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J. C. McCONNELL

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# REPRESENTATIONS OF SOLVABLE LIE ALGEBRAS.

## II. TWISTED GROUP RINGS

By J. C. Mc CONNELL

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### Introduction

This paper may be regarded as either an introduction to or a continuation of [4]. In [4] it was shown that the simple algebras which arise in the representation theory of solvable Lie algebras could be viewed as algebras of differential operators in which the multiplication is altered by a Lie 2-cocycle. In this paper we show that these cocycle twisted algebras of differential operators may also be viewed as (cocycle free) twisted group rings.

We first recall the main result of [4]. Let  $\mathfrak{g}$  be a completely solvable Lie algebra over a field  $k$  of characteristic zero and  $P$  be a prime ideal of  $U = U(\mathfrak{g})$ , the universal enveloping algebra of  $\mathfrak{g}$ . Let

$$E = E(U/P) = \{u + P : u + P \neq P \text{ and } [g, u] + P \in ku + P \text{ for all } g \in \mathfrak{g}\}.$$

$U/P$  has a simple quotient ring  $(U/P)_E$  with respect to  $E$ . It was shown in [4] that

$$(U/P)_E \cong (KS \otimes_K KG) \#_{\sigma} U(\mathfrak{a}),$$

where  $K$  is the centre of  $(U/P)_E$ ,  $KS$  is a commutative polynomial algebra,  $KS = K[y_1, \dots, y_n]$ ,  $n \geq 0$ ;  $KG$  is the group algebra of a free abelian group of finite rank  $m$ ,  $m \geq 0$ ,

$$KG = K[g_1, g_1^{-1}, \dots, g_m, g_m^{-1}];$$

$\mathfrak{a}$  is a subalgebra of the abelian Lie algebra  $\sum_{i,j} k \partial/\partial y_i + k g_j \partial/\partial g_j$  such that the ring of formal differential operators  $(KS \otimes_K KG) \# U(\mathfrak{a})$  is simple, (see Theorem 2.2);  $\sigma$  is a Lie 2-cocycle,  $\sigma \in Z^2(\mathfrak{a}, KS \otimes KG)$  and  $(KS \otimes_K KG) \#_{\sigma} U(\mathfrak{a})$  is the corresponding "twisted ring of differential operators." (See Theorem 2.1 and Remark.) The key result of this paper is that, under the conditions stated, the Lie cohomology group  $H^2(\mathfrak{a}, KS \otimes KG)$  is isomorphic to a subgroup of  $H^2(\mathfrak{a}, K)$ . From this it follows readily that such a simple algebra  $(KS \otimes KG) \#_{\sigma} U(\mathfrak{a})$  may be "turned upside down" and viewed as a twisted group ring,  $(A_n \otimes_K U(W)) \# KG$ , where  $A_n$  is a Weyl algebra with

centre  $K$ ,  $U(W)$  is the enveloping algebra of an abelian Lie algebra  $W$  and  $G$  is a finitely generated free abelian group of automorphisms of the coefficient ring  $A_n \otimes U(W)$ . Each element of  $G$  preserves the filtration on  $A_n \otimes U(W)$  and induces the identity automorphism on the associated graded algebra and  $(A_n \otimes U(W)) \# KG$  may be regarded as an algebra  $\mathcal{A}(V, \delta, G)$  constructed from a finite dimensional vector space  $V$ , an alternating bilinear form  $\delta$  on  $V$  and a finitely generated subgroup  $G$  of the dual space  $V^*$ .

It is shown that each such simple algebra  $\mathcal{A}(V, \delta, G)$  may be expressed as a  $(KS \otimes KG) \#_{\sigma} U(\mathfrak{a})$  and as a  $(U(\mathfrak{g})/P)_E$ . Thus there are bijections between the isomorphism classes of simple algebras,

$$\{(U(\mathfrak{g})/P)_E\}, \quad \{(KS \otimes KG) \#_{\sigma} U(\mathfrak{a})\} \quad \text{and} \quad \{\mathcal{A}(V, \delta, G)\}.$$

It is also shown that a simple algebra  $\mathcal{A}(V, \delta, G)$  is a subalgebra of  $A_n \otimes A'_m$ , where  $n = \text{rank } \delta/2$ ,  $m = \text{rank } G$  and  $A'_m$  is the localisation of  $A_m$  at the powers of the element  $x_1 x_2 \dots x_m$ .

A necessary and sufficient condition for  $\mathcal{A}(V, \delta, G)$  to be simple is that  $V^{\delta} \cap V^G = 0$ , where  $V^{\delta}$  is the orthogonal complement of  $V$  with respect to  $\delta$  and  $V^G = \cap \text{Ker } \lambda$ , ( $\lambda \in G$ ). The proof of this is elementary and is much easier than the proof of the corresponding theorem of [4] which gave sufficient conditions for  $(KS \otimes KG) \#_{\sigma} U(\mathfrak{a})$  to be simple. (See Theorems 2.2 and 4.7.)

Finally we consider when two simple algebras

$$\mathcal{A}_1 = \mathcal{A}(V, \delta, G) \quad \text{and} \quad \mathcal{A}_2 = \mathcal{A}(W, \gamma, H)$$

are isomorphic. We conjecture that these algebras are isomorphic if and only if there exists a vector space isomorphism  $\phi : V \rightarrow W$  such that (i)  $\phi$  is compatible with  $\delta$  and  $\gamma$  and (ii) the dual map  $\phi^*$  induces an isomorphism  $H \rightarrow G$ . If such a  $\phi$  exists then  $\mathcal{A}_1 \cong \mathcal{A}_2$ . Conversely, if  $\mathcal{A}_1 \cong \mathcal{A}_2$  then we show that there exists an isomorphism  $\phi$  such that (ii) holds but we are unable to show that  $\phi$  also satisfies (i) except in a special case. However we do have four integer valued isomorphism invariants associated with  $\mathcal{A}(V, \delta, G)$ , (or with a primitive ideal  $P$  of the universal enveloping algebra of a completely solvable Lie algebra  $\mathfrak{g}$ ) and if the conjecture is true then there is yet another integer valued isomorphism invariant. It is well known that if  $\mathfrak{g}$  is nilpotent then  $P$  determines a single integer  $n$ , where  $U/P \cong A_n$  and  $2n$  is the dimension of the orbit of  $\mathfrak{g}^*$  associated with  $P$ , [6] (4.2.1).

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### 1. Notation

$k$  is a field of characteristic zero. An algebra is an associative  $k$ -algebra with 1 and homomorphisms preserve 1. If  $A$  is an algebra,  $\text{Der } A$  is the Lie algebra of  $k$ -derivations of  $A$  and  $\text{Aut } A$  the group of  $k$ -algebra automorphisms of  $A$ .  $\text{End } A = \text{End}_k A$  and if  $A$  is commutative,  $\text{Diff } A$  is the subalgebra of  $\text{End } A$  generated by  $\text{Der } A$  and the multiplications by elements of  $A$ .

If  $\mathfrak{g}$  is a Lie algebra and  $M$  a  $\mathfrak{g}$ -module, then  $C^p(\mathfrak{g}, M)$  is the  $k$ -space of  $p$ -cochains,  $Z^p(\mathfrak{g}, M)$  the space of  $p$ -cocycles and  $H^p(\mathfrak{g}, M)$  the corresponding cohomology group. In particular  $C^2(\mathfrak{g}, k)$  is the space of alternating bilinear forms on  $\mathfrak{g}$ .  $M \rtimes \mathfrak{g}$  denotes the split extension of  $M$  by  $\mathfrak{g}$  and if  $\sigma \in Z^2(\mathfrak{g}, M)$ ,  $M \rtimes_{\sigma} \mathfrak{g}$  denotes the extension corresponding to  $\sigma$ . (See [1], Chapter 13 Section 8 and Chapter 14 Section 5.)

If  $A$  is an algebra and  $G$  a subgroup of  $\text{Aut } A$  then  $A \# kG$  denotes the corresponding twisted group ring which as a  $k$ -space is  $A \otimes_k kG$ , where  $kG$  is the group algebra of  $G$  over  $k$ , and multiplication is defined by

$$(a_1 \otimes g_1)(a_2 \otimes g_2) = a_1 g_1(a_2) \otimes g_1 g_2.$$

We will denote  $a \otimes g$  by  $ag$  as usual.  $A \# kG$  is also a free right  $A$ -module with the elements of  $G$  as a free basis.  $A \# kG$  has the universal property that if  $B$  is an algebra and  $\varphi$  is a homomorphism of  $A$  to  $B$  and  $\psi$  a group homomorphism of  $G$  into the group of units of  $B$  such that

$$\psi(g)\varphi(a) = \varphi(g(a))\psi(g),$$

for all  $a \in A$ ,  $g \in G$ , then there exists a unique ( $k$ -algebra) homomorphism of  $A \# kG$  to  $B$  which extends  $\varphi$  and  $\psi$ .

If  $V$  is a vector space and  $\delta$  an alternating bilinear form on  $V$  then a basis for  $V$  adapted to  $\delta$  is a basis

$$x_1, y_1, \dots, x_t, y_t, s_1, \dots, s_r$$

such that  $\delta(x_i, y_j) = \Delta_{ij}$  (where  $\Delta_{ij}$  is the Kronecker delta) and

$$\delta(x_i, x_j) = \delta(x_i, s_j) = \delta(y_i, y_j) = \delta(y_i, s_j) = \delta(s_i, s_j) = 0.$$

If  $A_n$  is a Weyl algebra with its usual filtration and  $U(W)$  the enveloping algebra of an abelian Lie algebra  $W$  with its usual filtration then  $A_n \otimes U(W)$  is a filtered algebra by

$$F_p(A_n \otimes U(W)) = \sum_{s=0}^p F_s(A_n) \otimes F_{p-s}(U(W)).$$

Thus if  $A_n = k[x_1, y_1, \dots, x_n, y_n]$  (with  $[x_i, y_j] = \Delta_{ij}$ ),  $U(W) = k[s_1, \dots, s_r]$  and  $V$  denotes the subspace of  $A_n \otimes U(W)$  spanned by  $\{x_1, y_1, \dots, x_n, y_n, s_1, \dots, s_r\}$  then

$$F_0(A_n \otimes U(W)) = k1 \quad \text{and} \quad F_1(A_n \otimes U(W)) = k1 \oplus V.$$

If  $\theta$  is an automorphism of  $A_n \otimes U(W)$  such that  $\theta$  preserves the filtration and induces the identity automorphism on the associated graded algebra then there exists a unique  $\lambda \in V^*$  such that for  $v \in V$ ,  $\theta(v) = v + \lambda(v)$ . Conversely, if  $\lambda \in V^*$  then there exists a unique automorphism  $\theta$  of  $A_n \otimes U(W)$  defined by  $\theta(v) = v + \lambda(v)$ ,  $v \in V$ , and  $\theta$  preserves the filtration and induces the identity automorphism on the associated graded algebra.

