BIRGER IVERSEN

Local Chern classes

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LOCAL CHERN CLASSES

BY BIRGER IVERSEN

The purpose of this paper is to give a construction of local Chern classes as conjectured by Grothendieck [6] (XIV 7.2).

The construction is given in the framework of complex vector bundles on topological spaces where it appears as a generalization of the relative Chern classes obtained from the “difference construction” in K-theory notably used by Atiyah ([1]-[4]).

It will be clear that the constructions performed work equally well in other theories, especially the etale cohomology of algebraic geometry.

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1. Introduction

By a complex $K^*$ of vector bundles on a topological space $X$ we understand a finite complex of $\mathbb{C}$-vector bundles each having constant rank. By the support of $K^*$ we understand the complement to the set of points $x \in X$ for which $K_x^*$ is an exact complex of vector spaces.

For a space $X$, $H^*(X; \mathbb{Z})$ denotes integral cohomology in the sense of sheaf theory, $\hat{H}^*(X; \mathbb{Z}) = \prod_i H^i(X; \mathbb{Z})$. For a closed subset we use interchangeably

$$H^*_Z(X; \mathbb{Z}) = H^*(X, X-Z; \mathbb{Z})$$

for cohomology with support in $Z$. 
A theory of local Chern classes consists in assigning to a complex $K'$ on $X$ with support in $Z$ a cohomology class
\[ c^Z(K') \in \hat{H}_Z(X; Z) \]
with the following two properties

1. For a continuous map $f: X \to Y$, closed subsets $Z \subseteq X$, $V \subseteq Y$ with $f(X - Z) \subseteq Y - V$ and a complex $L'$ on $Y$ with support in $V$:
\[ c^Z(f^*L') = f^*c^V(L'). \]

2. Let $r: \hat{H}_Z(X; Z) \to \hat{H}'(X; Z)$ denote the canonical map.
   Then
\[ r(c^Z(K')) + 1 = \prod_i c_i(K^{2i})c_i(K^{2i-1})^{-1}. \]

The main result of this paper is

**Theorem 1.3.** — A theory of local Chern classes exists and is unique.

As usual we introduce a local Chern character
\[ ch^Z(K') \in \hat{H}_Z(X; Q) \]
with the following properties:

1. **Functoriality.** — $f^*ch^Y(L') = ch^Z(f^*L').$
2. $r(ch^Z(K')) = \sum (-1)^i ch(K^i).$
3. **Decalage.** — $ch^Z(K'[1]) = -ch^Z(K').$
4. **Additivity.** — For complexes $K'$ and $L'$ on $X$ with support in $Z$:
\[ ch^Z(K' \oplus L') = ch^Z(K') + ch^Z(L'). \]
5. **Multiplicativity.** — Let $K'$ and $L'$ be complexes on $X$ with support in $Z$ and $V$, respectively. Then
\[ ch^Z(K' \otimes L') = ch^Z(K')ch^Y(L'). \]

The proof of 1.3 is given in paragraphs 2 and 3 while paragraphs 4 and 5 derives multiplicative and additive properties of $c^Z$ and $ch^Z.$

In paragraph 6 we derive Riemann-Roch formulas for the Thom class and paragraph 7 initiates applications to algebraic geometry.
In cases where $X$ is an oriented topological manifold of dimension $n$, Poincaré duality

$$H^i_Z(X; \mathbb{Z}) \cong H_{n-i}(Z; \mathbb{Z})$$

transforms our local cohomology classes into homology classes. In cases where $X$ is a smooth algebraic variety $\mathbb{C}$, this should be compared with the homology classes constructed by means of MacPherson's graph construction [5] compare [10], [14], [16].

It should also be mentioned that Illusie ([13] V.6) has constructed local Chern classes "à la Atiyah" in Hodge cohomology.

I should like to thank K. Suominen for stimulating my interest in these matters.

2. The canonical complex

Throughout this paragraph we shall work with the following data.

A topological space $X$, a sequence of vector bundles $(K^i)_{i \in \mathbb{Z}}$ on $X$ with $K^i = 0$ except for finitely many $i \in \mathbb{Z}$.

$$v_i = \text{rank } K^i.$$

We shall assume that there exists a sequence $(\lambda_i)_{i \in \mathbb{Z}}$ of integers with

$$\lambda_i + \lambda_{i+1} = v_i, \quad i \in \mathbb{Z},$$

$$\lambda_i \geq 0, \quad i \in \mathbb{Z}.$$

Put $K = \oplus_{i \in \mathbb{Z}} K^i$. The flag manifold whose sections are flags in $K$ of nationality $v_i$ will be denoted $Fl_v$. The fixed flag defined by

$$F_i = \oplus_{t \leq i} K^t$$

is denoted $F_i$.

