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ORBITAL INTEGRALS AND A FAMILY OF GROUPS ATTACHED TO A REAL REDUCTIVE GROUP (1)

By Diana SHELSTAD

1. Introduction

In this paper we pursue one of the questions suggested by the formulations in [7] (*cf.* [10]). Our concern will be with transferring orbital integrals from one group (of **R**-rational points on a connected reductive linear algebraic group defined over **R**) to another. In [9] we considered "stable" orbital integrals and obtained a transfer which will be our starting point. We recall some details. Suppose that f is a Schwartz function on the group G, that T is a Cartan subgroup of G and that γ is a regular element in T. Then, following Langlands, we have defined

$$\Phi_f^1(\gamma)=\sum_{\omega}\int_{G/T} f(g\gamma^{\omega}g^{-1})\,d\bar{g},$$

where $d\bar{g}$ is a G-invariant measure on G/T (whose normalization we ignore for the present) and ω ranges over the set $\mathscr{D}(T)$ [7] which we may identify simply as the quotient of the imaginary Weyl group for T by the subgroup of those elements realized in G . . . recall that any element of the imaginary Weyl group stabilizes T. Our interest in these stable orbital integrals lies in the fact that the distributions $f\to\Phi_f^1(\gamma)$ generate the characters attached to L-packets of tempered irreducible representations of G (*cf.* [9]).

Suppose that for each Cartan subgroup T we are given a function Φ^T on the regular elements in T. Then a theorem of [9] provides necessary and sufficient conditions for the existence of a Schwartz function f on G such that $\Phi^T=\Phi_f^1$ for each T. On the other hand, if we fix an L-group (= associate group [8]) for G then we are provided with a quasi-split group G* and an inner twist ψ from **G**, the underlying algebraic group for G, to **G***. The map ψ determines embeddings of each Cartan subgroup (of G) in G*; these embeddings induce an injection of the set $t(G)$ of conjugacy classes of Cartan subgroups of G in $t(G^*)$. Recall that $t(G)$ is partially ordered (*cf.* [3]); the image of $t(G)$ in $t(G^*)$ forms an "initial segment" of $t(G^*)$ [9]. We say that an element γ' of G* originates from the regular element γ of G if γ' is the image of γ under one of the embeddings in G* of the Cartan subgroup containing γ.

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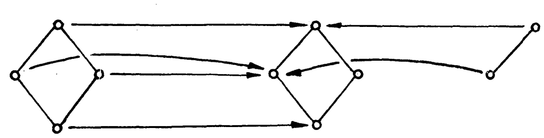
Now combining this and the characterization theorem, we can effect a transfer of stable orbital integrals from G to G^* in the following sense: given a Schwartz function f on G there is a Schwartz function f' on G^* such that $\Phi^1_{f'}(\gamma')=\Phi^1_f(\gamma)$ if γ' originates from the regular element γ in G, with $\Phi^1_{f'}(\gamma')=0$ if γ' does not originate in G.

We come then to our present problem. First, we replace Φ^1_f by an "unstable" orbital integral. If $\varkappa$ assigns to each ω in $\mathscr{D}(T)$ a value, either 1 or -1, then again following Langlands, we set

$$\Phi^{\varkappa}_f(\gamma)=\sum_{\omega\in\mathscr{D}(T)}\varkappa(\omega)\int_{G/T}f(gt^{\omega}g^{-1})\,dg,$$

for regular γ in T. Global considerations (for example, the suitable grouping of some terms on one side of the Trace Formula (*cf.* [5], § 5, for SL_2) suggest that we consider those $\varkappa$ described in [7]; we recall the appropriate definitions and observations in Paragraphs 2,3. Briefly, as described in [7], $\mathscr{D}(T)$ can be embedded in a quotient of the module generated by the coroots of T in G and $\varkappa$ is a quasicharacter on this quotient . . . the domain of $\varkappa$ is thus larger than $\mathscr{D}(T)$. From now on we assume that $\varkappa$ is of such type and call $\Phi^{\varkappa}_f$ a $\varkappa$-orbital integral. In Paragraph 4 we will describe the invariance, smoothness and "jump" properties (which we find easier to work with than "germ expansions") of $\varkappa$-orbital integrals.

The triple (G, T, $\varkappa$) determines, via an L-group construction, a quasi-split group H of same rank as G, but possibly of lower dimension [7]. We will recall the construction in Paragraph 5, remarking now only the fact that T can be embedded in H and $\mathscr{D}_H(T)$ transferred to G; $\varkappa$ is trivial on the image of $\mathscr{D}_H(T)$. An imprecise version of a question of Langlands asks whether the $\varkappa$-orbital integrals for G transfer to stable orbital integrals on H. To proceed to a more careful formulation we observe that the L-group construction provides not only H but also some ancillary data, including a quasi-split group G^* and an inner twist ψ from G to G^*. The data yield embeddings of the Cartan subgroups of H in G^* and a map from $t(H)$ into $t(G^*)$; recalling the map of $t(G)$ into $t(G^*)$ determined by ψ we obtain then a notion of a Cartan subgroup of H originating in G. For example, using the notation of [3] for $t(\;)$, we may have:



and obtain three conjugacy classes of Cartan subgroups in H originating in G (case G nonsplit, noncompact form of type C_2 . . . H of type $A_1\times A_1$). Suppose that T' originates from T (our given Cartan subgroup). Then the transfer of $\Phi^{\varkappa}_f$ to T' depends on the choice of map from T' to T. Thus we have to qualify our notion of an element γ' of H originating from a regular element γ of G. We will do this by choosing a set $\mathscr{I}=\{i_m: T'_m\to T_m,\ m=0, 1, \ldots, N\}$ of embeddings such that T_0 is our given group T and $T'_0, \ldots, T'_N$ form a complete set of representatives for the conjugacy classes which originate in G (*see*

Paragraph 6 for technical assumptions). We then say that γ' originates from $\gamma \in T_m$ with respect to $\mathscr{I}$ if γ' is stably conjugate to $i_m^{-1}(\gamma)$; that is, if γ' is obtained from $i_m^{-1}(\gamma)$ by the action of an element of $\mathscr{A}(T'_m)$ (*cf.* [7], recalled also in Paragraph 2). Also attached to $\mathscr{I}$ is a transfer of $\varkappa$ to each of the Cartan subgroups $T_1, \ldots, T_N$ (*cf.* Paragraph 7).

We come then to the main problem, that of finding a factor Δ so that for each Schwartz function f on G there is a Schwartz function f' on H satisfying:

(1) $\Phi^1_{f'}(\gamma') = \Delta(\gamma)\Phi^\varkappa_f(\gamma)$ if γ' originates from the regular element γ in G with respect to $\mathscr{I}$ and

(2) $\Phi^1_{f'} \equiv 0$ on those Cartan subgroups of H which do not originate in G.

Thus Δ is to be a function on the regular elements of $\bigcup_{m=0}^{N} T_m$. On each Cartan subgroup T_m we fix a system of positive imaginary roots. We may consider, at least formally,

$$\prod_{\substack{\alpha \text{ positive} \\ \text{imaginary} \\ \text{not from H}}} (e^{\alpha/2} - e^{-\alpha/2}) \prod_{\substack{\alpha \text{ positive} \\ \text{not imaginary} \\ \text{not from H}}} |e^{\alpha/2} - e^{-\alpha/2}|$$

(the conditions on α are made precise in Paragraph 7).

This expression can be interpreted as a function Δ_m on T_m if half the sum of the positive imaginary roots "not from H" lifts to a character on $\mathbf{T}_m$. That will be the major part of our assumption (8.1). In prescribing a candidate for Δ we insert parameters $\varepsilon_0, \ldots, \varepsilon_N$, each equal to 1 or -1; thus our candidate will be the function $\Delta^G_H = \Delta^G_H(\varepsilon_0, \ldots, \varepsilon_N)$ defined by $\{\varepsilon_m \Delta_m;\ m=0, \ldots, N\}$. The existence (for some choice of $\varepsilon_0, \ldots, \varepsilon_N$) of a "transfer of orbital integrals" in the sense of the last paragraph is then independent of our choice of $\mathscr{I}$ and the systems of positive imaginary roots. In Theorem 8.3 we show that $\gamma' \to \Delta^G_H(\gamma)\Phi^\varkappa_f(\gamma)$ is well-defined (although, in general, neither Δ^G_H nor $\Phi^\varkappa_f$ alone transfers to H in this way).

Our main result, Theorem 10.2, is a set of necessary and sufficient conditions on the choices for $\varepsilon_0, \ldots, \varepsilon_N$ in order that $\Delta^H_G = \Delta^H_G(\varepsilon_0, \ldots, \varepsilon_N)$ provide a transfer of orbital integrals. Suppose that the classes of T'_m and T'_n are adjacent in the lattice $t(H)$. Then we attach to the pair (m, n) a signature $\varepsilon_\varkappa(m, n)$ obtained from values of $\varkappa$ and a signature $\varepsilon_+(m, n)$ obtained by evaluating some determinants. Our conditions are:

$$\varepsilon_m \varepsilon_n = \varepsilon_\varkappa(m, n)\, \varepsilon_+(m, n).$$

In Paragraph 11 we begin a study of the consistency of these equations as the pair (m, n) varies. After some remarks, suggesting a general procedure, and two examples we can conclude that if the derived group of $\mathbf{G}$ is isogenous to a product of groups each of rank at most two, then there is indeed a choice of $\varepsilon_0, \ldots, \varepsilon_N$ for which $\Delta^H_G(\varepsilon_0, \ldots, \varepsilon_N)$ provides a transfer of orbital integrals.

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NOTATION. — We continue with the notation of [9], except that now σ denotes complex conjugation and we further generalize the notion of Cayley transform (*cf.* Paragraph 3). By the character module of a torus we will mean the group of rational characters, with multiplication written additively; roots will be rational characters, rather than linear functionals on the Lie algebra (as in [9] and the present section).

2. The set $\mathscr{D}(\mathrm{T})$

Let $\mathbf{T}$ be a maximal torus in $\mathbf{G}$, defined over $\mathbf{R}$. We recall from [7] that

$$\mathscr{A}(\mathrm{T})=\{g\in\mathbf{G} : \mathrm{ad}\, g/\mathbf{T} \text{ is defined over } \mathbf{R}\},$$

and $\mathscr{D}(\mathrm{T})=\mathrm{G}\backslash\mathscr{A}(\mathrm{T})/\mathbf{T}$. If $\mathbf{M}$ is the centralizer in $\mathbf{G}$ of the maximal $\mathbf{R}$-split torus in $\mathbf{T}$ then $\mathscr{A}(\mathrm{T})=\mathrm{G}\,.\,\mathrm{Norm}(\mathbf{M}, \mathbf{T})$ [9]. Hence we may as well regard $\mathscr{D}(\mathrm{T})$ as $\Omega(\mathrm{M}, \mathrm{T})\backslash\Omega(\mathbf{M}, \mathbf{T})$, $\Omega(\mathbf{M}, \mathbf{T})$ being the Weyl group of $\mathbf{T}$ in $\mathbf{M}$ (. . . the "imaginary Weyl group of $\mathbf{T}$") and $\Omega(\mathrm{M}, \mathrm{T})$ the subgroup of $\Omega(\mathbf{M}, \mathbf{T})$ consisting of those elements which can be realized in M.

We need to recall some facts from [7]. We will use σ to denote the non-trivial element of the Galois group of $\mathbf{C}$ over $\mathbf{R}$ and $\mathrm{H}^*(\ )$ to denote the cohomology of $\langle 1, \sigma\rangle$. If $g\in\mathscr{A}(\mathrm{T})$ then $\sigma(g^{-1})g\in\mathbf{T}$ so that $g\to(1\to 1, \sigma\to\sigma(g^{-1})g)$ yields a map of $\mathscr{A}(\mathrm{T})$ into the 1-cocycles for $\mathbf{T}$. This map induces a bijection between $\mathscr{D}(\mathrm{T})$ and those elements of $\mathrm{H}^1(\mathbf{T})$ which are annihilated by the natural map of $\mathrm{H}^1(\mathbf{T})$ into $\mathrm{H}^1(\mathbf{G})$. Such elements of $\mathrm{H}^1(\mathbf{T})$ lie in a subgroup $\mathscr{E}(\mathrm{T})$ obtained as follows. Let $\mathbf{G}^{\sim}$ be the simply-connected covering group of the derived group of $\mathbf{G}$, π the natural homomorphism of $\mathbf{G}^{\sim}$ into $\mathbf{G}$ and $\mathbf{T}^{\sim}$ the inverse image of $\mathbf{T}$ under π. Then $\mathscr{E}(\mathrm{T})$ is the image of $\mathrm{H}^1(\mathbf{T}^{\sim})$ under the homomorphism into $\mathrm{H}^1(\mathbf{T})$ induced by π.

To continue with [7], we denote the character module of $\mathbf{T}$ by $\mathrm{L}(\mathbf{T})$ and set $\mathrm{L}^{\vee}(\mathbf{T})=\mathrm{Hom}(\mathrm{L}(\mathbf{T}), \mathbf{Z})$. In the usual manner we identify $\mathrm{L}^{\vee}(\mathbf{T}^{\sim})$ with the submodule $\langle\Xi^{\vee}\rangle$ of $\mathrm{L}^{\vee}(\mathbf{T})$ generated by the set $\Xi^{\vee}$ of coroots for $\mathbf{T}$ in $\mathbf{G}$. Tate-Nakayama duality then establishes a canonical isomorphism between $\mathscr{E}(\mathrm{T})$ and the image under the natural homomorphism of $\mathrm{H}^{-1}(\langle\Xi^{\vee}\rangle)$ into $\mathrm{H}^{-1}(\mathrm{L}^{\vee}(\mathbf{T}))$ or, just as well, between $\mathscr{E}(\mathrm{T})$ and the quotient of $\{\lambda^{\vee}\in\langle\Xi^{\vee}\rangle : \sigma\lambda^{\vee}=-\lambda^{\vee}\}$ by

$$\mathscr{L}(\mathrm{T})=\{\lambda^{\vee}\in\langle\Xi^{\vee}\rangle : \lambda^{\vee}=\sigma\mu^{\vee}-\mu^{\vee}, \text{ some } \mu^{\vee} \text{ in } \mathrm{L}^{\vee}(\mathbf{T})\}.$$

Hence $\mathscr{D}(\mathrm{T})$ is identified as a collection of cosets of $\mathscr{L}(\mathrm{T})$ in $\langle\Xi^{\vee}\rangle$; we shall call this the T-N identification.

As for realizing T-N explicitly we will need only an (unpublished) observation of Langlands; we state it as a proposition as we will use it in several places. Recall that a root α is imaginary if and only if $\sigma\alpha=-\alpha$ or, equivalently, $\sigma\alpha^{\vee}=-\alpha^{\vee}$. Assume now that α is imaginary; ω_α, the Weyl reflection with respect to α, lies in $\Omega(\mathbf{M}, \mathbf{T})$. In the case α is compact (*cf.* [9]) ω_α lies in $\Omega(\mathrm{M}, \mathrm{T})$.

PROPOSITION 2.1. — *In the case α is noncompact the image under* T−N *of* $\Omega(\mathrm{M}, \mathrm{T})\omega_\alpha$ *is* $\alpha^{\vee}+\mathscr{L}(\mathrm{T})$.

The proof is straightforward. Indeed, fix a homomorphism (over $\mathbf{R}$) of SL_2 in $\mathbf{G}$ as in [9]. Then the image of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ is a 1-cocycle of $\mathbf{T}$ attached to ω_α in the manner earlier. It is now a matter of reviewing the $\mathrm{T}-\mathrm{N}$ identification explicitly (*cf.* [6]); we omit the details.

3. Characters on $\mathscr{D}(\mathrm{T})$

Following [7] we will consider a quasicharacter $\varkappa$ on $\langle \Xi^\vee \rangle$, trivial on $\mathscr{L}(\mathrm{T})$. Note that the restriction of $\varkappa$ (as quasicharacter on $\langle \Xi^\vee \rangle / \mathscr{L}(\mathrm{T})$) to $\mathscr{D}(\mathrm{T})$ takes only the values ± 1. We will often refer to $\varkappa$ as a "character on $\mathscr{D}(\mathrm{T})$" [although, in general, the domain is larger and $\varkappa$ is not determined by its restriction to $\mathscr{D}(\mathrm{T})$].

If $\omega \in \mathscr{A}(\mathrm{T})$ then clearly

$$\varkappa^\omega(\lambda^\vee) = \varkappa(\omega^{-1}\lambda^\vee), \qquad \lambda^\vee \in \langle \Xi^\vee \rangle,$$

defines a character on $\mathscr{D}(\mathrm{T}^\omega)$; here, as usual, ω acts on $\langle \Xi^\vee \rangle$ by the contragredient of the adjoint action. On the other hand, we will often write $\varkappa(\omega)$ for the value of $\varkappa$ on the coset $\Omega(\mathrm{M}, \mathrm{T})\,\omega$ in $\mathscr{D}(\mathrm{T})$. If $\omega_1 \in \mathscr{A}(\mathrm{T})$ and $\omega_2 \in \mathscr{A}(\mathrm{T}^{\omega_1})$ then $\omega_2\omega_1 \in \mathscr{A}(\mathrm{T})$ and:

Proposition 3.1:

$$\varkappa(\omega_2\omega_1) = \varkappa(\omega_1)\,\varkappa^{\omega_1}(\omega_2).$$

Proof. — Note that

$$\sigma(\omega_1^{-1}\omega_2^{-1})\,\omega_2\omega_1 = \sigma(\omega_1^{-1})\,\omega_1\,\omega_1^{-1}(\sigma(\omega_2^{-1})\,\omega_2)\,\omega_1.$$

Since the $\mathrm{T}-\mathrm{N}$ identification respects the action of $\mathscr{A}(\mathrm{T})$ the assertion is now clear.