The following notation is standard throughout the paper.  $kS$  will denote a commutative polynomial algebra over  $k$  with  $n \geq 0$  generators,  $kS = k[y_1, \dots, y_n]$ , and  $kG$  will denote the group algebra over  $k$  of a free abelian group of finite rank  $m$ ,  $m \geq 0$ ,

$kG = k[g_1, g_1^{-1}, \dots, g_m, g_m^{-1}]$ .  $\mathfrak{c}$  denotes the abelian Lie subalgebra of  $\text{Der}(kG \otimes kS)$ ,

$$\mathfrak{c} = \sum_{i=1}^n k\partial/\partial y_i + \sum_{j=1}^m kg_j\partial/\partial g_j.$$

This situation may be described more intrinsically as follows. The units in  $kS \otimes kG$  are just the units in  $kG$ , i. e. are scalar multiples of the elements of  $G$ . Consider  $kS$  with its usual graded algebra structure, (induced by the degree in  $y_1, \dots, y_n$ ) and let  $\text{Gr}_p(kS)$  denote the  $p$ th subspace in this grading. Then  $\mathfrak{c}$  is the subspace of  $\text{Der}(kS \otimes kG)$  such that,

(i) if  $c \in \mathfrak{c}$  and  $h$  is a unit in  $kS \otimes kG$ , then,  $c(h) \in kh$   
and

$$(ii) \quad c(\text{Gr}_p(kS)) \subseteq \text{Gr}_{p-1}(kS) \quad \text{for } p \geq 1.$$

$\mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{c}_2$ , where  $\mathfrak{c}_1 = \sum_i k\partial/\partial y_i$  and  $\mathfrak{c}_2 = \sum_j kg_j\partial/\partial g_j$ . For  $f = 1, 2$ ,  $\pi_f$  denotes the projection of  $\mathfrak{c} = \mathfrak{c}_1 \oplus \mathfrak{c}_2$  onto  $\mathfrak{c}_f$ . If  $g \in G$ ,  $\lambda_g \in \mathfrak{c}^*$  is defined by  $c(g) = \lambda_g(c)g$  for all  $c \in \mathfrak{c}$ .

A subspace  $\mathfrak{a}$  of  $\mathfrak{c}$  is said to satisfy condition (Sim) when the following two conditions hold :

- (i)  $\pi_1(\mathfrak{a}) = \mathfrak{c}_1$ ,
- (ii) if  $g, h \in G$  with  $g \neq h$  then  $\lambda_g|_{\mathfrak{a}} \neq \lambda_h|_{\mathfrak{a}}$ . (See Theorems 2.2 and 3.2.)

(Note that  $\mathfrak{c}_1$  (respectively  $\mathfrak{c}_2$ ) as defined above corresponds to  $\mathfrak{c}_2$  (respectively  $\mathfrak{c}_1$ ) as defined in [4] Section 5.)

$\mathfrak{a} \cap \mathfrak{c}_2$  will be denoted by  $\mathfrak{a}^G$ , i. e.  $\mathfrak{a}^G = \text{Ker}(\pi_1|_{\mathfrak{a}})$ .

## 2. A Simplicity Theorem

The following theorem is [4] (Theorem 2.8).

**THEOREM 2.1.** — *Let  $A$  be a commutative algebra,  $\mathfrak{g}$  a Lie algebra,  $\theta$  a Lie algebra homomorphism of  $\mathfrak{g}$  into  $\text{Der } A$  and  $\sigma$  a Lie 2-cocycle,  $\sigma \in Z^2(\mathfrak{g}, A)$ . Let  $I = I_\sigma$  be the ideal of  $U = U(A \times_\sigma \mathfrak{g})$  generated by*

$$\mathcal{S} = \{1_U - 1_A, a.b - ab : a, b \in A\},$$

where  $a.b$  denotes the product of  $a$  and  $b$  in  $U$  and  $ab$  their product in  $A$ .

- (1) *If  $\chi$  is a homomorphism of  $U$  into an algebra  $B$  then  $\chi|_A$  is an algebra homomorphism from  $A$  to  $B$  if and only if  $I \subset \text{Ker } \chi$ .*
- (2) *If  $\psi$  is the canonical homomorphism from  $U$  to  $U/I$  then  $\psi|_{A \times_\sigma \mathfrak{g}}$  is a monomorphism and if  $g_1, \dots, g_n$  is a basis for  $\mathfrak{g}$  and we identify  $A \times_\sigma \mathfrak{g}$  with its image under  $\psi$  then  $U/I$  is a free left  $A$ -module with the standard monomials in  $g_1, \dots, g_n$  as a free basis. ■*

*Remark.* —  $A \#_{\sigma} U(\mathfrak{g})$  has the universal property that if  $B$  is an algebra, and  $\theta$  is a Lie algebra homomorphism of  $A \times_{\sigma} \mathfrak{g}$  into  $B$  such that  $\theta|_A$  is an algebra homomorphism then  $\theta$  may be extended to an algebra homomorphism of  $A \#_{\sigma} U(\mathfrak{g})$  into  $B$ .

**THEOREM 2.2.** — *Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  and  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$ . Then  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$  is a simple algebra if and only if  $\mathfrak{a}$  satisfies condition (Sim).*

[Thus the simplicity of  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$  depends on the action of  $\mathfrak{a}$  and not on the choice of  $\sigma$ .]

*Proof.* — If  $\pi_1(\mathfrak{a}) \neq \mathfrak{c}_1$  then there exists  $0 \neq y \in \sum_i k y_i$  such that  $a(y) = 0$  for all  $a \in \mathfrak{a}$ .

Thus  $y$  is a central non unit in  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$ . If there exists  $g, h \in G$  with  $g \neq h$  and  $\lambda_g|_{\mathfrak{a}} = \lambda_h|_{\mathfrak{a}}$  then  $1 - gh^{-1}$  is a central nonunit in  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$ . Thus (Sim) is a necessary condition for  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$  to be simple. That (Sim) is also sufficient was proved in [4], Section 5, Theorems 5.2 and 5.3.

However in Theorem 4.7 we will give another proof of this, which is entirely elementary for the case  $\sigma = 0$  and the same proof can be used when  $\sigma \neq 0$  modulo Theorems 3.1 and 3.2. ■

**DEFINITION 2.3.** — Let  $A$  be an algebra and  $\mathfrak{g}$  a Lie subalgebra of  $\text{Der } A$ . If, for all  $a_1, a_2 \in A$ ,  $g(a_1) = g(a_2)$  for all  $g \in \mathfrak{g}$  implies that  $a_1 - a_2 \in k$  then we say that  $\mathfrak{g}$  separates the points of  $A$  modulo  $k$ .

**THEOREM 2.4.** — *Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$ . Then  $\mathfrak{a}$  satisfies (Sim) if and only if  $\mathfrak{a}$  separates the points of  $kS \otimes kG$  modulo  $k$ .*

*Proof.* — If  $\mathfrak{a}$  does not separate points modulo  $k$  then, as in the proof of Theorem 2.2,  $(kS \otimes kG) \# U(\mathfrak{a})$  contains central nonunits and so  $\mathfrak{a}$  does not satisfy condition (Sim), by Theorem 2.2. Conversely, if  $\mathfrak{a}$  separates points modulo  $k$  then  $\mathfrak{a}$  must satisfy conditions (i) and (ii) of (Sim). ■

### 3. Cohomology

**THEOREM 3.1.** — *Let  $A$  be a commutative algebra,  $\mathfrak{g}$  a Lie algebra and  $\theta$  a Lie algebra homomorphism of  $\mathfrak{g}$  into  $\text{Der } A$ . Let  $\sigma, \tau \in Z^2(\mathfrak{g}, A)$ . If  $\sigma, \tau$  are cohomologous cocycles then there exists an isomorphism*

$$\varphi' : A \#_{\sigma} U(\mathfrak{g}) \rightarrow A \#_{\tau} U(\mathfrak{g})$$

*which extends the identity map of  $A$  to  $A$ .*

*Proof.* — Since  $\sigma, \tau$  are cohomologous there is a Lie algebra isomorphism  $\varphi : A \times_{\sigma} \mathfrak{g} \rightarrow A \times_{\tau} \mathfrak{g}$  which extends the identity map on  $A$ .  $\varphi$  may be extended uniquely to an algebra isomorphism (again denoted by  $\varphi$ ) of  $U(A \times_{\sigma} \mathfrak{g}) \rightarrow U(A \times_{\tau} \mathfrak{g})$ . Consider the following diagram where  $I, \psi$  are as in Theorem 2.1,

$$\begin{array}{ccccccc} 0 & \rightarrow & I_{\sigma} & \rightarrow & U(A \times_{\sigma} \mathfrak{g}) & \xrightarrow{\psi_{\sigma}} & A \#_{\sigma} U(\mathfrak{g}) \rightarrow 0 \\ & & \downarrow & & \downarrow \varphi & \downarrow \psi_{\tau} & \downarrow \varphi' \\ 0 & \rightarrow & I_{\tau} & \rightarrow & U(A \times_{\tau} \mathfrak{g}) & \rightarrow & A \#_{\tau} U(\mathfrak{g}) \rightarrow 0 \end{array}$$

Since  $\varphi : I_\sigma \rightarrow I_\tau$ ,  $\varphi$  induces an isomorphism of  $A \#_\sigma U(\mathfrak{g}) \rightarrow A \#_\tau U(\mathfrak{g})$  which we denote by  $\varphi'$ . Since  $\psi_\sigma$ ,  $\psi_\tau$  and  $\varphi$  induce the identity map on  $A$ ,  $\varphi'$  also has this property. ■

The rest of this section is devoted to the proof of the following

**THEOREM 3.2.** — *Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  which satisfies condition (Sim). Let  $\mathfrak{a}^G = \mathfrak{a} \cap \mathfrak{c}_2$ . Then for  $p > 0$ , there is an isomorphism of Lie cohomology groups,*

$$H^p(\mathfrak{a}, kS \otimes kG) \cong H^p(\mathfrak{a}^G, k).$$

**LEMMA 3.3 (Rees).** — *Let  $R$  be a commutative ring and  $x \in R$  a non zerodivisor on  $R$ . Let  $M$  be an  $R$ -module with  $xM = 0$  and  $N$  an  $R$ -module such that  $x$  is a non zerodivisor on  $N$ .*

$$(1) \quad \begin{aligned} \text{Ext}_R^p(M, N) &\cong \text{Ext}_{R/Rx}^{p-1}(M, N/xN) && \text{for } p \geq 1, \\ &= 0 && \text{for } p = 0. \end{aligned}$$

(2) *If the map  $N \rightarrow N$  given by  $n \mapsto xn$  is surjective (as well as injective) then  $\text{Ext}_R^p(M, N) = 0$  for  $p \geq 0$ .*

*Proof* (1) is the Rees Reduction Theorem [7] and (1) implies (2). ■

The following lemma is a dual version of the Rees Reduction Theorem for the case when the map  $N \rightarrow N$  given by  $n \mapsto xn$  is surjective instead of injective. (Compare the concept of a cosequence as introduced by Matlis [3].)

**LEMMA 3.4.** — *Let  $R$  be a commutative ring and  $x \in R$  a non zerodivisor on  $R$ . Let  $M$  be an  $R$ -module such that  $xM = 0$  and  $N$  an  $R$ -module such that  $xN = N$ . Then for  $p \geq 0$ ,*

$$\text{Ext}_R^p(M, N) \cong \text{Ext}_{R/Rx}^p(M, \text{Ann}_N x),$$

where  $\text{Ann}_N x = \{n \in N : xn = 0\}$ .