**Definition 2.1.** — $T \subseteq Fl_v$ denote the closed subspace whose sections are flags $D_i$ with the property that

$$F_{i-1} \subseteq D_i \subseteq F_{i+1}, \quad i \in \mathbb{Z}.$$

The canonical projection is denoted $p: T \rightarrow X$. The restriction to $T$ of the canonical flag on $Fl_v$ will be denoted $E_i$. On $T$ we have a canonical complex $C^i$ given by

$$C^i = E_i/p^*F_{i-1},$$

$$\partial^i : E_i/p^*F_{i-1} \rightarrow E_{i+1}/p^*F_{i}$$
is induced by the inclusion $E_i \subseteq E_{i+1}$. \( \delta^{i+1} \delta^i = 0 \) since

$$p^* F_{i-1} \subseteq E_i \subseteq p^* F_{i+1}, \quad i \in \mathbb{Z}. $$

Finally $T_\psi$ is the complement in $T$ of the support of $C$, and $p_\psi: T_\psi \rightarrow X$ denotes the restriction of $p$ to $T_\psi$.

**Lemma 2.2.** — A section of $T$ over $X$ represented by a flag $D$, is a section of $T_\psi$ if and only if for all $x \in X$:

$$\text{rank}(D_{i,x} \cap F_{i,x}/F_{i-1,x}) = \lambda_i.$$  

**Proof.** — By definition $D$, represents a section of $T_\psi$ if and only if the complex

$$\cdots \rightarrow D_{i-1}/F_{i-2} \rightarrow D_i/F_{i-1} \rightarrow D_{i+1}/F_i \rightarrow \cdots$$

has exact fibres for all $x \in X$. Note that $D_i/F_{i-1}$ has rank $v_i$, and the lemma follows from the definition of $(\lambda_i)_{i \in \mathbb{Z}}$.

**Theorem 2.3.** — Let $i_\psi: T_\psi \rightarrow T$ denote the inclusion. Then

$$i_\psi^*: H^*(T; \mathbb{Z}) \rightarrow H^*(T_\psi; \mathbb{Z})$$

is surjective.

**Proof.** — Define

$$G_h = \prod_i \text{Grass}_{\lambda_i}(K^i) \rightarrow X,$$

where $p_i: \text{Grass}_{\lambda_i}(K^i) \rightarrow X$ is the fibre space whose sections are rank $\lambda_i$-subbundles of $K^i$.

$$f_h : T_\psi \rightarrow G_h$$

denotes the map which on the level of sections (compare 2.2) transforms

$$D_i \mapsto (D_i \cap F_i/F_{i-1})_{i \in \mathbb{Z}}.$$

We shall first prove

$$(2.4) \quad f_h^* : H^*(G_h; \mathbb{Z}) \rightarrow H^*(T_\psi; \mathbb{Z})$$

is an isomorphism.

We shall prove that $f_h$ is a fibration with fibres of type $A^d$ ($A^d$: affine space of dimension $d = \sum \lambda_i^2$). For this assume $X = P^i$. The fibre of $f_h$ above $B' \in G_h$ consists of sequences $(G^i)_{i \in \mathbb{Z}}$, where $G^i$ is a $\lambda_{i+1}$-plane in $2\lambda_{i+1}$-space $B^{i+1}/B^i$ intersection the $\lambda_{i+1}$-plane $F_i/B^i$ in zero.
Next define

\[ G_v = \prod_i \text{Grass}_i(K^i \oplus K^{i+1}) \]

and maps

- \( f_v : T \to G_v, \quad D_i \mapsto (D_i/F_{i-1})_{i \in \mathbb{Z}}; \)
- \( g : G_h \to G_v, \quad B_i \mapsto (B^i \oplus B^{i+1})_{i \in \mathbb{Z}}; \)
- \( s_h : G_h \to T_{\psi}; \)
- \( B'_{i \in \mathbb{Z}} \mapsto (\oplus K^i \oplus B^i \oplus B^{i+1})_{i \in \mathbb{Z}}, \)

where in each case the transformation on the level of sections is given.