Suppose that α is an imaginary root of $\mathbf{T}$ in $\mathbf{G}$. Provided that there is a noncompact root among the elements $\omega\alpha$, ω in the imaginary Weyl group of $\mathbf{T}$ [or, just as well, ω in $\Omega_0(\mathbf{G}, \mathbf{T})$, the elements realized in $\mathscr{A}(\mathrm{T})$], we can find $s \in \mathbf{G}$ such that $\sigma(s^{-1})\,s$ realizes the Weyl reflection ω_α [9]. In the case that α itself is noncompact we have called s a Cayley transform with respect to α [9]. It is convenient now to drop this requirement on α: thus, as long as α is imaginary and $\sigma(s^{-1})\,s$ realizes ω_α we will call *s a Cayley transform with respect to α*. The assertions of Proposition 2.7 in [9] remain true; in particular, $\mathbf{T}_s$, the image of $\mathbf{T}$ under s, is defined over $\mathbf{R}$.

Proposition 3.2. — *If $\varkappa(\alpha^\vee) = 1$ then*

$$\varkappa^s(\lambda^\vee) = \varkappa(s^{-1}\lambda^\vee), \qquad \lambda^\vee \in \langle \Xi_s^\vee \rangle,$$

defines a character on $\mathscr{D}(\mathrm{T}_s)$.

Here $\Xi_s^\vee$ denotes the set of coroots for $\mathbf{T}_s$ in $\mathbf{G}$.

Proof. — We have only to show that if $\mu^\vee \in L^\vee(\mathbf{T}_s)$ and $\sigma\mu^\vee - \mu^\vee \in \langle \Xi_s^\vee \rangle$ then $\varkappa^s(\sigma\mu^\vee - \mu^\vee) = 1$. But

$$s^{-1}\sigma\mu^\vee - s^{-1}\mu^\vee = \sigma\sigma(s^{-1})\,s(s^{-1}\mu^\vee) - s^{-1}\mu^\vee = \sigma\omega_{\alpha^\vee}(s^{-1}\mu^\vee) - s^{-1}\mu^\vee,$$

which differs from $\sigma(s^{-1}\mu^\vee) - s^{-1}\mu^\vee$ by an integral multiple of $\alpha^\vee$. Hence the proposition is proved.

Finally, we include some simple computations needed in the next section.

PROPOSITION 3.3. — *Suppose that* α *is a noncompact imaginary root for which* $\varkappa(\alpha^\vee) = 1$. *Then:*

(i) $\varkappa^{\omega_\alpha} = \varkappa$;

(ii) $\varkappa(\omega\omega_\alpha) = \varkappa(\omega)$, $\omega \in \mathscr{A}(\mathrm{T})$; *and*

(iii) *if s is a Cayley transform with respect to* α *then* $\varkappa^s(\omega) = \varkappa(s^{-1}\,\omega\, s)$ *for any* $\omega \in \mathscr{A}(\mathrm{T}_s)$ *which normalizes* $\mathbf{T}_s$.

Proof:

(i) $\varkappa^{\omega_\alpha}(\lambda^\vee) = \varkappa(\lambda^\vee)\,\varkappa(\omega_\alpha\lambda^\vee - \lambda^\vee) = \varkappa(\lambda^\vee)$, $\lambda^\vee \in \langle \Xi^\vee \rangle$;

(ii) $\varkappa(\omega\omega_\alpha) = \varkappa(\omega_\alpha)\,\varkappa^{\omega_\alpha}(\omega) = \varkappa(\alpha^\vee)\,\varkappa(\omega) = \varkappa(\omega)$ (*cf.* Props. 3.1, 2.1);

(iii) Proposition 4.6 of [9] and Proposition 3.1 show that it is enough to prove (iii) in the case where ω realizes the Weyl reflection with respect to an imaginary root β of $\mathbf{T}_s$.

Suppose that β is compact. Then $\varkappa^s(\omega_\beta) = 1$. Proposition 4.6 of [9] shows that either $\omega_{s^{-1}\beta}$ or $\omega_{s^{-1}\beta}\,\omega_\alpha$ is realized in G. Since $\varkappa(\omega_{s^{-1}\beta}\,\omega_\alpha) = \varkappa(\omega_{s^{-1}\beta})$ (ii) we obtain $\varkappa^s(\omega_\beta) = \varkappa(\omega_{s^{-1}\beta}) = \varkappa(s^{-1}\,\omega_\beta\, s)$.

Suppose that β is noncompact. Again an argument as in Proposition 4.6 of [9] shows that if $\omega_{s^{-1}\beta}$ is realized in G then so is ω_β. Hence if ω_β is not realized in G we get

$$\varkappa^s(\omega_\beta) = \varkappa^s(\beta^\vee) = \varkappa(s^{-1}\beta^\vee) = \varkappa(\omega_{s^{-1}\beta}) = \varkappa(s^{-1}\,\omega_\beta\, s).$$

On the other hand, if ω_β is realized in G we may argue as in the previous paragraph and the proof is completed.

4. Definition and properties of $\Phi_f^\varkappa$

We come then to orbital integrals. Fix a Schwartz function f on G. As in [9], if T is a Cartan subgroup of G, dt a Haar measure on T, dg a Haar measure on G and γ a regular element of T we set

$$\Phi_f(\gamma, dt, dg) = \int_{\mathrm{G}/\mathrm{T}} f(g\gamma g^{-1})\, d\bar{g},$$

$d\bar{g}$ denoting the quotient measure arising from dt and dg. Recall that if $\omega \in \mathscr{A}(\mathrm{T})$ then $\Phi_f(\gamma^\omega, (dt)^\omega, dg)$ depends only on the class of ω in $\mathscr{D}(\mathrm{T})$. Hence we may define

$$\Phi_f^\varkappa(\gamma, dt, dg) = \sum_{\omega \in \mathscr{D}(\mathrm{T})} \varkappa(\omega)\, \Phi_f(\gamma^\omega, (dt)^\omega, dg)$$

(*cf.* [5]); recall that $\gamma^{\omega}=\omega\gamma\omega^{-1}$. It is clear that

$$\Phi_f^{\varkappa}(\gamma, \alpha\, dt, \beta\, dg)=\beta/\alpha\ \Phi_f^{\varkappa}(\gamma, dt, dg), \qquad \alpha, \beta>0.$$

PROPOSITION 4.1:

$$\Phi_f^{\varkappa^{\omega}}(\gamma^{\omega}, (dt)^{\omega}, dg)=\varkappa(\omega)\ \Phi_f^{\varkappa}(\gamma, dt, dg), \qquad \omega\in\mathscr{A}(\mathrm{T}).$$

Proof:

$$\sum_{\omega'\in\mathscr{D}(T^{\omega})} \varkappa^{\omega}(\omega')\ \Phi_f((\gamma^{\omega})^{\omega'}, ((dt^{\omega})^{\omega'}, dg) = \sum_{\omega'\in\mathscr{D}(\mathrm{T}^{\omega})} \frac{\varkappa^{\omega}(\omega')}{\varkappa(\omega'\omega)}\ \varkappa(\omega'\omega)\ \Phi_f(\gamma^{\omega'\omega}, (dt)^{\omega'\omega}, dg)$$

$$=\varkappa(\omega) \sum_{\omega''\in\mathscr{D}(\mathrm{T})} \varkappa(\omega'')\ \Phi_f(\gamma^{\omega''}, (dt)^{\omega''}, dg),$$

as desired, since Proposition 3.1 shows that

$$\frac{\varkappa^{\omega}(\omega')}{\varkappa(\omega'\omega)}=\frac{1}{\varkappa(\omega)}=\varkappa(\omega).$$

Fix a system $\mathbf{I}^+$ of positive roots for $\mathbf{T}$ in $\mathbf{M}$; that is, a system of positive imaginary roots for $\mathbf{T}$. As in [9] we define

$$\mathrm{R_T}(\gamma)=\left|\det(\mathrm{Ad}\gamma-1)_{\mathfrak{g}/\mathfrak{m}}\right|^{1/2} \prod_{\alpha\in\mathrm{I}^+}(1-\alpha(\gamma^{-1})),$$

and then set

$$\Psi_f^{\varkappa}(\gamma)=\Psi_f^{\varkappa}(\gamma, dt, dg)=\mathrm{R_T}(\gamma)\ \Phi_f^{\varkappa}(\gamma, dt, dg).$$

PROPOSITION 4.2. — *$\Psi_f^{\varkappa}$ extends to a Schwartz function on*

$$\mathrm{T}_0=\{\gamma\in\mathrm{T}\colon \alpha(\gamma)\neq 1,\ \alpha\in\mathrm{I}^+\}.$$

Proof. — The assertion follows immediately from [2], for $\Psi_f^{\varkappa}(\gamma)=\sum_{\omega\in\mathscr{D}(\mathrm{T})}\varkappa(\omega)\ \Psi_f^{\omega}(\gamma)$ where $\Psi_f^{\omega}(\gamma)=\mathrm{R_T}(\gamma)\ \Phi_f(\gamma^{\omega})$ which can be written as $c\,\Lambda(\gamma)\ {}'\mathrm{F}_f(\gamma^{\omega})$ where c is a constant, Λ a unitary character on T and ${}'\mathrm{F}_f$ is the function of Harish-Chandra [12]; here we are using representatives ω [for the classes in $\mathscr{D}(\mathrm{T})$] which lie in Norm$(\mathbf{M}, \mathbf{T})$.

Thus, like the function ${}'\mathrm{F}_f$, $\Psi_f^{\varkappa}$ (and each derivative) "jumps" across each wall $\alpha=1$, $\alpha\in\mathrm{I}^+$. We discuss these "jumps" following the usual procedure (*cf.* [2]): α will be a root in I^+, γ_0 an element of T such that $\alpha(\gamma_0)=1$ and $\beta(\gamma_0)\neq 1$ if $\beta\neq\pm\alpha$, γ_{ν} will denote $\gamma_0 \exp i\nu \mathrm{H}_{\alpha}$, where H_{α} is the coroot (as element of $\underset{\sim}{\mathfrak{t}}$, the Lie algebra of $\mathbf{T}$) attached to α, and D will be an invariant differential operator on T.

LEMMA 4.3. — *If $\varkappa(\alpha^{\vee})=-1$ then*

$$\lim_{\nu\downarrow 0} \mathrm{D}\,\Psi_f^{\varkappa}(\gamma_{\nu})=\lim_{\nu\uparrow 0} \mathrm{D}\,\Psi_f^{\varkappa}(\gamma_{\nu}).$$

Proof. — If all the roots $\omega\alpha$, ω an element of the imaginary Weyl group, are compact then the result follows immediately from [2].

Suppose now that α is noncompact. By [2] again, (and an earlier paper cited in [2]), we have only to show that under the assumption $D^{\omega_\alpha}=D$ the jump for $\grave{D}$, as defined in [9], paragraph 4, is zero. Recall that $\grave{D}$, introduced because of the awkward transformation of R_T under the imaginary Weyl group, is the image of D under the automorphism induced by $H\to H+\mathfrak{s}(H)I$, $H\in\underline{\mathfrak{t}}$, where $\mathfrak{s}=1/2\sum_{\alpha\in I^+}\log\alpha$. Because $\varkappa(\alpha^\vee)=-1$ we have $\varkappa(\omega_\alpha)=-1$ (Prop. 2.1) and so ω_α is not realized in G. Hence to compute

$$\lim_{v\downarrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v)-\lim_{v\uparrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v),$$

we may replace $\Psi_f^\varkappa$ by Ψ where

$$\Psi(\gamma)=R_T(\gamma)\sum_\delta(\varkappa(\delta)\,\Phi_f(\gamma^\delta)+\varkappa(\delta\omega_\alpha)\,\Phi_f(\gamma^{\delta\omega_\alpha})),$$

and δ, an element of Norm $(\mathbf{M}, \mathbf{T})$ satisfying $\delta\alpha=\alpha$, ranges over a complete set of representatives for the classes in $\mathscr{D}(T)$ containing such an element (*cf.* [9], § 4). But $\varkappa^{\omega_\alpha}(\delta)=\varkappa(\delta)$. To prove this, a simple argument shows that it is enough to consider the case that δ is a reflection; then the proof is immediate (*cf.* Paragraph 3). Thus we have

$$\varkappa(\delta\omega_\alpha)=\varkappa(\omega_\alpha)\,\varkappa(\delta)=\varkappa(\alpha^\vee)\,\varkappa(\delta)=-\varkappa(\delta).$$

Hence $\Psi(\gamma^{\omega_\alpha})=(\iota-\omega_{\alpha}.\iota)(\gamma)\,\Psi(\gamma)$. Since $D^{\omega_\alpha}=D$ we obtain immediately that

$$\lim_{v\downarrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v)-\lim_{v\uparrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v)=0,$$

as desired.

Finally, suppose that α is compact but that $\omega\alpha$ is noncompact. Then since

$$\Psi_f^\varkappa(\gamma)=\varkappa(\omega)\,\Psi_f^{\varkappa^\omega}(\gamma^\omega),$$

[using the positive system $(I^+)^\omega$ to define $\Psi_f^{\varkappa^\omega}$] the proof is easily completed.

We come then to the other possibility, namely $\varkappa(\alpha^\vee)=1$. We have already observed that, regardless of the value of $\varkappa(\alpha^\vee)$, if all $\omega\alpha$ are compact then

$$\lim_{v\downarrow 0}D\Psi_f^\varkappa(\gamma_v)=\lim_{v\uparrow 0}D\Psi_f^\varkappa(\gamma_v).$$

For the remaining case we proceed in steps. Suppose first that α is noncompact and that s is a Cayley transform with respect to α, *standard* in the sense of [9], Paragraph 2. Since $\varkappa(\alpha^\vee)=1$ the character $\varkappa^s$ is well-defined (Prop. 3.2). We claim that

$$\lim_{v\downarrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v)-\lim_{v\uparrow 0}\hat{D}\Psi_f^\varkappa(\gamma_v)=2\,i\,\hat{D}^s\Psi_f^{\varkappa^s}(\gamma_0). \tag{1}$$

The notation is that of [9]. Thus we assume that the system I^+ is adapted to α (... if β is imaginary and $\langle \beta, \alpha \rangle > 0$ then $\beta \in I^+$); R_{T_s}, and hence $\Psi_f^{\varkappa^s}$, is defined relative to $I_s^+ = \{\beta : s^{-1}\beta \in I^+\}$; $\hat{D}$ and $\widehat{D^s}$ are defined relative to I^+ and I_s^+ respectively. For the choice of Haar measure on T_s we refer to [9].

To prove the claim we again recall the computations of [9], Paragraph 4. First, on the left-hand side of (1) $\Psi_f^{\varkappa}$ may be replaced by Ψ where $\Psi(\gamma) = \sum_{\omega} \varkappa(\omega) \Psi_f(\gamma^{\omega})$ with ω an element of Norm $(\mathbf{M}, \mathbf{T})$ satisfying $\omega\alpha = \pm\alpha$, ranging over a complete set of representatives for those classes in $\mathscr{D}(T)$ which contain such an element. Fix ω such that $\omega\alpha = \alpha$. Then by Proposition 4.5 of [9] we have

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^{\omega}(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^{\omega}(\gamma_v) = \mathrm{id}(\alpha) \, \widehat{D^s} \, \Psi_f^{s\omega s^{-1}}(\gamma_0),$$

where $d(\alpha) = 2$ if ω_α can be realized in G and $d(\alpha) = 1$ otherwise. If ω_α can be realized in G then we obtain

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^{\varkappa}(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^{\varkappa}(\gamma_v) = 2i \sum_{\omega\alpha = \alpha} \varkappa(\omega) \, \widehat{D^s} \, \Psi_f^{s\omega s^{-1}}(\gamma_0) = 2i \, \widehat{D^s} \, \Psi_f^{\varkappa^s}(\gamma_0),$$

since, by Proposition 3.3, $\varkappa^s(s\omega s^{-1}) = \varkappa(\omega)$. If ω_α is not realized in G and $\omega\alpha = \alpha$ then $\omega\omega_\alpha$ and ω lie in distinct classes of $\mathscr{D}(T)$. However $\varkappa(\omega\omega_\alpha) = \varkappa(\omega)$. We now argue again as in [9]. First, we may assume that $D^{\omega_\alpha} = D$. Then it follows that the term on the left-hand side of (1) coming from $\omega\omega_\alpha$ equals that for ω. By applying Lemma 4.6 of [9] we obtain the formula (1).

