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WEIL-CHÂTELET GROUPS OVER LOCAL FIELDS : ADDENDUM

BY JAMES S. MILNE

By using the structure theorems for the Néron minimal model of an abelian variety with semi-stable reduction, as presented in [2], it is possible to complete the proof of the following theorem. (Notations are as in [3].)

THEOREM. — *Let A be an abelian variety over a local field K (with finite residue field) and let $\hat{A}$ be the dual abelian variety. Then the pairings*

$$H^r(K, A) \times H^{1-r}(K, \hat{A}) \rightarrow H^2(K, \mathbf{G}_m) \approx \mathbf{Q}/\mathbf{Z},$$

as defined by Tate [4], are non-degenerate for all r .

After [3], we need only consider the case where K has characteristic $p \neq 0$. Also we have only to prove that the map

$$\theta_K(A)_p : H^1(K, A)_p \rightarrow (\hat{A}(K)^{(p)})^*$$

is injective, and it suffices to do this after making a finite separable field extension. Thus we may assume that A and $\hat{A}$ have semi-stable reduction ([2], § 3.6) and that

$$A_p(K) = A_p(\bar{K}), \quad \hat{A}_p(K) = \hat{A}_p(\bar{K}).$$

Let $\mathcal{A}$ be the Néron minimal model of A over R . The Raynaud group $\mathcal{A}^\natural$ of $\mathcal{A}$ over R is a smooth group scheme over R such that : (a) there are canonical isomorphisms $\overline{\mathcal{A}} \xrightarrow{\sim} \overline{\mathcal{A}^\natural}$ and $\overline{\mathcal{A}^0} \xrightarrow{\sim} \overline{\mathcal{A}^{\natural 0}}$ (where $\overline{\mathcal{H}}$ denotes the formal completion of a scheme $\mathcal{H}$ over R) and (b) there is an exact sequence $0 \rightarrow \mathfrak{T} \rightarrow \mathcal{A}^{\natural 0} \rightarrow \mathcal{B} \rightarrow 0$ in which $\mathcal{B}$ is an abelian scheme and $\mathfrak{T}$ is a torus ([2], § 7.2). $\mathcal{N} = (\mathcal{A}^{\natural 0})_p$ is identified through the isomorphism in (a) with the maximal finite flat subgroup scheme of the quasi-finite group scheme $\mathcal{A}_p^0$. If we write $B = \mathcal{B} \otimes_R K$, $N = \mathcal{N} \otimes_R K$, ..., then we get a filtration $A_p = \mathcal{A}_p^0 \otimes_R K \supset N \supset T_p \supset 0$ of A_p in which $N/T_p \approx B_p$.

Let $\mathcal{A}'$, $\mathcal{B}'$, $\mathcal{N}'$, ... be the schemes corresponding, as above, to $\hat{A}$. The canonical non-degenerate pairing $A_p \times \hat{A}_p \rightarrow \mathbf{G}_m$ respects the filtrations on A_p and $\hat{A}_p$, i. e. N and T_p are the exact annihilators of T'_p and N' respectively. Indeed, the induced pairing $N \times N' \rightarrow \mathbf{G}_m$ has a canonical extension to a pairing $\mathcal{N} \times \mathcal{N}' \rightarrow \mathbf{G}_{m,R}$ ([2], § 1.4). This pairing is trivial on $\mathfrak{T}_p$ and $\mathfrak{T}'_p$ and the quotient pairing $\mathcal{B}_p \times \mathcal{B}'_p \rightarrow \mathbf{G}_{m,R}$ is the non-degenerate pairing defined by a Poincaré divisorial correspondence on $(\mathcal{B}, \mathcal{B}')$ ([2], § 7.4, 7.5). This shows that T_p (resp. T'_p) is the left (resp. right) kernel in the pairing $N \times N' \rightarrow \mathbf{G}_m$. The pairing $A_p/T_p \times N' \rightarrow \mathbf{G}_m$ is right non-degenerate. But A_p/T_p has rank $p^{2n-\mu}$ where $n = \dim(A)$ and $\mu = \dim(\mathfrak{T})$ and N' has rank $p^{\mu+2\alpha}$, where $\alpha = \dim(\mathcal{B})$ (cf. [2], § 2.2.7). This shows that the pairing is also left non-degenerate (because $n = \mu + \alpha$), which completes the proof of our assertion.