We have the following diagram

\[
\begin{array}{ccc}
T & \xleftarrow{f_v} & T_{\psi} \\
| & & | \\
f_v \downarrow & & f_h \downarrow s_h \\
G_v & \xleftarrow{g} & G_h
\end{array}
\]

with

\[ f_v s_h = g, \quad f_h s_h = 1 \]

\((f_v s_h \neq g f_h).\)

Let us grant (2.5 below) that \( g^* \) is surjective.

- \( s_h^* f_h^* = 1 \) and whence by 2.4;
- \( f_h^* s_h^* = 1 \), on the other hand;
- \( s_h^* i_h^* f_v^* = g^* \) and whence;
- \( i_h^* f_v^* = f_h^* g^* \). Thus \( i_h^* \) surjective.

Q. E. D.

**Lemma 2.5.** — The map

\[ g : \prod_i \text{Grass}_i K^i \to \prod_i \text{Grass}_i K^i \oplus K^{i+1}, \]

\[ B'_{i \in \mathbb{Z}} \mapsto (B^i \oplus B^{i+1})_{i \in \mathbb{Z}} \]

induces a surjective map \( g^* \) on integral cohomology.

**Proof.** — Let \( P^i \) denote the canonical \( \lambda_i \)-bundle on \( \text{Grass}_i(K^i) \). Consider

\[ H'(\prod_i \text{Grass}_i K^i; \mathbb{Z}) \]

as a \( H'(X; \mathbb{Z}) \)-algebra. As is well known this algebra is generated by the homogeneous components of

\[ \text{pr}_i^* c.(P^i), \quad i \in \mathbb{Z}. \]
Consider the composite of \( g \) and the \( i \)'th projection
\[
\prod_i \text{Grass}_i K^i \to \text{Grass}_i K^i \oplus K^{i+1}
\]
to see that
\[
\text{pr}_i \ast \text{c.}(P^i) \text{pr}_{i+1} \ast \text{c.}(P^{i+1})
\]
and the inverse to that element belongs to the image of \( g^\ast \). It is now clear by decreasing induction that \( \text{pr}_i \ast \text{c.}(P_i) \) and \( \text{pr}_i \ast \text{c.}(P_i)^{-1} \) belong to the image of \( g^\ast \).

Q. E. D.

**Proposition 2.6.** — The \( H^\ast(X; Z) \)-module \( H^\ast(T_{\varphi}; Z) \) is finitely generated free and for any map \( X' \to X \).

\[
H^\ast(T_{\varphi}; Z) \otimes_{H^\ast(X; Z)} H^\ast(X'; Z) \to H^\ast(T_{\varphi} \times_X X'; Z)
\]
is an isomorphism.

*Proof.* — By 2.4 we may replace \( T_{\varphi} \) by a product of Grassmannian bundles for which this is well known.

Q. E. D.

**3. Construction of the local Chern class**

With the notation of paragraph 2 let \( (\partial^i)_{i \in \mathbb{Z}} \) be a family of linear maps \( \partial^i: K^i \to K^{i+1} \) with \( \partial^{i+1} \partial^i = 0, i \in \mathbb{Z} \). Define a flag \( s_\ast (\partial^\ast) \) in \( K = \bigoplus_i K^i \) as follows: \( s_\ast (\partial^\ast) \) is the graph of the map

\[
\bigoplus_{i \leq i} K^i \to \bigoplus_{i > i} K^i,
\]

\((\ldots, k_{i-2}, k_{i-1}, k_i) \mapsto (\partial^i k_i, 0^i, \ldots)\).

Clearly,

\[
F_{i-1} \subseteq s_\ast (\partial^\ast) \subseteq F_{i+1}, \quad i \in \mathbb{Z}.
\]

Thus we may interpret \( s_\ast (\partial^\ast) \) as a section of \( p: T \to X \)

\[
s_\ast (\partial^\ast): X \to T.
\]

Clearly

\[
(3.1) \quad s_\ast (\partial^\ast) \ast C' = (K', \partial^\ast).
\]
Let now $Z \subseteq X$ denote a closed subset such that $\text{Supp} \,(K', \partial') \subseteq Z$ then
\[ s_*(\partial')(X - Z) \subseteq T_{\varphi}. \]

