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ISOPERIMETRIC CONSTANTS AND THE FIRST EIGENVALUE OF A COMPACT RIEMANNIAN MANIFOLD

BY SHING-TUNG YAU *

Given a compact Riemannian manifold M_1 the Poincaré inequality tells us that for any smooth function f defined on M with $\int_M f = 0$, we can estimate $\int_M f^2$ in terms of $\int_M |\nabla f|^2$. By the minimax-principle, one knows that the best possible constant in the Poincaré inequality is given by the first eigenvalue of the Laplacian. While this constant has analytic importance, it also gives strong insight in the geometry of the manifold.

The upper estimate of the first eigenvalue has been discussed by many authors. J. Hersch [11] obtained an upper bound for manifolds homeomorphic to the two sphere. J. Cheeger [5], I. Chavel and E. Feldman [4] (*see* also Mazet [12] for generalization) obtained upper bound for manifolds with non-negative Ricci curvature. The comparison theorem of S. Y. Cheng [8] gives s sharp upper bound for general Riemannian manifold in terms of the lower bound of the Ricci curvature and the diameter.

While these progress had been made on the upper bound, not too many is known about the lower bound of the first eigenvalue. The best result is due to Lichnerowicz (*see* [1]) who gives a computable sharp lower bound for manifolds whose Ricci curvature is bounded from below by a positive constant. J. Cheeger [6] also gives a lower bound for general manifolds in terms of some isoperimetric constants. These constants of Cheeger, however, are not readily computable. Cheng [7] has observed that if the manifold is a two dimensional convex surface, then the isoperimetric constant has a lower bound in terms of the diameter.

In this paper, we give a computable lower bound of the first eigenvalue of a general Riemannian manifold in terms of the lower bound of the Ricci curvature, the upper bound of the diameter and the lower bound of the volume. Actually, we have obtained the lower bound for the isoperimetric constants similar to those of Cheeger. We can also obtain a Sobolev inequality for a general Riemannian manifold.

The first half of this paper is devoted to studying the two dimensional surfaces where better estimates for the isoperimetric constants are obtained. The second half of the paper is the proof of the main theorem. We shall come back to discuss the main theorem in a latter occasion.

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1. Isoperimetric constants for two dimensional manifolds with genus > 1

Let M be an n-dimensional compact manifold (with boundary). Then following [6], we define the *isoperimetric constant* $h(\mathrm{M})$ of M as follows:

(a) When $\partial \mathrm{M} = \varphi$, set

$$h(\mathrm{M}) = \inf \frac{\mathrm{A}(\mathrm{H})}{\min(\mathrm{V}(\mathrm{M}_1), \mathrm{V}(\mathrm{M}_2))} \qquad \text{where} \quad \mathrm{A}(\ ),$$

denotes the $(n-1)$-dimensional measure, $\mathrm{V}(\ )$ denotes the n-dimensional measure and the inf is taken over all smoothly embedded hypersurfaces H dividing M into two submanifolds M_1, M_2 with common boundary $\partial \mathrm{M}_1 = \partial \mathrm{M}_2 = \mathrm{H}$.

(b) If $\partial \mathrm{M} \neq \varphi$, set

$$h(\mathrm{M}) = \inf \frac{\mathrm{A}(\mathrm{H})}{\mathrm{V}(\mathrm{M}_1)} \qquad \text{where} \quad \mathrm{H} \cap \partial \mathrm{M} = \varphi$$

and M_1 is the unique submanifold of M with $\partial \mathrm{M}_1 = \mathrm{H}$.

THEOREME 1. — *Let* M *be any compact manifold. Then:*

(a) *If* $\partial \mathrm{M} = \varphi$, $\inf \left(\int_{\mathrm{M}} |\nabla f| \right) \left(\int_{\mathrm{M}} |f| \right)^{-1} = h(\mathrm{M})$ *where* inf *is taken over all* C^1-*function with* $\int_{\mathrm{M}} f = 0$.

(b) *If* $\partial \mathrm{M} \neq \varphi$, $h(\mathrm{M}) = \inf \left(\int_{\mathrm{M}} |\nabla f| \right) \left(\int_{\mathrm{M}} |f| \right)^{-1}$ *where* inf *is taken over all* C^1-*function vanishing on the boundary of* M.

Proof. — We only consider the case $\partial \mathrm{M} = \varphi$. The other case $\partial \mathrm{M} \neq \varphi$ is similar.

As above, let H divide M into two parts M_1 and M_2. Suppose $\mathrm{Vol}(\mathrm{M}_1) \leqq \mathrm{Vol}(\mathrm{M}_2)$. Then for $\varepsilon > 0$, define a function f_ε as follows. For $x \in \mathrm{M}_1$, we define $f_\varepsilon (x = r/\varepsilon$ when x has distance $r \leqq \varepsilon$ from H and $f_\varepsilon(x) = 1$ when x has distance $r > \varepsilon$ from H. For $x \in \mathrm{M}_2$, we define $f_\varepsilon(x) = -cr/\varepsilon$ when x has distance $r \leqq \varepsilon$ from H and $f_\varepsilon(x) = -c$ when x has distance $r > \varepsilon$ from H. Here c is chosen so that $\int_{\mathrm{M}} f_\varepsilon = 0$.

Clearly when $\varepsilon \to 0$, c tends to the constant $\mathrm{Vol}(\mathrm{M}_1)/\mathrm{Vol}(\mathrm{M}_2)$. Furthermore, $\int_{\mathrm{M}} |\nabla f_\varepsilon|$ tends to $(1+c)\,\mathrm{A}(\mathrm{H})$ and $\int_{\mathrm{M}} |f_\varepsilon|$ tends to $\mathrm{Vol}(\mathrm{M}_1) = c\,\mathrm{Vol}(\mathrm{M}_2)$. Therefore,

$$\lim_{\varepsilon \to 0} \left(\int_{\mathrm{M}} |\nabla f_\varepsilon| \right) \left(\int_{\mathrm{M}} |f_\varepsilon| \right)^{-1} \leqq h(\mathrm{M}).$$

The other inequality of theorem 1 follows from a direct application of the co-area formula (*see* [14], Theorem 3.2.22, and [6]).

COROLLARY (Cheeger [6]). — *Let* λ_1 *be the first eigenvalue of the Laplacian of* M *with the first eigenfunction f satisfying* $f^* df \mid \partial \mathrm{M} = 0$. *Then* $\lambda_1 \geqq (1/4)\, h(\mathrm{M})^2$.

Proof. — This follows by applying the Schwarz inequality and

$$\lambda_1 \int_{\mathrm{M}} f^2 = \int_{\mathrm{M}} f \Delta f = \int_{\mathrm{M}} |\nabla f|^2$$

to theorem 1.

Because of this corollary, it is interesting to find a lower estimate of $h(\mathrm{M})$ in terms of some other more computable geometric invariants.

In this section, we shall be only interested in *two dimensional* manifolds so that H can be written as a disjoint union of smoothly embedded circles $\bigcup_{i=1}^{m} \mathrm{S}_i$.

Our first observation is that we can assume both M_1 and M_2 are connected. This is based on the following argument.

First of all, assume $\partial \mathrm{M} = \varphi$. Suppose we have found a number $c(\mathrm{M})$ depending only on some geometric quantity of M such that whenever both M_1 and M_2 are connected,

$$\frac{\sum_{i=1}^{m} l(\mathrm{S}_i)}{\min(\mathrm{V}(\mathrm{M}_1), \mathrm{V}(\mathrm{M}_2))} \geqq c(\mathrm{M}). \tag{1.1}$$

Then we claim that (1.1) holds without any assumption on M_1 and M_2. We shall prove this claim by induction on m.