*Proof.* — By ([1], p. 348, Case 4 or p. 118, Case 4) with  $\Gamma = R/Rx$  and  $\Lambda = R$ , there is an edge homomorphism  $\theta$  in a spectral sequence,

$$\theta : \text{Ext}_{R/Rx}^p(M, \text{Hom}_R(R/Rx, N)) \rightarrow \text{Ext}_R^p(M, N)$$

and by [1] (p. 349 or p. 118, Proposition 4.1.4)  $\theta$  is an isomorphism if

$$\text{Ext}_R^q(R/Rx, N) = 0 \quad \text{for } q > 0.$$

Since  $x$  is a non zerodivisor on  $R$ ,

$$0 \rightarrow Rx \rightarrow R \rightarrow R/Rx \rightarrow 0$$

is an  $R$ -projective resolution of  $R/Rx$  and so  $\text{Ext}_R^q(R/Rx, N) = 0$  for  $q > 1$ . Applying  $\text{Hom}_R(-, N)$  to this exact sequence, we obtain

$$\begin{array}{ccccc} \text{Hom}_R(R, N) & \rightarrow & \text{Hom}_R(Rx, N) & \rightarrow & \text{Ext}_R^1(R/Rx, N) \rightarrow 0, \\ \parallel & & \Delta & & \parallel \\ N & \xrightarrow{\quad} & N & & \end{array}$$

where  $\Delta$  is defined by  $n \mapsto xn$ . Since  $xN = N$ ,  $\Delta$  is surjective and  $\text{Ext}_R^1(R/Rx, N) = 0$ . Thus  $\theta$  is an isomorphism. Since  $R/Rx$  is generated by  $1 + Rx$  and  $x \cdot (1 + Rx) = 0 + Rx$ ,  $\text{Hom}_R(R/Rx, N) \cong \text{Ann}_N x$ . ■

*Proof of Theorem 3.2.* — Let  $\mathfrak{a}'$  be a subspace of  $\mathfrak{a}$  such that  $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^G$ . Let  $\sigma \in C^p(\mathfrak{a}^G, k)$ .  $\sigma$  may be extended to an element of  $C^p(\mathfrak{a}, k)$  by requiring that  $\sigma(x_1, \dots, x_p) = 0$  if there exists an  $i$ ,  $1 \leq i \leq p$ , such that  $x_i \in \mathfrak{a}'$ . The canonical embedding  $k \rightarrow k1 \subset kS \otimes kG$  enables us to regard  $\sigma \in C^p(\mathfrak{a}, k)$  as an element of  $C^p(\mathfrak{a}, kS \otimes kG)$  and clearly  $\sigma \in Z^p(\mathfrak{a}, kS \otimes kG)$ . We now show that such a  $\sigma$  cannot be a coboundary. For each  $x \in \mathfrak{a}^G$ , the image of  $x$  (viewed as an operator on  $kS \otimes kG$ ) is contained in  $\sum_{g \neq 1} kS \otimes kg$ . For each  $f \in C^{p-1}(\mathfrak{a}, kS \otimes kG)$  and  $x_1, \dots, x_p \in \mathfrak{a}^G$ ,

$$\delta f(x_1, \dots, x_p) = \sum_i (-1)^{i-1} x_i(f(x_1, \dots, \hat{x}_i, \dots, x_p))$$

and so  $\delta f(x_1, \dots, x_p) \in \sum_{g \neq 1} kS \otimes kg$ . Thus  $\sigma \neq \delta f$ . Hence the induced homomorphism of

$$H^p(\mathfrak{a}^G, k) \cong C^p(\mathfrak{a}^G, k) \rightarrow H^p(\mathfrak{a}, kS \otimes kG)$$

is injective. It remains to show that this homomorphism is surjective.

Let  $U$  be the universal enveloping algebra of  $\mathfrak{a}$ , so  $U$  is a commutative polynomial algebra. By [1], Chapter 13, Section 8,

$$H^p(\mathfrak{a}, kS \otimes kG) \cong \text{Ext}_U^p(k, kS \otimes kG).$$

For  $g \in G$ ,  $kS \otimes kg$  is an  $\mathfrak{a}$ -submodule of  $kS \otimes kG$  and

$$kS \otimes kG = \bigoplus_{g \in G} kS \otimes kg.$$

Since  $k$  is a finitely generated  $U$ -module and  $U$  is noetherian,  $k$  has a projective resolution by finitely generated free  $U$ -modules. If  $F$  is a finitely generated free module then

$$\text{Hom}_U(F, kS \otimes kG) = \bigoplus_{g \in G} \text{Hom}_U(F, kS \otimes kg).$$

Thus

$$\text{Ext}_U^p(k, kS \otimes kG) \cong \bigoplus_{g \in G} \text{Ext}_U^p(k, kS \otimes kg).$$

The action of  $U$  of  $kS \otimes kg$  is induced by the action of  $\mathfrak{a}$  on  $kS \otimes kg$  which is given by : for  $a \in \mathfrak{a}$  and  $f \in kS$ ,

$$a(f \otimes g) = a(f) \otimes g + f \otimes a(g) = (a(f) + \lambda_g(a)f) \otimes g.$$

We show first that if  $g \neq 1_G$  then  $\text{Ext}_U^p(k, kS \otimes kg) = 0$  for  $p \geq 0$ . If  $g \neq 1_G$  then, since  $\mathfrak{a}$  satisfies (Sim), there exists  $a_1 \in \mathfrak{a}$  with  $\lambda_g(a_1) \neq 0$ . Choose a basis  $\{x_i\}$  for  $\mathfrak{a}$  with  $x_1 = a_1$ . Then there exist  $c_1, \dots, c_n$ ,  $\mu \in k$  with  $\mu = \lambda_g(a_1) \neq 0$  such that

$$x_1(f \otimes g) = \left( \left( \sum_{i=1}^n c_i \partial / \partial y_i + \mu I \right) f \right) \otimes g,$$



where  $I = 1_{kS}$ . The map  $kS \rightarrow kS$  given by  $f \mapsto (\sum_i c_i \partial/\partial y_i + \mu I)f$  is bijective. (It is clearly injective and is easily seen to be surjective by induction on the degree of  $f$ .) Thus, if  $g \neq 1_G$ , then for  $p \geq 0$ ,

$$\text{Ext}_{U_0}^p(k, kS \otimes kg) = 0$$

by Lemma 3.3 (ii).

We now consider  $\text{Ext}_{U_0}^p(k, kS \otimes k1_G)$  and we will identify the  $U$ -modules  $kS \otimes k1_G$  and  $kS$  by the map  $f \otimes 1_G \mapsto f$  for  $f \in kS$ . Since  $\mathfrak{a}$  satisfies (Sim), we may choose a basis  $x_1, \dots, x_t$  for  $\mathfrak{a}$  such that for  $f \in kS$ :

$$\begin{aligned} x_i \cdot f &= \partial f / \partial y_i & \text{for } 1 \leq i \leq n, \\ &= 0 & \text{for } n+1 \leq i \leq t. \end{aligned}$$

(Thus  $x_{n+1}, \dots, x_t$  are a basis for  $\mathfrak{a}^G$ .) Since  $k$  has characteristic zero,  $x_1 \cdot kS = kS$  and  $\text{Ann}_{kS} x_1 = k[y_2, \dots, y_n]$ . So by Lemma 3.4,

$$\text{Ext}_{U_0}^p(k, kS) \cong \text{Ext}_{U_1}^p(k, k[y_2, \dots, y_n]),$$

where  $U_1 = U/Ux_1 \cong k[x_2, \dots, x_t]$ . By induction on  $n$ , we obtain

$$\text{Ext}_{U_0}^p(k, kS) \cong \text{Ext}_{U_n}^p(k, k),$$

where

$$U_n = U/Ux_1 + \dots + Ux_n \cong k[x_{n+1}, \dots, x_t].$$

Thus

$$H^p(\mathfrak{a}, kS \otimes kG) = \text{Ext}_{U_0}^p(k, kS \otimes kG) \cong \text{Ext}_{U_n}^p(k, k) \cong H^p(\mathfrak{a}^G, k).$$

Since these isomorphisms are all  $k$ -vector space isomorphisms the theorem is proved. ■

*Remark.* — In the case when  $\text{rank } G = 0$  and so  $kS \otimes kG = k[y_1, \dots, y_n]$  and  $\mathfrak{a} = \sum_{i=1}^n k \partial/\partial y_i$ , this theorem is classical, (Lemma of Poincaré). See [2] (Theorem 2.2 and following remark).

NOTE 3.5. — Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  with  $\dim \mathfrak{a} > 0$ . In this case the converse of Theorem 3.2 is also true, i. e. the monomorphism  $C^p(\mathfrak{a}^G, k) \rightarrow H^p(\mathfrak{a}, kS \otimes kG)$ ,  $p \geq 1$ , is an isomorphism for all  $p \geq 1$  only if  $\mathfrak{a}$  satisfies (Sim). (This is easily seen by reversing the steps of the proof of Theorem 3.2.) Thus there are three equivalent conditions to (Sim), namely the simplicity of an algebra, Theorem 2.2; the separating points condition, Theorem 2.4; and, when  $\dim \mathfrak{a} > 0$ , the minimality of some cohomology groups. Yet another condition was given in [4] which briefly is as follows.

$$\mathfrak{c}_2 = \sum_{j=1}^m kg_j \partial/\partial g_j$$

may be viewed as the space of diagonal matrices on  $kg_1 + \dots + kg_m$  by

$$\lambda_1 g_1 \partial/\partial g_1 + \dots + \lambda_m g_m \partial/\partial g_m \rightarrow \text{diagonal matrix } (\lambda_1, \dots, \lambda_m).$$

It was shown in [4] Section 5, that condition (ii) of (Sim) is equivalent to the condition that the algebraic hull of  $\pi_2(\mathfrak{a})$  in  $\mathfrak{c}_2$  is  $\mathfrak{c}_2$ , [where the algebraic hull of  $\pi_2(\mathfrak{a})$  is the usual algebraic hull of a space of diagonal matrices].

#### 4. Twisted Group Rings

In this section we construct the algebras  $\mathcal{A}(V, \delta, G)$  mentioned in the introduction. Let  $V$  be a finite-dimensional vector space and  $\delta$  an alternating bilinear form on  $V$ . Let

$$V^\delta = \{v \in V : \delta(v, v') = 0 \text{ for all } v' \in V\}.$$

Then  $\text{rank } \delta = \dim V - \dim V^\delta$  is even, equal to  $2l$  say where  $l \geq 0$ . With this notation we have

LEMMA 4.1. — Consider  $V$  as an abelian Lie algebra,  $k$  as a trivial  $V$ -module and  $\delta$  as an element of  $Z^2(V, K)$ . Then

$$k \#_\delta U(V) \cong A_l \otimes U(V^\delta),$$

where  $A_l$  is a Weyl algebra and  $U(V^\delta)$  is the universal enveloping algebra of the abelian Lie algebra  $V^\delta$ .

*Proof.* — Choose a basis for  $V$  adapted to  $\delta$  and use Theorem 2.1 (2). ■

$k \#_\delta U(V)$  will be denoted by  $U_\delta(V)$  and should be regarded as the algebra containing  $V$  as a subspace and generated by the elements of  $V$  subject to the relations : for all  $v, v' \in V$ ,  $vv' - v'v = \delta(v, v')$ .

As noted in Section 1, if  $\lambda \in V^*$  then there corresponds a unique automorphism  $\theta_\lambda$  of  $U_\delta(V)$  defined by  $\theta_\lambda(v) = v + \lambda(v)$ ,  $v \in V$ . Let  $\Phi : V^* \rightarrow \text{Aut } U_\delta(V)$  be given by  $\lambda \mapsto \theta_\lambda$ . If  $\lambda, \mu \in V^*$  then  $\theta_{\lambda+\mu} = \theta_\lambda \theta_\mu$  so  $\Phi$  is a group monomorphism of the additive group  $V^*$  into  $\text{Aut } U_\delta(V)$ . If  $G$  is a subgroup of  $\Phi(V^*)$  then the inverse image of  $G$  under  $\Phi$  will be denoted by  $G$  and if  $H$  is a subgroup of  $V^*$  then  $\Phi(H)$  will be denoted by  $H$ . If  $G$  is a finitely generated subgroup of  $V^*$  then  $G$  is a torsionfree group (since  $k$  has characteristic zero) and hence is a free abelian group of finite rank.