Consider the exact sequence, (2.3):
\[ 0 \to \check{H}^r(T, T_{\varphi}; Z) \to \check{H}^r(T; Z) \to \check{H}^r(T_{\varphi}; Z) \to 0. \]

The image by $i_{\varphi}^*$ of the cohomology class
\[ c_*(C') - 1 = \prod c_i(C^{2i}) c_j(C^{2i-1})^{-1} - 1 \]
is zero since $C'$ is exact on $T_{\varphi}$. Let
\[ \gamma_T \in \check{H}^r(T, T_{\varphi}; Z) \]
denote the cohomology class characterized by
\[ (3.2) \quad r_*(\gamma_T) + 1 = c_*(C'). \]

**Definition 3.3.** — Consider the map induced by $s_*(\partial')$
\[ s_*(\partial')^* : \check{H}^r(T, T_{\varphi}; Z) \to \check{H}^r(T; Z) \]
and define the local Chern class of $(K', \partial')$ supported in $Z$ by
\[ c^Z(K', \partial') = s_*(\partial')^* \gamma_T. \]

**Proof of 1.3.** — Follows from 3.1 and 3.2.

Q. E. D.

As above we consider the cohomology class
\[ \gamma_X \in \check{H}^r(T, T_{\varphi}; Q) \]
characterized by
\[ (3.4) \quad r_*(\gamma_X) = \sum (-1)^i \text{ch}(C^i). \]

**Definition 3.5:**
\[ \text{ch}^Z(K', \partial') = s_*(\partial')^* \gamma_X. \]

The local Chern character thus defined satisfies clearly 1.4-6. Let us remark that \( \text{ch}^Z \) can be derived directly from \( c^Z \) by means of the theory of $\lambda$-rings, compare paragraph 5.
4. Properties of the local Chern character

In this paragraph we shall prove the multiplication property 1.8 of $\text{ch}^Z$. The proof of the additive property 1.7 is similar but simpler and will not be given. Finally, we give some variants of the additive property.

Proof of 1.8. — Let us first note that 1.8 is true if the canonical map

$$H^z_{Z/nY}(X; Z) \rightarrow H^z(X; Z)$$

is injective. We are going to reduce the problem to this case. Let $T = T(K')$ and $S = T(L')$ with a slight abuse of notation. It will now suffice to prove that

$$H^z(T \times S; Z) \rightarrow H^z(S \times T \cup T \times S; Z)$$

is surjective. Here and in the following all products are formed in the category of spaces/\text{X}. $H^z(-)$ denotes integral cohomology. Let us first recall that if $Z \subseteq Y$ is a closed subset of the space $Y$ and if $U \subseteq Y$ is an open subset, then there is a canonical exact sequence

$$\rightarrow H^z_{Z-U}(X) \rightarrow H^z_Z(X) \rightarrow H^z_Z(U) \rightarrow H^z(X) \rightarrow.$$ 

Put $X = S - S_\psi$ and $Y = T - T_\psi$. It follows from 2.6 that the following commutative diagram is exact $[\otimes$ is formed in the category of $H(X)$-modules$]$:

$$\begin{array}{cccccc}
0 & & H^z(S_\psi) & \otimes & H^z_Y(T) & \\
\downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow H^z_X(S \times T) & \rightarrow & H^z(S \times T) & \rightarrow & H^z(S_\psi \times T) & \\
\downarrow & & \downarrow & & \downarrow & \\
0 \rightarrow H^z_X(S_\psi \times T_\psi) & \rightarrow & H^z(S_\psi \times T_\psi) & \rightarrow & H^z_Y(S \times T_\psi) & \\
\downarrow & & \downarrow & & & \\
0 & & 0 & & & \\
\end{array}$$

From this follows that

$$H^z_X(S \times T) \rightarrow H^z_X(S \times T_\psi)$$

is surjective by remarking that $H^z(S) \otimes H^z_Y(T) \rightarrow H^z(S_\psi) \otimes H^z_Y(T)$ is surjective, taking into account the map from $H^z(S) \otimes H^z_Y(T)$ into the kernel of $H^z(S \times T) \rightarrow H^z(S \times T_\psi)$. Next, apply the above long exact sequence to $(S \times T, S \times T_\psi, X \times T)$ to get the exact sequence

$$\rightarrow H^z_X(S \times T) \rightarrow H^z_X(S \times T) \rightarrow H^z_X(S \times T_\psi) \rightarrow$$

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from which we conclude that
\[ H_{X \times Y}^*(S \otimes T) \to H_{X \times T}^*(S \otimes T) \]
is injective. From the following exact sequence and 2.6
\[ \to H_{X \times T}^*(S \otimes T) \to H^*(S \otimes T) \to H^*(S_{\mathbb{P}} \otimes T) \to \]
follows that
\[ H_{X \times T}^*(S \otimes T) \to H^*(S \otimes T) \]
is injective. Compose the last two results and write still another long exact sequence to derive the result.

