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CLOSEDNESS OF REGULAR 1-FORMS ON ALGEBRAIC SURFACES ⁽¹⁾

By Niels O. NYGAARD

Introduction

Let X/k be a proper, smooth surface over a perfect field k . If k has characteristic 0 it follows from Hodge theory and the Lefschetz principle that all regular 1-forms on X are closed, i. e. that the differential

$$d : H^0(X, \Omega_{X/k}^1) \rightarrow H^0(X, \Omega_{X/k}^2),$$

vanishes.

In characteristic $p > 0$ the situation is more complicated indeed Mumford [11] and more recently Raynaud have constructed surfaces with regular 1-forms which are not closed ⁽²⁾. It therefore becomes interesting to look for conditions on X that will ensure the closedness of regular 1-forms. We relate this question to an invariant defined and studied by Artin and Mazur in [1], the formal Brauer group, $\widehat{\text{Br}}_X$, specially we show that if $\widehat{\text{Br}}_X$ is pro-representable by a p -divisible formal group (Barsotti-Tate group) then all the regular 1-forms are closed, and indeed the whole Hodge to de Rham spectral sequence degenerates at E_1 . In a subsequent paper [13] we shall further develop the techniques employed in the proof of the above statement, and show how these can be used to prove the Rydakov-Shafarevitch theorem, that K 3 surfaces have no global vector fields.

We also consider a smooth family of surfaces $f : X \rightarrow S$ over an irreducible base scheme of characteristic p , here we show that if there is just one fiber X_s with p -divisible formal Brauer group then the differential

$$d : f_* \Omega_{X/S}^1 \rightarrow f_* \Omega_{X/S}^2,$$

is zero.

⁽¹⁾ This work was supported in part by the Danish Research Council.

⁽²⁾ Examples have also been constructed by W. Lang [17].

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Acknowledgement

I should like to thank L. Illusie for very useful correspondence during the preparation of this paper. I also thank the referee for pointing out a considerable strengthening of the methods developed in 2.

1. Some properties of the slope spectral sequence

For the construction and the basic properties of the slope spectral sequence we refer to Bloch [3]. Bloch's construction has been generalized and the restrictions on the relation between the dimension and characteristic has been removed (Illusie [9]), so the restriction in Bloch's paper will be ignored.

The notation will be as in [3]; the proof of the properties listed below will appear in [9].

Let F , V and d denote respectively the Frobenius, the Verschiebung and the differential in the pro-complex $C_{\bullet, X}$, then:

$$(1.1) \quad FV = VF = p.$$

$$(1.2) \quad dF = pFd, \quad Vd = p dV.$$

$$(1.3) \quad FdV = d.$$

(1.4) F , V and p are injective as maps of pro-sheaves i. e. the transition maps in the pro-system of kernels are 0.

(1.5) Let $n = \dim X$ then F is an automorphism of the pro-sheaf $C_{\bullet, X}^n$.

2. Surfaces over a perfect field

In this section we show that if the formal Brauer group of X/k is pro-representable by a p -divisible formal group then the Hodge to de Rham spectral sequence degenerates at E_1 . If we further assume that $H_{\text{crys}}^2(X/W)$ is torsion free then the Hodge symmetry

$$h^{i,j} = \dim_k H^j(X, \Omega_{X/k}^i) = \dim_k H^i(X, \Omega_{X/k}^j) = h^{j,i},$$

holds as well.

The following proposition has also been proved by Berthelot (private communication) using results of Mazur and Messing.

(2.1) PROPOSITION. — *Let X/k be a smooth proper variety over a perfect field k of characteristic $p > 0$. Assume that $H_{\text{crys}}^2(X/W)$ is torsion free, then the Picard scheme $\underline{\text{Pic}}(X)$ is reduced.*

Proof. — Consider the exact sequence of Zariski sheaves on X :

$$0 \rightarrow \mathcal{W}_r(\mathcal{O}_X) \xrightarrow{\vee} \mathcal{W}_{r+1}(\mathcal{O}_X) \rightarrow \mathcal{O}_X \rightarrow 0,$$

which gives rise to an exact sequence of finite length $W(k)$ -modules

$$\rightarrow H^i(X, \mathcal{W}_r(\mathcal{O}_X)) \xrightarrow{\vee} H^i(X, \mathcal{W}_{r+1}(\mathcal{O}_X)) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^{i+1}(X, \mathcal{W}_r(\mathcal{O}_X)),$$

and hence (using Mittag-Leffler) an exact sequence of $W(k)$ -modules

$$\rightarrow H^i(X, \mathcal{W}(\mathcal{O}_X)) \xrightarrow{\vee} H^i(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^i(X, \mathcal{O}_X) \rightarrow H^{i+1}(X, \mathcal{W}(\mathcal{O}_X)).$$

By [12], p. 196, $\underline{\text{Pic}}(X)$ is reduced if and only if the connecting homomorphism, in the exact sequence above, vanishes, this is equivalent to

$$H^1(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^1(X, \mathcal{O}_X),$$

being surjective.

