times (\rho \times \nu\rho)^{k_0 - i} \times \nu\rho^{k_1 - k_0}$$
$$= \langle \Delta \rangle^i \times (\langle \Delta \rangle + \langle \Delta \rangle^t)^{k_0 - i} \times \nu\rho^{k_1 - k_0}.$$

Apply the binomial formula for $(\langle \Delta \rangle + \langle \Delta \rangle^t)^{k_0 - i}$ and use (*a*).

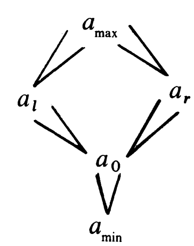
(*c*) Follows immediately from (*a*).

Note that (*c*) gives one more confirmation of Conjecture 9.17.

11.4. *Example.* — Representations with support $\{\rho, \nu\rho, \nu\rho, \nu^2\rho\}$. The set $\mathscr{O}(\varphi)$ in this case consists of 5 elements $a_{\max}$, a_l, a_r, a_0, $a_{\min}$, where

$$a_{\max} = \{\{\rho\}, \{\nu\rho\}, \{\nu\rho\}, \{\nu^2\rho\}\}, \qquad a_l = \{[\rho, \nu\rho], \{\nu\rho\}, \{\nu^2\rho\}\},$$
$$a_r = \{\{\rho\}, \{\nu\rho\}, [\nu\rho, \nu^2\rho]\}, \qquad a_0 = \{[\rho, \nu\rho], [\nu\rho, \nu^2\rho]\}, \qquad a_{\min} = \{[\rho, \nu^2\rho], \nu\rho\}.$$

One has



PROPOSITION. — *(a) One has* $m(a_0; a_{\max})=2$. *All other coefficients* $m(b; a)$, *where* $a, b \in \mathcal{O}(\varphi)$, $b<a$, *equal* 1.

(b) One has

$$\mathcal{D}(\langle a_{\min}\rangle)=\langle[\rho, \nu\rho]\rangle+\langle[\rho, \nu\rho], \{\nu\rho\}\rangle+\langle[\rho, \nu^2\rho]\rangle+\langle a_{\min}\rangle;$$

$$\mathcal{D}(\langle a_0\rangle)=\langle\{\rho\}, \{\nu\rho\}\rangle+\langle\{\rho\}, [\nu\rho, \nu^2\rho]\rangle+\langle a_0\rangle;$$

$$\mathcal{D}(\langle a_l\rangle)=\rho+\langle\{\rho\}, \{\nu^2\rho\}\rangle+\langle[\rho, \nu\rho]\rangle+\langle[\rho, \nu^2\rho]\rangle + 2.\langle[\rho, \nu\rho], \{\nu^2\rho\}\rangle+\langle\{\rho\}, \{\nu\rho\}, \{\nu^2\rho\}\rangle+\langle a_l\rangle;$$

$$\mathcal{D}(\langle a_r\rangle)=\nu\rho+\langle\{\nu\rho\}, \{\nu\rho\}\rangle+\langle[\nu\rho, \nu^2\rho]\rangle+\langle\{\nu\rho\}, [\nu\rho, \nu^2\rho]\rangle + \langle\{\rho\}, \{\nu\rho\}, \{\nu\rho\}\rangle+\langle a_r\rangle;$$

$$\mathcal{D}(\langle a_{\max}\rangle)=\langle\emptyset\rangle+\nu\rho+\nu^2\rho+2\langle\{\nu\rho\}, \{\nu^2\rho\}\rangle+\langle[\nu\rho, \nu^2\rho]\rangle + \langle\{\nu\rho\}, \{\nu\rho\}, \{\nu^2\rho\}\rangle+\langle\{\rho\}, \{\nu\rho\}, \{\nu^2\rho\}\rangle+\langle a_{\max}\rangle.$$

(c) One has

$$\langle a_{\max}\rangle^t=\langle a_{\min}\rangle, \quad \langle a_l\rangle^t=\langle a_r\rangle, \quad \langle a_r\rangle^t=\langle a_l\rangle,$$
$$\langle a_0\rangle^t=\langle a_0\rangle, \quad \langle a_{\min}\rangle^t=\langle a_{\max}\rangle.$$

So Conjecture 9.17 *holds for our* $\mathcal{O}(\varphi)$.