Q. E. D.

**Proposition 4.1.** — Let \( K'' \) denote a finite double complex on the topological space \( X \). Suppose \( Z \) is a closed subset of \( X \) such that \( K''_p \), has support in \( Z \) for all \( p \in \mathbb{Z} \). Then
\[ \text{ch}^2(\text{tot } K'') = \sum (-1)^p \text{ch}^2(K''_p), \]
where \( \text{tot } K'' \) denotes the total single complex associated to \( K'' \).

**Proof.** — We shall first change notation and let \( K'' \) denote the double indexed family of vector bundles on \( X \) underlying the above double complex. Let \( C(K'') \) denote the fibre space over \( X \) whose sections are pairs \( (\partial', \partial'') \) of endomorphisms of \( K'' \) such that \( (K'', \partial', \partial'') \) form a double complex. Let \( E'' \) denote the canonical double complex on \( C(K'') \) and \( C_{\mathbb{P}} \) the complement of the support of \( \text{tot } E'' \).

Consider now a fixed pair \( (\partial', \partial'') \) as above and assume that \( (K'', 0, \partial'') \) has support in \( Z \). Consider the map of spaces \( X: \)
\[ \theta: X \times \mathbb{A}^1 \to C(K'') \]
which on the section level is given by
\[ t \mapsto (K'', t \partial', \partial''). \]
Clearly
\[ \theta(X - Z) \subseteq C_{\mathbb{P}} \]
and
\[ \theta^*(\text{tot } E'') = \text{tot}(K'', t \partial', \partial''). \]
Conclusion by (1.6), (1.7) and a simple homotopy argument.

Q. E. D.
Corollary 4.2. — Consider an exact sequence of complexes of vector bundles on $X$:

$$0 \to K' \to L' \to M' \to 0$$

and suppose all three complexes have support in the closed subset $Z$ of $X$. Then

$$\text{ch}^Z(L') = \text{ch}^Z(K') + \text{ch}^Z(M').$$

Proof. — Consider an appropriate double complex and apply 4.1 twice.

Q. E. D.

Corollary 4.3. — Let $f: K' \to L'$ be a linear map of complexes on $X$ and let $K'$ and $L'$ have support in $Z$. If for all $x \in X$:

$$H'(f_x): H'(K'_x) \to H'(L'_x)$$

is an isomorphism, then

$$\text{ch}^Z(K') = \text{ch}^Z(L').$$

Proof. — Construct the mapping cone and apply 4.2.

Q. E. D.

5. Formulas without denominators

Let $Z$ be a closed subspace of the space $X$ and consider the commutative graded ring with $1$:

$$Z \oplus \tilde{H}_Z^c(X; Z^+).$$

To this we associate

$$1 + \tilde{H}_Z^c(X; Z)^+ = 1 + \prod_{l \geq 1} H^2_{2l}(X; Z)$$

which is an abelian group under cup product. Recall that $1 + \tilde{H}_Z^c(X; Z)^+$ comes equipped with a product $\star$ with the property

$$(1 + x_m + \text{higher terms}) \star (1 + y_n + \text{higher terms}) =$$

$$1 - \frac{(n+m-1)!}{(m-1)!(n-1)!} x_m y_n + \text{higher terms}$$

(5.1)

[6] (0, App. § 3).
If \( K' \) is a complex on \( X \) with support in \( Z \), we put

\[
\tilde{c}^Z(K') = 1 + c^Z(K'),
\]

\[
\tilde{c}^Z(K') \in 1 + \hat{H}^*_e(X; Z)^+.
\]