Define the pro-complex $*C_{\bullet, X}$ by

$$*C_{\bullet, X} = 0 \rightarrow C_{\bullet+1, X}^0 \xrightarrow{F^d} C_{\bullet, X}^1 \xrightarrow{d} \dots C_{\bullet, X}^{\dim X} \rightarrow 0,$$

since

$$dF^d = (pF)dd \quad \text{by (1.2),}$$

this is indeed a complex.

Now define

$$\tilde{V} : C_{\bullet, X} \rightarrow *C_{\bullet, X},$$

by

$$\tilde{V}^i : C_{\bullet, X}^i \rightarrow *C_{\bullet, X}^i = \begin{cases} V : C_{\bullet, X}^0 \rightarrow C_{\bullet+1, X}^0 & \text{if } i=0, \\ \text{id} : C_{\bullet, X}^i \rightarrow C_{\bullet, X}^i & \text{if } i>0. \end{cases}$$

It is clear by (1.3) that $\tilde{V}$ is a map of complexes, and since $C_{\bullet, X}^0 \simeq \mathcal{W}(\mathcal{O}_X)$ we get an exact sequence of pro-complexes

$$0 \rightarrow C_{\bullet, X} \xrightarrow{\tilde{V}} *C_{\bullet, X} \rightarrow \mathcal{O}_X \rightarrow 0.$$

Passing to hypercohomology we obtain an exact sequence of pro-modules

$$\mathbf{H}^i(X, C_{\cdot, X}) \rightarrow \mathbf{H}^i(X, *C_{\cdot, X}) \rightarrow \mathbf{H}^i(X, \mathcal{O}_X) \rightarrow \mathbf{H}^{i+1}(X, C_{\cdot, X}).$$

Since $\mathbf{H}^j(X, C_{r, X}^i)$ has finite length over $\mathcal{W}(k)$ for all i, j, r ([3], III, Prop. (1.1)) it follows from the hypercohomology spectral sequences that $\mathbf{H}^j(X, C_{\cdot})$ and $\mathbf{H}^j(X, *C_{\cdot})$ are pro-systems of modules of finite lengths so by Mittag-Leffler we get an exact sequence

$$\mathbf{H}_{\text{crys}}^i(X/W) \rightarrow \varprojlim \mathbf{H}^i(X, *C_{\cdot, X}) \rightarrow \mathbf{H}^i(X, \mathcal{O}_X) \rightarrow \mathbf{H}_{\text{crys}}^{i+1}(X/W).$$

Since $\mathbf{H}_{\text{crys}}^2(X/W)$ is assumed torsion free the connecting homomorphism

$$\mathbf{H}^1(X, \mathcal{O}_X) \rightarrow \mathbf{H}_{\text{crys}}^2(X/W),$$

in the exact sequence above vanishes, i. e.

$$\varprojlim \mathbf{H}^1(X, C_{\cdot, X}) \rightarrow \mathbf{H}^1(X, \mathcal{O}_X),$$

is surjective.

We have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \rightarrow & C_{\cdot, X} & \xrightarrow{\gamma} & *C_{\cdot, X} & \rightarrow & \mathcal{O}_X & \rightarrow 0 \\ & \downarrow & & \downarrow & & \parallel & \\ 0 \rightarrow & \mathcal{W}_{\cdot}(\mathcal{O}_X) & \xrightarrow{\gamma} & \mathcal{W}_{\cdot+1}(X) & \rightarrow & \mathcal{O}_X & \rightarrow 0. \end{array}$$

hence a commutative diagram

$$\begin{array}{ccccccc} \mathbf{H}_{\text{crys}}^1(X/W) & \rightarrow & \varprojlim \mathbf{H}^1(X, *C_{\cdot, X}) & \rightarrow & \mathbf{H}^1(X, \mathcal{O}_X) & \rightarrow 0 \\ \downarrow & & \downarrow & & \parallel & \\ \mathbf{H}^1(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{\gamma} & \mathbf{H}^1(X, \mathcal{W}(\mathcal{O}_X)) & \rightarrow & \mathbf{H}^1(X, \mathcal{O}_X), & \end{array}$$

it follows that

$$\mathbf{H}^1(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow \mathbf{H}^1(X, \mathcal{O}_X),$$

is surjective as desired.