If $m = 1$, then both M_1 and M_2 are connected and (1.1) is valid. Therefore, it remains for us to prove that if (1.1) is valid up to m, it is still valid for $m+1$. If both M_1 and M_2 are connected, then the last statement is valid by hypothesis. Otherwise we can assume M_1 is disconnected and $\mathrm{M}_1 = \mathrm{N} \cup \mathrm{P}$ where $\partial \mathrm{N} = \mathrm{S}_1 \cup \ldots \cup \mathrm{S}_k$ and $\partial \mathrm{P} = \mathrm{S}_{k+1} \cup \ldots \cup \mathrm{S}_{m+1}$ where $1 \leqq k \leqq m$.

Since $\mathrm{M} = \mathrm{N} \cup (\mathrm{P} \cup \mathrm{M}_2)$, we can apply the induction hypothesis and conclude that

$$\sum_{i=1}^{k} l(\mathrm{S}_i) \geqq c(\mathrm{M}) \min(\mathrm{V}(\mathrm{N}), \mathrm{V}(\mathrm{P}) + \mathrm{V}(\mathrm{M}_2)). \tag{1.2}$$

Similarly, we have

$$\sum_{i=k+1}^{m+1} l(\mathrm{S}_i) \geqq c(\mathrm{M}) \min(\mathrm{V}(\mathrm{P}), \mathrm{V}(\mathrm{N}) + \mathrm{V}(\mathrm{M}_2)). \tag{1.3}$$

Combining (1.2) and (1.3), we conclude immediately

$$\sum_{i=1}^{m+1} l(\mathrm{S}_i) \geqq c(\mathrm{M}) \min(\mathrm{V}(\mathrm{M}_1), \mathrm{V}(\mathrm{M}_2)), \tag{1.4}$$

which proves our claim.

In case $\partial M \neq \varphi$, the argument is similar and will be omitted. For a latter purpose, we record the following theorem of Burago and Zalgaller ([3], p. 100).

THEOREM 2. — *Let* M *be a two dimensional manifold which is homeomorphic to the disk. Then*

$$V(M) \leqq l(\partial M) R + \frac{1}{2}\left(\int_M K^+ - 2\pi\right) R^2, \tag{1.5}$$

where K^+ *is the positive part of the curvature of* M *and* $R = \sup_{x \in M} d(x, \partial M)$ *is the radius of the largest disk inscribed in* M.

COROLLARY (A. Huber, [10]). — *Let* M *be a two dimensional manifold which is homeomorphic to the disk. Then*

$$2\left(2\pi - \int_M K^+\right) V(M) \leqq [l(\partial M)]^2. \tag{1.6}$$

Basing on this theorem of Huber, we derive the following:

PROPOSITION 1. — *Let* M *be a two dimensional compact manifold. Let* $\bar{h}(M)$ *be a constant associated to* M *which is defined in the same way as* $h(M)$ *except that we require all the* S_i' s *are non-homotopically trivial and* M_1, M_2 *are connected. Then*

$$h(M) \geqq \min\left[\bar{h}(M), \sqrt{2\left(2\pi - \int_M K^+\right)} \cdot \frac{1}{\sqrt{V(M)}}\right]. \tag{1.7}$$

Proof. — If all the S_i' s are non-homotopically trivial, then (1.7) is trivial. Otherwise, assume S_1 is homotopically trivial.

Let D be the unit disk and $f : D \to M$ be a continuous map such that f maps ∂D bijectively onto S_1. Let $\pi : \tilde{M} \to M$ be the universal covering of M. Then we have a continuous map $\tilde{f} : D \to \tilde{M}$ such that $\pi \circ \tilde{f} = f$. Since $f \mid \partial D$ is injective, $\tilde{f}(\partial D)$ is a Jordan curve in $\tilde{M}$ such that $\pi \mid \tilde{f}(\partial M)$ is injective.

By the Jordan curve theorem, $\tilde{f}(\partial D)$ bounds a domain $\tilde{D}$ in $\tilde{M}$ such that $\tilde{D}$ is homeomorphic to the disk. If $\tilde{D}$ is not a component of both $\pi^{-1}(M_1)$ and $\pi^{-1}(M_2)$, then for some $i > 1$, a component of $\pi^{-1}(S_i)$ lies completely in $\tilde{D}$. This component must be an embedded circle which bounds a domain $\tilde{D}_i$ in $\tilde{D}$. Continuing in this way, we can find a domain $\tilde{D}_j$ in $\tilde{D}$ such that $\tilde{D}_j$ is homeomorphic to the disk and $\tilde{D}_j$ covers M_1 or M_2. Therefore, either M_1 or M_2 is a disk and $\partial M_1 = \partial M_2 = S_1$. (We use the fact that M_1 and M_2 are connected.) The inequality (1.7) then follows by applying Huber's theorem to this domain.

COROLLARY. — *Let* M *be a compact two dimensional Riemannian manifold. Let* $l(M)$ *be the infinimum of the lengths of all homotopically non-trivial closed geodesics in* M. *Then*

$$h(M) \geqq \frac{1}{\sqrt{V(M)}} \min\left[\frac{l(M)}{\sqrt{V(M)}}, \sqrt{2\left(2\pi - \int_M K^+\right)}\right]. \tag{1.8}$$

2. Isoperimetric inequalities for doubly connected region

In this section, we shall consider doubly connected régions. They are fundamental to the study of surfaces because they are the "handles" for building up surfaces. We start with the following:

PROPOSITION 2. — *Let* M *be a doubly connected surface bounded by two simple closed curves* σ_1 *and* σ_2. *Suppose the curvature of* M *is non-positive. Then*

$$V(M) \leqq [l(\sigma_1)+l(\sigma_2)]\left[\frac{1}{2}l(\sigma_1)+\frac{1}{2}l(\sigma_1)+d(M)\right], \tag{2.1}$$

where $d(M)$ *is the shortest distance between* σ_1 *and* σ_2.

Proof. — It is well-known that every doubly connected surface is conformally equivalent to an annulus $\{z \mid R_1 \leqq |z| \leqq R_2\}$ in the plane. Hence, we can transform part of the problem to the annulus. We first note the following

LEMMA 1. — *Let* ρ *be a smooth subharmonic function defined on the annulus* $\{z \mid R_1 \leqq |z| \leqq R_2\}$. *Then the function*

$$\log(f(r)) = \log\left[\int_0^{2\pi}[\exp\rho(r,\theta)]\,r\,d\theta\right]$$

is a convex function of $\log r$ *so that*

$$f(r) \leqq \max(f(R_1), f(R_2)). \tag{2.2}$$

Proof. — Let $r = e^s$. Then direct computations shows:

$$\begin{aligned}\frac{d^2}{ds^2}(\log f) &= \frac{1}{f}\int_0^{2\pi} e^{\rho+s}\left[\left(\frac{\partial\rho}{\partial s}+1\right)^2+\frac{\partial^2\rho}{\partial s^2}\right]d\theta-\frac{1}{f^2}\left[\int_0^{2\pi} e^{\rho+s}\left(\frac{\partial\rho}{\partial s}+1\right)d\theta\right]^2 \\ &\geqq \frac{1}{f}\int_0^{2\pi} e^{\rho+s}\left[\left(\frac{\partial\rho}{\rho s}+1\right)^2-\frac{\partial^2\rho}{\partial\theta^2}\right]d\theta-\frac{1}{f^2}\left[\int_0^{2\pi} e^{\rho+s}\left(\frac{\partial\rho}{\rho s}+1\right)d\theta\right.^2 \\ &= \frac{1}{f}\int_0^{2\pi} e^{\rho+s}\left[\left(\frac{\partial\rho}{\partial s}+1\right)^2+\left(\frac{\partial\rho}{\partial\theta}\right)^2\right]d\theta-\frac{1}{f^2}\left[\int_0^{2\pi} e^{\rho+s}\left(\frac{\partial\rho}{\partial s}+1\right)d\theta\right]^2,\end{aligned} \tag{2.3}$$

where the last inequality is a consequence of partial integration.

By Schwarz's inequality it is clear that $(d^2/ds^2)(\log f) \geqq 0$ and lemma 1 is proved.

With lemma 1 we can prove proposition 3 in the following way.