We continue with the assumption that α is noncompact but allow s to be any Cayley transform with respect to α. Then s may be written as $\omega_0 s_0$, where s_0 is a standard transform (with respect to α) and $\omega_0 \in \mathscr{A}(T_{s_0})$. We know that

$$\varkappa^{s_0}(\omega_0) \, \Phi_f^{\varkappa^s}(\gamma^{\omega_0}) = \Phi_f^{\varkappa^{s_0}}(\gamma), \qquad \gamma \in (T_{s_0})_{\mathrm{reg}}.$$

Also, by definition,

$$R_{T_s}(\gamma^{\omega_0}) = R_{T_{s_0}}(\gamma), \qquad \gamma \in T_{s_0},$$

and $\widehat{D^s} = (\widehat{D^{s_0}})^{\omega_0}$. Hence

$$\lim_{v \downarrow 0} \hat{D} \Psi_f^{\varkappa}(\gamma_v) - \lim_{v \uparrow 0} \hat{D} \Psi_f^{\varkappa}(\gamma_v) = 2i \varkappa^{s_0}(\omega_0) \, \widehat{D^s} \, \Psi_f^{\varkappa^s}(\gamma_0^s).$$

Now we come to the general case. Thus we will assume that $\sigma(s^{-1})s$ realizes ω_α, with α possibly compact. Suppose that $\omega\alpha$ is noncompact. Then s may be written in the form $\omega_0 s_0 \omega$ where s_0 is a standard transform with respect to $\omega\alpha$ and $\omega_0 \in \mathscr{A}(T_{s_0})$. But

$$\Psi_f^{\varkappa}(\gamma) = \varkappa(\omega) \, \Psi_f^{\varkappa^{\omega}}(\gamma^{\omega}), \qquad \gamma \in T_{\mathrm{reg}},$$

the "R-" function in the definition of $\Psi_f^{\varkappa^\omega}$ being relative to $(I^+)^{\omega}$, a system adapted to $\omega\alpha$. It is then easy to check that

$$\hat{D} \Psi_f^{\varkappa}(\gamma_v) = \varkappa(\omega) \, \widehat{D^{\omega}} \, \Psi_f^{\varkappa^{\omega}}(\gamma_v^{\omega}),$$

$\widehat{\mathrm{D}}^{\omega}$ being defined relative to $(\mathrm{I}^+)^{\omega}$. We have then

$$(2) \qquad \lim_{v \downarrow 0} \hat{\mathrm{D}}\,\Psi_f^{\varkappa}(\gamma_v) - \lim_{v \uparrow 0} \hat{\mathrm{D}}\,\Psi_f^{\varkappa}(\gamma_v) = 2\,i\,\varkappa(\omega)\,\varkappa^{s_0\omega}(\omega_0)\,\widehat{\mathrm{D}^s}\,\Psi_f^{\varkappa^s}(\gamma_0^s),$$

where terms on the right-hand side are defined relative to the positive system $((\mathrm{I}^+)^{\omega})_{s_0}^{\omega_0} = \mathrm{I}_s^+$.

We wish to give an (intrinsic) interpretation of $\varkappa(\omega)\,\varkappa^{s_0\omega}(\omega_0)$ as a "$\varkappa$-signature" for s. We continue to assume that α is an imaginary root, that $\varkappa(\alpha^{\vee}) = 1$ and that s is an element of $\mathbf{G}$ such that $s_\sigma = \sigma(s^{-1})\,s$ realizes ω_α. We write $\mathbf{G}_\alpha$ for the image of the appropriate real form of SL_2 under one of the standard homomorphisms attached to α (*cf.* [9]); $\mathbf{G}_\alpha$ is independent of the choices made in defining such a homomorphism. Our first observation is that we may modify s_σ by an element of $\mathbf{G}_\alpha$ to obtain a 1-cocycle for $\mathbf{T}$ trivial in $\mathbf{G}$ (... we are considering the cohomology of just $\langle 1, \sigma \rangle$, as before). Indeed, suppose that α is noncompact. Then $s = \omega_0\,s_0$ where $s_0 \in \mathbf{G}_\alpha$ and $\omega_0 \in \mathscr{A}(\mathrm{T}_{s_0})$. We write ω_0 as $g_0\,\omega_0'$ where $g_0 \in \mathrm{G}$ and ω_0' normalizes $\mathbf{T}_{s_0}$ and centralizes the maximal $\mathbf{R}$-split torus in $\mathbf{T}_{s_0}$. Then setting $\omega_1 = s_0^{-1}\,\omega_0'\,s_0$ we have $\omega_1^{-1}\,\mathbf{G}_\alpha\,\omega_1 = \mathbf{G}_\alpha$ and hence

$$s_\sigma = \sigma(\omega_1^{-1})\,\sigma(s_0^{-1})\,s_0\,\omega_1 = \sigma(\omega_1^{-1})\,\omega_1\,\omega_1^{-1}\,\sigma(s_0^{-1})\,s_0\,\omega_1,$$

where $\sigma(\omega_1^{-1})\,\omega_1$ is a 1-cocycle for $\mathbf{T}$ (and 1-coboundary for $\mathbf{G}$) and $\omega_1^{-1}\,\sigma(s_0^{-1})\,s_0\,\omega$ is an element of $\mathbf{G}_\alpha$. Now suppose that α is compact but that $\omega\alpha$ is noncompact. Then we may write s as $\omega_0\,s_\omega\,\omega$ where $s_\omega \in \mathbf{G}_{\omega\alpha}$ is such that $\sigma(s_\omega^{-1})\,s_\omega$ realizes $\omega_{\omega\alpha}$ and $\omega_0 \in \mathscr{A}(\mathrm{T}_{s_\omega})$. Decomposing ω_0 as before we find that we may assume that $\omega_0 = 1$. Then

$$s_\sigma = \sigma(\omega^{-1})\,\omega\,.\,\omega^{-1}\,\sigma(s_\omega^{-1})\,s_\omega\,\omega,$$

where $\sigma(\omega^{-1})\,\omega$ is a 1-cocycle for $\mathbf{T}$ (and 1-coboundary for $\mathbf{G}$) and $\omega^{-1}\,\sigma(s_\omega^{-1})\,s_\omega\,\omega \in \mathbf{G}_\alpha$. This justifies our claim.

Suppose now that we decompose s_σ in two ways, say $s_\sigma = w_1\,t_1 = w_2\,t_2$, where $w_1, w_2 \in \mathbf{G}_\alpha$ and t_1, t_2 are 1-cocycles for $\mathbf{T}$ and 1-coboundaries for $\mathbf{G}$. We claim that the images of (the cohomology classes of) t_1, t_2 under T-N differ by an element of $\mathbf{Z}\,\alpha^{\vee}$ and hence $\varkappa(t_1) = \varkappa(t_2)$. To prove the claim we have only to note that the classes of t_1 and t_2 differ by an element of $\mathrm{H}^1(\mathbf{T} \cap \mathbf{G}_\alpha)$; such an element maps under T-N into $\mathbf{Z}\,\alpha^{\vee}$.

It is now immediate that if $s_\sigma \in t_\sigma\,\mathbf{G}_\alpha$ where t_σ is a 1-cocycle of $\mathbf{T}$ trivial in $\mathbf{G}$ then we may define the "$\varkappa$-signature" $\varepsilon_\varkappa(s)$ of s as $\varkappa(t_\sigma)$.

In (2) we wrote s as $\omega_0\,s_0\,\omega$ where $\omega\alpha$ is noncompact, $s_0 \in \mathbf{G}_{\omega\alpha}$ and $\omega_0 \in \mathscr{A}(\mathrm{T}_{s_0})$. To compute s_σ we may assume that ω_0 normalizes $\mathbf{T}_{s_0}$. Set $\omega_1 = s_0^{-1}\,\omega_0\,s_0\,\omega$. Then

$$\varepsilon_\varkappa(s) = \varkappa(\sigma(\omega_1^{-1})\,\omega_1) = \varkappa(\omega_1)$$

in our usual notation. On the other hand

$$\varkappa(\omega)\,\varkappa^{s_0\omega}(\omega_0) = \varkappa(\omega)\,\varkappa^{\omega}(s_0^{-1}\,\omega_0\,s_0) = \varkappa(\omega_1).$$

Hence $\varkappa(\omega)\,\varkappa^{s_0\omega}(\omega_0) = \varepsilon_\varkappa(s)$. We conclude:

LEMMA 4.4. — *Suppose* $\varkappa(\alpha^{\vee}) = 1$. *Then:*

(i) *if all* $\omega\alpha$ *are compact we have*

$$\lim_{v \downarrow 0} \mathrm{D}\,\Psi_f^{\varkappa}(\gamma_v) = \lim_{v \uparrow 0} \mathrm{D}\,\Psi_f^{\varkappa}(\gamma_v);$$

(ii) *if* $\sigma(s^{-1})s$ *realizes* ω_α *then*

$$\lim_{v\downarrow 0} \hat{\mathrm{D}}\,\Psi_f^{\varkappa}(\gamma_v) - \lim_{v\uparrow 0} \hat{\mathrm{D}}\,\Psi_f^{\varkappa}(\gamma_v) = 2\,i\,\varepsilon_\varkappa(s)\,\widehat{\mathrm{D}^s}\,\Psi_f^{\varkappa^s}(\gamma_0^s).$$

Here, we recall, the terms $\hat{\mathrm{D}}$, $\Psi_f^{\varkappa}$ are defined relative to a system I^+ of positive imaginary roots adapted to α and the terms $\widehat{\mathrm{D}^s}$, $\Psi_f^{\varkappa^s}$ relative to $\mathrm{I}_s^+ = \{\beta : s^{-1}\beta \in \mathrm{I}^+\}$.

5. (T, $\varkappa$)-groups

To establish notation and introduce the groups of [7] we recall some more material from [7] (and [8]). Our data will be a connected reductive group $\mathbf{G}$ over $\mathbf{R}$, a maximal torus $\mathbf{T}$ in $\mathbf{G}$, also defined over $\mathbf{R}$, and a quasicharacter $\varkappa$ on the module generated by the coroots for $\mathbf{T}$, trivial on the submodule $\mathscr{L}(\mathrm{T})$ (*cf.* Paragraph 2).

We begin with an L-group ${}^{\mathrm{L}}\mathrm{G}$ for $\mathbf{G}$. Thus fix a pair $(\mathbf{G}^*, \psi)$, where $\mathbf{G}^*$ is a group quasi-split over $\mathbf{R}$ and ψ: $\mathbf{G} \to \mathbf{G}^*$ is an isomorphism (over $\mathbf{C}$) such that $\sigma(\psi^{-1})\psi$ is inner. In $\mathbf{G}^*$ fix a Borel subgroup $\mathbf{B}^*$ over $\mathbf{R}$ and a maximal torus $\mathbf{T}^*$ over $\mathbf{R}$, contained in $\mathbf{B}^*$. To abbreviate notation we use L for the character module for $\mathbf{T}^*$ and $\mathrm{L}^\vee$ for its dual; $\Sigma \subset \mathrm{L}$ will be the set of simple roots for $\mathbf{T}^*$ in $\mathbf{B}^*$ and $\Sigma^\vee$ the corresponding set of coroots. Fix a triple $({}^{\mathrm{L}}\mathrm{G}^0, {}^{\mathrm{L}}\mathrm{B}^0, {}^{\mathrm{L}}\mathrm{T}^0)$, where ${}^{\mathrm{L}}\mathrm{G}^0$ is a connected reductive group over $\mathbf{C}$, ${}^{\mathrm{L}}\mathrm{B}^0$ is a Borel subgroup of ${}^{\mathrm{L}}\mathrm{G}^0$ and ${}^{\mathrm{L}}\mathrm{T}^0$ is a maximal torus contained in ${}^{\mathrm{L}}\mathrm{B}^0$, such that the character module for ${}^{\mathrm{L}}\mathrm{T}^0$ is $\mathrm{L}^\vee$ and the set of simple roots for ${}^{\mathrm{L}}\mathrm{T}^0$ in ${}^{\mathrm{L}}\mathrm{B}^0$ is $\Sigma^\vee$. For each $\alpha^\vee \in \Sigma^\vee$ fix a root vector $\mathrm{X}_{\alpha^\vee}$ in the Lie algebra of ${}^{\mathrm{L}}\mathrm{G}^0$. The element σ acts on $\mathbf{T}^*$, L, $\mathrm{L}^\vee$ and ${}^{\mathrm{L}}\mathrm{T}^0$; we denote also by σ the action on ${}^{\mathrm{L}}\mathrm{G}^0$ which extends that on ${}^{\mathrm{L}}\mathrm{T}^0$ and satisfies $\sigma\,\mathrm{X}_{\alpha^\vee} = \mathrm{X}_{\sigma\alpha^\vee}$, $\alpha^\vee \in \Sigma^\vee$. The semi-direct product of ${}^{\mathrm{L}}\mathrm{G}^0$ by the Weil group of $\mathbf{C}/\mathbf{R}$, with $1 \times \sigma$ acting by σ and $\mathbf{C}^* \times 1$ acting trivially, defines an object in the category $\mathscr{G}^\vee(\mathbf{R})$ of [8]; this object will be our L-group ${}^{\mathrm{L}}\mathrm{G}$.

Next, we use the pair $(\mathrm{T}, \varkappa)$ to construct another object ${}^{\mathrm{L}}\mathrm{H}$ in $\mathscr{G}^\vee(\mathbf{R})$. We denote by σ_{T} the action of σ on $\mathbf{T}$, $\mathbf{L}(\mathbf{T})$ and $\mathrm{L}^\vee(\mathbf{T})$. Fix $x \in \mathbf{G}^*$ such that $\psi_x = \mathrm{ad}\,x \circ \psi$ maps $\mathbf{T}$ to $\mathbf{T}^*$. Thus ψ_x induces an isomorphism of $\mathrm{L}^\vee(\mathbf{T})$ with $\mathrm{L}^\vee$ by which we transfer σ_{T} to $\mathrm{L}^\vee$; by the same means we transfer $\varkappa$ to a quasicharacter on $\langle \Sigma^\vee \rangle$; this new quasicharacter, $\varkappa^*$, is trivial on $\mathscr{L} = \{\lambda^\vee \in \langle \Sigma^\vee \rangle : \lambda^\vee = \mu^\vee - \sigma_{\mathrm{T}}\mu^\vee, \text{ some } \mu^\vee \in \mathrm{L}^\vee\}$ and so is σ_{T}-invariant. Let ${}^{\mathrm{L}}\mathrm{H}^0$ be the connected reductive subgroup of ${}^{\mathrm{L}}\mathrm{G}^0$ generated by ${}^{\mathrm{L}}\mathrm{T}^0$ and the 1-parameter subgroups defined by those roots of ${}^{\mathrm{L}}\mathrm{T}^0$ in ${}^{\mathrm{L}}\mathrm{G}^0$ on which $\varkappa^*$ is trivial. Fix a Borel subgroup of ${}^{\mathrm{L}}\mathrm{H}^0$ containing ${}^{\mathrm{L}}\mathrm{T}^0$ and let $\Sigma_{\mathrm{H}}^\vee$ be the set of simple roots for ${}^{\mathrm{L}}\mathrm{T}^0$ in this group. Since $\varkappa^*$ is σ_{T}-invariant the set of all roots of ${}^{\mathrm{L}}\mathrm{T}^0$ in ${}^{\mathrm{L}}\mathrm{H}^0$ is preserved by σ_{T}. We write σ_{T} as a product $\omega\,.\,\sigma_{\mathrm{H}}$, with $\omega \in \Omega({}^{\mathrm{L}}\mathrm{H}^0, {}^{\mathrm{L}}\mathrm{T}^0)$, the Weyl group of ${}^{\mathrm{L}}\mathrm{T}^0$ in ${}^{\mathrm{L}}\mathrm{H}^0$, and σ_{H} induced by an automorphism of $\Sigma_{\mathrm{H}}^\vee$. For each $\alpha^\vee \in \Sigma_{\mathrm{H}}^\vee$ choose a root vector $\mathrm{Y}_{\alpha^\vee}$ in the Lie algebra of ${}^{\mathrm{L}}\mathrm{H}^0$; we denote also by σ_{H} that extension of σ_{H} to ${}^{\mathrm{L}}\mathrm{H}^0$ satisfying $\sigma_{\mathrm{H}}\,\mathrm{Y}_{\alpha^\vee} = \mathrm{Y}_{\sigma_{\mathrm{H}}\alpha^\vee}$, $\alpha^\vee \in \Sigma_{\mathrm{H}}^\vee$. The semi-direct product of ${}^{\mathrm{L}}\mathrm{H}^0$ by the Weil group of $\mathbf{C}/\mathbf{R}$, with $1 \times \sigma$ acting by σ_{H} and $\mathbf{C}^* \times 1$ acting trivially, defines an object ${}^{\mathrm{L}}\mathrm{H}$ in $\mathscr{G}^\vee(\mathbf{R})$; the isomorphism class of ${}^{\mathrm{L}}\mathrm{H}$ in $\mathscr{G}^\vee(\mathbf{R})$ depends only on $\mathbf{G}$, $\mathbf{T}$ and $\varkappa$.