The next proposition was pointed out by the referee, the proof is based on an idea by Deligne.

(2.2) PROPOSITION. — Assume that the differentials in the E_1 term of the slope spectral sequence vanish then it degenerates at E_1 .

Proof. — We show by induction that the differentials in the E_s term vanish so assume that the differentials in the E_t terms $t=1, \dots, s-1$ are zero.

This implies that $E_s^{i,j} = H^j(X, C_X^i)$ for all i, j so we must show that

$$d : H^j(X, C_X^i) \rightarrow H^{j-s+1}(X, C_X^{i+s}),$$

vanishes.

Consider the commutative diagram of pro-complexes

$$\begin{array}{ccccccccccccccccccc} C_{\cdot, X}^{\cdot} = 0 & \rightarrow & C_{\cdot, X}^0 & \xrightarrow{d} & \dots & \rightarrow & C_{\cdot, X}^i & \xrightarrow{d} & C_{\cdot, X}^{i+1} & \xrightarrow{d} & \dots & \xrightarrow{d} & C_{\cdot, X}^{i+s} & \rightarrow & \dots & \rightarrow & C_{\cdot, X}^{\dim X} & \rightarrow 0, \\ & & \uparrow p^i V & & & & \uparrow V & & \parallel & & & & \parallel & & & & & \parallel & \\ \tilde{C}_{\cdot, X}^{\cdot} = 0 & \rightarrow & C_{\cdot, -1, X}^0 & \xrightarrow{d} & \dots & \rightarrow & C_{\cdot, -1, X}^i & \xrightarrow{dV} & C_{\cdot, -1, X}^{i+1} & \xrightarrow{d} & \dots & \xrightarrow{d} & C_{\cdot, -1, X}^{i+s} & \rightarrow & \dots & \rightarrow & C_{\cdot, -1, X}^{\dim X} & \rightarrow 0, \\ & & \parallel & & & & \parallel & & \downarrow F & & & & \downarrow p^i F & & & & & \downarrow p^{(\dim X - i)F} & \\ C_{\cdot, -1, X}^{\cdot} = 0 & \rightarrow & C_{\cdot, -1, X}^0 & \rightarrow & \dots & \rightarrow & C_{\cdot, -1, X}^i & \rightarrow & C_{\cdot, -1, X}^{i+1} & \rightarrow & \dots & \rightarrow & C_{\cdot, -1, X}^{i+s} & \rightarrow & \dots & \rightarrow & C_{\cdot, -1, X}^{\dim X} & \rightarrow 0. \end{array}$$

Consider the hyper cohomology sequences then we have a commutative diagram

$$\begin{array}{ccc} E_s^{i,j}(C_{\cdot, X}^{\cdot}) & \xrightarrow{d_s} & E_s^{i+s, j-s+1}(C_{\cdot, X}^{\cdot}) \\ \uparrow V & & \uparrow \pi \\ E_s^{i,j}(\tilde{C}_{\cdot, X}^{\cdot}) & \xrightarrow{d_s} & E_s^{i+s, j-s+1}(\tilde{C}_{\cdot, X}^{\cdot}) \\ \downarrow \delta & & \downarrow p^i F \\ E_s^{i,j}(C_{\cdot, -1, X}^{\cdot}) & \xrightarrow{d_s} & E_s^{i+s, j-s+1}(C_{\cdot, -1, X}^{\cdot}). \end{array}$$

Passing to the limit we get a commutative diagram

(2.3)

$$\begin{array}{ccc} E_s^{i,j} & \xrightarrow{d_s} & E_s^{i+s, j-s+1}, \\ \downarrow V & & \uparrow \pi \\ E_s^{i,j}(\tilde{C}) & \xrightarrow{d_s} & E_s^{i+s, j-s+1}(\tilde{C}), \\ \downarrow \delta & & \downarrow p^i F \\ E_s^{i,j} & \xrightarrow{d_s} & E_s^{i+s, j-s+1}. \end{array}$$

If the differentials in the preceding terms vanish π and δ are identities so we have a commutative diagram

$$\begin{array}{ccc} H^j(X, C_X^i) & \xrightarrow{d_s} & H^{j-s+1}(X, C_X^{i+s}), \\ \uparrow V & & \downarrow p^i F \\ H^j(X, C_X^i) & \xrightarrow{d_s} & H^{j-s+1}(X, C_X^{i+s}). \end{array}$$

By iteration we get

$$d_s = p^{sn} F^n d_s V^n \quad \text{for all } n, \text{ hence,}$$

$$\text{Im } d_s \subset \bigcap_n p^n H^{j-s+1}(X, C_X^{i+s}) = 0.$$

(2.4) THEOREM. — *Let X/k be a surface, proper and smooth over k with k perfect of characteristic $p > 0$, then the slope spectral sequence degenerates at E_1 if and only if $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is a finitely generated $\mathcal{W}(k)$ module.*

Proof. — Assume that the slope spectral sequence degenerates at E_1 then $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is a quotient of $H^2_{\text{crys}}(X/W)$ hence is finitely generated. The proof of the other implication rests on the following Lemma.