Let τ be a geodesic segment joining σ_1 and σ_2 such that $l(\tau)$ realizes the distance between σ_1 and σ_2. Then cut M along this geodesic segment so that the resulting surface $\overline{M}$ is simply connected.

Let $p \in \overline{M}$ be such that $d(p, \partial\overline{M}) = \sup_{x \in \overline{M}} d(x, \partial\overline{M})$. Then we claim that

$$2d(p, \partial\overline{M}) \leqq \max(l(\sigma_1), l(\sigma_2)). \tag{2.4}$$

In fact, let β be a closed curve passing through p which is not homologous to zero in M and which has least length among all these curves. Clearly, β must cut the geodesic τ at some point and hence

$$2d(p, \partial\overline{M}) \leqq l(\beta). \tag{2.5}$$

On the other hand, lemma 1 shows that passing through p, we can find a closed curve which is non-homologous to zero and which has length not gearter than $\max(l(\sigma_1), l(\sigma_2))$. The claim follows from this observation.

Using theorem 2, we see that

$$\begin{aligned} V(\overline{M}) &\leqq \frac{1}{2} l(\beta)[l(\sigma_1)+l(\sigma_2)+2d(\tau)] \\ &\leqq [l(\sigma_1)+l(\sigma_2)]\left[\frac{1}{2}l(\sigma_1)+\frac{1}{2}l(\sigma_2)+d(M)\right]. \end{aligned} \tag{2.6}$$

Proposition 2 follows immediately from this.

COROLLARY. — *Let* M *be a compact doubly connected surface with non-positive curvature. Then*

$$h(M) \geqq \min\left[\frac{1}{2d(M)}, \frac{2d(M)}{V(M)}, \frac{\sqrt{4\pi}}{\sqrt{V(M)}}\right]. \tag{2.7}$$

Proof. — It is elementary to see from Proposition 2 that

$$\bar{h}(M) \geqq \min\left(\frac{1}{2d(M)}, \frac{2d(M)}{V(M)}\right), \tag{2.8}$$

so that (2.7) follows from (1.7).

By giving further restriction on the curvature, one can remove the dependence of $d(M)$ in (2.1). First we prove

LEMMA 2. — *Let* M *be a simply connected two dimensional manifold bounded by a simple closed curve* σ_1. *Suppose the curvature of* M *is bounded from above by* $-c$ *with* $c > 0$. *Then*

$$V(M) \leqq \frac{2}{\sqrt{c}} l(\sigma_1) \sin h^{-1}\left(\frac{\sqrt{c}\, l(\sigma_1)}{\pi}\right). \tag{2.9}$$

Proof. — We shall prove (2.9) by applying theorem 2 again.

Let $p \in M$ be a point such that $d(p, \partial K) = R$ is the radius of the largest disk that can be inscribed in M. Let D be the disk of radius R and center p. Then an application of the Gauss-Bonnet theorem shows that every point in D can be joined by a unique minimal geodesic to p. (One has also to use the fact that M is simply connected.)

By an application of the comparison theorem (*see* [2]) one can now prove that the area of D is not less than $(2\pi/\sqrt{c})\int_0^R (\sin h\sqrt{c}\,x)\,dx$. Applying theorem 2, we conclude therefore

$$\frac{2\pi}{\sqrt{c}}\int_0^R (\sin h\sqrt{c}\,x)\,dx \leqq l(\sigma_1)\,\mathrm{R}. \tag{2.10}$$

Since

$$\frac{1}{2}\sin h\frac{\sqrt{c}\,\mathrm{R}}{2} \leqq \frac{1}{\mathrm{R}}\int_0^R (\sin h\sqrt{c}\,x)\,dx, \tag{2.11}$$

we conclude from (2.10) that

$$\mathrm{R} \leqq \frac{2}{\sqrt{c}}\sin h^{-1}\left(\frac{\sqrt{c}\,l(\sigma_1)}{\pi}\right). \tag{2.12}$$

The conclusion (2.9) then follows from theorem 2 and (2.12).

Let us now assume that M is doubly connected, bounded by two simple closed curves σ_1 and σ_2. Then by the minimizing procedure, we can find a simple closed curve σ in M which is non-homologous to zero and has the shortest length compared with any other such curves.

By using the Jordan curve theorem, one can prove that σ divides M into two parts M_1 and M_2. If σ touches both σ_1 and σ_2, then both M_1 and M_2 are essentially simply connected domains whose boundaries have length not greater than $\max(l(\sigma_1)+l(\sigma), l(\sigma_2)+l(\sigma))$. (Essentially simply connected domain means that we can slightly deform σ at certain points of its intersection with σ_i to form a simple closed curve.) Then we can conclude from lemma 2 that

$$\mathrm{V}(\mathrm{M}) \leqq \frac{4}{\sqrt{c}}[l(\sigma_1)+l(\sigma_2)]\sin h^{-1}\left[\frac{\sqrt{c}(l(\sigma_1)+l(\sigma_2))}{\pi}\right]. \tag{2.13}$$

It remains to discuss the case where σ intersects σ_1 only. Let M_1 be the part of M which contains σ_2. Then taking the normal of σ to be the one which points inside M_1, one can see from the extremal property of σ that σ has non-negative geodesic curvature everywhere.

The last fact enables us to assert the following statement from the Gauss-Bonnet theorem: For any point $p \in \mathrm{M}_1$, there is at most one shortest geodesic joining p and σ.

Let $\mathrm{N} = \{p \in \mathrm{M}_1 \mid d(p, \sigma) \leqq \inf_{x \in \sigma_2} d(x, \sigma)\}$. Then N is homeomorphic to the annulus $\{z \in \mathrm{C} \mid 1 \leqq |z| \leqq \mathrm{R}\}$ with the inner circle corresponding to σ. Furthermore, on the annulus, the metric of N is given by

$$ds^2 = dr^2 + f(r, \theta)^2\, d\theta^2. \tag{2.14}$$

The condition that σ has non-negative geodesic curvature tells us that

$$\frac{\partial f}{\partial r} \geqq 0, \tag{2.15}$$

for $r = 1$.

On the other hand, direct computation shows that the curvature of (2.14) is given by $-(1/f)(\partial^2 f/\partial r^2)$. Hence

$$\frac{\partial^2 f}{\partial r^2} \geqq cf. \tag{2.16}$$

Consider the following function

$$g(r) = \left[\int_1^r \int_0^{2\pi} f(t, \theta)\, d\theta\, dt\right]\left(\int_0^{2\pi} f(r, \theta)\, d\theta\right)^{-1}. \tag{2.17}$$

It must attain its maximum at some point $r_0 > 1$. At this point,

$$0 \leqq g'(r_0) = 1 - g(r_0)\left(\int_0^{2\pi} \frac{\partial f}{\partial r}(r_0, \theta)\, d\theta\right)\left(\int_0^{2\pi} f(r_0, \theta)\, d\theta\right)^{-1}. \tag{2.18}$$

An easy application of (2.15) and (2.16) shows that

$$\frac{\partial f}{\partial r}(r_0, \theta) \geqq c \int_1^{r_0} f(r, \theta)\, dr. \tag{2.19}$$

Hence, (2.18) implies

$$c g(r_0)^2 \leqq 1. \tag{2.20}$$

The formula (2.20) happens at the maximum point of g, we conclude therefore,

$$\int_1^r \int_0^{2\pi} f(t, \theta)\, d\theta\, dt \leqq \frac{1}{\sqrt{c}} \int_0^{2\pi} f(r, \theta)\, d\theta, \tag{2.21}$$

for all $1 \leqq r \leqq \mathrm{R}$.