We come then to the groups attached to $\mathbf{G}$: we call a quasi-split group $\mathbf{H}$ over $\mathbf{R}$ a <u>(T, $\varkappa$)-group for $\mathbf{G}$</u> if the object ${}^{\mathrm{L}}\mathrm{H}$ described above is an L-group for $\mathbf{H}$. Up to isomorphism over $\mathbf{R}$ there is exactly one (T, $\varkappa$)-group for $\mathbf{G}$.

6. Cartan subgroups

We change notation slightly to write $(T_0, \varkappa_0)$ for the fixed Cartan subgroup and quasicharacter; $\mathbf{H}$ will now be a $(T_0, \varkappa_0)$-group for $\mathbf{G}$. In this section we embed the Cartan subgroups of H in G^* . . . and some of them in G. The basis for our discussion is a result (unpublished) of Langlands.

For once and for all we fix (in the notation of the last section):

(i) ψ, $\mathbf{G}^*$, $\mathbf{B}^*$, $\mathbf{T}^*$ and hence ${}^L G^0$; $\{X_{\alpha^\vee}\}$ and hence ${}^L G$;

(ii) an element x of $\mathbf{G}^*$ such that $\psi_x = \operatorname{ad} x \circ \psi$ maps $\mathbf{T}_0$ to $\mathbf{T}^*$, and hence ${}^L H^0$;

(iii) a Borel subgroup of ${}^L H^0$ and hence the action of σ_H on ${}^L T^0$; $\{Y_{\alpha^\vee}\}$ and hence the action of σ_H on ${}^L H^0$ and the object ${}^L H$.

Recalling that $L^\vee$ is the dual of the character module for $\mathbf{T}^*$ we make the canonical identification of $\mathbf{T}^*$ with $L^\vee \otimes \mathbf{C}^*$. By construction, $L^\vee$ is also the dual of the character module for some torus in $\mathbf{H}$ defined over $\mathbf{R}$ (and containing a torus maximal among the $\mathbf{R}$-split tori in $\mathbf{H}$). Thus we can identify $\mathbf{T}^*$ (as complex torus) together with the action of σ_H (induced from that of σ_H on $L^\vee$) as a maximal torus in $\mathbf{H}$, defined over $\mathbf{R}$. Recall that the action of $\Omega(\mathbf{G}, \mathbf{T}^*)$ (respectively, $\Omega(\mathbf{H}, \mathbf{T}^*)$) on $L^\vee$ coincides with that of $\Omega({}^L G^0, {}^L T^0)$ [respectively, $\Omega({}^L H^0, {}^L T^0)$]. Hence $\Omega(\mathbf{H}, \mathbf{T}^*)$ is a subgroup of $\Omega(\mathbf{G}^*, \mathbf{T}^*)$. We remark that on $\mathbf{T}^*$, $\sigma_T = \omega_1 \sigma_H$, $\omega_1 \in \Omega(\mathbf{H}, \mathbf{T}^*)$; $\sigma_G = \omega_2 \sigma_T$, $\omega_2 \in \Omega(\mathbf{G}^*, \mathbf{T}^*)$ and so $\sigma_G = \omega_3 \sigma_H$, $\omega_3 \in \Omega(\mathbf{G}^*, \mathbf{T}^*)$.

We come now to the embeddings. Let $\mathbf{T}'$ be a maximal torus in $\mathbf{H}$ defined over $\mathbf{R}$. We pick $h \in \mathbf{H}$ such that $\operatorname{ad} h$ maps $\mathbf{T}'$ to $\mathbf{T}^*$. Composing $\operatorname{ad} h$ with the identity on $\mathbf{T}^*$ (as map over $\mathbf{C}$, from a subgroup of $\mathbf{H}$ to $\mathbf{G}^*$) we obtain an embedding $j(h)$ of $\mathbf{T}'$ in $\mathbf{G}^*$, defined over $\mathbf{C}$. According to Langlands (unpublished) there exists $g \in \mathbf{G}^*$ such that $j(g, h) = \operatorname{ad} g^{-1} \circ j(h)$ is defined over $\mathbf{R}$. (The proof proceeds as follows. Choose an element $\gamma = \exp X$, $X \in t'$, such that $h \gamma h^{-1} \in \mathbf{T}^*$ is regular in $\mathbf{G}^*$ and lies in the derived group of $\mathbf{G}^*$. Consider the natural projection of the simply-connected covering group $(\mathbf{G}^*)^\sim$ onto the derived group. There is an element γ_0 in the preimage of $h \gamma h^{-1}$ whose conjugacy class in $(\mathbf{G}^*)^\sim$ is defined over $\mathbf{R}$. By [11] this class contains an $\mathbf{R}$-rational point, say γ_1. Let $g(h \gamma h^{-1}) g^{-1}$ be the image of γ_1 in $\mathbf{G}^*$. Then $\operatorname{ad} g \circ j(h)$ is defined over $\mathbf{R}$.) If both $j(g, h)$ and $j(g', h')$ map $\mathbf{T}'$ into $\mathbf{G}^*$ over $\mathbf{R}$ then the action of $j(g', h') \circ j(g, h)^{-1}$ on $\mathbf{T}$, the image of $\mathbf{T}'$ under $j(g, h)$, can be realized by an element of $\mathbf{G}^*$. Clearly this element lies in $\mathscr{A}(T)$. Hence the image of $\mathbf{T}'$ is determined up to conjugacy under G^* (*cf.* [9]). It follows easily that if $\mathbf{T}'$ and $\mathbf{U}'$ are conjugate in $\mathbf{H}$ under H then their images in $\mathbf{G}^*$ are conjugate under G^*. We conclude then that the embeddings $j(\ ,\ )$ induce a map from the set $t(H)$ of conjugacy classes of Cartan subgroups of H to $t(G^*)$. This map preserves the usual ordering (*cf.* [9], § 2) and, in fact, maps adjacent classes to adjacent classes. However it need not be injective. On the other hand, our twist $\psi: \mathbf{G} \to \mathbf{G}^*$ induces an embedding of $t(G)$ in $t(G^*)$ (*cf.* [9]). Thus we have a map from a subset of $t(H)$ into $t(G)$ (preserving adjacency). The domain is non-empty for, according to [7], the image contains the conjugacy class of T_0, our fixed Cartan subgroup of G.

While the map above is canonical [given the data in (i), (ii), (iii)] we will need to examine the correspondence of individual Cartan subgroups, where the choices will be of importance (temporarily). First, we will say that a Cartan subgroup T' of H originates in G if its conjugacy class lies in the domain of the map into $t(G)$. Clearly, if T' originates in G then $\mathbf{T}'$ is embedded in $\mathbf{G}$, over $\mathbf{R}$, by a map $\psi^{-1}\circ \operatorname{ad} g''\circ j(g', h)$, $g', g''\in\mathbf{G}^*$, $h\in\mathbf{H}$; that is, by a map of the form

$$i(g, h)=\psi\circ \operatorname{ad} g^{-1}\circ \operatorname{id}\circ \operatorname{ad} h, \qquad g\in\mathbf{G}^*, \quad h\in\mathbf{H}.$$

Moreover, $i(g, h)=i(g', h')$ if and only if $g'=w_1 g$, $h'=w_2 h$ where both w_1, w_2 realize some element ω of $\Omega(\mathbf{H}, \mathbf{T}^*)$; both $i(g, h)$ and $i(g'', h'')$ embed $\mathbf{T}'$ in $\mathbf{G}$ (over $\mathbf{R}$) if and only if $i(g'', h'')=\operatorname{ad} g_0\circ i(g, h)$ for some $g_0\in\mathscr{A}(T)$, $\mathbf{T}$ denoting the image of $\mathbf{T}'$ under $i(g, h)$.

We now fix a set $\mathscr{I}=\{i_0, \ldots, i_N\}$ of these embeddings $i(g, h)$, denoting the domain of i_m by $\mathbf{T}'_m$ and the range by $\mathbf{T}_m$ ($\mathbf{T}_0$ remains our fixed torus). We assume:

(i) $T'_0, \ldots, T'_N$ form a complete set of non-conjugate groups among the Cartan subgroups of H originating in G;

(ii) i_0 is of the form $i(x,\ )$, where x is the element fixed in (ii) at the beginning of this section and

(iii) if T_m is conjugate to T_n then $T_m=T_n$.

That (ii) is possible is indicated in [7] (the argument is similar to that we reported earlier); (iii) is only for convenience.

We consider an embedding i_m: $\mathbf{T}'_m\to\mathbf{T}_m$. Write L_m for $L(\mathbf{T}_m)$, $L^\vee_m$ for $L^\vee(\mathbf{T}_m)$, Ξ_m for the roots of $\mathbf{T}_m$ in $\mathbf{G}$, $\Xi^\vee_m$ for the coroots and $\mathscr{L}_m$ for the module $\mathscr{L}(T_m)$. Clearly i_m induces isomorphism between $L(\mathbf{T}'_m)$ and L_m and between $L^\vee(\mathbf{T}'_m)$ and $L^\vee_m$. We claim that under these maps the coroots for $\mathbf{T}'_m$ are embedded in $\Xi^\vee_m$ and the roots in Ξ_m. Moreover these embeddings commute with the action of σ and if α' maps to α in Ξ_m and γ' to γ in T_m then $\alpha'(\gamma')=\alpha(\gamma)$. To obtain the embedding of the coroots we write i_m as $i(g_m, h_m)$; then $\operatorname{ad} h_m$ maps the coroots for $\mathbf{T}'_m$ to the coroots for $\mathbf{T}^*$ in $\mathbf{H}$ and $\operatorname{ad} g_m\circ\psi$ maps $\Xi^\vee_m$ to the coroots for $\mathbf{T}^*$ in $\mathbf{G}$. Now we need only recall that a coroot for $\mathbf{T}^*$ in $\mathbf{G}^*$ (respectively, $\mathbf{H}$) is a root for ${}^LT^0$ in ${}^LG^0$ (respectively, ${}^LH^0$). For the correspondence of roots, if we identify L with $(L^\vee)^\vee$ then a root α of $\mathbf{T}^*$ in $\mathbf{H}$ is identified with

$$\lambda^\vee\to\frac{2\langle\alpha^\vee, \lambda^\vee\rangle}{\langle\lambda^\vee, \lambda^\vee\rangle}, \qquad \lambda^\vee\in L^\vee,$$

where we use a positive definite bilinear form $\langle\ ,\ \rangle$ on $L^\vee\otimes\mathbf{Q}$ invariant under $\Omega({}^LG^0, {}^LT^0)$. But then α is also identified as a root of $\mathbf{T}^*$ in $\mathbf{G}^*$. The rest follows easily.

The map $\omega\to i_m\circ\omega\circ i_m^{-1}$ yields an embedding of $\Omega(\mathbf{H}, \mathbf{T}'_m)$ into $\Omega(\mathbf{G}, \mathbf{T}_m)$ compatible, in the obvious sense, with the map on roots. We will denote by $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ the subgroup of $\Omega(\mathbf{G}, \mathbf{T}_m)$ consisting of those elements which commute with σ; that is, those elements which can be realized in $\mathscr{A}(T_m)$. The map above embeds $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$.

We continue with a set $\mathscr{I}$ of embeddings satisfying (i)-(iii). If T' is any Cartan subgroup of H (originating in G) an *$\mathscr{I}$-embedding* of T' in G will be a map of the form $i_m\circ\operatorname{ad} h$ where h is an element of $\mathscr{A}(T')$ mapping T' to T'_m. Also we will say that an element γ' in T' *originates from a regular element γ of G with respect to $\mathscr{I}$* if γ is the image of γ' under some $\mathscr{I}$-embedding; γ' is then regular in H.

LEMMA 6.1. — *If γ originates from regular elements $\gamma_1 \in \mathrm{T}_m$ and $\gamma_2 \in \mathrm{T}_n$ (with respect to $\mathscr{I}$) then $\mathrm{T}_m = \mathrm{T}_n$ and there exists ω in the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ such that $\gamma_2 = \gamma_1^{\omega}$.*

Proof. — That $\mathrm{T}_m = \mathrm{T}_n$ follows from [9], Theorem 2.1 and the condition (iii) satisfied by $\mathscr{I}$. The rest is immediate.

We remark that whether or not an element of H originates in $\mathrm{G}_{\mathrm{reg}}$ is independent of the choice for $\mathscr{I}$; however the collection of elements (if non-empty) from which it originates is not.

7. Transferring $\varkappa_0$

We have fixed $\mathscr{I} = \{ i_0, \ldots, i_{\mathrm{N}} \}$. It is now an easy matter to transfer $\varkappa_0$ to a character $\varkappa_m$ on $\mathscr{D}(\mathrm{T}_m)$. Indeed, choose $h \in \mathbf{H}$ such that $\mathrm{ad}\, h$ maps $\mathbf{T}'_m$ to $\mathbf{T}'_0$. Then $\bar{h} = i_0 \circ \mathrm{ad}\, h \circ i_m^{-1}$ maps $\mathbf{T}_m$ to $\mathbf{T}_0$ and $\Xi_m^{\vee}$ to $\Xi_0^{\vee}$. Thus we have immediately a quasi-character $\varkappa_m$ on $\langle \Xi_m^{\vee} \rangle$. That $\varkappa_m$ is trivial on $\mathscr{L}_m$ follows from:

PROPOSITION 7.1. — *$\bar{h}$ maps $\mathscr{L}_m$ to $\mathscr{L}_0 + \mathrm{Ker}\, \varkappa_0$.*

Proof. — Let $\lambda^{\vee} \in \mathrm{L}_m^{\vee}$ be such that $\sigma\lambda^{\vee} - \lambda^{\vee} \in \langle \Xi_m^{\vee} \rangle$. Then

$$\bar{h}(\sigma\lambda^{\vee} - \lambda^{\vee}) = (\sigma(\bar{h}\lambda^{\vee}) - \bar{h}\lambda^{\vee}) + \sigma(\sigma(\bar{h})\bar{h}^{-1}(\bar{h}\lambda^{\vee}) - \bar{h}\lambda^{\vee}).$$

But $\sigma(\bar{h})\bar{h}^{-1}$ lies in the image of $\Omega(\mathbf{H}, \mathbf{T}'_0)$ in $\Omega(\mathbf{G}, \mathbf{T}_0)$. Hence the second term is a sum of coroots for $\mathbf{T}_0$ each coming from $\mathbf{H}$. This forces the second term to lie in Ker $\varkappa_0$ because, by choosing $i_0 = i(x,\ )$, we have arranged that $\alpha^{\vee} \in \Xi_0^{\vee}$ come from $\mathbf{H}$ if and only if $\alpha^{\vee} \in \mathrm{Ker}\, \varkappa_0$. It follows now that the first term lies in $\langle \Xi_0^{\vee} \rangle$ and hence in $\mathscr{L}_0$. This completes the proof.

We have to check that $\varkappa_m$ is well-defined. Suppose that h is replaced by h'. Then:

PROPOSITION 7.2. — *$\bar{h}'\lambda^{\vee} \in \bar{h}\lambda^{\vee} + \mathrm{Ker}\, \varkappa_0$, $\lambda^{\vee} \in \langle \Xi_m^{\vee} \rangle$.*

Proof:

$$\bar{h}'\lambda^{\vee} = \bar{h}\lambda^{\vee} + \bar{h}(\bar{h}^{-1}\bar{h}'\lambda^{\vee} - \lambda^{\vee}).$$

Since $\bar{h}^{-1}\bar{h}'$ lies in the image of $\Omega(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega(\mathbf{G}, \mathbf{T}_m)$ the assertion follows easily.

For future use we note:

PROPOSITION 7.3:

(i) *a coroot $\alpha^{\vee}$ in $\Xi_m^{\vee}$ lies in the image of the coroots for $\mathbf{T}'_m$ (that is, "comes from $\mathbf{H}$") if and only if $\varkappa_m(\alpha^{\vee}) = 1$;*

(ii) *if ω lies in the image of $\Omega(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega(\mathbf{G}, \mathbf{T}_m)$ then $\varkappa_m^{\omega} = \varkappa_m$.*

Proof. — The assertion in (i) is immediate since it is true for $m = 0$ (*cf.* the proof of Proposition 7.1).

For (ii), let $\lambda^{\vee} \in \langle \Xi_m^{\vee} \rangle$. Then $\omega\lambda^{\vee} - \lambda^{\vee}$ lies in the span of the image in $\Xi_m^{\vee}$ of the coroots for $\mathbf{T}'_m$. Hence, by (i), $\varkappa_m^{\omega} = \varkappa_m$.

The converse of (ii) is false. To clarify this, extend $\varkappa_0^*$ ($\varkappa_0$ shifted to $\langle \Sigma^\vee \rangle$) in some way to a quasicharacter on $L^\vee$. Identify this extension as an element of ${}^LT^0 \ldots {}^LH^0$ is the connected component of the identity in the centralizer in ${}^LG^0$ of this element. The condition $\varkappa_m^\omega = \varkappa_m$ is that the action of ω shifted to ${}^LT^0$ be realized in the (full) centralizer of our element.

We have defined $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ and $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ (Paragraph 6); i_m induces an embedding of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$. If ω_1, ω_2 lie in the image then we have defined $\varkappa_m(\omega_1), \varkappa_m(\omega_2)$; by Propositions 3.1 and 7.3, $\varkappa_m(\omega_1 \omega_2) = \varkappa_m(\omega_1)\varkappa_m(\omega_2)$. Clearly also $\varkappa_m(\omega_i) = \pm 1$. We will need further information.