With the notation of the corresponding formulas for \( \text{ch}^Z \), 1.4-8, we have

\[(5.2) \quad \tilde{c}^Z(f^*L') = f^*\tilde{c}^Y(L'), \]

\[(5.3) \quad r(\tilde{c}^Z(K')) = \prod_l c.(K^{2l}) c.(K^{2l-1})^{-1}, \]

\[(5.4) \quad \tilde{c}^Z(K'[1]) = \tilde{c}^Z(K')^{-1}, \]

\[(5.5) \quad \tilde{c}^Z(K' \oplus L') = \tilde{c}^Z(K')\tilde{c}^Z(L'), \]

\[(5.6) \quad \tilde{c}^Z \text{odd}(K' \otimes L') = \tilde{c}^Z(K') \star \tilde{c}^Y(L'). \]

These formulas are easily derived by the method developed in paragraph 4. From \textit{loc. cit.} follows

\[(5.7) \quad \text{Suppose } c^Z(K') = a_n + \text{higher terms, then } \text{ch}^Z(K') = 1/(-1)^n (n-1)! a_n + \text{higher terms}. \]

6. Riemann-Roch formula for the Thom class

Let \( \pi: E \rightarrow X \) denote a rank \( n \) vector bundle, and let \( \lambda_E \) denote the canonical complex on \( E \). Recall that \( (\lambda_E)^l = \Lambda^l \pi^*E \). The Koszul complex, i.e. the complex dual to \( \lambda_E \) will be denoted \( \lambda_E^\vee \).

**Proposition 6.1.** — With the above notation

\[ (-1)^n \text{Todd}(E^\vee) \text{ch}^X(\lambda_E) = \text{Todd}(E) \text{ch}^X(\lambda_E^\vee) = \text{Thom class of E}. \]

**Proof.** — Let \( \tilde{E} = \text{Proj}(E \oplus 1) \) and let \( H \) denote the canonical line bundle on \( \tilde{E} \). From the canonical imbedding ([1], p. 100):

\[ H^\vee \subseteq E \oplus 1 \]

we derive the canonical section

\[ s \in \Gamma(\tilde{E}, E \otimes H \otimes H). \]

The projection of \( s \) onto \( E \otimes H \) will be denoted

\[ t \in \Gamma(\tilde{E}, E \otimes H). \]
The zero's of \( t \) all lie on the canonical section \( X \rightarrow \tilde{E} \). Consider the commutative diagram

\[
\begin{array}{ccc}
\tau \in \hat{H}^*_X(E; \mathbb{Q}) & \rightarrow & \hat{H}^*(\tilde{E}; \mathbb{Q}) \\
\downarrow & \Downarrow & \downarrow \\
\tau \in \hat{H}^*_X(E; \mathbb{Q}) & \rightarrow & \hat{H}^*(E; \mathbb{Q})
\end{array}
\]

where \( \tau \) denotes the Thom class. Let us first prove that

\[ \tilde{r}(\tau) = c_n(E \otimes H). \]

For this let us note that \( \tilde{r} \) is injective. Namely, \( H^*(\tilde{E}; \mathbb{Q}) \rightarrow H^*(\tilde{E}-X; \mathbb{Q}) \) is surjective since the restriction to \( \tilde{E}-X \) of

\[ 1, c_1(H), \ldots, c_1(H)^{n-1} \]

form a basis for the \( H^*(X; \mathbb{Q}) \)-module \( H^*(E-X; \mathbb{Q}) \). Note that the restriction of \( c_n(E \otimes H) \) to \( \tilde{E}-X \) is zero because of the section \( t \). Let \( \sigma \in H^*_X(E) \) be such that

\[ \tilde{r}(\sigma) = c_n(E \otimes H). \]

We shall show that \( \sigma \) is the Thom class. For this it suffices to treat the case \( X = P^i \). In this case \( c_n(E \otimes H) = c_1(H)^n \) and the statement is clear.