We note in passing that if  $\lambda \in V^*$  then  $\theta_\lambda = \exp d_\lambda$ , (exponential  $d_\lambda$ ), where  $d_\lambda$  is the unique derivation of  $U_\delta(V)$  such that

$$\begin{array}{ccc} & d_\lambda & \\ U_\delta(V) & \xrightarrow{\quad} & U_\delta(V) \\ \uparrow & \lambda & \uparrow \\ V & \longrightarrow & k \end{array}$$

is commutative.  $d_\lambda$  is locally nilpotent and  $\exp d_\lambda$  is defined, (since  $k$  has characteristic zero), and is an automorphism of  $U_\delta(V)$  which coincides with  $\theta_\lambda$  since they agree on  $V$ .

Thus given a finite dimensional vector space  $V$ , an alternating bilinear form  $\delta$  on  $V$  and a finitely generated subgroup  $G$  of  $V^*$  we can form the twisted group ring  $U_\delta(V) \# kG$  and we will denote this algebra by  $\mathcal{A}(V, \delta, G)$ . Conversely an algebra  $(A_l \otimes U(W)) \# kG$ , where  $G$  is finitely generated and each element of  $G$  preserves

the filtration and induces the identity automorphism on the associated graded algebra, is an  $\mathcal{A}(V, \delta, G)$ , where

$$V = W + \sum_{i=1}^l kx_i + ky_i$$

and for  $v, v' \in V$ ,  $\delta(v, v') = [v, v']$ .

Two specialisations of  $\mathcal{A} = \mathcal{A}(V, \delta, G)$  are the following. If  $G = 0$  and  $\delta$  is non-singular on  $V$  then  $\mathcal{A} \cong A_p$ , where  $2p = \dim V$ . If  $\delta = 0$  and  $m = \text{rank } G = \dim V$  and  $G$  spans  $V^*$  then  $\mathcal{A} \cong A'_m$ . To see this let  $g_1, \dots, g_m$  be a set of generators of  $G$ , let  $\lambda_1, \dots, \lambda_m$  be the corresponding elements of  $V^*$ , which are a basis for  $V^*$ , and let  $v_1, \dots, v_m$  be a dual basis for  $V$  with  $\lambda_i(v_j) = \Delta_{ij}$ . So  $[g_i, v_j] = \Delta_{ij} g_i$  and

$$\mathcal{A} = k[g_1, g_1^{-1}, \dots, g_m, g_m^{-1}, v_1, \dots, v_m].$$

Set  $x_i = g_i$  and  $y_i = g_i^{-1} v_i$ . With the notation as defined in Section 1 we have

**THEOREM 4.2.** — *Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  which satisfies (Sim) and let  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$ . Then there exists a finite dimensional vector space  $V$ , an alternating bilinear form  $\delta$  on  $V$  and a finitely generated subgroup  $G$  of  $V^*$  such that*

$$(kS \otimes kG) \#_{\sigma} U(\mathfrak{a}) \cong \mathcal{A}(V, \delta, G).$$

*Proof.* — By Theorems 3.1 and 3.2 we may assume that  $\sigma \in C^2(\mathfrak{a}, k)$ . Also if  $\tau \in C^2(\mathfrak{a}, k)$  then  $\sigma$  and  $\tau$  determine isomorphic algebras if  $\sigma|_{\mathfrak{a}^G} = \tau|_{\mathfrak{a}^G}$ . Thus if  $\mathfrak{a}'$  is a subspace of  $\mathfrak{a}$  such that  $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}^G$  then we may assume that  $\mathfrak{a}'$  is  $\sigma$ -orthogonal to  $\mathfrak{a}$ . Let  $\text{rank}(\sigma|_{\mathfrak{a}^G})$  be  $2r$  and let  $W$  be the subspace of  $\mathfrak{a}^G$  which is the  $\sigma$ -orthogonal complement of  $\mathfrak{a}^G$ . Let  $V$  be the subspace  $S \oplus \mathfrak{a}$  of  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$ , where  $S$  denotes the space of homogeneous elements of degree one of  $kS$ .  $\delta \in C^2(V, k)$  is defined by

$$\delta(v, v') = [v, v'], \quad v, v' \in V.$$

By (Sim),  $\pi_1(\mathfrak{a}') = \mathfrak{c}_1$  and so the subalgebra generated by  $S$  and  $\mathfrak{a}'$  is  $A_n$ , where  $n$  is the Krull dimension of  $kS$ . By Theorem 2.1 (2), the subalgebra generated by  $\mathfrak{a}^G$  is  $A_r \otimes U(W)$  and the subalgebra generated by  $V$  is  $A_{n+r} \otimes U(W)$  and so is  $U_{\delta}(V)$ .

For  $a \in \mathfrak{a}$  and  $g \in G$ ,  $[a, g] = \lambda_g(a)g$  or equivalently  $ga = (a - \lambda_g(a))g$ .  $\lambda_g \in \mathfrak{a}^*$  and  $\lambda_g$  may be extended to an element of  $V^*$  by setting  $\lambda_g(S) = 0$ . Thus for  $v \in V$  and  $g \in G$ ,

$$gv = (v - \lambda_g(v))g \quad \text{or} \quad gvg^{-1} = v - \lambda_g(v).$$

Thus each  $g \in G$  determines an automorphism of  $U_{\delta}(V)$  given by  $u \mapsto gug^{-1}$  and if  $g, h \in G$  with  $g \neq h$ , then  $\lambda_g|_{\mathfrak{a}} \neq \lambda_h|_{\mathfrak{a}}$  by (Sim) and so  $g$  and  $h$  determine distinct automorphisms of  $U_{\delta}(V)$ . Since

$$(kS \otimes kG) \#_{\sigma} U(\mathfrak{a}) \cong kG \otimes kS \otimes U(\mathfrak{a}),$$

as vector spaces,  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$  is a free right  $U_{\delta}(V)$ -module with the elements

of  $G$  as a free basis. Thus

$$(kS \otimes kG) \#_{\sigma} U(\mathfrak{a}) \cong \mathcal{A}(V, \delta, G),$$

where  $G = \{ -\lambda_g : g \in G \}$  is a subgroup of  $V^*$ .

$\mathcal{A}(V, \delta, G)$  satisfies the following universal property.

**THEOREM 4.3.** — *Let  $\mathcal{A}(V, \delta, G)$  be as above. Let  $B$  be an algebra,  $\chi : V \rightarrow B$  be a  $k$ -module map and  $\psi : G \rightarrow (\text{Units of } B)$  be a group homomorphism such that*

- (i)  $[\chi(v), \chi(v')] = \delta(v, v'), v, v' \in V,$
- (ii)  $\psi(\lambda) \chi(v) = (\chi(v) + \lambda(v)) \psi(\lambda).$

*Then there exists a unique homomorphism  $\chi' : \mathcal{A}(V, \delta, G) \rightarrow B$  such that  $\chi' \mid V = \chi$  and for  $\lambda \in G$ ,  $\chi'(\theta_{\lambda}) = \psi(\lambda).$*

*Proof.* — By the remark following Theorem 2.1 and (i),  $\chi$  can be extended to an algebra homomorphism of  $U_{\delta}(V)$  to  $B$ . By (ii) and the universal property of a twisted group ring, there is an algebra homomorphism  $\chi'$  as required. ■

**LEMMA 4.4.** — *Consider an algebra  $\mathcal{A}(V, \delta, G)$ . If  $G$  spans  $V^*$  then for each  $r \in \mathcal{A}(V, \delta, G)$  with  $r \notin k$  there exists  $\lambda \in G$  such that  $\exp d_{\lambda}(r) \neq r$  and there exists  $g \in G$  such that  $gr \neq rg$ .*

*Proof.* — Let  $0 \neq \lambda \in G$ , denote  $\exp d_{\lambda}$  by  $g$ , set  $V' = \text{Ker } \lambda$  and choose  $v \in V$  so that  $V = V' \oplus kv$ . As  $k$ -modules,

$$U_{\delta}(V) \cong U_{\delta}(V') \otimes k[v].$$

For  $r \in U_{\delta}(V)$ ,  $grg^{-1} = \exp d_{\lambda}(r)$ . If  $r = \sum_{i=0}^s r_i \otimes v^i$ , where  $r_i \in U_{\delta}(V')$ ,  $i = 0, \dots, s$ , and  $r_s \neq 0$  then, since

$$\exp d_{\lambda}(v^i) = (v + \lambda(v))^i,$$

the coefficient of  $v^{s-1}$  in  $\exp d_{\lambda}(r) - r$  is  $s\lambda(v)r_s$ . So if  $\exp d_{\lambda}(r) = r$  then,  $\lambda(v)r_s \neq 0$ ,  $s = 0$  and  $r \in U_{\delta}(V')$ . Thus if  $r$  commutes with  $g$  for all  $g \in G$ , then

$$r \in \bigcap_{\lambda} U_{\delta}(\text{Ker } \lambda) = U_{\delta}(\bigcap_{\lambda} \text{Ker } \lambda),$$

by a Poincaré-Birkhoff-Witt argument. Since  $G$  spans  $V^*$ ,  $U_{\delta}(\bigcap_{\lambda} \text{Ker } \lambda) = k$ . ■

**NOTATION 4.5.** — Let  $V$  be a finite-dimensional vector space,  $\delta$  an alternating bilinear form on  $V$  and  $G$  a subgroup of  $V^*$ . Set  $V^G = \bigcap_{\lambda} \text{Ker } \lambda$ , ( $\lambda \in G$ ).  $V^{\delta}$  denotes the  $\delta$ -orthogonal complement of  $V$  and  $(V^G)^{\delta}$  the subspace of  $V^G$  which is  $\delta$ -orthogonal to  $V^G$ .

**THEOREM 4.6.** — *Let  $V$  be a finite-dimensional vector space,  $\delta$  an alternating bilinear form on  $V$  and  $G$  a finitely generated subgroup of  $V^*$ . The following conditions are equivalent.*

- (i)  $\mathcal{A}(V, \delta, G) \equiv U_\delta(V) \# kG$  is a simple algebra.
- (ii)  $\{r \in U(V^\delta) : \exp d_\lambda(r) = r \text{ for all } \lambda \in G\} = k$ .
- (iii)  $G|V^\delta$  spans  $(V^\delta)^*$ , i. e.  $V^G \cap V^\delta = 0$ .
- (iv)  $U(V^\delta)$  is the only non-zero ideal of  $U(V^\delta)$  which is invariant under  $G$ .
- (v)  $\mathcal{A}(V^\delta, 0, G|V^\delta) = U(V^\delta) \# kH$ , where  $H = G|V^\delta$ , is a simple subalgebra of  $\mathcal{A}(V, \delta, G)$ .

*Proof.* — (i)  $\rightarrow$  (ii)  $\rightarrow$  (iii)  $\rightarrow$  (ii)  $\rightarrow$  (iv)  $\rightarrow$  (i), (iv)  $\rightarrow$  (v)  $\rightarrow$  (iii), (i)  $\rightarrow$  (ii) If there exists an  $r \in U(V^\delta)$  with  $r \notin k$  such that  $\exp d_\lambda(r) = r$  for all  $\lambda \in G$  then  $r$  is a central non unit in  $\mathcal{A}(V, \delta, G)$  and so this algebra is not simple.