Let σ_3 be the curve which forms the other part of the boundary of N. Then (2.21) means

$$\mathrm{V}(\mathrm{N}) \leqq \frac{l(\sigma_3)}{\sqrt{c}}. \tag{2.22}$$

Our next step is to see that $l(\sigma_3) \leqq l(\sigma_2)$. Let $\tilde{\mathrm{N}}_1$ be the sets of points of M_1 which can be joined to σ by a shortest geodesic. Then $\tilde{\mathrm{N}}_1$ is a closed subset of M_1 where the metric tensor on $\tilde{\mathrm{N}}_1$ can be written in the form (2.14). We assert that $\partial\tilde{\mathrm{N}}_1$ is equal to $\partial\mathrm{M}$ by replacing certain parts of $\partial\mathrm{M}_1$ by minimal geodesics.

It suffices to prove the following statement: Let $q \in \partial\tilde{N}_1$ be a point in the interior of M_1 and let τ be the minimal geodesic joining q and σ. Suppose q_1 is the first point where τ intersects σ_2 and q_2 is the last point beyond q where τ intersects σ_2. Then the part of τ lies in between q_1 and q_2 is part of the boundary of $\tilde{N}_1$. In fact, if there were a point $\bar{q}$ in this part of τ which is an interior point of $\tilde{N}_1$, then we can join all points near $\bar{q}$ to σ by a minimal geodesic so that we have a strip covering τ. This contradicts the existence of q_1.

From our assertion, we conclude that $\partial\tilde{N}_1$ is rectifiable and $l(\partial\tilde{N}_1\setminus\sigma) \leqq l(\sigma_2)$. On the other hand, since on $\tilde{N}_1$, we have the representation (2.14) with $(\partial f/\partial r) \geq 0$, we obtain $l(\sigma_3) \leqq l(\partial\tilde{N}_1\setminus\sigma) \leqq l(\sigma_2)$.

Putting lemma 2, (2.22) and the last fact together, we have

$$V(M_1) \leqq \frac{l(\sigma_2)}{\sqrt{c}} + \frac{4}{\sqrt{c}}\, l(\sigma_2) \sin h^{-1}\left(\frac{2\sqrt{c}\, l(\sigma_2)}{\pi}\right). \tag{2.23}$$

THEOREM 3. — *Let* M *be a doubly connected compact two dimensional manifold bounded by two simple closed curves* σ_1 *and* σ_2. *Suppose the curvature of* M *is bounded from above by* $-c$ *with* $c > 0$. *Then*

$$\begin{aligned} V(M) \leqq\ & \frac{l(\sigma_1)+l(\sigma_2)}{\sqrt{c}} + \frac{4}{\sqrt{c}}\, l(\sigma_1) \sin h^{-1}\left(\frac{2\sqrt{c}\, l(\sigma_1)}{\pi}\right) \\ & + \frac{4}{\sqrt{c}}\, l(\sigma_2) \sin h^{-1}\left(\frac{2\sqrt{c}\, l(\sigma_2)}{\pi}\right). \end{aligned} \tag{2.24}$$

Remark. — An optimal result seems to be that $\sqrt{c}\, V(M)/[l(\sigma_1)+l(\sigma_2)]$ is bounded by an absolute constant. This can be checked for many important cases.

Finally, we generalize proposition 2 in the following manner.

THEOREM 4. — *Let* M *be a doubly connected compact two dimensional Riemannian manifold bounded by two simple closed curves* σ_1 *and* σ_2. *Suppose the curvature of* M *is bounded from above by a positive constant* $c > 0$. *Then if* $\int K^+ < \pi$ *and* $c\bar{d}(M)^2 < 2$ *with* $\bar{d}(M) = \sup\{d(x, \sigma_1), d(x, \sigma_2)\}$:

$$V(M) \leqq \frac{l(\sigma_1)+l(\sigma_2)}{2-c\bar{d}(M)^2}\left[l(\sigma_1)+l(\sigma_2)+2\bar{d}(M)\right] + \frac{l^2(\sigma_1)+l^2(\sigma_2)}{2\left(2\pi - \int_M K^+\right)}. \tag{2.25}$$

Proof. — The proof will depend on a combination of arguments that we gave in proposition 2 and theorem 3.

Let σ be a single closed curve in M which is non-homologous to zero in M and which has shortest length among these curves. As in theorem 3, we can assume σ intersects σ_1 only.

Let $\tilde{N}_1$ and M_1 be the sets defined as in theorem 3 and $\sigma^{\perp}$ be the part of the normal bundle of σ where all the normal vectors point inside M_1. Then we claim that every point in $\tilde{N}_1$ can be joined to σ by a *unique* minimal geodesic.

By the Rauch-Warner [15] comparison theorem and the fact that σ has non-negative geodesic curvature, we know that the focal length of σ is not less than $(\pi/2\sqrt{c}) \geqq \bar{d}(M)$. Let q be a point in $\tilde{N}_1$ which is closest to σ and which can be joined to σ by two distinct minimal geodesics τ_1 and τ_2. Then we claim that, by the argument of Klingberg [24] τ_1 and τ_2 form a smooth geodesic at q. In fact, let $\bar{\tau}_1, \bar{\tau}_2 : [0, l] \to \sigma^{\perp}$ be the preimagesof τ_1 and τ_2 under the exponential map where $\{ \tau_1(0), \tau_2(0) \} \subset \sigma$ and $\exp(\tau_1(l) = \exp(\tau_2(l)) = q$. Let $\bar{\beta}_1, \bar{\beta}_2$ be two curves in $\sigma^{\perp}$ which pass through $\tau_1(l), \tau_2(l)$ respectively and have distance equal to l from σ. Then by our knowledge about the focal length, the images of $\bar{\beta}_1$ and $\bar{\beta}_2$ under the exponential map are smooth. If τ_1 and τ_2 do not form a smooth geodesic at q, the images of $\bar{\beta}_1$ and $\bar{\beta}_2$ will meet transversally at q. Then it is not hard to find a point $\bar{q}$ near q which can be joined to σ by two distinct minimal geodesic and whose distance from σ is less than $d(q, \sigma)$. This contradicts the minimality of l. Hence, τ_1, τ_2 and part of σ form a simply connected region with exactly two corners. As these two corners have angle $\pi/2$, the Gauss-Bonnet theorem shows that $\int_M K^+ \geqq \pi$, a contradiction.

Therefore, $\tilde{N}_1$ is isometric to the domain $D = \{ z \in C \mid 1 \leqq |z| \leqq g(\arg z) \}$ with the metric $dr^2 + f^2(r, \sigma)\, d\sigma^2$. Here the curve σ is mapped to the circle $|z| = 1$.

For each θ, let r_0 be a point where $f(r, \theta)$ attains its maximum in $[1, g(\theta)]$. Then $(\partial f/\partial r)(r_0, \theta) \geqq 0$ and as in theorem 3,

$$
\begin{aligned}
(2.26) \qquad f(g(\theta), \theta) - f(r_0, \theta) &\geqq -c \int_{r_0}^{g(\theta)} \int_{r_0}^{t} f(s, \theta)\, ds\, dt \\
&\geqq -\frac{c}{2}(g(\theta) - r_0)^2 f(r_0, \theta).
\end{aligned}
$$

Therefore, if $1 > c(g(\theta) - 1)^2/2$,

$$
(2.27) \qquad \sup_{1 \leqq r \leqq g(\theta)} f(r, \theta) \leqq \frac{2}{[2 - c(g(\theta) - 1)^2]} f(g(\theta), \theta).
$$

It is easy to see from (2.26) that passing through any point in $\tilde{N}_1$, we can find a simple closed curve which is non-homologous to zero and which has length not greater than $(2/2 - c\bar{d}(M)^2)\, l(\partial\tilde{N}_1 \setminus \sigma)$. The proof of proposition 2 then shows

$$
(2.28) \qquad V(\tilde{N}_1) \leqq \frac{l(\partial\tilde{N}_1 \setminus \sigma)}{2 - c\bar{d}(M)^2} [l(\partial\tilde{N}_1) + 2d(M)].
$$

Since $l(\partial\tilde{N}_1 \setminus \sigma) \leqq l(\sigma_2)$ (*see* the proof of theorem 3), one can deduce (2.24) from from (2.28) as we did in theorem 3.