We shall now prove the first formula. Let \( \lambda^- \) denote the Koszul complex associated with the section \( t \) of \( E \otimes H \). The restriction of \( \lambda^- \) to \( E \) is \( \lambda^-_E \). Let us recall [8], Lemma 18 that for a rank \( n \) bundle \( N \) we have

\[ (6.2) \quad \text{ch}(\lambda^-_N) = c_n(N) \text{Todd}(N)^{-1}. \]

The formula will now follow by applying (1.5) to \( \lambda^- \)

\[ \tilde{r}(\text{ch}^xy^-) = \text{ch}(\lambda^-_E \otimes H) = c_n(E \otimes H) \text{Todd}(E \otimes H)^{-1}, \]

\[ \text{Todd}(E \otimes H)^{-1} \equiv \text{Todd}(E)^{-1} \mod c_1(H), \]

\[ c_n(E \otimes H) c_1(H) = 0 \]

as it follows from the fact that \( t \in \Gamma(\tilde{E}, E \otimes H \otimes H) \) has no zeros. Whence

\[ \tilde{r}(\text{ch}^xy^-) = c_n(E \otimes H) \text{Todd}(E)^{-1}. \]

Q. E. D.

Remark. — The above formula should be considered as generalizations of formulas used in [2], [3], [4].
7. Multiplicity in algebraic geometry

In this paragraph we shall work in the framework of [7] and prove a fundamental relation 7.1 between local Chern classes and the multiplicity of local algebra [15], compare [2], 6.2.

Let $V$ denote a smooth (connected) algebraic variety $\mathbb{C}$ and $X \subseteq V$ a closed subvariety of codimension $d$. The local fundamental class will be denoted

$$\text{cl}^X \in \mathbb{H}^{2d}(V; \mathbb{Z}).$$

The fundamental class of $X$, i.e. the image of $\text{cl}^X$ in $\mathbb{H}^{2d}(V; \mathbb{Z})$ will be denoted

$$\text{cl}(X) \in \mathbb{H}^{2d}(V; \mathbb{Z}).$$

For a coherent (algebraic) sheaf $M$ on $V$ with support in $X$, $l(M)$ denotes the length of the stalk of $M$ at the generic point of $X$.

**Theorem 7.1.** — Let $E'$ denote a complex of locally free coherent (algebraic) sheaves on $V$ with $\text{Supp} (E') \subseteq X$. Then

$$\text{ch}^X(E') = \sum (-1)^k l(H^k(E')) \text{cl}^X + \text{higher terms.}$$

**Proof.** — Let $O$ denote the local ring of $V$ at the generic point of $X$, $m$ denotes the maximal ideal of $O$. Let $K_m(O)$ denote the Grothendieck group of the category of finite complexes of finitely generated free $O$-modules with homology of finite length (modulo exact complexes). We are going to define a topological character

$$l : K_m(O) \to \mathbb{Z}.$$

Recall first that if $U$ is a Zariski open subset of $V$ with $X \cap U \neq \emptyset$, then the restriction map

$$\mathbb{H}^{2d}(V; \mathbb{Z}) \to \mathbb{H}^{2d}_{X \cap U}(U; \mathbb{Z})$$

is an isomorphism which carries $\text{cl}^X$ to $\text{cl}^{X \cap U}$. From this follows that there is a character $l$ as above such that for any complex $E'$ as in the theorem

$$\text{ch}^X(E') = l(E') \text{cl}^X + \text{higher terms.}$$

As is well known $K_m(O) \cong \mathbb{Z}$ since $O$ is a regular local ring [6]. Thus it will suffice to find a resolution $E'$ of $O/m$ by finitely generated free sheaves with $l(E') = 1$. Let us first consider the case $V = \mathbb{A}^d$, $X = \{0\}$. In this case we can take for $E'$ the standard Koszul complex. That $l(E') = 1$ follows from 6.1.
In the general case choose a Zariski open set U of V and $f_1, \ldots, f_d \in \Gamma(U, O_Y)$ which defines $X \cap U$. This defines a map

$$f : U \to \mathbb{A}^d$$

with $f^{-1}(\{0\}) = U \cap X$. It follows that

$$f^* : H^{2d}_{(\emptyset)}(\mathbb{A}^d; \mathbb{Z}) \to H^{2d}_{X \cap U}(U; \mathbb{Z})$$

is an isomorphism. The pull-back of the complex considered before will now do the job.

Q. E. D.

Remark 7.2. — Taking in particular a resolution of the structure sheaf $O_X$ of X we obtain by means of (5.7):

$$c_d(O_X) = (-1)^{d-1}(d-1)! \text{cl}(X)$$


Remark 7.3. Combining 7.1 and 1.8 we obtain Serre’s “alternating Tor-formula” [15] for the topological intersection number.

REFERENCES


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Birger IVERSEN,
Matematisk Institut,
Universitetsparken Ny Munkegade.
8000 Aarhus C,
Dannemark.