(ii)  $\rightarrow$  (iii). If  $r \in V^\delta$  then  $\exp d_\lambda(r) = r + \lambda(r)$ .

(iii)  $\rightarrow$  (ii) by Lemma 4.4.

(ii)  $\rightarrow$  (iv). Let  $I$  be a nonzero ideal of  $U(V^\delta)$  which is invariant under  $G$ . If  $0 \neq r \in I$  and  $r$  has minimal degree then  $I \ni \exp d_\lambda(r) - r$  which is zero or has degree less than  $r$ . Thus  $\exp d_\lambda(r) = r$  for all  $\lambda \in G$  and so  $I \cap k \neq 0$ .

(iv)  $\rightarrow$  (i). We use a variation of a classical "minimal length of a relation" argument. (The particular case when  $V^\delta = V$  may be deduced from [5], (Theorem 1.5 and the remark on pp 260-261).) Note first that if  $v \in V$ ,  $0 \neq u \in U_\delta(V)$  and  $g = \exp d_\lambda \in G$  then

$$[v, ug] = vug - u(v + \lambda(v))g = ([v, u] - \lambda(v)u)g$$

and if  $\lambda(v) \neq 0$  then  $[v, u] - \lambda(v)u \neq 0$  since  $\text{degree } [v, u] < \text{degree } u$ . We show that if  $J$  is a nonzero ideal of  $U_\delta(V) \# kG$  then  $J \cap U(V^\delta)$  is a nonzero ideal of  $U(V^\delta)$  which is invariant under  $G$ . Among the non-zero elements of  $J$ , choose  $r = \sum_{i=1}^s u_i g_i$  with the property firstly that  $s = \text{length } r$  is minimal and secondly that  $\text{degree } u_1$  is minimal. Without loss of generality we may assume that  $g_1 = 1_G$ . Firstly  $u_1 \in U(V^\delta)$ , since otherwise there exists  $0 \neq v \in V$  with  $\text{degree } [v, u_1] < \text{degree } u_1$  and  $[v, r] \in J$ , which contradicts the minimality of  $r$ . If  $\text{length } r = s > 1$ , suppose  $g_i = \exp d_{\lambda_i}$  and choose  $v \in V$  with  $v \notin \text{Ker } \lambda_2$ . Then

$$[v, r] = \sum_{i=2}^s ([v, u_i] - \lambda_i(v)u_i)g_i \neq 0,$$

since  $\lambda_2(v) \neq 0$ , and this contradicts the minimality of  $r$ . Thus  $J \cap U(V^\delta) \neq 0$ .

(iv)  $\rightarrow$  (v). That  $\mathcal{A}(V^\delta, 0, G|V^\delta)$  is simple follows from (iv)  $\rightarrow$  (i) by supposing that  $V = V^\delta$ . It remains to show that  $\mathcal{A}(V^\delta, 0, G|V^\delta)$  may be regarded as a subalgebra of  $\mathcal{A}(V, \delta, G)$ . Consider the canonical homomorphism  $G \rightarrow G|V^\delta$ .  $G|V^\delta$  is finitely generated and hence free abelian so the exact sequence

$$0 \rightarrow \text{Ker} \rightarrow G \rightarrow G|V^\delta \rightarrow 0$$

splits and we may regard  $G|V^\delta$  as a subgroup of  $G$ . Thus  $\mathcal{A}(V^\delta, 0, G|V^\delta)$  may be identified with a subalgebra of  $\mathcal{A}(V, \delta, G)$ . (This is the only place in the proof of the Theorem where the assumption that  $G$  is finitely generated is used).

(v)  $\rightarrow$  (iii) is the particular case of (i)  $\rightarrow$  (iii) corresponding to  $V = V^\delta$ . ■

We now give the alternative proof of part of Theorem 2.2.

**THEOREM 4.7.** *Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  which satisfies (Sim) and  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$ . Then  $(kS \otimes kG) \#_\sigma U(\mathfrak{a})$  is a simple algebra.*

*Proof.* — Consider first the case when  $\sigma = 0$ .

Then by Theorem 4.2,  $(kS \otimes kG) \# U(\mathfrak{a}) \cong \mathcal{A}(V, \delta, G)$ , where  $V^\delta = \mathfrak{a}^G$  and so  $V^\delta \cap V^G = 0$ .

If  $\sigma \neq 0$  then by Theorem 4.2 again,

$$(kS \otimes kG) \#_\sigma U(\mathfrak{a}) \cong \mathcal{A}(V, \gamma, G),$$

where  $V^\gamma \subseteq V^\delta = \mathfrak{a}^G$ . So  $V^\gamma \cap V^G = 0$ . In either case apply Theorem 4.6 (iii). ■

**THEOREM 4.8.** — *Let  $\mathcal{A} = \mathcal{A}(V, \delta, G)$  be a simple algebra. Then*

(i) *There exists an abelian by abelian completely solvable Lie algebra  $\mathfrak{g}$  and a prime ideal  $P$  of  $U = U(\mathfrak{g})$  such that*

$$\mathcal{A} \cong (U/P)_E.$$

(ii)  $\mathcal{A} \cong (kS \otimes kG) \#_\sigma U(\mathfrak{a})$ , for a suitable choice of  $S, G, \mathfrak{a} \subset \mathfrak{c}$  satisfying (Sim) and  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$ .

*Proof.* — (i) Let  $\dim(V^G)^\delta = q$  and  $\dim V^G = 2p + q$ . Since  $\mathcal{A}$  is simple,  $V^G \cap V^\delta = 0$  and we can choose a basis

$$x_1, y_1, \dots, x_p, y_p, s_1, \dots, s_t$$

for  $V$  adapted to  $\delta$  so that  $y_{p+1}, \dots, y_{p+q}$  is a basis for  $(V^G)^\delta$  and

$$x_1, y_1, \dots, x_p, y_p, y_{p+1}, y_{p+2}, \dots, y_{p+q}$$

is a basis for  $V^G$ .

Let  $g_1, \dots, g_m$  be generators for  $G$  corresponding to linear forms  $\lambda_1, \dots, \lambda_m \in V^*$ . (Recall that  $G$  is free abelian of rank  $m$ .) Consider the Lie algebra  $\mathfrak{g}$  of dimension  $2l + t + m + 1$  defined as follows.  $\mathfrak{g} = kw + \sum_{i=1}^m kg_i + V$ , i.e.  $\mathfrak{g}$  has a basis

$$w, g_1, \dots, g_m, y_1, \dots, y_{p+q}, y_{p+q+1}, \dots, y_t, x_1, \dots, x_p, s_1, \dots, s_t,$$

and relations  $[x_i, y_j] = \Delta_{ij} w$ ,

$$[g_i, v] = \lambda_i(v) g_i \quad \text{for } v \in V,$$

and all other commutators of basis elements are zero.

$w, g_1, \dots, g_m, y_1, \dots, y_{p+q}$  span an abelian ideal  $\mathfrak{h}$  of  $\mathfrak{g}$  and  $\mathfrak{g}/\mathfrak{h}$  is abelian.  $w \in \text{Centre } \mathfrak{g}$  and if  $E$  denotes the subsemi-group of  $U$  generated by  $g_1, \dots, g_m$  then

$$(U/(w-1))_E \cong \mathcal{A},$$

by Theorem 4.3 and the simplicity of  $\mathcal{A}$ . (The Heisenberg algebra of dimension  $2l+1$  is the subalgebra of  $\mathfrak{g}$  spanned by  $x_i, y_i$  and  $w$ ,  $1 \leq i \leq l$  and it is easy to see that  $\mathfrak{g}$  is a

subalgebra of the Lie algebra which is the direct sum of the Heisenberg algebra of dimension  $2l+1$  and  $m$  copies of the two dimensional solvable Lie algebra. Compare Theorem 4.9 below.)

(ii) In  $\mathcal{A}$ , set  $kS = k[y_1, \dots, y_{p+q}]$ ,  $kG = k[g_1, g_1^{-1}, \dots, g_m, g_m^{-1}]$  and let  $\mathfrak{a}$  be the space spanned by

$$x_1, x_2, \dots, x_p; x_{p+1}, \dots, x_{p+q}, x_{p+q+1}, \dots, x_l, y_{p+q+1}, \dots, y_l, s_1, \dots, s_t.$$

Then  $\mathfrak{a} \subset \mathfrak{c}$  and  $x_1, \dots, x_p$  spans  $\mathfrak{a} \cap \mathfrak{c}_1$  and

$$x_{p+q+1}, \dots, x_l, y_{p+q+1}, \dots, y_l, s_1, \dots, s_t$$

spans  $\mathfrak{a}^G$ . The 2-cocycle  $\sigma \in C^2(\mathfrak{a}, k)$  is the restriction of  $\delta$  to  $\mathfrak{a}$ . ■

In the next section we show that the integers  $p, q, 2l+t$  and  $m = \text{rank } G$  are isomorphism invariants for  $\mathcal{A} = \mathcal{A}(V, \delta, G)$ .  $\mathcal{A}$  may be presented as  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$  and conversely, (Theorems 4.2 and 4.8) and these integers appear in the two different presentations as follows :

$$q = \dim(V^G)^{\delta}, \quad p = \frac{1}{2}(\dim V^G - \dim(V^G)^{\delta}), \quad 2l+t = \dim V \quad (\text{and } 2l = \text{rank } \delta);$$

Krull dimension of  $kS$  is  $p+q$ ,

$$\dim(\mathfrak{a} \cap \mathfrak{c}_1) = p, \quad \dim \mathfrak{a} = 2l+t-p-q,$$

$$\dim \mathfrak{a}^G = 2l+t-2p-2q \quad [\text{and if } \sigma \in C^2(\mathfrak{a}, k) \text{ then } 2l-2p-2q = \text{rank}(\sigma|_{\mathfrak{a}^G})].$$

**THEOREM 4.9.** — *Let  $\mathcal{A} = \mathcal{A}(V, \delta, G)$  be a simple algebra. If  $\text{rank } \delta = 2l$  and  $\text{rank } G = m$  then  $\mathcal{A}$  is a subalgebra of  $A_l \otimes A'_m$ .*

*Proof.* — Choose a basis  $x_1, y_1, \dots, x_l, y_l, s_1, \dots, s_t$  for  $V$  adapted to  $\delta$ . Let  $g_1, \dots, g_m$  be generators for  $G$  and suppose that

$$[g_j, x_i] = \lambda_{ji} g_i, \quad [g_j, y_i] = \mu_{ji} g_j \quad \text{and} \quad [g_j, s_i] = \nu_{ji} g_j.$$

Let

$$A_l = k[X_1, Y_1, \dots, X_l, Y_l] \quad \text{with} \quad [X_i, Y_j] = \Delta_{ij}$$

and

$$A'_m = k[U_1, U_1^{-1}, V_1, \dots, U_m, U_m^{-1}, V_m] \quad \text{with} \quad [U_i, V_j] = \Delta_{ij} U_i.$$

Consider the vector space map,

$$V \rightarrow \sum_{i,j} kX_i + kY_i + kV_j \quad (i = 1, \dots, l, j = 1, \dots, m)$$

given by

$$x_i \rightarrow X_i + \sum_{j=1}^m \lambda_{ji} V_j,$$

$$y_i \rightarrow Y_i + \sum_{j=1}^m \mu_{ji} V_j,$$

$$s_i \rightarrow \sum_{j=1}^m \nu_{ji} V_j.$$

By the universal property of  $\mathcal{A}$ , Theorem 4.3, this map extends to an algebra homomorphism  $\mathcal{A} \rightarrow A_l \otimes A'_m$  mapping  $g_j \rightarrow U_j$  for  $1 \leq j \leq m$ . Since  $\mathcal{A}$  is simple this homomorphism is a monomorphism. ■

The class of simple subalgebras of  $A_l \otimes A'_m$ , each of which is the image of a simple algebra  $\mathcal{A}(V, \delta, G)$  as in the proof of Theorem 4.9, may be described intrinsically as follows.