3. Isoperimetric constant for the torus

In this section, we apply the isoperimetric inequality in paragraph 2 to a compact surface M with genus one.

Let $S_1, \ldots, S_k$ be disjoint simple closed curves in M whose union decomposes M into two connected domains M_1 and M_2. Then the addition formula for euler numbers shows that $\chi(M_1)+\chi(M_2) = 0$.

In order to compute the number $\bar{h}(M)$ we defined in paragraph 1, we can assume both M_1 and M_2 are not homeomorphic to the disk so that $\chi(M_1) = \chi(M_2) = 0$. Hence, both M_1 and M_2 are doubly connected surfaces bounded by two simple closed curves S_1 and S_2.

At this point, there are two ways to attack the problem. One is to apply theorem 4 to M_1 or M_2 directly. The other is to apply theorem 4 to find a lower bound for $l(S_1)$ and $l(S_2)$. Since the latter gives more information, we prefer to estimate a lower bound for $l(S_1)$ and $l(S_2)$.

Let σ be a shortest closed geodesic in M. Then we have to estimate $l(\sigma)$. We cut the torus along the simple closed curve σ to obtain a doubly connected surface $\overline{M}$ whose boundary consists of two isometric copies of σ. It is an easy exercise to see that the quantities $\bar{d}(\overline{M})$ and $d(\overline{M})$ defined in theorem 4 is bounded from above by $2\,d(M)$ where $d(M)$ is the diameter of M.

Let c be an upper bound of the curvature of M. Then if $\int_M K^+ < \pi$ and $c\,d(M)^2 < 1/2$, theorem 4 implies

$$V(M) \leqq \frac{2\,l(\sigma)}{1-2\,c\,d(M)^2}\left[l(\sigma)+2\,d(M)\right]+\frac{l(\sigma)^2}{\left(2\pi-\int_M K^+\right)}. \tag{3.1}$$

Since we also have $l(\sigma) \leqq 2\,d(M)$, we conclude

$$l(\sigma) \geqq \frac{V(M)}{2\,d(M)}\left[4(1-2\,c\,d(M)^2)^{-1}+\left(2\pi-\int_M K^+\right)^{-1}\right]^{-1}. \tag{3.2}$$

Under the assumption $\int_M K^+ < \pi$, the Gauss-Bonnet theorem shows that every non-trivial simple closed geodesic without corner is homotopically non-trivial. Hence, a result of Klingberg [24] and (3.2) show the following.

THEOREM 5. — *Let* M *be a compact surface with genus one. Suppose the curvature of* M *is bounded from above by a constant* $c > 0$ *such that* $c\,d(M)^2 < 1/2$ *and* $\int_M K^+ < \pi$. *Then the injectivity radius of* M *is bounded from below by* $\pi/\sqrt{c}$ *or half the quantity shown* *in the right hand side of* (3.2). *Furthermore,*

$$h(\mathrm{M}) \geqq \min\left\{\frac{1}{2} d(\mathrm{M})^{-1}\left[4(1-2c\,d(\mathrm{M})^2)^{-1}+\left(2\pi-\int_{\mathrm{M}}\mathrm{K}^+\right)^{-1}\right]^{-1}\right. \\ \left.\left[2\left(2\pi-\int_{\mathrm{M}}\mathrm{K}^+\right)(\mathrm{V}(\mathrm{M}))^{-1}\right]^{1/2}\right\}. \tag{3.3}$$

4. Isoperimetric inequalities for higher dimensional negatively curved manifolds

Let M be a n-dimensional complete simply connected manifold with sectional curvature bounded from above by $-\mathrm{K}$ where $\mathrm{K} \geqq 0$. Let r be the distance function from a fixed point $p \in \mathrm{M}$. Then it is well-known that r^2 is a smooth function. Furthermore, by using the comparison theorem, one can prove that

$$\Delta r^2 \geqq 2n, \tag{4.1}$$

when $\mathrm{K} = 0$, and

$$\Delta r^2 \geqq 2+2(n-1)\mathrm{K}^{1/2} r \coth \mathrm{K}^{1/2} r, \tag{4.2}$$

when $\mathrm{K} < 0$.

The inequalities have immediate applications to isoperimetric inequalities for domains in M. The inequality (4.2) is particularly interesting in connection with theorems in section 2.

PROPOSITION 3. — *Let* D *be a compact domain in a complete simply connected manifold with sectional curvature bounded from above by* $-\mathrm{K}$ *where* $\mathrm{K} > 0$. *Then*

$$\mathrm{Vol}(\mathrm{D}) \leqq \frac{\mathrm{A}(\partial \mathrm{D})}{(n-1)\sqrt{\mathrm{K}}}, \tag{4.3}$$

where $\mathrm{A}(\partial\mathrm{D})$ *is the area of* $\partial\mathrm{D}$.

Proof. — This follows because (4.2) implies

$$\Delta r \geqq (n-1)\sqrt{\mathrm{K}} \coth \sqrt{\mathrm{K}}\, r \geqq (n-1)\sqrt{\mathrm{K}}. \tag{4.4}$$

Integrating this inequality over D and applying the divergence theorem, (4.3) follows immediately.

In case $\mathrm{K} = 0$, Hoffman and Spruck [16], using the method of Michael and Simon [17], has been able to prove that

$$[\mathrm{Vol}(\mathrm{D})]^{(n-1)/n} \leqq c_n \mathrm{A}(\partial \mathrm{D}), \tag{4.5}$$

where $c_n = [2^{n-1}\, n\, \pi/(n-1)]\,(2\,\omega_n)^{-1/n}$ with ω_n = volume of the unit n-sphere.

Putting (4.3) and (4.5) together, we have

$$\mathrm{Vol}(\mathrm{D}) \leqq \min\left\{\frac{\mathrm{A}(\partial \mathrm{D})}{(n-1)\sqrt{\mathrm{K}}},\ (c_n)^{n/(n-1)}[\mathrm{A}(\partial \mathrm{D})]^{n/(n-1)}\right. . \tag{4.6}$$

Corollary 1. — *Let* M *be a compact manifold* (*with boundary*) *which is isometric to a domain of an n-dimensional simply connected complete Riemannian manifold with sectional curvature bounded from above by a negative constant* $-\mathrm{K}$. *Then*

$$h(\mathrm{M}) \geqq \max[(n-1)\sqrt{\mathrm{K}},\ c_n^{-1}(\mathrm{Vol}(\mathrm{D}))^{-1/n}]. \tag{4.7}$$

Corollary 2 (McKean [13]). — *Let* M *be an n-dimensional complete simply connected manifold with sectional curvature bounded from above by a negative constant* $-\mathrm{K}$. *Then* $\lambda_1(\mathrm{M}) = \sup \lambda_1(\mathrm{D})$, *where* $\lambda_1(\mathrm{D})$ *is the first eigenvalue* (*for either the Dirichlet problem or the Neuman problem*) *of the Laplacian for the compact subdomain* D, *is not less than* $[(n-1)^2/4]\,\mathrm{K}$.

Corollary 3. — *Let* M *be a closed manifold with sectional curvature bounded from above a non-positive constant* $-\mathrm{K}$. *Then for any point* $p \in \mathrm{M}$, *the* $(n-1)$-*dimensional measure of the cut locus of* p *is non-zero.*

5. Another isoperimetric constant

In this section, we shall be interested in another isoperimetric constant. While this constant is different from $h(\mathrm{M})$, it makes the computation of the first eigenvalue of the Laplacian of a compact manifold more tractable. As a result, this constant is more delicate than $h(\mathrm{M})$.

Given a compact manifold M (with boundary), $\mathrm{I}(\mathrm{M})$ *is defined to be* $\inf[\mathrm{A}(\partial \mathrm{M}_1 \cap \partial \mathrm{M}_2)/\min(\mathrm{Vol}(\mathrm{M}_1), \mathrm{Vol}(\mathrm{M}_2))]$ where the inf is taken over all decompositions $\mathrm{M} = \mathrm{M}_1 \cup \mathrm{M}_2$ with $\mathrm{Vol}(\mathrm{M}_1 \cap \mathrm{M}_2) = 0$. Similar to theorem 1, one has the following well-known.