(2.5) LEMMA. — Let $d : L \rightarrow M$ be a linear map of $\mathcal{W}(k)$ modules. Let F (resp. V) be a σ -linear (resp. σ^{-1} -linear) endomorphism of M (resp. L) [this means $F(\lambda x) = \lambda^\sigma F(x)$ and $V(\lambda y) = \lambda^{\sigma^{-1}} V(y)$ where $\lambda \in \mathcal{W}(k)$ and σ denotes the Frobenius endomorphism of $\mathcal{W}(k)$]. Assume that L and M are topological $\mathcal{W}(k)$ modules, d is continuous, M is separated and the topology on L is weaker than the V -topology (i.e. the topology defined by the submodules $\{V^n L\}$), assume moreover that $FdV = d$. Then if the chains

$$\ker d \subset \ker Fd \subset \dots \subset \ker F^n d \subset \dots \subset L,$$

$$\text{Im } d \subset \text{Im } Fd \subset \dots \subset \text{Im } F^n d \subset \dots \subset M,$$

stabilize one has $d=0$.

Proof. — Assume that both chains are stable at the n 'th level. Let $x \in \ker F^n d$, then $0 = F^n dx = F^{n+1} dVx$ so $Vx \in \ker F^{n+1} d = \ker F^n d$ i.e. $\ker F^n d$ is stable under V and so $V^n x \in \ker F^n d$ hence $dx = F^n dV^n x = 0$ and it follows that

$$\ker d = \ker Fd = \dots = \ker F^n d = \dots \subset L.$$

Now the commutative diagram

$$\begin{array}{ccc} L/\ker d & \xrightarrow{\sim} & \text{Im } F^n d, \\ \downarrow V & & \parallel \\ L/\ker d & \xrightarrow{\sim} & \text{Im } F^{n+1} d, \end{array}$$

shows that V induces an automorphism on $L/\ker d$ which is equivalent to $\ker d$ being dense in the V -topology. Since the original topology on L is weaker than the V -topology, $\ker d$ is also dense in the original topology. But d is continuous and M is separated hence $\ker d$ is also closed and so $\ker d = L$.

Let us go back to the proof of the Theorem. By (2.2) it is enough to show that the differentials in the E_1 term vanish. The E_1 term looks as below:

$$\begin{array}{ccccc} H^2(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_1^{0,2}} & H^2(X, C_X^1) & \xrightarrow{d_1^{1,2}} & H^2(X, C_X^2), \\ H^1(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_1^{0,1}} & H^1(X, C_X^1) & \xrightarrow{d_1^{1,1}} & H^1(X, C_X^2), \\ H^0(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_1^{0,0}} & H^0(X, C_X^1) & \xrightarrow{d_1^{1,0}} & H^0(X, C_X^2). \end{array}$$

Let us first show that the differentials in the bottom row are 0. This follows from the fact (1.4) that p is injective on $H^0(X, C_X^1)$ i.e. these modules are torsion free and the slope spectral sequence degenerates at E_1 modulo torsion ([3], III (3.2)). Next consider the differentials

$$d_1^{0,i} : H^i(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^i(X, C_X^1), \quad i = 1, 2.$$

The modules have separated and complete topologies being limits of the discrete spaces $H^i(X, \mathcal{W}_r(\mathcal{O}_X))$ and $H^i(X, C_{r,X}^1)$, clearly $d_1^{0,i}$ is continuous. The relation $F d_1^{0,i} V = d_1^{0,i}$ is satisfied by (1.3) and the exact sequences

$$H^i(X, \mathcal{W}(\mathcal{O}_X)) \xrightarrow{V} H^i(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^i(X, \mathcal{W}_r(\mathcal{O}_X)),$$

show that the V-topology is finer than the limit topology on $H^i(X, \mathcal{W}(\mathcal{O}_X))$ (they are actually identical), so by (2.5) we only have to show that the chains

$$\ker d_1^{0,i} \subset \ker F d_1^{0,i} \subset \dots \subset \ker F^n d_1^{0,i} \subset \dots \subset H^i(X, \mathcal{W}(\mathcal{O}_X)),$$

$$\text{Im } d_1^{0,i} \subset \text{Im } F d_1^{0,i} \subset \dots \subset \text{Im } F^n d_1^{0,i} \subset \dots \subset H^i(X, C_X^1),$$

stabilize. Now $H^i(X, \mathcal{W}(\mathcal{O}_X))$ is finitely generated, for $i=2$ it is the assumption and for $i=1$ it is always true as proved in [15], Proposition 4 so the first chain stabilizes.