In  $A_l \otimes A'_m$ , let  $W_1 = \sum_{i=1}^m k X_i + k Y_i$ ,  $W_2 = \sum_{j=1}^m k V_j$  and  $W = W_1 + W_2$ . Let  $\pi_1$  and  $\pi_2$  be the canonical projections of  $W$  onto  $W_1$  and  $W_2$  respectively. If  $w \in W_2$  then  $[w, U_i] = \lambda_i(w) U_i$ , where  $\lambda_i \in W_2^*$ . The map

$$w \rightarrow \text{Diag}(\lambda_1(w), \dots, \lambda_m(w))$$

is a linear transformation of  $W_2$  into the space of diagonal matrices acting on  $k U_1 + \dots + k U_m$  and we identify  $W_2$  with this space of diagonal matrices.

**THEOREM 4.10.** — *Let  $V$  be a subspace of  $W$ . Then the subalgebra of  $A_l \otimes A'_m$  generated by  $V$  and the units of  $A_l \otimes A'_m$  is a simple algebra if  $V$  satisfies*

- (i)  $\pi_1(V) = W_1$  and
- (ii) the algebraic hull of  $\pi_2(V) = W_2$ , [where  $\pi_2(V)$  and  $W_2$  are considered as spaces of diagonal matrices].

*Proof.* — By [4], Remark before Theorem 5.3, the algebraic hull of  $\pi_2(V)$  is  $W_2$  if and only if  $\lambda_1|_{\pi_2(V)}, \dots, \lambda_m|_{\pi_2(V)}$  are linearly independent over the rational field  $\mathbb{Q}$ . If  $g$  is a unit in  $A_l \otimes A'_m$  then  $g$  is a monomial in  $U_1, U_1^{-1}, \dots, U_m, U_m^{-1}$ . Let  $\lambda_g \in W_2^*$  be defined by  $[w, g] = \lambda_g(w) g$  for  $w \in W_2$ .

If  $g$  and  $h$  are units in  $A_l \otimes A'_m$  with  $g \neq h$  then  $\lambda_1|_{\pi_2(V)}, \dots, \lambda_m|_{\pi_2(V)}$  are linearly independent over  $\mathbb{Q}$  if and only if  $\lambda_g|_{\pi_2(V)} \neq \lambda_h|_{\pi_2(V)}$ . If the latter condition holds then the subalgebra generated by  $V$  and the units is of the form  $\mathcal{A}(V, \delta, G)$  and this is a simple algebra since  $V^\delta \cap V^G = 0$ . ■

**REMARK 4.11.** —  $A_l \otimes A'_m \cong \text{Diff}(k S \otimes k G) \subset \text{End}(k S \otimes k G)$ , where

$$k S = k[Y_1, \dots, Y_l] \quad \text{and} \quad k G = k[U_1, U_1^{-1}, \dots, U_m, U_m^{-1}].$$

The simple algebras of Theorem 4.10 are simple subalgebras of  $\text{Diff}(k S \otimes k G)$  which do not necessarily contain the multiplications by elements of  $k S \otimes k G$ . Is there a sense in which they are “dense subalgebras” of  $\text{Diff } k S \otimes k G$ ?

## 5. Isomorphism Theorems

We now consider when two simple algebras of the form  $\mathcal{A} = \mathcal{A}(V, \delta, G)$  are isomorphic. If  $G = 0$  then  $\mathcal{A} \cong A_n$ , where  $2n = \dim V$  and  $A_n \cong A_m$  if and only if  $n = m$ , by [6] (2.6, Proposition). If  $G \neq 0$  then, unlike  $A_n$ ,  $\mathcal{A}(V, \delta, G)$  has units which do not belong to  $k$ . So it is sufficient to consider the case when  $G \neq 0$  and we examine how  $\mathcal{A}(V, \delta, G)$  is built around its group of units.



LEMMA 5.1. — Consider  $\mathcal{A} = \mathcal{A}(V, \delta, G)$ , where  $G \neq 0$  and let  $V'$  be a subspace of  $V$  such that  $V = V^G \oplus V'$ .

- (i)  $C$ , the centraliser in  $\mathcal{A}$  of the group of units of  $\mathcal{A}$ , is  $U_\delta(V^G) \otimes kG$ .
- (ii) Let  $L$  be the Lie subalgebra of  $\mathcal{A}$  defined by  $r \in L \Leftrightarrow [r, g] \in kg$  for each unit  $g \in \mathcal{A}$ . Then  $L = C + V'$ .
- (iii)  $D$ , the centre of  $C$ , is  $U((V^G)^\delta) \otimes kG$ .
- (iv) The first two terms of the upper central series of  $D$  as an  $L$ -module, are  $k$  and  $k + (V^G)^\delta$ .

*Proof.* — (i) Clearly  $U_\delta(V^G) \otimes kG \subset C$  since the units of  $\mathcal{A}$  are just the scalar multiples of the elements of  $G$ . Now  $\mathcal{A} \cong U_\delta(V) \# kG \cong U_\delta(V^G) \otimes U_\delta(V') \otimes kG$ , as  $k$ -modules but not as algebras. Since  $G \mid V'$  spans  $(V')^*$ ,  $C \cap U_\delta(V') = k$  by Lemma 4.4. Thus, by a unique representation of elements argument,

$$C = U_\delta(V^G) \otimes k \otimes kG = U_\delta(V^G) \otimes kG.$$

(ii) Choose  $g_1, \dots, g_s \in G$ , where  $g_i$  corresponds to  $\lambda_i \in V^*$  such that the corresponding elements of  $(V')^*$ ,  $\lambda_1 \mid V', \dots, \lambda_s \mid V'$ , are a basis for  $(V')^*$ . Let  $v_1, \dots, v_s$  be a dual basis for  $V'$  with  $\lambda_i(v_j) = \Delta_{ij}$ . Then  $[g_i, v_j] = \Delta_{ij} g_i$ . We show first that

$$\{r \in \mathcal{A} : [g, r] \in C\} = C + Cv_1 + \dots + Cv_s.$$

This follows from

$$[g_i, cv_1^h \dots v_l^i \dots v_s^j] = icg_i v_1^h \dots v_l^{i-1} \dots v_s^j$$

modulo elements of lower degree, for  $1 \leq l \leq s$ . Also  $[g_i, c_1 v_1 + \dots + c_s v_s] = c_i g_i$  which belongs to  $kg_i$  if and only if  $c_i \in k$ . Thus  $L = C + V'$ .

(iii) Let  $V_1 = (V^G)^\delta$  and let  $V_2$  be a subspace of  $V^G$  such that  $V^G = V_1 \oplus V_2$ . Then  $\delta \mid V_2$  is non singular and  $U_\delta(V_2) \cong A_p$ , where  $p = (1/2) \dim V_2$ . Thus

$$C \cong A_p \otimes U(V_1) \otimes kG$$

and so the centre of  $C$  is  $U(V_1) \otimes kG$ .

(iv) As an  $L$ -module,  $D$  is a direct sum of  $L$ -submodules,  $D = \sum_{g \in G} U(V_1) \otimes kg$ . If  $g \neq 1_G$  then no nonzero element of  $U(V_1) \otimes kg$  is annihilated by  $\text{ad } L$ . [Compare the proof of (Theorem 4.6 (iv)  $\Rightarrow$  (i)).] Hence the upper central series for  $D$ , as an  $L$ -module, is just the upper central series for  $U(V_1)$  as an  $L$ -module and the assertion follows at once since  $V^G \cap V^\delta = 0$ . ■

LEMMA 5.2. — Let  $\mathfrak{a}$  be a subspace of  $\mathfrak{c}$  which satisfies (Sim) and  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$ . Let  $S_1$  denote the subspace  $k + ky_1 + \dots + ky_n$  of  $kS$ . Then

$$M = \{r \in (kS \otimes kG) \#_\sigma U(\mathfrak{a}) : [s, r] \in k \text{ for } s \in S_1 \text{ and } [g, r] \in kg \text{ for } g \in G\}$$

is the Lie algebra  $(kS \otimes kG) \times_\sigma \mathfrak{a}$  [considered as a subspace of  $(kS \otimes kG) \#_\sigma U(\mathfrak{a})$ ].

*Proof.* — Let  $L$  be as in Lemma 5.1. Then

$$M = \{r \in L : [s, r] \in k \text{ for } s \in S_1\} = (kS \otimes kG) \times_\sigma \mathfrak{a}$$

by a similar argument to that used in Lemma 5.1 (ii). ■

The following theorem should be compared with Theorem 3.1.

**THEOREM 5.3.** — *Let  $\mathfrak{a}$  and  $\mathfrak{b}$  be subspaces of  $\mathfrak{c}$  each of which satisfies (Sim). Let  $\sigma \in Z^2(\mathfrak{a}, kS \otimes kG)$  and  $\tau \in Z^2(\mathfrak{b}, kS \otimes kG)$ . Then there exists an isomorphism  $\theta$  of  $(kS \otimes kG) \# {}_\sigma U(\mathfrak{a})$  onto  $(kS \otimes kG) \# {}_\tau U(\mathfrak{b})$  which extends the identity map from  $kS \otimes kG$  to  $kS \otimes kG$  if and only if  $\mathfrak{a} = \mathfrak{b}$  and  $\sigma$  and  $\tau$  are cohomologous cocycles in  $Z^2(\mathfrak{a}, kS \otimes kG)$ .*

*Proof.* — Suppose  $\theta$  exists. By Lemma 5.2,  $\theta$  induces a Lie algebra isomorphism

$$(kS \otimes kG) \times {}_\sigma \mathfrak{a} \rightarrow (kS \otimes kG) \times {}_\tau \mathfrak{b}$$

which is the identity map on  $kS \otimes kG$ . For  $\xi \in kS \otimes kG$  and  $a \in \mathfrak{a}$ ,  $[a, \xi] = [\theta(a), \xi]$ . Thus, viewed as elements of  $\text{Der}(kS \otimes kG)$ ,  $a = \theta(a)$ . Since  $\mathfrak{b}$  and  $\mathfrak{a}$  are subspaces of  $\mathfrak{c}$ , this implies that  $\mathfrak{b} = \mathfrak{a}$  and that  $\theta$  induces the identity map from  $\mathfrak{a}$  to  $\mathfrak{a}$  in the following diagram of Lie algebras, where the horizontal maps are the obvious inclusions and projections :

$$\begin{array}{ccccccc} 0 & \rightarrow & kS \otimes kG & \rightarrow & (kS \otimes kG) \times {}_\sigma \mathfrak{a} & \rightarrow & \mathfrak{a} \rightarrow 0 \\ & & \downarrow 1 & & \downarrow \theta & & \downarrow 1 \\ 0 & \rightarrow & kS \otimes kG & \rightarrow & (kS \otimes kG) \times {}_\tau \mathfrak{a} & \rightarrow & \mathfrak{a} = 0. \end{array}$$

Thus  $\sigma$  and  $\tau$  are cohomologous cocycles. ■

**DEFINITION 5.4.** — Let  $\mathcal{A}(V, \delta, G)$  and  $\mathcal{A}(W, \gamma, H)$  be simple algebras. These algebras are said to be *locally isomorphic* if there exists a vector space isomorphism  $\varphi : V \rightarrow W$  such that

- (i)  $\delta(v_1, v_2) = \gamma(\varphi(v_1), \varphi(v_2))$ , for all  $v_1, v_2 \in V$  and
- (ii) the dual map  $\varphi^* : W^* \rightarrow V^*$  restricts to an isomorphism of  $H$  onto  $G$ .