Let $\mathbf{T}$ be a maximal torus, over $\mathbf{R}$, in a connected reductive group $\mathbf{G}$ over $\mathbf{R}$. Let Ω be the Weyl group of $\mathbf{T}$ in $\mathbf{G}$ and Ω_0 the subgroup of Ω consisting of those elements realized in $\mathscr{A}$ (T). Let $\mathbf{S}$ be the maximal $\mathbf{R}$-split torus in $\mathbf{T}$ and $\mathbf{M}$ be the centralizer of $\mathbf{S}$ in $\mathbf{G}$. The imaginary Weyl group of $\mathbf{T}$, denoted here by Ω_I, is the Weyl group of $\mathbf{T}$ in $\mathbf{M}$; we have $\Omega_I \subset \Omega_0$. Let $\mathscr{W}$ be the restricted Weyl group attached to the pair $(\mathbf{G}, \mathbf{S})$. Restriction to $\mathbf{S}$ defines a surjective homomorphism from Ω_0 to $\mathscr{W}$ (this follows easily from Theorem 2.1 of [9]); the kernel is Ω_I. We will classify the elements of Ω_0 according to image in $\mathscr{W}$. First we recall the structure of $\mathscr{W}$. According to [4], $\mathscr{W}$ is generated by the reflections with respect to certain (useful) roots of $(\mathbf{G}, \mathbf{S})$. To describe the reflections needed we assume $\mathbf{G}$ simple. For convenience we exclude for the present the case that $\mathbf{G}$ is of type G_2. Then if $\tilde{\alpha}$ is a root of $(\mathbf{G}, \mathbf{S})$ the set of roots proportional to $\tilde{\alpha}$ is $\{\pm\tilde{\alpha}\}$, $\{\pm 1/2\tilde{\alpha}, \pm\tilde{\alpha}\}$ or $\{\pm\tilde{\alpha}, \pm 2\tilde{\alpha}\}$ [4]. We assume that $1/2\tilde{\alpha}$ is not a root. We call $\tilde{\alpha}$ of type (A), (B′) or (C) accordingly as:

(A) $\tilde{\alpha}$ coincides with some (real) root of $(\mathbf{G}, \mathbf{T})$;

(B′) $\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{T})$; $2\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{S})$, or

(C) $\tilde{\alpha}$ is not a root of $(\mathbf{G}, \mathbf{T})$; $2\tilde{\alpha}$ is a root of $(\mathbf{G}, \mathbf{S})$.

Suppose that $\tilde{\alpha}$ is of type B′ and choose a root λ of $(\mathbf{G}, \mathbf{T})$ whose restriction to $\mathbf{S}$ is $\tilde{\alpha}$. Then $\lambda \neq \sigma\lambda$ (σ denotes complex conjugation) and $\lambda + \sigma\lambda$ is not a root so that $\langle \lambda, \sigma\lambda \rangle \geqq 0$. An argument on $\langle \lambda, \lambda \rangle$ shows that $\langle \lambda, \sigma\lambda \rangle$ is independent of the choice of λ. If $\langle \lambda, \sigma\lambda \rangle > 0$ then comparison with the definitions of [4] shows that $\tilde{\alpha}$ cannot be useful in the sense of [4]. We call $\tilde{\alpha}$ of type B if $\tilde{\alpha}$ is of type B′ and $\langle \lambda, \sigma\lambda \rangle = 0$ for each λ.

Suppose now that $\tilde{\alpha}$ is of type C. Choose a root λ of $(\mathbf{G}, \mathbf{T})$ whose restriction to $\mathbf{S}$ is $\tilde{\alpha}$ and a root μ whose restriction is $2\tilde{\alpha}$. If $\langle \lambda, \sigma\lambda \rangle > 0$ then $\langle \mu, \mu \rangle \geqq 3 \langle \lambda, \lambda \rangle$. Since we have excluded systems of type G_2 we conclude that $\langle \lambda, \sigma\lambda \rangle \leqq 0$, and moreover that $\lambda + \sigma\lambda$ is a root of $\mathbf{T}$.

The reflections ω_α, $\tilde{\alpha}$ of type A, B or C, generate $\mathscr{W}$. We call $\omega \in \Omega_0$ of type A (respectively, B, C) if its image in $\mathscr{W}$ is a reflection of type A (respectively, B, C).

We return to the tori $\mathbf{T}'_m$ in $\mathbf{H}$ and $\mathbf{T}_m$ in $\mathbf{G}$. Let $\mathbf{S}'_m$ be the maximal $\mathbf{R}$-split torus in $\mathbf{T}'_m$ and $\mathbf{S}_m$ the maximal $\mathbf{R}$-split torus in $\mathbf{T}_m$. Then i_m maps $\mathbf{S}'_m$ to $\mathbf{S}_m$ and induces an embedding of the set of roots of $(\mathbf{H}, \mathbf{S}'_m)$ in the set of roots of $(\mathbf{G}, \mathbf{S}_m)$ [since each root of $\mathbf{S}'_m$ (respectively, $\mathbf{S}_m$) is the restriction of a root of $\mathbf{T}'_m$ (respectively, $\mathbf{T}_m$)]. Let $\mathscr{W}'_m$ be the restricted Weyl group attached to $(\mathbf{H}, \mathbf{S}'_m)$ and $\mathscr{W}_m$ be the group for $(\mathbf{G}, \mathbf{S}_m)$. Then i_m

induces an embedding of $\mathscr{W}'_m$ in $\mathscr{W}_m$; the image of the reflection with respect to a root of $\mathbf{S}'_m$ is the reflection with respect to its image in the roots of $\mathbf{S}_m$ and also

$$\begin{array}{ccc} \Omega_0(\mathbf{H}, \mathbf{T}'_m) & \rightarrow & \Omega_0(\mathbf{G}, \mathbf{T}_m) \\ \downarrow & & \downarrow \\ \mathscr{W}'_m & \longrightarrow & \mathscr{W}_m \end{array}$$

is commutative.

We come then to computing $\varkappa_m$ as a character on the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$.

PROPOSITION 7.4. — *If ω lies in the image of the imaginary Weyl group of $\mathbf{T}'_m$ then $\varkappa_m(\omega)=1$.*

Proof. — Let $\omega=\omega_{\alpha_r}\ldots\omega_{\alpha_1}$ where $\alpha_1, \ldots, \alpha_r$ are imaginary roots of $\mathbf{T}_m$ coming from $\mathbf{H}$. Then

$$\varkappa_m(\omega)=\varkappa_m(\omega_{\alpha_1})\,\varkappa_m^{\omega_{\alpha_1}}(\omega_{\alpha_r}\ldots\omega_{\alpha_2})=\varkappa_m(\omega_{\alpha_r}\ldots\omega_{\alpha_2})$$

since $\varkappa_m(\omega_{\alpha_1})=1$ if α_1 is compact, $\varkappa_m(\omega_{\alpha_1})=\varkappa_m(\check{\alpha}_1)=1$ if α_1 is noncompact and, in either case, $\varkappa_m^{\omega_{\alpha_1}}=\varkappa_m$. Induction now completes the argument.

We conclude from this proposition that $\varkappa_m(\omega)$ depends only on the image of ω in $\mathscr{W}_m$. Assume now that $\mathbf{G}$ is simple. If $\mathbf{G}$ is of type G_2 then direct computation shows that $\varkappa_m(\omega)=1$ for all ω in the image of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$. Suppose that $\mathbf{G}$ is not of type G_2; then neither is any simple factor (of the simply-connected covering of the derived group) of $\mathbf{H}$. If $\tilde{\alpha}'$ is a root of $\mathbf{S}'_m$ of type A and $\boldsymbol{\alpha}$ is its image in $\mathbf{G}$ then $2\tilde{\alpha}$ is not a root (by an argument as in [13], § 1.1); $1/2\tilde{\alpha}$ may be a root. If $1/2\tilde{\alpha}$ is not a root then $\tilde{\alpha}$ is of type A; if $1/2\tilde{\alpha}$ is a root then $1/2\tilde{\alpha}$ is of type C. If $\tilde{\alpha}'$ is of type B then $1/2\tilde{\alpha}$ is not a root. Also $\tilde{\alpha}$ cannot be of type A; hence $\tilde{\alpha}$ is of type B or C. If $\tilde{\alpha}'$ is of type C then so also is $\tilde{\alpha}$.

LEMMA 7.5:

(i) *If ω is the image of an element of $\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ of type* A *or* C *then $\varkappa_m(\omega)=1$, and*

(ii) *if ω is the image of an element of type* B *then $\varkappa_m(\omega)=1$ if* ω is also of type B; *otherwise* $\varkappa_m(\omega)=-1$.

Proof. — In case (i) there is a real root λ such that ω_λ has the same image in $\mathscr{W}_m$ as ω. Hence $\varkappa_m(\omega)=\varkappa_m(\omega_\lambda)=1$ since ω_λ can be realized in G.

In case (ii), suppose that ω has image $\omega_{\tilde{\alpha}}$ in $\mathscr{W}_m$ and that λ is a root of $\mathbf{T}_m$ such that $\langle \lambda, \sigma\lambda \rangle=0$ and the restriction of λ to $\mathbf{S}_m$ is $\tilde{\alpha}$. Then ω has the same image in $\mathscr{W}_m$ as $\omega_\lambda\,\omega_{\sigma\lambda}$. If $\tilde{\alpha}$ is of type B then $\{\pm\lambda, \pm\sigma\lambda\}$ are the only roots of $\mathbf{T}_m$ in the plane determined by λ, $\sigma\lambda$. Hence $\omega_\lambda\,\omega_{\sigma\lambda}$ can be realized in G and $\varkappa(\omega)=1$. The only other possible type for $\tilde{\alpha}$ is C; then

$$\omega_\lambda\,\omega_{\sigma\lambda}=\omega_{\lambda-\sigma\lambda}\,\omega_{\lambda+\sigma\lambda}$$

and

$$\varkappa_m(\omega_\lambda\,\omega_{\sigma\lambda})=\varkappa_m(\omega_{\lambda-\sigma\lambda})=\varkappa_m((\lambda-\sigma\lambda)\check{})=-1$$

by Proposition 2.1, since $\lambda-\sigma\lambda$ is not from $\mathbf{H}$ and must be noncompact (by an examination of the root systems of type C_2). This completes the argument.

8. A factor and an assumption

Again we consider one of the embeddings i_m: $\mathbf{T}'_m \to \mathbf{T}_m$. We fix, for once and for all, a positive system for the imaginary roots of $\mathbf{T}_m$ in $\mathbf{G}$ and use the induced system for the imaginary roots of $\mathbf{T}'_m$ in $\mathbf{H}'$. Recalling the "R"-function of Paragraph 4 we set

$$\mathrm{R}_m(\gamma)=\prod_{\substack{\alpha \text{ imaginary}\\ \alpha>0}}(1-\alpha(\gamma^{-1}))\prod_{\substack{\alpha \text{ not imaginary}\\ \alpha>0}}\left|(\alpha(\gamma))^{1/2}-(\alpha(\gamma))^{-1/2}\right|,$$

for $\gamma\in\mathbf{T}_m$, α denoting a root of $\mathbf{T}_m$ in $\mathbf{G}$, and

$$\mathrm{R}'_m(\gamma')=\prod_{\substack{\alpha' \text{ imaginary}\\ \alpha'>0}}(1-\alpha'(\gamma'^{-1}))\prod_{\substack{\alpha' \text{ not imaginary}\\ \alpha'>0}}\left|(\alpha'(\gamma'))^{1/2}-(\alpha'(\gamma'))^{-1/2}\right|,$$

for $\gamma'\in\mathbf{T}'_m$, α' denoting a root of $\mathbf{T}'_m$ in $\mathbf{H}$; the second product in each expression is to be interpreted as in Paragraph 4. Next we set

$$\iota_m=\frac{1}{2}\sum_{\substack{\alpha \text{ imaginary}\\ \alpha>0}}\alpha \quad\text{and}\quad \iota'_m=\frac{1}{2}\sum_{\substack{\alpha' \text{ imaginary}\\ \alpha'>0}}\alpha';$$

$\iota_m\in\mathrm{L}_m\otimes\mathbf{Q}$ and $\iota'_m\in\mathrm{L}(\mathbf{T}'_m)\otimes\mathbf{Q}$. Using i_m we transfer ι'_m to $\mathrm{L}_m\otimes\mathbf{Q}$, again writing ι'_m. Our assumption will be

$$\iota_m-\iota'_m\in\mathrm{L}_m,$$

(8.1) and

$$\iota_0-\iota'_0-\bar{h}(\iota_m-\iota'_m)\in\langle\Xi_0\rangle,$$

for some $h\in\mathrm{H}$ such that $\bar{h}=i_0\circ\operatorname{ad}h\circ i_m^{-1}$ maps $\mathbf{T}_m$ to $\mathbf{T}_0$.

As before, Ξ_m is the set of roots of $\mathbf{T}_m$. Clearly (8.1) is independent of the choice of h and the positive systems for imaginary roots. The second part of the assumption is a consequence of the first in all but a few cases. Those cases where (8.1) fails will be dealt with in another paper.

On transferring R'_m to T_m (without change in notation) we may define a function Δ_m on the regular elements of T_m by

$$\Delta_m(\gamma)=\frac{(\iota_m-\iota'_m)(\gamma)\,\mathrm{R}_m(\gamma)}{\mathrm{R}'_m(\gamma)}.$$

LEMMA 8.2. — *If* ω *is in the image of* $\Omega_0(\mathbf{H},\mathbf{T}'_m)$ *in* $\Omega_0(\mathbf{G},\mathbf{T}_m)$ *then*

$$\Delta_m(\gamma^\omega)=\varkappa_m(\omega)\,\Delta_m(\gamma),\ \gamma\,\varepsilon\,(\mathrm{T}_m)_{\mathrm{reg}}.$$

Proof. — Suppose that ω is the image of ω'. Then

$$\Delta_m(\gamma^\omega)=\frac{\varepsilon(\omega)}{\varepsilon(\omega')}\Delta_m(\gamma),$$

where $\varepsilon(-)$ denotes the signature with respect to imaginary roots; that is, $\varepsilon(\omega)=(-1)^n$ where n is the number of positive imaginary roots α of $\mathbf{T}_m$ in $\mathbf{G}$ for which $\omega\alpha$ is negative, and $\varepsilon(\omega')$ is similarly defined relative to the imaginary roots of $\mathbf{T}'_m$ in $\mathbf{H}$. To show that $\varepsilon(\omega)/\varepsilon(\omega')=\varkappa_m(\omega)$ we proceed in steps. We remark first that $\varepsilon(-)$ does not depend on the choice of positive system for the imaginary roots.

(i) Consider the signature of ω with respect to all roots of $\mathbf{T}_m$ in $\mathbf{G}$ (and some choice of positive system). This signature coincides with the determinant of ω (on $\mathrm{L}_m\otimes\mathbf{C}$) since $\omega\in\Omega(\mathbf{G},\mathbf{T}_m)$; we denote it by $\det\omega$. Similarly we consider the signature $\det\omega'$ of ω' with respect to all roots of $\mathbf{T}'_m$. Clearly $\det\omega=\det\omega'$ since the result is true if we replace ω' by any reflection in $\Omega(\mathbf{H},\mathbf{T}'_m)$.

(ii) Because ω preserves real roots also, we can consider the signature $\eta(\omega)$ of ω relative to the real roots of $\mathbf{T}_m$ in $\mathbf{G}$ (. . . and similarly the real signature $\eta(\omega')$ of ω' in $\mathbf{H}$). We claim that $\det\omega=\varepsilon(\omega)\,\eta(\omega)$, $\det\omega'=\varepsilon(\omega')\,\eta(\omega')$. To prove this we choose systems of positive roots in the following way. Take a system of positive roots for $\mathbf{T}_m$ in $\mathbf{G}$ with the property that if $\alpha>0$ and $\sigma\alpha\neq-\alpha$ then $\sigma\alpha>0$. Use the induced systems for the real roots of $\mathbf{T}_m$, the imaginary roots of $\mathbf{T}_m$, all roots of $\mathbf{T}'_m$, the imaginary roots of $\mathbf{T}'_m$, etc. Since $\omega(\sigma\alpha)=\sigma(\omega\alpha)$ the claim follows.

We will prove the lemma [in (v)] by showing that $\eta(\omega)/\eta(\omega')=\varkappa_m(\omega)$.

(iii) To compute $\eta(-)$ we use restricted roots. As before, let $\mathbf{S}_m$ be the maximal $\mathbf{R}$-split torus in $\mathbf{T}_m$. Each root $\tilde{\alpha}$ of $(\mathbf{G},\mathbf{S}_m)$ is the restriction to $\mathbf{S}_m$ of some root of $\mathbf{T}_m$; we define $m(\tilde{\alpha})$ to be the number of roots of $\mathbf{T}_m$ whose restriction to $\mathbf{S}_m$ is $\tilde{\alpha}$. Recall that restriction to $\mathbf{S}_m$ also defines a surjective homomorphism from $\Omega_0(\mathbf{G},\mathbf{T}_m)$ to $\mathscr{W}_m$, the restricted Weyl group attached to $(\mathbf{G},\mathbf{S}_m)$. We denote by $\tilde{\omega}$ the image in $\mathscr{W}_m$ of $\omega\,\varepsilon\,\Omega_0(\mathbf{G},\mathbf{T}_m)$. Finally, we set $\tilde{\alpha}>0$ if $\tilde{\alpha}$ is the restriction of a positive root of $\mathbf{T}_m$, using an ordering for the roots of $\mathbf{T}_m$ as in (ii).