For the second we have

$$\text{Im } F^n d_1^{0,i} \subset \ker d_1^{1,i} \quad \text{for all } n,$$

so

$$\text{Im } F^n d_1^{0,i} / \text{Im } d_1^{0,i} \subset \ker d_1^{1,i} / \text{Im } d_1^{1,i} = E_2^{1,i},$$

and we have $E_2^{1,i} = E_\infty^{1,i}$ (since $\dim X = 2$) which is a subquotient of $H_{\text{crys}}^{i+1}(X/W)$ hence finitely generated so the chain

$$\text{Im } F d_1^{0,i} / \text{Im } d_1^{0,i} \subset \dots \subset \text{Im } F^n d_1^{0,i} / \text{Im } d_1^{0,i} \subset \dots \subset E_\infty^{1,i},$$

stabilizes which shows that the second chain is stable.

For the differential

$$d_1^{1,2} : H^2(X, C_X^1) \rightarrow H^2(X, C_X^2),$$

we use the fact that F is an automorphism of $H^2(X, C_X^2)$ to conclude that the chain of kernels stabilize, namely

$$\ker d_1^{1,2} = \ker F^n d_1^{1,2} \quad \text{for all } n.$$

The chain of images stabilizes because $H^2(X, C_X^2) / \text{Im } d_1^{1,2} = E_2^{2,2} = E_\infty^{2,2}$ is finitely generated. To conclude that $d_1^{1,2} = 0$ we only need to show that the V-topology is finer than the limit topology, this follows however from the commutative diagram

$$\begin{array}{ccc} H^1(X, C_X^i) & \xrightarrow{V} & H^j(X, C_X^i), \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & H^j(X, C_{r,X}^i). \end{array}$$

The only differential left is

$$d_1^{1,1} : H^1(X, C_X^1) \rightarrow H^1(X, C_X^2).$$

The chain of kernels stabilizes for the same reasons as above, and in order to show that the chain of images stabilizes it is enough to show that $H^1(X, C_X^2)/\text{Im } d_1^{1,1}$ is finitely generated, this follows however from the exact sequence

$$E_2^{0,2} \xrightarrow{d_2^{0,2}} H^1(X, C_X^2)/\text{Im } d_1^{1,1} = E_2^{2,1} \rightarrow E_3^{2,1} = E_\infty^{2,1},$$

because $E_2^{0,2} \subset H^2(X, \mathcal{W}(\mathcal{O}_X))$ is finitely generated. This concludes the proof of the Theorem.

Remark. — Some parts of the proof of (2.4) goes through without assuming that $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is finitely generated or that X is a surface; in particular that

$$H^1(X, \mathcal{W}(\mathcal{O}_X)) = E_\infty^{0,1},$$

and

$$H^0(X, C_X^1) = E_\infty^{1,0},$$

there results an exact sequence

$$0 \rightarrow H^0(X, C_X^1) \rightarrow H_{\text{crys}}^1(X/W) \rightarrow H^1(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow 0,$$

since $H^0(X, C_X^1)$ is torsion free by (1.4) and $H^1(X, \mathcal{W}(\mathcal{O}_X))$ is torsion free by [15], p. 32, we deduce the well known fact that $H_{\text{crys}}^1(X/W)$ is torsion free.

(2.6) COROLLARY. — *Let X/k be a smooth proper variety and assume that $\text{Pic}(X)$ is reduced then the differential*

$$d_1^{0,1} : H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \Omega_{X/k}^1),$$

vanishes.

Proof. — Let (E, d) denote the slope spectral sequence and (E', d') the Hodge to de Rham spectral sequence. Since $C_{1,X} \simeq \Omega_{X/k}$ ([3], II (3.1)) we have a map of spectral sequences

$$(E, d) \rightarrow (E', d'),$$

in particular a commutative diagram

$$\begin{array}{ccc} H^1(X, \mathcal{W}(\mathcal{O}_X)) & \xrightarrow{d_1^{0,1}} & H^1(X, C_X^1) \\ \downarrow & & \downarrow \\ H^1(X, \mathcal{O}_X) & \xrightarrow{d_1^{0,1}} & H^1(X, \Omega_X^1). \end{array}$$

By the remark above the horizontal map on top is zero, and the left hand vertical map is surjective since $\text{Pic}(X)$ is reduced, hence the corollary.

(2.6) Has also been proved by T. Oda in his Harvard thesis [14].