**THEOREM 5.5.** — *If  $\mathcal{A}(V, \delta, G)$  and  $\mathcal{A}(W, \gamma, H)$  are simple algebras which are locally isomorphic with respect to  $\varphi : V \rightarrow W$  then there exists an algebra isomorphism  $\psi : \mathcal{A}(V, \delta, G) \rightarrow \mathcal{A}(W, \gamma, H)$  which extends  $\varphi$ .*

*Proof.* — For  $\lambda \in G$  let  $g(\lambda)$  denote the corresponding element of  $G$  and similarly for  $H$ .

Set  $\psi(g(\lambda)) = h((\varphi^*)^{-1}(\lambda))$  and  $\psi(v) = \varphi(v)$ ,  $\lambda \in G$ ,  $v \in V$ . Then by the universal property (Theorem 4.3)  $\psi$  extends to an algebra homomorphism of  $\mathcal{A}(V, \delta, G)$  onto  $\mathcal{A}(W, \gamma, H)$  and  $\psi$  is injective since  $\mathcal{A}(V, \delta, G)$  is a simple algebra. ■

**CONJECTURE 5.6.** — If  $\mathcal{A}(V, \delta, G)$  and  $\mathcal{A}(W, \gamma, H)$  are simple algebras which are isomorphic (as algebras) then they are locally isomorphic.

We will prove this conjecture in the case when  $V^G = (V^G)^\delta$ , [i. e. when the centraliser in  $\mathcal{A}(V, \delta, G)$  of its group of units is a commutative algebra] and go some way towards answering it in the general case.

Let  $\mathcal{A} = \mathcal{A}(V, \delta, G)$ , where  $G \neq 0$ . By Lemma 5.1,  $C = U_\delta(V^G) \otimes kG$ ,  $L = U_\delta(V^G) \otimes kG + V'$ ,  $D = U((V^G)^\delta) \otimes kG$  and  $k + (V^G)^\delta$  are subsets of  $\mathcal{A}$  which

are independent of the presentation of  $\mathcal{A}$  as  $\mathcal{A}(V, \delta, G)$ . Denote  $\dim(V^G)^\delta$  by  $q$  and then  $\dim V^G = 2p + q$  for some integer  $p \geq 0$ . So  $C \cong A_p \otimes D$ . If  $I$  is an ideal of  $A_p \otimes D$  which is generated by an ideal of codimension one of  $D$  then  $A_p \cong C/I$  and the Krull dimension of  $C/I$  is  $p$  by [8]. The subalgebra of  $\mathcal{A}$  generated by the units is  $kG$  and the Krull dimension of  $kG$  is  $m = \text{rank } G$ . Also  $\dim L/C = \dim V'$  is uniquely determined by  $\mathcal{A}$  so we have four integer valued isomorphism invariants for  $\mathcal{A}$ , viz  $p, q, m$  and  $\dim V = \dim V' + 2p + q$ .

Suppose now that  $\chi$  is an isomorphism of  $\mathcal{A}(V, \delta, G)$  onto  $\mathcal{A}(W, \gamma, H)$ . We endeavour to define a vector space isomorphism  $\phi : V \rightarrow W$  which satisfies the conditions stated in the conjecture. Let  $V_1 = (V^G)^\delta$  and  $V_2$  be a subspace of  $V^G$  such that  $V^G = V_2 \oplus V_1$  and define  $W_1$  and  $W_2$  similarly. The essential difficulty is that the only information that we have on  $\chi(U_\delta(V_2))$  is that  $\chi(C_V) = C_W$ , i. e. :

$$\chi(U_\delta(V_2)) \otimes (U(W_1) \otimes kH) = U_\gamma(W_2) \otimes (U(W_1) \otimes kH).$$

By Lemma 5.1,  $\chi(V_1) \subset k + W_1$ . Let  $\pi$  be the projection of  $k + W_1$  onto  $W_1$  with  $\text{Ker } \pi = k$  and define  $\phi$  on  $V_1$  by  $\phi = \pi\chi$ . Extend  $\phi$  to  $V^G$  by  $\phi(V_2) = W_2$  and  $\phi$  is compatible with  $\delta|V_2$  and  $\gamma|W_2$ . Let  $V^C$  be the  $\delta$ -orthogonal complement of  $V_2$  in  $V$ . Then  $V = V_2 \oplus V^C$ . Since no element of  $V_1$  is orthogonal to  $V^C$ , we may choose a subspace  $V_3$  of  $V^C$  such that  $V_1 \cap V_3 = 0$ ,  $\delta|V_3 = 0$  and  $\delta$  is non singular on  $V_1 \oplus V_3$ . Finally let  $V_4$  be the  $\delta$ -orthogonal complement of  $V_1 + V_2 + V_3$  in  $V$ . So

$$V = V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

Thus  $V_3 + V_4$  is a complementary subspace  $V'$  to  $V^G = V_1 + V_2$  as in Lemma 5.1. Let  $L_V, L_W, C_V, C_W$  be as in Lemma 5.1.  $\chi(V_3)$  is a subspace of  $L_W$  such that  $C_W \cap \chi(V_3) = 0$ . Let  $W'$  be the subspace of  $W$  such that

$$W' + C_W/C_W = \chi(V_3) + C_W/C_W.$$

For  $v \in V_3$ , choose  $w' \in W'$  such that  $\chi(v) + C_W = w' + C_W$ . Then, for

$$v_1 \in V_1, \quad \delta(v_1, v) = [\chi(v_1), \chi(v)] = [\chi(v_1), w'] = \gamma(\phi(v_1), w').$$

Hence  $\gamma|W'$  is non singular. Hence there exists  $W_3 \subset W'$  such that

$$W' = W_1 \oplus W_2 \oplus W_3,$$

$W_3$  is  $\gamma$ -orthogonal to  $W_2$  and  $\gamma|W_3 = 0$ . We now extend the domain of  $\phi$  from  $V_1 + V_2$  to  $V_1 + V_2 + V_3$  by, for  $v_3 \in V_3$ , let  $w_3 \in W_3$  be the unique element such that  $\chi(v_3) + C_W = w_3 + C_W$  and define  $\phi(v_3) = w_3$ . So  $\phi : V_1 + V_2 + V_3 \rightarrow W_1 + W_2 + W_3$  and  $\phi$  is compatible with  $\delta$  and  $\gamma$  by construction.

Let  $W_4$  be the  $\gamma$ -orthogonal complement of  $W_1 + W_2 + W_3$  in  $W$ . Define

$$\phi : V_4 \rightarrow W_4 \quad \text{by} \quad \phi(v_4) = w_4 \quad \text{where} \quad \chi(v_4) + C_W = w_4 + C_W.$$

We now show that  $\phi : V \rightarrow W$  satisfies the second condition in Conjecture 5.6. Let  $\lambda \in G$  and  $g_\lambda$  be the corresponding element of  $G$ . Then  $\chi(g_\lambda) = \alpha h_\mu$  where  $\alpha \in k$  and  $h_\mu \in H$

corresponds to  $\mu \in \mathbf{H}$ . Then, by the construction, we have that for all  $v \in V$ ,  $\lambda(v)g_\lambda = [g_\lambda, v]$ , so after applying  $\chi$  we have

$$\lambda(v)\alpha h_\mu = [\alpha h_\mu, \chi(v)] = [\alpha h_\mu, \varphi(v)] = \mu(\varphi(v))\alpha h_\mu,$$

since  $\chi(v) + C_W = \varphi(v) + C_W$ . Thus  $\lambda(v) = \mu(\varphi(v))$  for all  $v \in V$ , i. e.  $\lambda = \mu\varphi$  or  $\varphi^*\mu = \lambda$ . Thus  $\varphi^* : W^* \rightarrow V^*$  induces an isomorphism  $\varphi^* : \mathbf{H} \rightarrow \mathbf{G}$ . However we are not able to prove that  $\varphi|_{V_4}$  is compatible with  $\delta|_{V_4}$  and  $\gamma|_{W_4}$ . This difficulty becomes clearer when we write  $\mathcal{A}(V, \delta, \mathbf{G})$  in the form  $(kS \otimes kG) \#_\sigma U(\mathbf{a})$  as in

Theorem 4.8 (ii) with  $V_2 = \sum_{i=1}^p kx_i + ky_i$ . Then

$$\mathbf{a} = \sum_{i=1}^p kx_i + V_3 + V_4,$$

$V_4$  is  $\mathbf{a}^G$  and

$$\mathcal{A}(V, \delta, \mathbf{G}) \cong (kS \otimes kG) \#_\sigma U(\mathbf{a}),$$

where  $\sigma = \delta|_{\mathbf{a}}$  and so  $\sigma \in C^2(\mathbf{a}, k)$ . Similarly, since there exists  $\varphi : V \rightarrow W$  which satisfies condition (ii) of conjecture 5.6,

$$\mathcal{A}(W, \gamma, \mathbf{H}) \cong (kS \otimes kG) \#_\tau U(\mathbf{a}),$$

where  $\tau = \gamma|_{\mathbf{a}}$ . Thus if  $\mathcal{A}(V, \delta, \mathbf{G})$  and  $\mathcal{A}(W, \gamma, \mathbf{H})$  are isomorphic then they are cocycle twisted variants of the same “untwisted algebra”  $(kS \otimes kG) \# U(\mathbf{a})$ , (but  $\sigma$  and  $\tau$  need not be cohomologous). The next theorem shows that the “untwisted algebra” is independent of the presentation.

**THEOREM 5.7.** — *Let  $\mathbf{a}$  and  $\mathbf{a}'$  be subspaces of  $\mathbf{c}$  and  $\mathbf{c}'$  respectively each of which satisfies (Sim) and  $\sigma \in Z^2(\mathbf{a}, kS \otimes kG)$  and  $\tau \in Z^2(\mathbf{a}', kS' \otimes kG')$ .*

(i) *Consider the following diagram*

$$\begin{array}{ccccccc} 0 & \rightarrow & kS \otimes kG & \rightarrow & (kS \otimes kG) \times_{\sigma} \mathbf{a} & \rightarrow & \mathbf{a} \rightarrow 0 \\ & & \alpha \downarrow & & \beta \downarrow & & \eta \downarrow \\ 0 & \rightarrow & kS' \otimes kG' & \rightarrow & (kS' \otimes kG') \times_{\tau} \mathbf{a}' & \rightarrow & \mathbf{a}' \rightarrow 0 \end{array}$$

*If there exist an algebra isomorphism  $\alpha$  and Lie algebra isomorphisms  $\beta, \eta$  such that this diagram is commutative, (where the horizontal map are the canonical inclusions and projections), then there exists an algebra isomorphism*

$$\beta' : (kS \otimes kG) \#_\sigma U(\mathbf{a}) \rightarrow (kS' \otimes kG') \#_\tau U(\mathbf{a}')$$

*which extends  $\beta$ .*

(ii) *If there is an algebra isomorphism*

$$\theta : (kS \otimes kG) \#_\sigma U(\mathbf{a}) \rightarrow (kS' \otimes kG') \#_\tau U(\mathbf{a}')$$

*then there exists an algebra isomorphism  $\alpha$  and Lie algebra isomorphisms  $\beta$ ,*

$\eta$  such that

$$\begin{array}{ccccccc} 0 \rightarrow kS \otimes kG & \rightarrow & (kS \otimes kG) \times \mathfrak{a} & \rightarrow & \mathfrak{a} & \rightarrow & 0 \\ & \searrow \alpha & & \searrow \beta & & \searrow \eta & \\ 0 \rightarrow kS' \otimes kG' & \rightarrow & (kS' \otimes kG') \times \mathfrak{a}' & \rightarrow & \mathfrak{a}' & \rightarrow & 0 \end{array}$$

is commutative and hence there is an isomorphism

$$\beta' : (kS \otimes kG) \# U(\mathfrak{a}) \rightarrow (kS' \otimes kG') \# U(\mathfrak{a}')$$

which extends  $\beta$ .