Proof. — First of all

$$\langle a_{\min}\rangle=\pi(a_{\min})=\langle[\rho, \nu^2\rho]\rangle\times\nu\rho \quad (\textit{see} \text{ e.g. } 8.5).$$

To compute $\mathcal{D}(\langle a_{\min}\rangle)$ one has only to apply 3.5 and the fact that $\mathcal{D}$ is the ring homomorphism.

By 4.6 and 7.1 one has

$$\pi(a_0)=\langle a_0\rangle+\langle a_{\min}\rangle \quad \text{so} \quad m(a_{\min}; a_0)=1.$$

One may easily compute $\mathcal{D}(\langle a_0\rangle)$ using that

$$\mathcal{D}(\langle a_0\rangle)=\mathcal{D}(\pi(a_0))-\mathcal{D}(\pi(a_{\min})).$$

Furthermore

$$\pi(a_l)=\langle[\rho, \nu\rho]\rangle\times\nu\rho\times\nu^2\rho=\langle[\rho, \nu\rho]\rangle\times(\langle[\nu\rho, \nu^2\rho]\rangle + \langle[\nu\rho, \nu^2\rho]^t\rangle)=\pi(a_0)+\langle[\rho, \nu\rho]\rangle\times\langle[\nu\rho, \nu^2\rho]\rangle^t.$$

Set $\omega=\langle[\rho, \nu\rho]\rangle\times\langle[\nu\rho, \nu^2\rho]\rangle^t$. By 8.4 $\langle a_l\rangle$ occurs in $\mathcal{J}\,H^0(\omega)$ with multiplicity 1. By 7.1 $\mathcal{J}\,H(\omega)$ may contain only $\langle a_l\rangle$, $\langle a_0\rangle$ and $\langle a_{\min}\rangle$. On the other hand one may easily compute $\mathcal{D}(\omega)$ using 3.5 and 9.6 and check that $\mathcal{D}(\langle a_0\rangle)$ and $\mathcal{D}(\langle a_{\min}\rangle$, which we have already computed, are not contained in $\mathcal{D}(\omega)$. We conclude that $\langle a_l\rangle=\omega$, so $m(a_{\min}; a_l)=m(a_0; a_l)=1$. Similarly it is proved that

$$\langle a_r\rangle=\langle[\rho, \nu\rho]\rangle^t\times\langle[\nu\rho, \nu^2\rho]\rangle.$$

This allows one to compute $\mathcal{D}(\langle a_r\rangle)$ and to conclude that $m(a_{\min}; a_r)=m(a_0; a_r)=1$.

By 9.7 and 8.4 $\langle a_{\max}\rangle=\langle[\rho, \nu^2\rho]\rangle^t\times\nu\rho$. This allows one to compute $\mathscr{D}(\langle a_{\max}\rangle)$. By 9.13:

$$\langle[\rho, \nu^2\rho]\rangle^t=\pi(\{\rho\}, \{\nu\rho\}, \{\nu^2\rho\})-\pi(\{\rho\}, [\nu\rho, \nu^2\rho])-\pi([\rho, \nu\rho], \{\nu^2\rho\})+\pi([\rho, \nu^2\rho]).$$

Multiply this equality by $\nu\rho$:

$$\langle a_{\max}\rangle=\pi(a_{\max})-\pi(a_r)-\pi(a_l)+\pi(a_{\min}).$$

Therefore

$$(\star)\quad \pi(a_{\max})=\pi(a_l)+\pi(a_r)-\pi(a_{\min})+\langle a_{\max}\rangle$$
$$=(\langle a_l\rangle+\langle a_0\rangle+\langle a_{\min}\rangle)+(\langle a_r\rangle+\langle a_0\rangle+\langle a_{\min}\rangle)-\langle a_{\min}\rangle+\langle a_{\max}\rangle$$
$$=\langle a_{\max}\rangle+\langle a_l\rangle+\langle a_r\rangle+2.\langle a_0\rangle+\langle a_{\min}\rangle.$$

Parts (*a*) and (*b*) are proven. All equalities in (*c*) besides $\langle a_0\rangle^t=\langle a_0\rangle$ follow immediately from the already proven equalities

$$\langle a_{\max}\rangle=\langle[\rho, \nu^2\rho]\rangle^t\times\nu\rho, \qquad \langle a_{\min}\rangle=\langle[\rho, \nu^2\rho]\rangle\times\nu\rho,$$
$$\langle a_l\rangle=\langle[\rho, \nu\rho]\rangle\times\langle[\nu\rho, \nu^2\rho]\rangle^t, \qquad \langle a_r\rangle=\langle[\rho, \nu\rho]\rangle^t\times\langle[\nu\rho, \nu^2\rho]\rangle.$$

Evidently, $\pi(a_{\max})^t=\pi(a_{\max})$. Thus one may deduce $\langle a_0\rangle^t=\langle a_0\rangle$ from ($\star$).