(2.7) COROLLARY. — *Let X/k be a smooth proper surface. Assume that $\text{Br}_X^\wedge$ is pro-represented by a p -divisible formal group then the Hodge to de Rham spectral sequence degenerates at E_1 .*

Proof. — By [1], Corollary (4.3), the (covariant) Dieudonné module of $\text{Br}_X^\wedge$ is $H^2(X, \mathcal{W}(\mathcal{O}_X))$ so $\text{Br}_X^\wedge$ p -divisible implies that $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is finitely generated and free [10], and hence by (2.4) the slope spectral sequence degenerates at E_1 .

Since $H^2(X, \mathcal{W}(\mathcal{O}_X))$ is free

$$H^1(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^1(X, \mathcal{O}_X),$$

is surjective.

$$H^2(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^2(X, \mathcal{O}_X),$$

is surjective because $H^3(X, \mathcal{W}(\mathcal{O}_X)) = 0$, and

$$H^0(X, \mathcal{W}(\mathcal{O}_X)) \rightarrow H^0(X, \mathcal{O}_X),$$

because $H^1(X, \mathcal{W}(\mathcal{O}_X))$ is free ([15], p. 32) it follows that

$$d_1^{0,i} : H^i(X, \mathcal{O}_X) \rightarrow H^i(X, \Omega_{X/k}^1),$$

is zero $i=0, 1, 2$, by Serre duality the rest of the differentials in the E_1 term vanish. A similar argument shows that the higher differentials vanish as well.

(2.8) PROPOSITION. — *With the assumptions of (2.7) assume further that $H_{\text{crys}}^2(X/W)$ is torsion free then:*

(i) $\dim_k H_{\text{DR}}^i(X/k) = \dim_k H_{\text{crys}}^i(X/W) \otimes K$, $i=0, 1, 2, 3, 4$, where K is the fraction field of $\mathcal{W}(k)$;

(ii) $h^{i,j} = \dim_k H^j(X, \Omega_{X/k}^i) = \dim_k H^i(X, \Omega_{X/k}^j) = h^{j,i}$.

Proof. — (i) follows from the exact sequences:

$$0 \rightarrow H_{\text{crys}}^i(X/W) \otimes k \rightarrow H_{\text{DR}}^i(X/k) \rightarrow \text{Tor}_1^{W(k)}(H_{\text{crys}}^{i+1}(X/W), k) \rightarrow 0,$$

plus the fact that $H_{\text{crys}}^3(X/W)$ is also torsion free (by Poincaré duality).

To prove (ii) it is enough to show

$$h^{0,1} = h^{1,0},$$

the other equalities then follow from Serre duality.

$$h^{0,1} = \dim_k H^1(X, \mathcal{O}_X) = \dim \underline{\text{Pic}}^0(X),$$

since $\underline{\text{Pic}}^0(X)$ is reduced.

$$\dim H_{\text{DR}}^1(X/k) = h^{0,1} + h^{1,0}, \quad \text{by (2.7)}$$

and

$$\dim_K H_{\text{crys}}^1(X/W) \otimes K = 2 \dim \underline{\text{Pic}}^0(X),$$

and the equality follows from (i).

3. Surfaces over an irreducible scheme

In this section we consider a smooth proper S -scheme $f: X \rightarrow S$ with geometrically irreducible fibers of dimension 2; S an irreducible $\mathbf{F}_p$ -scheme such that $f_* \mathcal{O}_X = \mathcal{O}_S$.

(3.1) LEMMA. — *Let A be a local domain of characteristic p with maximal ideal $\mathcal{M}$ and residue field k . Let $\hat{A}$ be the completion at $\mathcal{M}$ and L the fraction field of $\hat{A}$. Assume that $G = \text{Spf } A[[t_1, \dots, t_n]]$ is a connected formal Lie group such that G_k^- is p -divisible, then the formal Lie group G_L is p -divisible.*

Proof. — Let the power series $f_1, \dots, f_n$ define multiplication by p in G , then $\ker p: G_L \rightarrow G_L$ is represented by $L[[t_1, \dots, t_n]]/(f_1, \dots, f_n)$ and it is enough to show that this is a finite dimensional L -vectorspace ([6], p. 47). Since $G_k^- = \text{Spf } k[[t_1, \dots, t_n]]$ is p -divisible $G_{A/\mathcal{M}^r} = \text{Spf } A/\mathcal{M}^r[[t_1, \dots, t_n]]$ is p -divisible for all $r \geq 1$ ([6], p. 62) so $A/\mathcal{M}^r[[t_1, \dots, t_n]]/(f_1, \dots, f_n)$ is a finitely generated $A/\mathcal{M}^r$ -module. Let $e_1, \dots, e_s \in A[[t_1, \dots, t_n]]/(f_1, \dots, f_n)$ such that

$$\{\bar{e}_1, \dots, \bar{e}_s\} \subset k[[t_1, \dots, t_n]]/(f_1, \dots, f_n),$$

is a set of generators, it follows from Nakayama's Lemma that

$$\{\bar{e}_1, \dots, \bar{e}_s\} \subset A/\mathcal{M}^r[[t_1, \dots, t_n]]/(f_1, \dots, f_n),$$

generates for all $r \geq 1$.