Theorem 6. — $\mathrm{I}(\mathrm{M}) = \inf\left(\int_{\mathrm{M}} |\nabla f|\right)\left(\inf_{\beta \in \mathrm{R}} \int_{\mathrm{M}} |f-\beta|\right)^{-1}$ *where the* inf *is taken over all* C^1-*function defined on* M.

Proof. — Let f be any C^1-function defined on M. Then define $f^+ = \max(f-k, 0)$, $f^- = -\min(f-k, 0)$ where $k \in \mathrm{R}$ is chosen so that

$$\mathrm{Vol}\{x \mid f^+(x) > 0\} \leqq \frac{1}{2}\mathrm{Vol}(\mathrm{M}) \quad \text{and} \quad \mathrm{Vol}\{x \mid f^-(x) > 0\} \leqq \frac{1}{2}\mathrm{Vol}(\mathrm{M}).$$

As in theorem 1, one can apply the co-area formula to obtain

$$\int_{\mathrm{M}} |\nabla f^+| \geqq \mathrm{I}(\mathrm{M}) \int_{\mathrm{M}} |f^+| \quad \text{and} \quad \int_{\mathrm{M}} |\nabla f^-| \geqq \mathrm{I}(\mathrm{M}) \int_{\mathrm{M}} |f^-|.$$

Therefore

$$\int_M |\nabla f| \geqq \mathrm{I}(M) \int_M |f-k|. \tag{5.1}$$

The converse of theorem 9 can be proved in the same way as we did in theorem 1. In fact, if $\mathrm{Vol}(M_1) \leqq \mathrm{Vol}(M_2)$, we can define a function f_ε as follows. For $x \in M_1$, we define $f_\varepsilon(x) = 1$. For $x \in M_2$, we define $f_\varepsilon(x) = 1-(r/\varepsilon)$ when x has distance $r \leqq \varepsilon$ from $\partial M_1 \cap \partial M_2$ and $f_\varepsilon(x) = 0$ when x has distance $> \varepsilon$ from $\partial M_1 \cap \partial M_2$. Clearly we can choose k_ε so that $\int_M |f_\varepsilon - k_\varepsilon| = \inf_\beta \int_M |f_\varepsilon - \beta|$ and $k_\varepsilon \to 0$. Therefore,

$$\frac{\mathrm{A}(\partial M_1 \cap \partial M_2)}{\min(\mathrm{Vol}(M_1), \mathrm{Vol}(M_2))} \geqq \lim_{\varepsilon \to 0} \left(\int_M |\nabla f_\varepsilon| \right) \left(\inf_\beta \int_M |f_\varepsilon - \beta| \right)^{-1}.$$

COROLLARY 1. — *Let* M *be a compact manifold* (*with boundary*). *Then for any* C^1-*function* f *defined on* M, *we have*

$$\int_M |\nabla f|^2 \geqq \frac{\mathrm{I}(M)^2}{4} \int_M (f-k)^2, \tag{5.2}$$

for any $k \in \mathbf{R}$ *satisfying*

$$\mathrm{Vol}\{x \,|\, f(x) \geqq k\} \geqq \frac{1}{2}\mathrm{Vol}(M) \quad \textit{and} \quad \mathrm{Vol}\{x \,|\, f(x) \leqq k\} \geqq \frac{1}{2}\mathrm{Vol}(M).$$

In particular, if $\int_M f = 0$,

$$\int_M |\nabla f|^2 \geqq \frac{\mathrm{I}(M)^2}{4} \int_M f^2. \tag{5.3}$$

Proof. — Clearly,

$$\int_M f^+ f^- = 0. \tag{5.4}$$

Hence, by using the co-area formula again,

$$\begin{aligned} \int_M (f-k)^2 &= \int_M (f^+ + f^-)^2 \leqq \int_M (f^+)^2 + \int_M (f^-)^2 \\ &\leqq \frac{1}{\mathrm{I}(M)} \left[\int_M |\nabla (f^+)^2| + \int_M |\nabla (f^-)^2| \right] \leqq \frac{2}{\mathrm{I}(M)} \int_M (f^+ + f^-) |\nabla f|. \end{aligned} \tag{5.5}$$

The inequality (5.3) follows from (5.5) by Schwarz's lemma. Inequality (5.3) follows from (5.2) becauce if $\int_M f = 0$,

$$\int_M f^2 = \inf_k \int_M (f-k)^2. \tag{5.6}$$

6. A lower estimate of I (M)

In this section, we give a lower estomate of I (M) for a general compact Riemannian mamnifold. In particular, we obtain a lower estimate for the first eigenvalueof the Laplacian of a general Riemannian manifold.

Let M be an n-dimensional manifold (with boundary) such that for some $p \in \mathrm{M}$, every other point in M can be joined to p by a minimal geodesic. Then the exponential map at p is surjective and we can identify a *domain* in the tangent space at p with the open set in M which is within the cut locus of p. *Let* S (p) *be the unit sphere in the tangent space* $\mathrm{T}_p(\mathrm{M})$. Then we can write the *domain* in polar coordinate as

$$\mathrm{D}(p) = \{(r, \theta) \mid \theta \in \mathrm{S}(p),\ 0 \leqq r \leqq r(\theta)\}, \tag{6.1}$$

where $r(\theta)$ is a function defined on S (p).

For every point $q \in \mathrm{D}(p)$, we can write the volume element of M *as* $\sqrt{g}\,(p, q)\, r^{n-1}\, dr\, d\theta$. For every measurable subset E of D (p), we define the *cone* of p *over* E to be

$$\mathrm{C}_p(\mathrm{E}) = \{(r, \theta) \mid \text{for some } \bar{r},\ (\bar{r}, \theta) \in \mathrm{E}\}. \tag{6.2}$$

LEMMA 4. — *Let h be a Lipschitz function defined on* D (p). *Then for* $\mathrm{E} = \{x \in \mathrm{M} \mid h(x) = 0\}$, *we have*

$$\mu[\mathrm{S}(p) \cap \mathrm{C}_p(\mathrm{E})]\,|h(p)| \leqq \int_{\mathrm{C}_p(\mathrm{E})} \left|\frac{\partial h}{\partial r}\right| [\sqrt{g}\,(p, x)\,\overline{p, x}^{\,n-1}]^{-1}\, dx, \tag{6.3}$$

where μ *denotes the euclidean measure on* S (p), *dx denotes the volume element of* M *and* $\overline{p, x}$ *denotes the distance between p and x.*

Proof. — Let $(r, \theta) \in \mathrm{C}_p(\mathrm{E})$ be an arbitrary point. Then for some $\bar{r} \leqq r(\theta)$, we have $h(\bar{r}, \theta) = 0$ and

$$|h(r, \theta)| \leqq \int_0^{r(\theta)} \left|\frac{\partial h}{\partial r}\right| dr. \tag{6.4}$$

Fixing r, and integrating (6.4) over the set $\mathrm{S}(p) \cap \mathrm{C}_p(\mathrm{E})$, we obtain

$$\int_{\theta \in \mathrm{S}(p) \cap \mathrm{C}_p(\mathrm{E})} |h(r, \theta)| \leqq \int_{\theta \in \mathrm{S}(p) \cap \mathrm{C}_p(\mathrm{E})} \int_0^{r(\theta)} \left|\frac{\partial h}{\partial r}\right| dr\, d\theta. \tag{6.5}$$

By definition, the volume element dx is given by $\sqrt{g}\,(p, x)\, r^{n-1}\, dr\, d\theta$ and so (6.5) implies

$$\int_{\theta \in \mathrm{S}(p) \cap \mathrm{C}_p(\mathrm{E})} |h(r, \theta)| \leqq \int_{\mathrm{C}_p(\mathrm{E})} \left|\frac{\partial h}{\partial r}\right| [\sqrt{g}\,(p, x)\,\overline{p, x}^{\,n-1}]^{-1}\, dx. \tag{6.6}$$

Letting $r \to 0$ in (6.6), we obtain (6.3).