*Proof.* — (i) follows immediately from the universal property of  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$ , see Theorem 2.1 and the remark which follows it. Note also that  $\alpha|_{kS}$  is a filtered algebra isomorphism from  $kS$  to  $kS'$  since the terms of the usual filtration of  $kS$  are precisely the terms of the upper central series of  $kS \otimes kG$  as a  $(kS \otimes kG) \times_{\sigma} \mathfrak{a}$ -module [compare Lemma 5.1 (iv)].

(ii) Denote  $G'$  by  $H$  where necessary. Consider  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a})$ . Let  $\mathfrak{b}$  be a subspace of  $\mathfrak{a}$  such that  $\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{a}^G$ . Without loss of generality we may suppose that  $\sigma \in C^2(\mathfrak{a}, k)$  and that  $\mathfrak{b}$  is  $\sigma$ -orthogonal to  $\mathfrak{a}$ . Similarly, let  $\mathfrak{a}' = \mathfrak{b}' \oplus (\mathfrak{a}')^H$  and suppose that  $\tau \in C^2(\mathfrak{a}', k)$  and  $\mathfrak{b}'$  is  $\tau$ -orthogonal to  $\mathfrak{a}'$ . Let  $S$  and  $S'$  be the spaces of homogenous elements of degree one of  $kS$  and  $kS'$  respectively.  $(kS \otimes kG) \#_{\sigma} U(\mathfrak{a}) \cong \mathcal{A}(V, \delta, G)$ , where  $V = S \oplus \mathfrak{a}$  and  $\delta$  is defined by  $\delta(a, s) = [a, s]$ ,  $a \in \mathfrak{a}$  and  $s \in S$ ,  $\delta|_S = 0$  and  $\delta|_{\mathfrak{a}} = \sigma$ . Similarly

$$(kS' \otimes kG') \#_{\tau} U(\mathfrak{a}') \cong \mathcal{A}(W, \gamma, H),$$

where  $W = S' \oplus \mathfrak{a}'$ . Define  $\delta_1 \in C^2(V, k)$  by requiring that  $\delta_1$  coincides with  $\delta$  on  $S \oplus \mathfrak{b}$  and  $\mathfrak{a}^G$  is  $\delta_1$ -orthogonal to  $V$ . Similarly define  $\gamma_1 \in C^2(W, k)$  by requiring that  $\gamma_1$  coincides with  $\gamma$  on  $S' \oplus \mathfrak{b}'$  and  $(\mathfrak{a}')^H$  is  $\gamma_1$ -orthogonal to  $W$ . Then

$$(kS \otimes kG) \# U(\mathfrak{a}) \cong \mathcal{A}(V, \delta_1, G) \quad \text{and} \quad (kS' \otimes kG') \# U(\mathfrak{a}') \cong \mathcal{A}(W, \gamma_1, H).$$

Since  $\mathcal{A}(V, \delta, G) \cong \mathcal{A}(W, \gamma, H)$ , there is a vector space isomorphism  $\varphi : V \rightarrow W$  which satisfies (ii) of Conjecture 5.6 and  $\varphi$  is compatible with  $\delta$  and  $\gamma$  except possibly from  $V_4 = \mathfrak{a}^G$  to  $W_4 = (\mathfrak{a}')^H$ . Thus  $\varphi$  satisfies both (i) and (ii) of Conjecture 5.6 with respect to  $\delta_1$  and  $\gamma_1$ . Recall that  $\varphi : V_1 \rightarrow W_1$  and also that  $\varphi : V_2 \rightarrow W_2$  may be chosen arbitrarily subject to being compatible with  $\delta$  and  $\gamma$ . Thus we may suppose that  $\varphi$  restricts to a vector space isomorphism from  $S$  to  $S'$ . Since  $V = S \oplus \mathfrak{a}$  and  $W = S' \oplus \mathfrak{a}'$ ,  $\varphi$  induces a vector space isomorphism  $\xi : \mathfrak{a} \rightarrow \mathfrak{a}'$  by  $\xi(a) + S' = \varphi(a) + S'$ ,  $a \in \mathfrak{a}$ . The required maps  $\alpha$ ,  $\beta$  and  $\eta$  are now defined by  $\alpha|_S = \varphi$ ,  $\alpha|_{kG}$  is induced by  $(\varphi^*)^{-1}$ ,  $\beta$  is the extension of  $\alpha$  defined by  $\beta(a) = \xi(a)$ ,  $a \in \mathfrak{a}$ , and  $\eta = \xi$ . ■

Theorem 5.7 is unsatisfactory since one would like to prove that if there is an algebra isomorphism  $\theta$  as in 5.7 (ii), then there exist  $\alpha$ ,  $\beta$  and  $\eta$  which make the diagram in 5.7 (i) commutative. However, this is equivalent to an affirmative answer to Conjecture 5.6.

EXAMPLE 5.8. — Let  $\text{rank } G = 2$  and

$$kG = k[g, g^{-1}, h, h^{-1}] \quad \text{and} \quad \mathbf{a} = kg \partial/\partial g + kh \partial/\partial h.$$

Set  $a_1 = g \partial/\partial g$  and  $a_2 = h \partial/\partial h$ . Then

$$H^2(\mathbf{a}, kG) \cong H^2(\mathbf{a}, k) = C^2(\mathbf{a}, k),$$

by Theorem 3.2.

$$C^2(\mathbf{a}, k) \cong k \quad \text{via} \quad \sigma \mapsto \sigma(a_1, a_2), \quad \sigma \in C^2(\mathbf{a}, k).$$

Define  $\tau \in C^2(\mathbf{a}, k)$  by  $\tau(a_1, a_2) = 1$ . It is easy to see that  $kG \#_{\sigma} U(\mathbf{a}) \cong kG \#_{\tau} U(\mathbf{a})$  if  $\sigma \neq 0$ . By [4], Example 5.8 or by Theorem 5.9 below,  $kG \#_{\tau} U(\mathbf{a})$  is not isomorphic to  $kG \# U(\mathbf{a})$  and so the 2-cocycle twisted variants of  $kG \# U(\mathbf{a})$  fall into exactly two isomorphism classes.

THEOREM 5.9. — If  $\mathcal{A}(V, \delta, G)$  and  $\mathcal{A}(W, \gamma, H)$  are simple algebras which are isomorphic and  $\delta|_{V^G} = 0$ , or equivalently the centraliser of the group of units of  $\mathcal{A}(V, \delta, G)$  is a commutative algebra, then  $\mathcal{A}(V, \delta, G)$  and  $\mathcal{A}(W, \gamma, H)$  are locally isomorphic.

*Proof.* — Let  $\chi$  be an isomorphism. It is sufficient to show that the map  $\varphi: V \rightarrow W$  which was constructed earlier is compatible with  $\delta$  and  $\gamma$ . Let  $v, v' \in V_4$ .

Then  $\chi(v) = \varphi(v) + r$ ,  $\chi(v') = \varphi(v') + r'$  for some  $r, r' \in C_W$ . Now  $\delta(v, v') = [v, v']$ , so, after applying  $\chi$ , we have

$$\begin{aligned} \delta(v, v') &= [\chi(v), \chi(v')] \\ &= [\varphi(v) + r, \varphi(v') + r'] \\ &= [\varphi(v), \varphi(v')] + [r, \varphi(v')] + [\varphi(v), r'], \end{aligned}$$

since  $C_W$  is commutative.

Now

$$[\varphi(v), \varphi(v')] = \gamma(\varphi(v), \varphi(v')) \in k$$

and

$$[r, \varphi(v')] + [\varphi(v), r'] \in \sum_{g \neq 1} U(V_1) \otimes kg.$$

So  $[r, \varphi(v')] + [\varphi(v), r'] = 0$  and  $\varphi$  satisfies the required property. ■

EXAMPLE 5.10. — We now consider the algebras  $\mathcal{A}(V, 0, G)$  in the special case when  $\dim V = 1$  and  $\text{rank } G = 2$ . Let  $v$  be a basis of  $V$ . If  $\lambda_1, \lambda_2 \in V^*$  then  $\lambda_1$  and  $\lambda_2$  are linearly independent over  $\mathbb{Q}$  if and only if  $\lambda_1(v)$  and  $\lambda_2(v)$  are linearly independent over  $\mathbb{Q}$ . Let  $\lambda_1, \lambda_2, \mu_1, \mu_2 \in V^*$  have the property that  $\lambda_1$  and  $\lambda_2$  (respectively  $\mu_1$  and  $\mu_2$ ) are linearly independent over  $\mathbb{Q}$  and denote the subgroup of  $V^*$  generated by  $\{\lambda_1, \lambda_2\}$  and  $\{\mu_1, \mu_2\}$  by  $G_1$  and  $G_2$  respectively. By Theorem 5.9,  $\mathcal{A}(V, 0, G_1) \cong \mathcal{A}(V, 0, G_2)$  if and only if there exists  $0 \neq \rho \in k$  and a  $2 \times 2$  unimodular matrix  $\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$  with integer coefficients such that

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \rho \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

Thus if  $k$  is  $\mathbf{R}$  or  $\mathbf{C}$  then there are infinitely many non isomorphic simple algebras  $\mathcal{A}(V, 0, \mathbf{G})$  with  $\dim V = 1$  and  $\text{rank } \mathbf{G} = 2$ .

REMARK 5.11. — If Conjecture 5.6 is false then there are simple algebras  $\mathcal{A}(V, \delta, \mathbf{G})$  and  $\mathcal{A}(W, \gamma, \mathbf{H})$  which are isomorphic but not locally isomorphic. Let

$$V = V_1 + V_2 + V_3 + V_4$$

be a decomposition of  $V$  as considered earlier with  $\dim V_2 = 2p > 0$ . Set  $V' = V_1 + V_3 + V_4$ . Then

$$\begin{aligned} \mathcal{A}(V, \delta, \mathbf{G}) &\cong U_\delta(V_2) \otimes \mathcal{A}(V', \delta, \mathbf{G} \mid V') \\ &\cong A_p \otimes \mathcal{A}(V', \delta, \mathbf{G} \mid V'). \end{aligned}$$

By a similar decomposition,

$$\mathcal{A}(W, \gamma, \mathbf{H}) \cong A_p \otimes \mathcal{A}(W', \gamma, \mathbf{H} \mid W').$$

Now  $\mathcal{A}(V', \delta, \mathbf{G} \mid V')$  and  $\mathcal{A}(W', \gamma, \mathbf{H} \mid W')$  cannot be locally isomorphic by the hypothesis, and so are not isomorphic by Theorem 5.9. But

$$A_p \otimes \mathcal{A}(V', \delta, \mathbf{G} \mid V') \cong A_p \otimes \mathcal{A}(W', \gamma, \mathbf{H} \mid W')!$$

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J. C. McCONNELL,  
School of Mathematics,  
University of Leeds,  
Leeds, LS 2 9JT,  
England.