Let M be the A -module generated by $\{e_1, \dots, e_s\}$ then

$$M/\mathcal{M}^r M = A/\mathcal{M}^r[[t_1, \dots, t_n]]/(f_1, \dots, f_n),$$

$$\hat{A}[[t_1, \dots, t_n]]/(f_1, \dots, f_n) = \varprojlim A/\mathcal{M}^r[[t_1, \dots, t_n]]/(f_1, \dots, f_n) = \varprojlim M/\mathcal{M}^r M = \hat{M}.$$

Since M is finitely generated $\hat{M} = M \otimes \hat{A}$ is finitely generated over $\hat{A}$ so

$$\hat{A}[[t_1, \dots, t_n]]/(f_1, \dots, f_n),$$

and hence

$$L[[t_1, \dots, t_n]]/(f_1, \dots, f_n),$$

is finitely generated.

(3.2) THEOREM. — Assume that there is a closed point $s_0 \in S$ such that the geometric fibre $Y = X_{s_0}$ has p -divisible formal Brauer group, then the differential

$$d : f_* \Omega_{X/S}^1 \rightarrow f_* \Omega_{X/S}^2,$$

is zero.

Proof. — By the smoothness of f , $f_* \Omega_{X/S}^2$ is a locally free sheaf on S so the set

$$F = \{s \in S \mid d_s : (f_* \Omega_{X/S}^1)_s \rightarrow (f_* \Omega_{X/S}^2)_s \text{ is zero}\},$$

is a closed set. We are going to show that the generic points is in F .

$\text{Pic}^0 X/S$ is representable by [7], Theorem (3.1) and since we have assumed that Br_Y is pro-representable by a p -divisible formal group $H^2(Y, \mathcal{W}(\mathcal{O}_Y))$ is free so $H^1(Y, \mathcal{O}_Y) \rightarrow H^2(Y, \mathcal{W}(\mathcal{O}_Y))$ is zero hence $\text{Pic}^0(Y)$ is smooth ([12], p. 196).

By [8], Theorem (3.5) there is a non-empty open set $s_0 \in \mathcal{U} \subset S$ such that $\text{Pic}^0 X/\mathcal{U}$ is smooth and hence $\widehat{\text{Br}}_{X/\mathcal{U}}$ is representable by a formal group which is formally smooth since the fibre dimension is 2 ([1], Cor. (4.1)).

Let $\{G[n]\}_n$ be the inductive system of locally free finite groups associated to the formal Lie group $\widehat{\text{Br}}_{X/\mathcal{U}}$ ([6], Prop. (2.6)). Locally on $\mathcal{U}$ each $G[n]$ is isomorphic to $\text{Spec } \mathcal{O}_{\mathcal{U}}[t_1, \dots, t_d]/(t_1^{p^n}, \dots, t_d^{p^n})$ where d is the rank of the conormal bundle of $\widehat{\text{Br}}_{X/\mathcal{U}}$ ([6], Prop. (2.1)).

We can assume $S = \text{Spec } R$ where $R = \mathcal{O}_{S, s_0}$, hence over $\text{Spec } R$, $\widehat{\text{Br}}_{X/R}$ is isomorphic to $\varinjlim G[n]$ with each

$$G[n] \simeq \text{Spec } R[t_1, \dots, t_d]/(t_1^{p^n}, \dots, t_d^{p^n}).$$

Since X/R is smooth the functor $\widehat{\text{Br}}_{X/R}$ is isomorphic to the sheaf $R^2 f_* \hat{\mathbf{G}}_m$ on the big etale site of $\text{Spec } R$ ([1], Prop. (1.7)). By general theorems about sheaf cohomology ([16], Prop. (5.1)) this implies that the formal Brauer group commutes with all base changes. In terms of the inductive system this means that

$$\widehat{\text{Br}}_{X_T} \simeq \varinjlim G[n] \otimes_R T \simeq \varinjlim \text{Spec } \mathcal{O}_T[t_1, \dots, t_d]/(t_1^{p^n}, \dots, t_d^{p^n}),$$

for every R -scheme T .