LEMMA 5. — *$\sqrt{g}(p, q) = \sqrt{g}(q, p)$ whenever they are well-defined.*

The referee says that this is a folk result which seems to be proved explicitly in the literature only in the real analytic case (*see* [21], formula (2) on page 19 or [22], proposition 1.4). A proof for the C^∞ case follows easily from the formula

$$B_c(s) = A(s)\int_s^c A^{-1}(t)[A^{-1}(t)]^* \, dt \quad \text{on page 31 of [18].} \tag{6.7}$$

Now let M be any general n-dimensional manifold (with boundary). (We drop the restriction that we assume at the beginning of this section.) Let M_p be the subset of M which can be joined to p by a smooth minimal geodesic. Then applying lemma 4 to M_p, we have

$$\omega_p(E)\,|h(p)| \leqq \int_M \left|\frac{\partial h}{\partial r}\right| [\sqrt{g}(p, x)\,\overline{p, x}^{n-1}]^{-1}\, dx, \tag{6.8}$$

where $\omega_p(E) = \mu[S(p) \cap C_p(E \cap M_p)]$.

Integrating (6.8), we obtain

$$\begin{aligned} \int_M |h(p)|\, dp &\leqq \int_M \int_M \left|\frac{\partial h}{\partial r}(x)\right| [\omega_p(E)\sqrt{g}(p, x)\,\overline{p, x}^{n-1}]^{-1}\, dx\, dp \\ &\leqq \left(\int_M |\nabla h|\, dx\right) \sup_x \left(\int_M [\omega_p(E)\sqrt{g}(p, x)\,\overline{p, x}^{n-1}]^{-1}\, dp\right). \end{aligned} \tag{6.9}$$

To evaluate the last integral, we use the exponential map to identify a domain in the tangent space at x with the open set in M which is *within the cut locus of* x. Using lemma 5 and the same notation as before, one sees

$$\int_M [\omega_p(E)\sqrt{g}(p, x)\,\overline{p, x}^{n-1}]^{-1}\, dp = \int_{S(x)} \int_0^{r(\theta)} [\omega_{(r,\theta)}(E)]^{-1}\, dr\, d\theta. \tag{6.10}$$

LEMMA 6 :

$$[\omega_p(E)]^{-1} \leqq \mathrm{Vol}(E \cap M_p)^{-1} \sup_{\theta \in S(p)} \int_0^{r(\theta)} \sqrt{g}(p, (r, \theta))\, r^{n-1}\, dr.$$

Proof. — By definition,

$$\begin{aligned} \mathrm{Vol}(E \cap M_p) &\leqq \mathrm{Vol}(C_p(E \cap M_p)) \\ &= \int_{\theta \in S(p) \cap C_p(E \cap M_p)} \int_0^{r(\theta)} \sqrt{g}(p, (r, \theta))\, r^{n-1}\, dr\, d\theta \\ &\leqq \omega_p(E) \sup_{\theta \in S(p)} \int_0^{r(\theta)} \sqrt{g}(p, (r, \theta))\, r^{n-1}\, dr. \end{aligned} \tag{6.11}$$

Putting (6.9), (6.10) and (6.11) together, we have

$$\int_M |h| \leqq \left(\int_M |\nabla h|\right)\left[\sup_p \sup_{\theta \in S(p)} \int_0^{r(\theta)} \sqrt{g}(p, (r, \theta))\, r^{n-1}\, dr\right] \times \left\{\sup_x \int_M [\mathrm{Vol}(E \cap M_p)\sqrt{g}(x, p)\overline{x, p}^{n-1}]^{-1}\, dp\right\}. \tag{6.12}$$

In the inequality (6.12), h is an arbitrary Lipschitz function and E is the zero of h.

Suppose f is any Lipschitz function defined on M. Then for some number k,

$$\begin{cases} \mathrm{Vol}\{x \mid f(x) < k\} \leqq \dfrac{1}{2}\mathrm{Vol}(M), \\ \mathrm{Vol}\{x \mid f(x) > k\} \leqq \dfrac{1}{2}\mathrm{Vol}(M). \end{cases} \tag{6.13}$$

For this k, we define

$$f_1(x) = \begin{cases} f(x) - k & \text{if } f(x) \geqq k, \\ 0 & \text{otherwise;} \end{cases} \tag{6.14}$$

$$f_2(x) = \begin{cases} f(x) - k & \text{if } f(x) \leqq k, \\ 0 & \text{otherwise.} \end{cases} \tag{6.15}$$

Clearly, $f(x) - k = f_1(x) + f_2(x)$. Furthermore, the volume of both the zero sets of f_1 and f_2 are not less than (1/2) Vol (M).

Since

$$\int_M |f - k| = \int_M |f_1| + \int_M |f_2| \quad \text{and} \quad \int_M |\nabla f| = \int_M |\nabla f_1| + \int_M |\nabla f_2|,$$

it follows from (6.12) that

$$\int_M |f - k| \leqq \left(\int_M |\nabla f|\right)\left[\sup_p \sup_{\theta \in S(p)} \int_0^{r(\theta)} \sqrt{g}(p, (r, \theta))\, r^{n-1}\, dr\right] \times \left\{\sup_E \sup_x \int_M [\mathrm{Vol}(E \cap M_p)\sqrt{g}(x, p)\overline{x, p}^{n-1}]^{-1}\, dp\right\}, \tag{6.16}$$

where E ranges over all subsets of M with Vol (E) $\geqq$ (1/2) Vol (M).

When M is compact, $M = M_p$:

$$\sup_E \sup_x \int_M [\mathrm{Vol}(E \cap M_p)\sqrt{g}(x, p)\overline{x, p}^{n-1}]^{-1}\, dp \leqq 2\,\mathrm{Vol}(M)^{-1} \sup_x \int_{S(x)} \int_0^{r(\theta)} dr\, d\theta \leqq 2\,\mathrm{Vol}(M)^{-1}\, d(M)\, \alpha(n), \tag{6.17}$$

where $\alpha(n)$ is the volume of the $(n-1)$-dimensional unit sphere.

Putting (6.17) into (6.16), we have

$$(6.18)\quad \mathrm{I(M)}^{-1} \leqq 2\alpha(n)\,d(\mathrm{M})\,\mathrm{Vol(M)}^{-1}\left[\sup_{p}\ \sup_{\theta\in\mathrm{S}(p)}\int_0^{r(\theta)}\sqrt{g}(p,(r,\theta))\,r^{n-1}\,dr\right].$$

To state the main theorem, we shall use the following *convention* : $\sqrt{-\mathrm{K}^{-1}}\sinh\sqrt{-\mathrm{K}}\,r$ *is interpreted as r when* $\mathrm{K}=0$, $\sqrt{\mathrm{K}^{-1}}\sin\sqrt{\mathrm{K}}\,r$ *when* $\mathrm{K}>0$. Then when the Ricci curvature of M is bounded from below by $(n-1)\,\mathrm{K}$, the comparison theorem (*see* [2]) says that $\sqrt{g}(p,(r,\theta))\,r^{n-1} \leqq (\sqrt{-\mathrm{K}^{-1}}\sin h\sqrt{-\mathrm{K}}\,r)^{n-1}$. It follows therefore from (6.10) that

$$(6.19)\qquad \mathrm{I(M)}^{-1} \leqq \alpha(n)\,d(\mathrm{M})\,\mathrm{Vol(M)}^{-1}\int_0^{d(\mathrm{M})}(\sqrt{-\mathrm{K}^{-1}}\sin h\sqrt{-\mathrm{K}}\,r)^{n-1}\,dr.$$

THEOREM 7. — *Let* M *be a compact n-dimensional manifold without boundary whose Ricci curvature bounded from below by* $(n-1)\,\mathrm{K}$. *Then* (6.19) *holds with* $\alpha(n)$ *equal to the volume of the unit* $(n-1)$*-sphere. Since* $\lambda_1(\mathrm{M}) \geqq \mathrm{I(M)}^2/4$, *we find a lower bound of* $\lambda_1(\mathrm{M})$ *in terms of* $d(\mathrm{M})$, $\mathrm{Vol(M)}$ *and* K.