Consider the commutative diagram :

$$\begin{array}{ccc} \mathcal{A}^0(R)^{(p)} & \longrightarrow & H^1(R, \mathcal{A}_p^0) \\ \downarrow & & \downarrow \\ A(K)^{(p)} & \longrightarrow & H^1(K, A_p) \end{array}$$

in which the horizontal maps are boundary maps in the cohomology sequences for multiplication by p on A and $\mathcal{A}^0$. $H^1(R, \mathcal{A}_p^0) \approx H^1(R, \mathcal{N})$ because $\mathcal{A}_p^0/\mathcal{N}$ is smooth over R with zero special fibre and so has zero cohomology groups ([1], § 11.7) (including in dimension 0). The top arrow is an isomorphism because $H^1(R, \mathcal{A}^0) = 0$ (*loc. cit.*). The cokernel of the left vertical arrow is $\Phi_0(k)^{(p)}$, where Φ_0 is the group of connected components of $\mathcal{A} \otimes_R k$ (cf. [2], § 11.1). Using all of this, one can extract from the top diagram on p. 275 of [3] (with $m = p$) an exact commutative

diagram :

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Phi_0(k)^{(p)} & \longrightarrow & H^1(K, A_p)/H^1(R, \mathcal{U}) & \longrightarrow & H^1(K, A)_p \longrightarrow 0 \\
 & & \downarrow & & \downarrow \psi_1 & & \downarrow \psi_2 \\
 & & 0 & \longrightarrow & H^1(R, \mathcal{U}')^* & \longrightarrow & \mathcal{A}'^0(R)^{(p)*} \longrightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

It is easy to see that $\theta_K(A)_p$ is an isomorphism if and only if

$$[\ker \psi_2] = [\hat{A}(K)^{(p)}/\mathcal{A}'^0(R)^{(p)}], \quad \text{i. e.} \quad [\ker \psi_2] = [\Phi_0(k)^{(p)}].$$

We shall show that

$$[\ker \psi_1] = p^{2\mu}, \quad [\Phi_0(k)^{(p)}] = p^\mu = [\Phi'_0(k)^{(p)}],$$

and as $[\ker \psi_2][\Phi_0(k)^{(p)}] = [\ker \psi_1]$, this completes the proof.

Consider first the situation : M is a finite group scheme over K and $\mathcal{U}$ and $\mathcal{U}'$ are finite flat group schemes over R with given embeddings $N \rightarrow M$, $N' \rightarrow \hat{M}$. If $\mathcal{U} = \mathcal{O}_p$ for some abelian scheme $\mathcal{O}$ over R and $M = N$, $\mathcal{U}' = \hat{\mathcal{U}}$, then

$$\psi : H^1(K, M)/H^1(R, \mathcal{U}) \rightarrow H^1(R, \mathcal{U}')^*,$$

the map defined by the cup-product pairing

$$H^1(K, M) \times H^1(K, \hat{M}) \rightarrow H^2(K, \mathbf{G}_m),$$

is an isomorphism [3]. If $\mathcal{U} = \mu_p$, $M = N$, and $\mathcal{U}' = 0$, then $[\ker \psi] = p$ because [3]

$$H^1(K, \mu_p)/H^1(R, \mu_p) \approx H^1(R, \mathbf{Z}/p\mathbf{Z})^* \approx H^1(k, \mathbf{Z}/p\mathbf{Z})^*.$$

If $M = \mathbf{Z}/p\mathbf{Z}$, $\mathcal{U} = 0$, and $\mathcal{U}' = \mu_p$, then $[\ker \psi] = p$ because [3] $\ker \psi = H^1(R, \mathbf{Z}/p\mathbf{Z})$. It follows from this, and the above discussion of the structures of A_p and $\hat{A}_p$, that $[\ker \psi_1] = p^{2\mu}$.

Finally, let $\Phi = \mathcal{A}'/\mathcal{A}'^0$. It is a finite étale group scheme over R such that $\Phi \otimes_R k = \Phi_0$, and there is an exact sequence

$$0 \rightarrow \mathcal{U} \rightarrow \mathcal{A}'_p \rightarrow \Phi_p \rightarrow 0.$$

$\mathcal{A}'_p(R) \approx \mathcal{A}_p(R)$, because $\mathcal{A}'_p$ and $\mathcal{A}_p$ differ only by a scheme with empty special fibre, and $\mathcal{A}_p(R) \approx \mathcal{A}_p(K)$. It follows that $\Phi_p(K) = A_p(K)/N(K)$ has p^μ elements. But

$$\Phi(K) \approx \Phi(R) \approx \Phi_0(k) \quad \text{and so} \quad [\Phi_0(k)^{(p)}] = [\Phi_0(k)_p] = p^\mu.$$

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