For any $\tau\in\mathscr{W}_m$ we define

$$\tilde{\eta}(\tau)=\sum_{\tilde{\alpha}>0}(-1)^{m(\tilde{\alpha})\,n_\tau(\tilde{\alpha})},$$

where $n_\tau(\tilde{\alpha})=0$ if $\tau\tilde{\alpha}>0$ and $n_\tau(\tilde{\alpha})=1$ if $\tau\tilde{\alpha}<0$. If $\omega\in\Omega_0(\mathbf{G},\mathbf{T}_m)$ then $\eta(\omega)=\tilde{\eta}(\tilde{\omega})$.

(iv) To compute $\tilde{\eta}$, we note that $\tilde{\eta}$ is a quadratic character on $\mathscr{W}_m$ since η is a quadratic character on $\Omega_0(\mathbf{G},\mathbf{T}_m)$. We will then need to calculate just $\tilde{\eta}(\omega_{\tilde{\alpha}})$ assuming $\omega_{\tilde{\alpha}}\in\mathscr{W}_m$ (and $\tilde{\alpha}>0$).

If $\tilde{\beta}>0$ and $\omega_{\tilde{\alpha}}(\tilde{\beta})<0$, set $\tilde{\gamma}=-\omega_{\tilde{\alpha}}(\tilde{\beta})$. Then $\tilde{\gamma}>0$ and $\omega_{\tilde{\alpha}}(\tilde{\gamma})<0$; also $m(\tilde{\gamma})=m(\tilde{\beta})$. Since $\tilde{\gamma}=\tilde{\beta}$ if and only if $\tilde{\beta}$ is proportional to $\tilde{\alpha}$ we conclude that

$$\tilde{\eta}(\omega_{\tilde{\alpha}})=(-1)^{a(\tilde{\alpha})}\qquad\text{where}\quad a(\tilde{\alpha})=\sum_{\substack{\tilde{\beta}>0\\ \tilde{\beta}\ \text{prop. to}\ \tilde{\alpha}}}m(\tilde{\beta}).$$

To determine the parity of $a(\tilde{\alpha})$ we may assume **G** semisimple and simply-connected (by replacing the group by the simply-connected covering of its derived group) and consider each simple factor of **G** separately. It is convenient to exclude factors of type G_2 and deal with them separately later. Thus we assume that **G** is simple and not of type G_2. Suppose that $\tilde{\alpha}$ is a root of $\mathbf{S}_m$ for which $1/2\,\tilde{\alpha}$ is not a root. To generate $\mathscr{W}_m$ we need only $\omega_{\tilde{\alpha}}$, for those $\tilde{\alpha}$ which are of type A, B or C (*cf.* Paragraph 7).

If $\tilde{\alpha}$ is of type A then $m(\tilde{\alpha})$ is odd and $2\tilde{\alpha}$ is not a root. Thus $\tilde{\eta}(\omega_{\tilde{\alpha}})=-1$. If $\tilde{\alpha}$ is of type B then $m(\tilde{\alpha})$ is even and $\tilde{\eta}(\omega_{\tilde{\alpha}})=1$. If $\tilde{\alpha}$ is of type C then again $m(\tilde{\alpha})$ is even. However, $m(2\tilde{\alpha})$ is odd. Hence $\tilde{\eta}(\omega_{\tilde{\alpha}})=-1$.

(v) We come now to the proof of the lemma. A straight forward argument shows that we may assume that **G** is simple. If then **G** is of type G_2 direct computation shows that

$$\frac{\eta(\omega)}{\eta(\omega')}=\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')}=\varkappa_m(\omega)=1 \quad \text{for all } \omega.$$

Suppose that **G** is not of type G_2; then neither is any simple factor (of the simply-connected covering of the derived group) of **H**. It is enough to consider ω' of type A, B or C (*cf.* § 7). If ω' is of type A or C then we know that ω is of type A or C. Hence

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')}=\varkappa_m(\omega)=1.$$

If ω' is of type B then we have that ω is of type B or C. If ω is of type B then again

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')}=\varkappa_m(\omega)=1.$$

However, if ω is of type C then

$$\frac{\tilde{\eta}(\tilde{\omega})}{\tilde{\eta}(\tilde{\omega}')}=-1;$$

this is exactly the case where $\varkappa_m(\omega)=-1$. The lemma is therefore proved.

We now define a function $\Delta_H^G=\Delta_H^G(\varepsilon_0, \ldots, \varepsilon_N)$ on the regular elements in $\bigcup_{m=0}^{N} T_m$ by $\Delta_H^G(\gamma)=\varepsilon_m\,\Delta_m(\gamma)$, if γ is a regular element in T_m; ε_m is a constant, either 1 or -1. We also write just $\Phi_f^{\varkappa}(\gamma, \ , \ )$ for $\Phi_f^{\varkappa_m}(\gamma, \ , \ )$.

We summarize our choices once again: a set $\mathscr{I}=\{i_m\colon \mathbf{T}'_m\to\mathbf{T}_m,\ m=0, \ldots, N\}$ of embeddings of tori as in Paragraph 6, on each T_m a positive system for the imaginary roots, and parameters $\varepsilon_0, \ldots, \varepsilon_N$.

Let f be a Schwartz function on G and assume fixed Haar measures on $T_0, \ldots, T_N$ (denoted generically by dt) and G (denoted dg). If $\gamma'\in H$ originates from the regular element γ of G with respect to $\mathscr{I}$ set

$$\Phi(\gamma')=\Delta_H^G(\gamma)\,\Phi_f^{\varkappa}(\gamma, dg, dt).$$

Then:

THEOREM 8.3. — *Φ is a well-defined function on the elements of* H *which originate from regular elements of* G. *If γ' is such an element and lies in the Cartan subgroup* T′ *of* H *then*

$$\Phi((\gamma')^{\omega'})=\Phi(\gamma') \qquad \textit{for} \quad \omega'\in\mathscr{A}(T').$$

Proof. — This follows from Lemmas 4.1, 6.1 and 8.2.

9. Transferring orbital integrals

We continue with the notation of the last section. Our aim now is to write down conditions on $\varepsilon_0, \ldots, \varepsilon_N$ (necessary for generic f) to ensure the existence of a Schwartz function f' on H such that

$$\Phi^1_{f'}(\gamma', dt', dh)=\Delta^G_H(\gamma)\,\Phi^{\varkappa}_f(\gamma, dt, dg), \tag{1}$$

if γ' originates from the regular element γ of G and

$$\Phi^1_{f'}(\ ,\ ,\ )=0, \tag{2}$$

on Cartan subgroups H of G which do not originate in G.

Here dt' is to be obtained from dt via an $\mathscr{I}$-embedding; for each measure dg we pick a Haar measure dh on H subject only to the conditions: if $(dh)'$ corresponds to $(dg)'$ and $(dg)'=\beta dg$, $\beta>0$, then $(dh)'=\beta dh$, and if H is a torus then $dh=dt'$.

Before proceeding, we note that a change in $\mathscr{I}$ or the positive systems for imaginary roots causes at most a sign change on the right-hand side of (1); this change may as well be effected by adjusting $\varepsilon_0, \ldots, \varepsilon_N$ instead.

Let T′ be a Cartan subgroup of H. Then we set:

$$\Phi^{T'}(\gamma', dt', dh')=\begin{cases}\Delta^G_H(\gamma)\,\Phi^{\varkappa}_f(\gamma, dt, dg),\\ \text{if } \gamma'\in T' \text{ originates from } \gamma\in T_{\mathrm{reg}},\\ 0 \text{ if } T' \text{ does not originate in } G.\end{cases}$$

Then:

$$\Phi^{T'}(\gamma', \alpha dt', \beta dh)=\beta/\alpha\,\Phi^{T'}(\gamma', dt', dh), \qquad \alpha, \beta>0, \tag{I}$$

and

$$\Phi^{(T')^{\omega'}}((\gamma')^{\omega'}, (dt')^{\omega'}, dh)=\Phi^{T'}(\gamma', dt', dh), \qquad \omega\in\mathscr{A}(T'). \tag{II}$$

We want to check whether $\{\Phi^{T'}\}$ satisfies the remaining conditions of [9], Theorem 4.7. From (I) and (II) above it follows that we may fix dt and dh and assume that either T′ is one of the Cartan subgroups T'_m and $\gamma=i_m(\gamma')$ or T′ does not originate in G.

We dispose first of the case that T′ does not originate in G. Then nor does $(T')^{s'}$, for any Cayley transform s' with respect to a noncompact imaginary root of $\mathbf{T}'$ (*cf.* [9], § 2). Hence III, III *a*, III *b* of Theorem 4.7 in [9] are satisfied.

We will write Φ_m for $\Phi^{T'_m}$ and set

$$\Psi_m(\gamma) = R'_m(\gamma')\,\Phi_m(\gamma') = \varepsilon_m(\iota_m - \iota'_m)(\gamma)\,R_m(\gamma)\,\Phi_f^{\varkappa_m}(\gamma);$$

$\Psi_m(\gamma)$ is defined on the regular elements in T_m. For the next few paragraphs we omit the subscript m from $\mathbf{T}_m$, $\mathbf{T}'_m$, Φ_m, Ψ_m, i_m, R_m, R'_m, ι_m and ι'_m; we write I for the set of imaginary roots for $\mathbf{T}_m$, I′ for the imaginary roots of $\mathbf{T}'_m$ and sometimes identify I′ with its image in I.

From Proposition 4.2 we obtain that Ψ is a Schwartz function on

$$T^{(0)} = \{\gamma \in T : \alpha(\gamma) \neq 1,\ \alpha \in I\}.$$

To satisfy III of Theorem 4.7 in [9] we have to show that Ψ extends to a Schwartz function on

$$T^{(1)} = \{\gamma \in T : \alpha(\gamma) \neq 1,\ \alpha \in I'\}.$$

According to a standard argument (*cf.* [13], § 8.4) it is sufficient to show:

PROPOSITION 9.1. — *If* $\alpha \in I - I'$ *and* $\gamma_0 \in T$ *is such that* $\beta(\gamma_0) = 1$ *only if* $\beta = \pm\alpha$ *then*

$$\lim_{\nu \downarrow 0} D\Psi(\gamma_\nu) = \lim_{\nu \uparrow 0} D\Psi(\gamma_\nu),$$

where $\gamma_\nu = \gamma_0 \exp i\nu H_\alpha$, H_α *denoting the coroot attached to* α (*as element of* $\mathfrak{t}$), *and* D *is any invariant differential operator on* $\mathbf{T}$.

Proof. — Since $\varkappa(\alpha^\vee) = -1$ this follows immediately from Lemma 4.3.

We come next to III *a* of Theorem 4.7 in [9]. Because H is quasi-split this condition is vacuous. Indeed:

LEMMA 9.2. — *If* $\mathbf{G}$ *is a quasi-split group over* $\mathbf{R}$ *then the following is true for any Cartan subgroup* T *of* G: *if* α *is an imaginary root for* $\mathbf{T}$ *then there exists* ω *in the imaginary Weyl group for* $\mathbf{T}$ *such that* $\omega\alpha$ *is noncompact.*

Proof. — We may assume that $\mathbf{G}$ is semisimple and simply-connected. By [9], Proposition 4.11 it is sufficient to show that for each imaginary root α of $\mathbf{T}$ there exists $g \in \mathbf{G}$ such that $\sigma(g^{-1})g$ realizes ω_α, the Weyl reflection with respect to α.

Let $\gamma_0 \in T$ be such that $\alpha(\gamma_0) = 1$ and $\beta(\gamma_0) \neq 1$ if $\beta \neq \pm\alpha$. Set $\mathbf{C}$ to be the connected component of the identity in the centralizer of γ_0 in $\mathbf{G}$; recall that $\mathbf{C}$ is of type A_1 and C contains T as fundamental Cartan subgroup. Let $\psi: \mathbf{C} \to \mathbf{C}'$ be an inner twist taking $\mathbf{C}$ to a quasi-split form $\mathbf{C}'$ and such that the restriction of ψ to $\mathbf{T}$ is defined over $\mathbf{R}$ (*cf.* [8]). Let s be a Cayley transform with respect to a (noncompact) root $\psi(\alpha)$ of $\psi(\mathbf{T})$ and set $\lambda = \operatorname{ad} s \circ \psi$. Then clearly the automorphism $\sigma(\lambda^{-1})\lambda$ of $\mathbf{T}$ realizes ω_α. Now choose an $\mathbf{R}$-rational point t in the image of $\mathbf{T}$ under λ such that $\gamma = \lambda^{-1}(t)$ is regular in $\mathbf{G}$. Then $\sigma(\gamma) = \gamma^{\omega_\alpha}$ so that the conjugacy class of γ in $\mathbf{G}$ is defined over $\mathbf{R}$. But then, by [11], this class contains an $\mathbf{R}$-rational point, say $g\gamma g^{-1}$. Clearly $(\sigma(g^{-1})g)\gamma(\sigma(g^{-1})g)^{-1}=\gamma^{\omega_\alpha}$. Since γ is regular in $\mathbf{T}$ this implies that $\sigma(g^{-1})g$ realizes ω_α and so the lemma is proved.

We come then to the condition III b of [9], Theorem 4.7. Suppose that α' is a noncompact root in I′. Then α, the image of α' in I, may be compact... in fact it may happen that each $\omega\alpha$, ω in the imaginary Weyl group of $\mathbf{T}$, is compact.

PROPOSITION 9.3. — *Let s' be a Cayley transform with respect to α'. Then we can find a noncompact root among the $\omega\alpha$ if and only if $(\mathrm{T}')_{s'}$ originates in* G.

Proof. — Suppose that $(\mathrm{T}')_{s'}$ originates in G. Then an $\mathscr{I}$-embedding $i^{(s')}$ of $(\mathbf{T}')_{s'}$ in $\mathbf{G}$ yields a map $i^{(s')}\circ \mathrm{ad}\, s'\circ i^{-1}$ on $\mathbf{T}$ which can be realized by an element of $\mathbf{G}$, say s. Clearly $\sigma(s^{-1})s$ realizes ω_α and we are done. Conversely, suppose that $\omega\alpha$ is noncomapct in $\mathbf{G}$ and that s is a Cayley transform with respect to α (in our general sense). Then $i^{(s)}=\mathrm{ad}\, s\circ i\circ \mathrm{ad}(s')^{-1}$ is defined over $\mathbf{R}$; by choosing s suitably we can ensure that $i^{(s)}$ is an $\mathscr{I}$-embedding. Hence $(\mathrm{T}')_{s'}$ originates in G and the proposition is proved.

Suppose now that α' (noncompact in H) is a root for which all $\omega\alpha$ are compact. Suppose that $\gamma_0'\in\mathrm{T}'$ is such that $\pm\alpha'$ are the only roots in I′ annihilating γ_0'. It is possible that $\beta(\gamma_0)=1$ where β lies outside I′ (as usual, γ_0 is the image of γ_0'); nevertheless, for small ν, $\gamma_\nu=\gamma_0\exp i\nu\mathrm{H}_\alpha$ lies in $\mathrm{T}^{(1)}$ so that $\Psi(\gamma_\nu)$ is well-defined. To satisfy condition III b for the present α' we have to show

$$\lim_{\nu\downarrow 0}\mathrm{D}\Psi(\gamma_\nu)=\lim_{\nu\uparrow 0}\mathrm{D}\Psi(\gamma_\nu),$$

for each D. If γ_0 is annihilated by no root outside I′ then it is immediate (*cf.* Lemmas 4.3 and 4.4). To obtain this formula in general we have only to apply the usual argument (*cf.* [13], § 8.4).

The remaining case provides us with the conditions on $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\mathrm{N}$. Here we have a noncompact imaginary root α' for which some root in the imaginary Weyl group orbit of α is noncompact. Suppose that γ_0' is an element for which $\alpha'(\gamma_0')=1$ and $\beta'(\gamma_0')\neq 1$ if $\beta'\neq\pm\alpha'$. Once again a straight forward argument shows that we may assume that $\pm\alpha$ are the only roots which annihilate γ_0.

We return to writing $\mathbf{T}_m$ for $\mathbf{T}$, Φ_m for Φ, etc. Fix a Cayley transform s' with respect to α'. Recall that $(\mathbf{T}_m')_{s'}$ originates in $\mathbf{G}$ (Prop. 9.3). Whatever our choice for s', $(\mathbf{T}_m')_{s'}$ originates from the same torus, say $\mathbf{T}_n$, among $\mathbf{T}_0, \ldots, \mathbf{T}_\mathrm{N}$. Since to verify III b we are free to make any choice for s' we may assume that $(\mathbf{T}_m')_{s'}$ is $\mathbf{T}_n'$. Thus we have:

$$\begin{array}{c} \mathbf{T}_m'\xrightarrow{i_m}\mathbf{T}_m \\ {\scriptstyle s'}\big\downarrow\quad\ \ \\ \mathbf{T}_n'\xrightarrow{i_n}\mathbf{T}_n. \end{array}$$

We denote by s the map $i_n\circ s'\circ i_m^{-1}$; s can be realized by an element of $\mathbf{G}$ and $\sigma(s^{-1})s$ realizes ω_α.