From (6.10), we also obtain :

THEOREM 7. — *Let* M *be a compact n-dimensional manifold* (*with boundary*). *Then for* $\omega = \inf_p \inf_{\mathrm{E}} \omega_p(\mathrm{E})$ *where* E *ranges over sets with volume* $\leqq (1/2)\,\mathrm{Vol(M)}$, $\mathrm{I(M)}^{-1} \leqq \alpha(n)\,d(\mathrm{M})\,\omega$.

In order to obtain a useful estimate, we weaken theorem 7 and proceed as follows: For each $p\in\mathrm{M}$, let $\mathrm{B}_p(r)$ be the geodesic ball of radius r around p. Then for

$$\overline{x,p} \leqq \frac{1}{2}d(p,\partial\mathrm{M}),\qquad \mathrm{B}_x\left[\frac{1}{2}d(p,\partial\mathrm{M})\right]\subset\mathrm{M}_x.$$

In particular for every point $x\in\mathrm{B}_p[(1/3)\,d(p,\partial\mathrm{M})]$, $\mathrm{B}_p[(1/3)\,d(p,\partial\mathrm{M})] \subseteq \mathrm{M}_x$. Hence for every Lipschitz function f defined on M, we have

$$(6.20)\qquad \left(\inf_{\beta}\int_{\mathrm{B}_p[(1/3)\,d(p,\partial\mathrm{M})]}|f-\beta|\right)\left(\int_{\mathrm{B}_p[(2/3)\,d(p,\partial\mathrm{M})]}|\nabla f|\right)^{-1}$$
$$\leqq \frac{4}{3}\alpha(n)\,d(p,\partial\mathrm{M})\,\mathrm{Vol}\left[\mathrm{B}_p\left(\frac{1}{3}d(p,\partial\mathrm{M})\right)\right]^{-1}$$
$$\times\left[\sup_{\overline{q,p}\leqq(1/3)\,d(p,\partial\mathrm{M})}\ \sup_{\theta\in\mathrm{S}(q)}\int_0^{r(\theta)}\sqrt{g}(q,(r,\theta))\,r^{n-1}dr\right].$$

LEMMA 7. — *Let* σ *be a minimal geodesic segment in a Riemannian manifold* M *with dimension n. Suppose at each point* $\sigma(t)$, *the Ricci curvature of* M *in direction* $\sigma'(t)$ *is given by* $\mathrm{K}(\sigma'(t))$. *Then*

$$(6.21)\qquad \sqrt{g}(\sigma(0),\sigma(r)) \leqq \exp\left[\int_0^r\left(-\frac{1}{t^2}\int_0^t\mathrm{K}(\sigma'(\tau))\,\tau^2\,d\tau\right)dt\right].$$

Proof. — Denote the distance function from $\sigma(0)$ by r. Then as in [1] p. 137, we have

$$\frac{\partial}{\partial r}(\log\sqrt{g}) = \Delta r - \frac{n-1}{r}, \tag{6.22}$$

when r is smooth.

On the other hand, by the fundamental theorem for the index form, one can derive (*cf.* [19]) :

$$\Delta r \leqq \frac{n-1}{r} - \frac{1}{r^2}\int_0^r K(t)\, t^2\, dt. \tag{6.23}$$

The inequality (6.21) follows immediately from (6.22) and (6.23). Putting (6.21) into (6.20), we have :

THEOREM 8. — *Let* M *be a compact manifold with boundary. Let* $p \in M$ *be a point such that* $d(p, \partial M) = r$. *Let*

$$G = \sup_{\sigma} \exp\left[\int_0^l \left(-\frac{1}{t^2}\int_{\sigma}^{t} K(\sigma'(\tau))\,\tau^2\, d\tau\right) dt\right]$$

where σ *ranges over all minimal geodesic segment with length* $l \leqq (2/3)\, r$ *and* $\overline{p, \sigma(0)} \leqq r/3$. *Then for all Lipschitz function* f,

$$\inf_{\beta}\int_{B_p(r/3)} |f-\beta| \leqq \frac{4\, r^{n+1}}{3\, n}\alpha(n)\,\mathrm{Vol}\left[B_p\left(\frac{r}{3}\right)\right]^{-1} G \int_{B_p(2r/3)} |\nabla f|. \tag{6.24}$$

Remark. — The main point of (6.24) is that when G is uniformly bounded and Vol $[B_p(r)]$ grows up at least like r^n, then (6.24) is exactly similar to the Poincaré inequality (without compact support) in euclidean space. This enables one to push the standard R^n-arguments to manifolds. (*See* the conclusion at the end of next section for introducing the constant G.)

7. Remarks

The estimate of I (M) in section 6 depends on a lower bound of the volume of M. We suspect that this may not be necessary. (However, examples show that dependence on the lower bound of the Ricci curvature and the upper bound of the diameter of the manifold is essential. This is easily illustrated by the flat torus.)

In case M is a minimal submanifold in a complete simply connected manifold N with *non-positive curvature*, one can obtain a lower estimate of the volume of a geodesic ball of M as follows.

Let $p \in M \subset N$ be an arbitrary point. Let R be the distance function of N from p. Then straightforward computation (using the comparison theorem, *see* [2]), shows that when we restrict R^2 to M, we have

$$\Delta R^2 \geqq 2\, n. \tag{7.1}$$

Let $B_p(r)$ be a geodesic ball of radius r in M. Then integrating (7.1) on this ball, we have

$$(7.2)\qquad 2r\,\mathrm{Vol}[\partial B(r)] \geqq 2\int_{\partial B_p(r)} R\,|\nabla R| \geqq 2n\,\mathrm{Vol}[B_p(r)].$$

Since Vol $[B(r)] = (\partial/\partial r)$ (Vol $[B_p(r)]$), (7.2) implies Vol $[B_p(r)]\, r^{-n}$ is non-decreasing and hence

$$(7.3)\qquad \mathrm{Vol}[B_p(r)] \geqq \alpha(n)\, r^n.$$

A consequence of this inequality is that a complete minimal submanifold of a complete simply connected manifold with non-positive curvature has infinite volume. (*The immersion is not assumed to be proper.*) Another consequence is that if M is a compact minimal submanifold of a complete manifold with non-positive curvature, then π_1 (M) is non-trivial and it grows up at least polynomially with order $\geqq$ dim M. (For definitions and arguments, *see* Milnor [20]).

Putting (7.3) into (6.24), one sees that for a minimal submanifold in a complete simply connected manifold with non-positive curvature, we have

$$(7.4)\qquad \inf_{\beta}\int_{B_p(r/3)} |f-\beta| \leqq c_n\, r\, G\int_{B_p(2r/3)} |\nabla f|.$$

where c_n depends only on n = dim M.

Since Sobolev inequality was proved for manifolds of the above type in [17] and [16], one can apply the standard De Giorgi-Nash-Moser result (*see* [23]) to prove a "Harnack inequality" on such manifolds. Such a "Harnack inequality" will imply "Liouville's theorem" on such manifolds if the manifold M is complete and G is uniformly bounded on M. In particular, if M is a complete minimal submanifold in a complete non-positively curved manifold and if for some $\varepsilon > 0$, $K(\sigma'(t))\, t^{2+\varepsilon}$ is uniformly bounded, then M admits no non-constant bounded harmonic function.

Remark. -- The inequality (7.3) is also obtained by H. Alexender (private communication from L. Simon) in case N is the euclidean space.

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