11.5. *Remark.* — Example 11.4 is included here for two reasons. The first that the proof of Proposition 11.4 is a good illustration of the technique developped in this work. The second is that this example gives counter-examples to many possible general conjectures about coefficients $m(b; a)$, derivatives of irreducible representation, and so on. I. N. Bernstein has pointed out to me that a similar example exists in the theory of Verma modules for GL(4) (*see* [4], p. 9).

REFERENCES

[1] I. N. BERNSTEIN and A. V. ZELEVINSKY, *Induced Representations of Reductive* $\mathfrak{p}$*-Adic Groups I* (*Ann. scient. Éc. Norm. Sup.*, 4e serie, t. 10, 1977, pp. 441-472).

[2] I. N. BERNSTEIN and A. V. ZELEVINSKY, *Representations of the Group* GL(n, F), *where* F *is a Local Non-Archimedean Field* (*Uspekhi Mat. Nauk*, Vol. 31, No. 3, 1976, pp. 5-70).

[3] I. N. BERNSTEIN and A. V. ZELEVINSKY, *Induced Representations of the Group* GL(n) *Over a* $\mathfrak{p}$*-Adic Field* (*Funkc. Anal. i Priložen.*, Vol. 10, No. 3, 1976, pp. 74-75).

[4] I. N. BERNSTEIN, I. M. GELFAND and S. I. GELFAND, *The Structure of Representations Generated by Vectors of Highest Weight* (*Funkc. Anal. i Prilozen*, Vol. 5, No. 1, 1971, pp. 1-9).

[5] N. BOURBAKI, *Algèbre*, Chap. 1 à 3, Hermann, Paris, 1970.

[6] W. CASSELMAN, *Introduction to the Theory of Admissible Representations of* $\mathfrak{p}$*-Adic Reductive Groups*, preprint.

[7] P. DELIGNE, *Formes modulaires et représentations de* GL(2), *in Modular Functions of One Variable II* (*Lecture Notes in Math.*, 349, Springer-Verlag, 1973).

[8] I. M. GELFAND and D. A. KAZHDAN, *Representations of* GL(n, K), in *Lie Groups and Their Representations*, Akadèmiai Kiado, Budapest, 1974.

[9] R. GODEMENT and H. JACQUET, *Zeta-Functions of Simple Algebras* (*Lecture Notes in Math.*, 260, Springer-Verlag, 1972).
[10] M. HALL, *Combinatorial Theory*, Blaisdell Publ. Comp., 1967.
[11] A. W. KNAPP and G. ZUCKERMAN, *Classification of Irreducible Tempered Representations of Semisimple Lie Groups* (*Proc. Nat. Acad. Sc. U.S.A.*, Vol. 73, No. 7, 1976, pp. 2178-2180.
[12] R. LANGLANDS, *On the Classification of Irreducible Representations of Real Algebraic Groups*, mimeographed notes, Inst. for Adv. Study, 1973.
[13] G. I. OLSHANSKY, *Intertwining Operators and Complementary Series in the Class of Representations of the General Group of Matrices Over a Locally Compact Division Algebra, Induced from Parabolic Subgroups* (*Mat. Sb.*, Vol. 93, No. 2, 1974, pp. 218-253).
[14] A. V. ZELEVINSKY, *Classification of Irreducible Non-Cuspidal Representations of the Group* GL_n *Over a* $\mathfrak{p}$*-Adic Field* (*Funkc. Anal. i Priložen*, Vol. 11, No. 1, 1977, pp. 67-68).
[15] A. V. ZELEVINSKY, *The Representation Ring of Groups* $GL(n)$ *Over a* $\mathfrak{p}$*-Adic Field* (*Funkc. Anal. i Priložen*, Vol. 11, No. 3, 1977, pp. 78-79).

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