Let η be the fraction field of R and L the fractional field of $\hat{R}$, κ is the residue field. By assumption

$$\widehat{\text{Br}}_Y = \varinjlim \bar{\kappa}[t_1, \dots, t_d]/(t_1^{p^n}, \dots, t_d^{p^n}),$$

is p -divisible hence (3.1) gives that

$$\mathrm{Br}_{\hat{X}_L} = \varinjlim \mathrm{Spec} L[t_1, \dots, t_d]/(t_1^p, \dots, t_d^p),$$

is p -divisible and so also $\mathrm{Br}_{\hat{X}_L}$ is p -divisible.

(2.4) Then implies that

$$d : H^0(X_L, \Omega_{X_L}^1) \rightarrow H^0(X_L, \Omega_{X_L}^2),$$

is zero, and by faithfully flat descent

$$d : H^0(X_\eta, \Omega_{X_\eta}^1) \rightarrow H^0(X_\eta, \Omega_{X_\eta}^2),$$

is zero which shows $\mathrm{spec} \eta \in F$.

(3.3) COROLLARY. — *With the assumptions of (3.2) assume that all the sheaves $R^j f_* \Omega_{X/S}^i$ are locally free on S then the spectral sequence*

$$E_1^{i,j} = R^j f_* \Omega_{X/S}^i \Rightarrow H_{\mathrm{DR}}^*(X/S),$$

degenerates at E_1 .

Proof. — In this case the set

$$F = \{s \in S \mid (R^j f_* \Omega_{X/S}^i)_s \Rightarrow H_{\mathrm{DR}}^*(X/S)_s \text{ degenerates at } E_1\},$$

is closed and the Proof of (3.2) shows that F contains the generic point.

REFERENCES

- [1] M. ARTIN and B. MAZUR, *Formal Groups Arising from Algebraic varieties* (Ann. Scient. Éc. Norm. Sup., 4^e série, t. 10, 1977, pp. 87-132).
- [2] P. BERTHELOT, *Cohomologie cristalline des schémas de caractéristique $p > 0$* (Lecture Notes in Mathematics, No. 407, Springer-Verlag, Berlin, 1974).
- [3] S. BLOCH, *Algebraic K-theory and Crystalline Cohomology* (Publ. Math. Inst. Hautes Études Sc., No. 47, pp. 188-268).
- [4] P. CARTIER, *Modules associés à un groupe formel commutatif. Courbes typiques* (C.R. Acad. Sc., Paris, t. 265, série A, 1967, pp. 129-132).
- [5] M. DEMAZURE, *Lectures on p -divisible groups* (Lecture Notes in Mathematics, No. 302, Springer-Verlag, Berlin, 1972).
- [6] A. GROTHENDIECK, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, Les presses de l'Université de Montréal, Montréal, 1974.
- [7] A. GROTHENDIECK, *Techniques de descente et théorèmes d'existence en géométrie algébrique. V. Les schémas de Picard : théorèmes d'existence* (Séminaire Bourbaki, No. 232, 1961/1962).
- [8] A. GROTHENDIECK, *Techniques de descente et théorèmes d'existence en géométrie algébrique. VI. Les schémas de Picard : propriétés générales* (Séminaire Bourbaki, No. 237, 1961/1962).
- [9] L. ILLUSIE, *Complexe de de Rham-Witt et cohomologie cristalline* (in preparation).
- [10] M. LAZARD, *Commutative Formal Groups* (Lecture Notes in Mathematics, No. 443, Springer-Verlag, Berlin, 1975).

- [11] D. MUMFORD, *Pathologies of Modular Algebraic Varieties* (*Amer. J. Math.*, Vol. 23, 1961).
- [12] D. MUMFORD, *Lectures on Curves on an Algebraic Surface*, Princeton University Press, Princeton, 1966.
- [13] N. NYGAARD, *A p -adic Proof of the Non-existence of Vector Fields on K^3 Surfaces* (in preparation).
- [14] T. ODA, *The First de Rham Cohomology Group and Dieudonné Modules* (*Ann. Scient. Éc. Norm. Sup.*, 4^e série, t. 2, 1969, pp. 63-135).
- [15] J-P. SERRE, *Sur la topologie des variétés algébriques en caractéristique p* (*Symp. Int. de Top. Alg.*, Mexico, 1958, pp. 24-53).
- [16] J-L. VERDIER, *Cohomologie dans les topos* (in *SGA 4*, Fasc. 2, *Lecture Notes in Mathematics*, No. 270, Springer Verlag, Berlin, 1972).
- [17] W. LANG, *Quasi-Elliptic Surfaces in Char. 3* (*Thesis*, Harvard, 1978).

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