We need now to label our chosen systems of positive imaginary roots; we denote by I_m^+ the system for $\mathbf{T}_m$ and by $(I_m^+)'$ the induced system for $\mathbf{T}'_m$. Because I_m^+ need not be adapted to α we pick a system J^+ which is. Then the induced system $(J^+)'$ is adapted to α'. We denote by R^* the R-function defined by J^+ and by ι^* one-half the sum of the roots in J^+. We write

$$R=\varepsilon(I_m^+, J^+)(\iota^*-\iota)R^*; \qquad \varepsilon(I_m^+, J^+)=\pm 1.$$

Similarly we define $(R^*)'$, $(\iota^*)'$ and $\varepsilon((I_m^+)', (J^+)')$. As before, we will often transfer functions and operators from T'_m to T_m without change in notation. We have to compute

$$\lim_{\nu\downarrow 0}\hat{D}((R^*)'\Phi_m)(\gamma_\nu)-\lim_{\nu\uparrow 0}\hat{D}((R^*)'\Phi_m)(\gamma_\nu),$$

... III b will be satisfied if and only if the result is

$$2i\widehat{D^s}((R^*)'_{s'}\Phi_n)(\gamma_0^s).$$

We summarize our calculations in:

PROPOSITION 9.4:

(a) $\hat{D}((R^*)'\Phi_m)=\varepsilon_m\varepsilon(I_m^+, J^+)\varepsilon((I_m^+)', (J^+)')(\iota^*-(\iota^*)')\hat{D}(R^*\Phi_f^{\varkappa_m})$,

(b) $(\iota^*-(\iota^*)')(\gamma_0)=(\iota_s^*-(\iota^*)'_{s'})(\gamma_0^s)$,

(c) $(\iota_s^*-(\iota^*)'_{s'})(\gamma_0^s)\widehat{D^s}(R_s^*\Phi_f^{\varkappa_n})(\gamma_0^s)=\varepsilon_n\varepsilon(I_n^+, J_s^+)\varepsilon((I_n^+)', (J^+)'_{s'})\widehat{D^s}((R^*)'_{s'}\Phi_n)(\gamma_0^s)$.

Note that (b) utilizes the second part of our assumption (8.1). Lemma 4.4 now shows that III b is satisfied provided

$$\varepsilon_m\varepsilon_n=\varepsilon_{\varkappa_m}(s)\,\varepsilon(I_m^+, J^+)\,\varepsilon((I_m^+)', (J^+)')\,\varepsilon(I_n^+, J_s^+)\,\varepsilon((I_n^+)', (J^+)'_{s'}). \tag{9.5}$$

Recall that $\varepsilon_{\varkappa_m}(s)$, the $\varkappa_m$-signature of s, was defined in Paragraph 4.

10. Transferring orbital integrals (cont.)

We come now to some explicit calculations and our main result (Theorem 10.2). Suppose that T'_m and T'_n are a pair among $\{T'_0, \ldots, T'_N\}$ for which there is some Cayley transform (in our general sense) from T'_m to T'_n. This means just that the conjugacy class of T'_n succeeds that of T'_m in the lattice $t(H)$ (more briefly, "T'_n succeeds T'_m"). The left-hand side of (9.5) depends, apparently, on the choice (α') of root to define the Cayley transform, choice (s') of Cayley transform and choice (J^+) of positive system adapted to the image in G of that root. We will check that the choices have no effect. Let

$$\varepsilon_{\varkappa_0}(m, n)=\varepsilon_{\varkappa_m}(s),$$

$$\varepsilon_+(m, n)=\varepsilon(I_m^+, J^+)\,\varepsilon((I_m^+)', (J^+)')\,\varepsilon(I_n^+, J_s^+)\,\varepsilon((I_n^+)', (J^+)'_{s'}).$$

Although we have omitted it in notation, $\varepsilon_{\varkappa_0}(m, n)$ and $\varepsilon_+(m, n)$ may depend on the choices above... it is only their product which we claim to be independent. Our equations (9.5) are now:

$$\varepsilon_m \varepsilon_n = \varepsilon_{\varkappa_0}(m, n)\, \varepsilon_+(m, n). \tag{10.1}$$

Summarizing Paragraph 9 we have:

THEOREM 10.2. — *If* $\varepsilon_m \varepsilon_n = \varepsilon_{\varkappa_0}(m, n)\,\varepsilon_+(m, n)$ *whenever* T'_n *succeeds* T'_m $(m, n=0, 1, \ldots, N)$ *then the factor* $\Delta_G^H = \Delta_G^H(\varepsilon_0, \ldots, \varepsilon_N)$ *has the property that for each Schwartz function f on* G *there exists a Schwartz function f' on* H *such that:*

(1) $$\Phi_{f'}^1(\gamma', dt', dh) = \Delta_H^G(\gamma)\, \Phi_f^{\varkappa}(\gamma, dt, dg),$$

if γ' originates from $\gamma \in G_{\text{reg}}$ and

(2) $$\Phi_{f'}^1(\ ,\ ,\ ) \equiv 0,$$

on Cartan subgroups of H *which do not originate in* G.

The notation has been explained in Paragraph 9. The converse is also true: if the equations are not satisfied then we can find functions f for which there is no f' satisfying (1) and (2). Of interest for character identities is the following: if both f' and f'' are attached to f as in the theorem then any of the (tempered) characters χ_φ of [9] takes the same value on f' and f'' and, conversely, we can always replace f' by a function on which each χ_φ takes the same value (*cf.* [9], Lemma 5.3).

It remains now to prove our claim of the first paragraph; α', α, s', s, J^+ and $(J^+)'$ are as at the end of Paragraph 9.

PROPOSITION 10.2:

$$\varepsilon(I_m^+, J^+)\, \varepsilon(I_n^+, J_s^+) = \frac{1}{2}(|\{\beta : \langle \beta, \alpha \rangle \neq 0, \text{ both } \beta \text{ and } \omega_\alpha(\beta) \in I_m^+\}|)$$
$$+ |\{\beta : \beta \in I_m^+, \langle \beta, \alpha \rangle = 0 \text{ and } \beta \notin I_n^+\}|.$$

The proof is straightforward; we omit the details.

COROLLARY 10.3. — *Neither* $\varepsilon(I_m^+, J^+)\, \varepsilon(I_n^+, J_s^+)$ *nor* $\varepsilon((I_m^+)', (J^+)')\, \varepsilon((I_n^+)', (J^+)'_{s'})$ *depends on the choice for* J^+.

We will need the following:

LEMMA 10.5. — *Let* **G** *be a connected reductive group over* **R**, T *a Cartan subgroup of* G *and α, β imaginary roots of* T *for which there exist Cayley transforms. Suppose that the image of* **T** *under some (and hence every) Cayley transform with respect to α is* G*-conjugate to the image under some transform with respect to β. Then there exists ω in $\Omega_0(\mathbf{G}, \mathbf{T})$ [that is, an element ω of $\Omega(\mathbf{G}, \mathbf{T})$ realized in $\mathscr{A}(T)$] mapping α to β.*

It is clear that, conversely, if α and β are so related then the image of **T** under a Cayley transform with respect to α is G-conjugate to the image under any Cayley transform with respect to β. Lemma 9.2 thus says that if $\mathbf{G}$ is quasi-split then the $\mathscr{A}(T)$-orbits of imaginary roots of T parametrize the successors in the lattice $t(G)$ of the conjugacy class of T.

Proof. — As usual, we denote by $\mathbf{G}^{\sim}$ the simply-connected covering of the derived group of $\mathbf{G}$: two maximal tori in $\mathbf{G}$, defined over $\mathbf{R}$, are stably conjugate if and only if their preimages in $\mathbf{G}^{\sim}$ are stably conjugate in $\mathbf{G}^{\sim}$ and so the natural projection induces a bijection between $t(G^{\sim})$ and $t(G)$. Hence it is enough to prove the lemma in the case $\mathbf{G}$ is simply-connected, semi simple... clearly, we can then assume $\mathbf{G}$ simple, as well. Finally, by the results of Paragraph 2 in [9] we can assume $\mathbf{G}$ quasi-split.

The rest of our proof is a case-by-case study. In several places we will use the following. Let T_0 be a fundamental Cartan subgroup of G and $\Delta=\{\alpha_1, \ldots, \alpha_r\}$ an ordered set of imaginary roots for T_0 with the property that $\mathbf{T}=s\mathbf{T}_0 s^{-1}$, where $s=s_r s_{r-1} \ldots s_1$, s_1 is a Cayley transform with respect to α_1 and, for $i\geqq 2$, s_i is a Cayley transform with respect to $s_{i-1} \ldots s_1 \alpha_i$. Then $\alpha_0=s^{-1}\alpha$, $\beta_0=s^{-1}\beta$ are imaginary roots of T_0, perpendicular to Δ. Suppose that there exists $\omega_0\in\Omega(\mathbf{G}, \mathbf{T}_0)$ such that $\beta_0=\omega_0\alpha_0$, $\sigma\omega_0=\omega_0\sigma$ and ω_0 fixes $\alpha_1, \ldots, \alpha_r$. Then clearly $\omega=s\omega_0 s^{-1}$ has the properties required in the lemma.

We summarize now the (elementary) argument for each type. The roots for $\mathbf{T}_0$ are labelled as in [1]; we transfer roots from $\mathbf{T}_0$ to $\mathbf{T}$ (via s) without change in notation.

(A_n) We have only to consider SL_{n+1} and special unitary groups (of maximal index). In the case of SL_{n+1} only the roots $e_{2i-1}-e_{2i}$ of (the usual) T_0 are imaginary and it is easy to find ω_0. In the case of unitary groups all the roots of T_0 are imaginary and again ω_0 is easily found.

(B_n, C_n) We give an argument for C_n which adapts immediately to the case B_n. Consider each pair of (imaginary) roots in T_0 as possibilities for $\{\alpha_0, \beta_0\}$. In the cases $\{2e_i, 2e_j\}$, $\{e_i\pm e_j\}$, $\{e_i-e_j, e_i-e_k\}$, $j\neq k$, and $\{e_i-e_j, e_k-e_l\}$ with i, j, k, l distinct and e_i+e_j, $e_k+e_l\notin\Delta$ the choice of ω_0 is easy. In the case $\{e_i-e_j, e_k-e_l\}$ with i, j, k, l distinct and both e_i+e_j, e_k+e lying in Δ, we have on $\mathbf{T}$ that $\sigma e_i=e_j$, $\sigma e_k=e_l$ so that $\omega=\omega_{e_i-e_k}\omega_{e_j-e_l}$ commutes with σ and maps α to β. Next we observe that $\{e_i-e_j, e_k-e_l\}$ with i, j, k, l distinct and $e_i+e_j\in\Delta$, $e_k+e_l\notin\Delta$, is not a possibility (by counting the number of long imaginary roots in the images of $\mathbf{T}$ under Cayley transforms with respect to e_i-e_j, e_k-e_l). Similarly $\{e_i-e_j, 2e_k\}$ is not possible. The remaining pairs are similarly dealt with.

(D_n) Here we have to consider (i) the groups $\mathrm{Spin}(2m, 2m)$, $\mathrm{Spin}(2m, 2m+2)$ (where fundamental Cartan subgroups are compact) and (ii) $\mathrm{Spin}(2m+1, 2m+1)$, $\mathrm{Spin}(2m-1, 2m+1)$ (where fundamental Cartan subgroups are not compact). Again we examine each pair of imaginary roots in $\mathbf{T}_0$. In the case $\{e_i-e_j, e_i-e_l\}$, $j\neq l$, ω_0 is easily found. In the case $\{e_i\pm e_j\}$, suppose that there is some e_k not appearing in the roots in Δ. Then for both (i) and (ii) the choice of ω_0 is easy. In the same case, suppose that every e_k appears in a root of Δ and that for some pair (l, p) both e_l+e_p and e_l-e_p lie in Δ. Then on $\mathbf{T}$, $\sigma e_l=e_l$ and $\sigma e_j=-e_j$ so that $\omega_{e_j+e_l}\omega_{e_j-e_l}$ will do for ω. Finally, suppose that every index appears in the roots of Δ (except i, j) and that if $e_l\pm e_p$ belongs to Δ then $e_l\mp e_p$ does not. Then we must be in the case of a well-known example for

Spin $(2m, 2m)$ (*cf.* [12]) where twists by e_i+e_j, e_i-e_j lead to non-conjugate Cartan subgroups.

Next we consider the case $\{e_i-e_j, e_k-e_l\}$ with i, j, k, l distinct. If either both or neither e_i+e_j, e_k+e_l belong to Δ then we find ω as before (*cf.* the argument for C_n). We claim that if exactly one of these roots belongs to Δ then $\{e_i-e_j, e_k-e_l\}$ is not a possibility. We justify this by performing Cayley transforms on $\mathbf{T}$ with respect to e_i-e_j and e_k-e_l and then calculating the "root spaces" attached to the images (*cf.* [12]); these spaces are easily seen to be non-conjugate in the sense of [12] [for both types (i), (ii)].

The remaining cases are now easily examined.

(E_6) There are two groups to consider: the simply-connected split form, whose fundamental Cartan subgroup is not compact and the simply-connected quasi-split form with compact fundamental Cartan subgroup. We investigate the second first.

If both roots e_i+e_j, $\pm 1/2\,(e_i+e_j)\pm\ldots$ are imaginary then clearly we can find an element of $\mathscr{A}(\mathrm{T})$ mapping the former to the latter. A simple inductive argument then shows that we can assume that Δ contains only roots of the form $e_i\pm e_j$. We have now only to show that for any pair among $\{e_i\pm e_j\}$, $1\leqq j<i\leqq 5$, we can find an ω as desired. For pairs $\{e_i-e_j, e_i-e_l\}$, j, l distinct, this is immediate. In the case of $\{e_i\pm e_j\}$, there is some e_k not appearing in the roots of Δ and so we can argue as for the case D_n. In the case of $\{e_i-e_j, e_k-e_l\}$, with i, j, k, l distinct we again argue as before if either both or neither of e_i+e_j, e_k+e_l belong to Δ. Suppose that $e_i+e_j\in\Delta$, $e_k+e_l\notin\Delta$. Then $\Delta=\{e_i+e_j\}$ and the root $1/2\,(e_i-e_j+e_k-e_l+\ldots)$ is imaginary in $\mathbf{T}$ and perpendicular to neither e_i-e_j nor e_k-e_l. Hence ω is easily found. The remaining cases are handled similarly.

To investigate the other form of type E_6 we make the appropriate definition of "inverse Cayley transform" with respect to a *real* root of $\mathbf{T}$ (generalizing the usual notion). It follows easily that we have only to check that if there are inverse Cayley transforms with respect to the real roots α, β which lead to conjugate Cartan subgroups then β is of the form $\omega\alpha$, with $\omega\in\mathscr{A}(\mathrm{T})$ (or, just as well, with ω in G). To make this check we set up the analogue of Δ among the (real) roots of the split Cartan subgroup of G. As before, we can assume that this set contains only roots of the form $e_i\pm e_j$ and consider candidates for α, β. The argument is analogous to that of the previous paragraph; we omit the details.

(E_7) We can assume that Δ contains only roots of the form $e_i\pm e_j$, for if $\mathrm{T}=\mathrm{T}_0$, any two roots of T can be connected by an element of $\mathscr{A}(\mathrm{T})$ and so we can restrict our attention to the case Δ contains e_8-e_7. We have then to consider just pairs from $\{e_i\pm e_j\}$, $1\leqq j<i\leqq 6$, as candidates for α, β. For a pair $\{e_i-e_j, e_i-e_l\}$, i, j, l distinct, ω is easily found. Consider a pair $\{e_i+e_j, e_i-e_j\}$. Our previous arguments show how to find ω in all but the case where Δ has three elements e_8-e_7, $e_k\star e_l$, $e_m\star e_n$ where k, l, m, n are distinct from i, j and $\star$ denotes some choice of $\pm$. For this $\mathbf{T}$ we perform Cayley transforms by e_i-e_j and e_i+e_j and count the number of real roots in the images; this enables us to exclude this case. Next we consider a pair $\{e_i-e_j, e_k-e_l\}$, with i, j, k, l distinct. Again if either both or neither e_i+e_j, e_k+e_l lie in Δ then we can find $\omega\ldots$ and similar arguments apply if we change either or both signs in $\{e_i-e_j, e_k-e_l\}$. The remaining case requires several arguments; we find it easier to use numerical indices. Suppose that $\Delta=\{e_8-e_7, e_1+e_2\}$. We exclude the pair

$\{e_2-e_1, e_4-e_3\}$ by counting the number of imaginary roots in the image of **T** under a Cayley transform with respect to e_2-e_1, e_4-e_3, respectively. It follows similarly that $\{e_2-e_1, e_4+e_3\}$ is not a possibility. Suppose now that $\Delta=\{e_8-e_7, e_1+e_2, e_6-e_5\}$. Then the pair $\{e_2-e_1, e_4+e_3\}$ is excluded (... this time counting real roots in the images). On the other hand, consider $\{e_2-e_1, e_4-e_3\}$. The root $1/2((e_8-e_7)+(e_2-e_1)+(e_4-e_3)+(e_6-e_5))$ is imaginary and perpendicular to neither e_2-e_1 nor e_4-e_3. Hence we can find ω in $\mathscr{A}(\mathrm{T})$ mapping e_2-e_1 to e_4-e_3. Suppose that $\Delta=\{e_8-e_7, e_1+e_2, e_6\pm e_5\}$. Then for each pair $\{e_2-e_1, e_4-e_3\}$, $\{e_2-e_1, e_4+e_3\}$ we can construct a root as above and so obtain ω. We can now easily complete the argument.

(E_8) Once again we can assume that Δ contains only roots of the form $e_i\pm e_j$ and investigate just pairs among $\{e_i\pm e_j\}$. The arguments are similar to those for E_7 and so we omit the details.

(F_4) For the pairs $\{e_i\pm e_j\}$, $\{e_i-e_j, e_i-e_l\}$, $\{e_i-e_j, e_k-e_l\}$ and $\{1/2(e_1\pm e_2\ldots), e_i\}$, i, j, k, l distinct, we find ω easily. The pairs $\{e_i-e_j, e_k\}$ are eliminated (by counting short imaginary roots in the images of **T** under...) and the argument then easily completed.

(G_2) Here we need only observe that if **T** is compact then the G-conjugacy class of image of **T** under a Cayley transform depends just on the length of the root used.

Lemma 10.5 is thus proved.

Returning to $\varepsilon_{\varkappa_0}(m, n)$ and $\varepsilon_+(m, n)$ we have now that we may replace α' only by $\omega'\alpha'$, $\omega'\in\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ [since we have required α' to be noncompact there is further restriction on ω' (*cf.* [9], Lemma 4.2) but we do not need this explicitly]. Thus s' may be replaced only by $t'=\omega'_0 s'\omega'$ where $\omega'\in\Omega_0(\mathbf{H}, \mathbf{T}'_m)$ and $\omega'_0\in\Omega_0(\mathbf{H}, \mathbf{T}'_n)$; s is then replaced by $\omega_0 s\omega$ where ω is the image of ω' in $\Omega_0(\mathbf{G}, \mathbf{T}_m)$ and ω_0 the image of ω'_0 in $\Omega_0(\mathbf{G}, \mathbf{T}_n)$. A straightforward computation shows that $\varepsilon_{\varkappa_0}(m, n)$ is multiplied by $\varkappa_m(\omega)\varkappa_n(\omega_0)$ and $\varepsilon_+(m, n)$ by $\varepsilon(\omega)/\varepsilon(\omega').\varepsilon(\omega_0)/\varepsilon(\omega'_0)$ (in the notation of Lemma 8.2). Hence, by the proof of Lemma 8.2, $\varepsilon_{\varkappa_0}(m, n)\,\varepsilon(m, n)$ is unchanged.

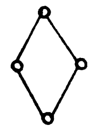
11. Application of Theorem 10.2

As an immediate corollary of Theorem 10.2 we obtain:

PROPOSITION 11.1. — *If the ordering on* $t(\mathrm{H})$ *is linear* (*that is, if the derived group of* **H** *is trivial, of type* A_n *or of type* E_6) *or if* G *has split rank one then given some* ε_M *there is a choice for* $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{\mathrm{M}-1}, \varepsilon_{\mathrm{M}+1}, \ldots, \varepsilon_\mathrm{N}$ *for which the factor* $\Delta_\mathrm{G}^\mathrm{H}(\varepsilon_0, \ldots, \varepsilon_\mathrm{N})$ *provides a transfer of orbital integrals in the sense of Theorem* 10.2.

We would like to remove this assumption on **H** (... or **G**). Here we just describe some reductions and, as application, check that the conclusion of Proposition 11.1 remains valid under the assumption that the derived group of **G** is isogenous to a product of groups each of which has rank at most two, with **H** (or, more precisely, $\varkappa_0$) arbitrary. Recall that we admit only those pairs (**G**, **H**) which satisfy the condition (8.1); in particular, for each T_m one half of the sum of the positive imaginary roots not coming from **H** defines a character on T_m.

We have thus to investigate the consistency of the equations (10.1) as the pair (m, n) varies. The following observation will allow us to consider just consistency around subsets of $t(\mathrm{H})$ of the form:



PROPOSITION 11.2. — *Suppose that* $\mathbf{G}$ *is a connected reductive group over* $\mathbf{R}$ *and that* T' *and* T'' *are Cartan subgroups of* G *succeeded by the same Cartan subgroup. Then both* T' *and* T'' *succeed some Cartan subgroup.*

Proof. — A straightforward argument brings us to the case where $\mathbf{G}$ is simply-connected, simple and quasi-split (*cf.* the proof of Lemma 10.6). We have then only to examine the possibilities for $t(\mathrm{G})$. This is easily done using the lists in [12]; we omit the details.

Suppose now that T'_m is fixed and ε_m chosen as 1. Suppose also that T'_{n_1} and T'_{n_2} are non-conjugate Cartan subgroups which succeed T'_m and that ε_{n_1} and ε_{n_2} are defined so that (10.1) holds; that is,

$$\varepsilon_{n_i}=\varepsilon_{\varkappa_0}(m, n_i)\,\varepsilon_+(m, n_i) \qquad \text{for} \quad i=1, 2.$$

Finally, suppose that T'_p succeeds both T'_{n_1} and T'_{n_2}. Then both

$$\varepsilon_{\varkappa_0}(m, n_i)\,\varepsilon_{\varkappa_0}(n_i, p)\,\varepsilon_+(m, n_i)\,\varepsilon_+(n_i, p), \qquad i=1, 2,$$

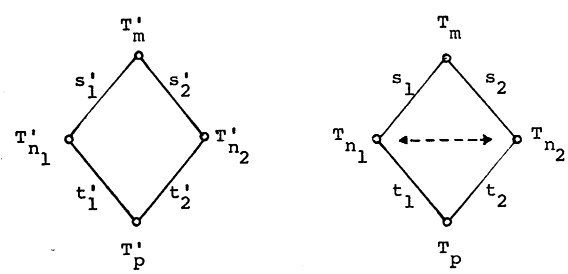
are candidates for ε_p. Proposition 11.2 and a simple inductive argument allow us to conclude:

LEMMA 11.3. — *Given some* ε_M *there is a choice for* $\varepsilon_0, \ldots, \varepsilon_{\mathrm{M}-1}, \varepsilon_{\mathrm{M}+1}, \ldots, \varepsilon_\mathrm{N}$ *for which* $\Delta_\mathrm{G}^\mathrm{H}(\varepsilon_0, \ldots, \varepsilon_\mathrm{N})$ *provides a transfer of orbital integrals if and only if*

$$\varepsilon_{\varkappa_0}(m, n_1)\,\varepsilon_{\varkappa_0}(n_1, p)\,\varepsilon_+(m, n_1)\,\varepsilon_+(n_1, p)=\varepsilon_{\varkappa_0}(m, n_2)\,\varepsilon_{\varkappa_0}(n_2, p)\,\varepsilon_+(m, n_2)\,\varepsilon_+(n_2, p),$$

for each 4-*tuple* (m, n_1, n_2, p) *as above.*

To compute terms, let α'_i be a noncompact root of T_m for which there is a Cayley transform, say s'_i, with respect to α'_i taking $\mathbf{T}'_m$ to $\mathbf{T}'_{n_i}$ $(i=1, 2)$; let α_i, s_i be the images in $\mathbf{G}$. Similarly, let β'_i be a noncompact root of T'_{n_i} for which there is a Cayley transform, say t'_i, with respect to β'_i taking $\mathbf{T}'_{n_i}$ to $\mathbf{T}_p$, and β_i, t_i be the images in $\mathbf{G}$. Thus we have



Note that it may happen that $T_{n_1} = T_{n_2}$. Choose a positive system I^+ for the imaginary roots of T_p and system J_i^+ for T_m adapted to α_i such that $(J_i^+)_{s_i}$ is adapted to β_i and $((J_i^+)_{s_i})_{t_i} = I^+$, $i = 1, 2$. Then the product of all terms in (11.4) of the form $\varepsilon_+(\ ,\ )$ is

$$\varepsilon_+ = \varepsilon(J_1^+, J_2^+)\,\varepsilon((J_1^+)', (J_2^+)'),$$

[*see* Paragraph 10 for the definition of $\varepsilon_+(\ ,\ )$]. Note that ε_+ depends only on the isogeny class of the derived part of $\mathbf{G}$. The same is true for the remaining terms in (11.4), for these are the signatures of the Cayley transforms s_i, t_i ($i = 1, 2$): to compute the signature of, say s_1, choose $\tilde{s}_1$ in the preimage of s_1 in $\tilde{\mathbf{G}}$ (the simply-connected covering of the derived group of $\mathbf{G}$). Then $\tilde{s}_1$ is a Cayley transform in $\mathbf{G}^\sim$ and its signature (regarding $\varkappa_m$ as a character for $\mathbf{G}^\sim$) is the same as that of s_1. Indeed if $\sigma(\tilde{s}_1^{-1})\tilde{s}_1 \in \tilde{t}_\sigma \mathbf{G}_\alpha^\sim$ then $\sigma(s_1^{-1})s_1 \in t_\sigma \mathbf{G}_\alpha$, where t_σ is the image of $\tilde{t}_\sigma$ in $\mathbf{G}$; by definition, $\varkappa_m(\tilde{t}_\sigma) = \varkappa_m(t_\sigma)$. We will write ε_* for the product of the signatures of the s_i, t_i ($i = 1, 2$).

Our second observation is that we need only verify (11.4) in the case that α_1, α_2 are roots for the same simple factor of $\mathbf{G}^\sim$. It remains then to examine the various simple types... here we will examine just the simple systems of rank two (only for the split forms of type C_2, G_2 is there something to prove). For the reduction, we argue as follows. Suppose that α_i is a root for the simple factor $\mathbf{G}_i^\sim$ of $\mathbf{G}^\sim$, $i = 1, 2$. Recalling the comment of the third paragraph of Paragraph 9 we may assume that i_{n_1}, i_{n_2} and i_p have been chosen in such a way that we may take $\alpha_1' = \beta_2'$, $\alpha_2' = \beta_1'$ and $\tilde{s}_1 = \tilde{t}_2$, $\tilde{s}_2 = \tilde{t}_1$ with $\tilde{s}_i$ lying in the factor $\mathbf{G}_i^\sim$ of $\mathbf{G}^\sim$.

Then clearly the $\varkappa_{n_1}$-signature of $\tilde{t}_1$ is the same as the $\varkappa_m$-signature of $\tilde{s}_2$ and the $\varkappa_{n_2}$-signature of $\tilde{t}_2$ is the same as the $\varkappa_m$-signature of $\tilde{s}_1$. This implies that $\varepsilon_* = 1$. On the other hand, the positive systems J_1^+, J_2^+ are equal so that $\varepsilon_+ = 1$ also, as desired.

Suppose now that the Lie algebra of the derived group of G is the split form of type G_2. There will be consistency problems only if H also has split rank two. Since such an H must contain (a copy of) the fundamental Cartan subgroup of G we may restrict our attention to the case that T_0 is a fundamental Cartan subgroup. We list the roots of $\mathbf{T}_0$ as $\alpha = e_1 - e_2$, $\beta = -2e_1 + e_2 + e_3$, $\alpha + \beta$, $3\alpha + \beta$, $3\alpha + 2\beta$ and their negatives, and the dual system as $\alpha^\vee = e_1 - e_2$, $\beta^\vee = 1/3(-2e_1 + e_2 + e_3)$, etc. The possibilities for $\varkappa_0$ are given in the following table:

	$\alpha^\vee$	$\beta^\vee$	$\alpha^\vee + 3\beta^\vee$	$2\alpha^\vee + 3\beta^\vee$	$\alpha^\vee + \beta^\vee$	$\alpha^\vee + 2\beta^\vee$
1	1	1	1	1	1	1
$\varkappa_0^1$	1	−1	−1	−1	−1	1
$\varkappa_0^2$	−1	1	−1	1	−1	−1
$\varkappa_0^3$	−1	−1	1	−1	1	−1

The characters $\varkappa_0^2$, $\varkappa_0^3$ are of the form $(\varkappa_0^1)^\omega$, $\omega \in \Omega(\mathbf{G}, \mathbf{T}_0)$. It follows that we need consider only the case $\varkappa_0 = \varkappa_0^1$. Then, on fixing embeddings of the Cartan subgroups of H into G according to the prescription of Paragraph 6, we can identify α and $3\alpha + 2\beta$ as the roots from $\mathbf{H}$... $\mathbf{H}$ is thus a group of type $A_1 \times A_1$. Note that the condition (8.1) is satisfied. As usual, we will denote the preimage of α by α' and the preimage of $3\alpha + 2\beta$

by $(3\alpha+2\beta)'$. On the Cartan subgroup T_0' both α' and $(3\alpha+2\beta)'$ are noncompact; we may assume that we have labelled the roots of $\mathbf{T}_0$ so that both α and $3\alpha+2\beta$ are noncompact. We compute first the term ε_+. For J_1^+ we must take the positive system with simple roots $3\alpha+2\beta$ and $-(\alpha+\beta)$ and for J_2^+ the system with simple roots α and β; it follows that $\varepsilon_+=1$ (to conform with our earlier notation we write α as α_1 and $3\alpha+2\beta$ as α_2). In computing the signature of s_1, we have only to write s_1 as $s_1'\omega$, where s_1' is a standard transform with respect to α_1 followed by a real conjugation and $\omega\in\Omega(\mathbf{G},\mathbf{T}_0)$ fixes α_1. Then the $\varkappa_0$-signature of s_1 is $\varkappa_0(\omega)$ (*cf.* Paragraph 4). But the only possibilities for ω are 1 and ω_{α_2}, both of which are annihilated by $\varkappa_0$. Similarly all the other signatures to be computed are one and so we obtain $\varepsilon_*=1$ and (11.4) is satisfied.

The case that the Lie algebra of the derived group of G is of type C_2 is more instructive. Again we may assume that T_0 is the fundamental Cartan subgroup. We list the roots of T_0 as $\alpha=e_1-e_2$, $\beta=2e_2$, $\alpha+\beta$, $2\alpha+\beta$ and their negatives and the dual system as $\alpha^\vee=e_1-e_2$, $\beta^\vee=e_2$, etc. The possibilities for $\varkappa_0$ are:

	$\alpha^\vee$	$\beta^\vee$	$\alpha^\vee+2\beta^\vee$	$\alpha^\vee+\beta^\vee$
1	1	1	1	1
$\varkappa_0^1$	1	-1	1	-1
$\varkappa_0^2$	-1	1	-1	-1
$\varkappa_0^3$	-1	-1	-1	1

Only $\varkappa_0^1$ gives a group $\mathbf{H}$ of rank 2. In this case we can identify the roots α and $\alpha+\beta$ as the roots of $\mathbf{H}$. . . $\mathbf{H}$ is again of type $A_1\times A_1$ and the assumption (8 . 1) is satisfied. We may as well take T_0, or, more precisely, its Lie algebra, as in [12] [we are assuming that $\mathfrak{g}$ is $\mathfrak{sp}(2, \mathbf{R})$] and label the roots in the usual way. Then, on $\mathbf{T}_0$, α is compact and $\alpha+\beta$ noncompact, whereas the preimages α', $(\alpha+\beta)'$ are both noncompact. Again to conform with earlier notation we write α as α_1 and $\alpha+\beta$ as α_2. For J_1^+ we must take the system with simple roots $\alpha+\beta$ and $-\beta$ and for J_2^+ the system with simple roots α and β. It follows that $\varepsilon_+=-1$. As before, the signatures of s_2, t_1 and t_2 are all easily shown to be one. We have then to show that s_1 has negative signature. If we write s_1 as $s_1'\omega$, where s_1' is a standard transform with respect to $\underline{\underline{\alpha_2}}$ (noncompact) followed perhaps by a real conjugation and $\omega\in\Omega(\mathbf{G},\mathbf{T}_0)$ takes α_1 to α_2 then $\varkappa_0(\omega)$ is the signature of s_1 (*cf.* Paragraph 4). Clearly ω is either ω_β or $\omega_{2\alpha+\beta}\omega_\alpha$. But $\varkappa_0(\omega_{2\alpha+\beta}\omega_\alpha)=\varkappa_0(\omega_{2\alpha+\beta})$ since α comes from $\mathbf{H}$; both β and $2\alpha+\beta$ are noncompact so that

$$\varkappa_0(\omega_\beta)=\varkappa_0(\beta^\vee)=-1$$

and

$$\varkappa_0(\omega_{2\alpha+\beta})=\varkappa_0((2\alpha+\beta)^\vee)=\varkappa_0(\alpha^\vee+\beta^\vee)=-1.$$

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