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Tunneling of the Kawasaki dynamics at low temperatures in two dimensions

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Abstract. Consider a lattice gas evolving according to the conservative Kawasaki dynamics at inverse temperature β on a two dimensional torus $\Lambda_L = \{0, \dots, L-1\}^2$. We prove the tunneling behavior of the process among the states of minimal energy. More precisely, assume that there are n^2 particles, $n < L/2$, and that the initial state is the configuration in which all sites of the square $\{0, \dots, n-1\}^2$ are occupied. We show that in the time scale $e^{2\beta}$ the process evolves as a Markov process on Λ_L which jumps from any site $\mathbf{x}$ to any other site $\mathbf{y} \neq \mathbf{x}$ at a strictly positive rate which can be expressed in terms of the hitting probabilities of simple Markovian dynamics.

Résumé. On considère un gaz sur réseau évoluant selon la dynamique de Kawasaki à température inverse β sur le tore bi-dimensionel $\Lambda_L = \{0, \dots, L-1\}^2$. Nous étudions l'évolution du processus parmi les états d'énergie minimale.

Supposons la présence de n^2 particules, $n < L/2$ et qu'à l'état initial les sites du carré $\{0, \dots, n-1\}^2$ soient tous occupés. Nous montrons qu'à l'échelle de temps $e^{2\beta}$ le processus évolue comme une chaîne de Markov sur Λ_L qui saute d'un site $\mathbf{x}$ vers un site $\mathbf{y} \neq \mathbf{x}$ à un taux strictement positif qui peut-être exprimé en terme de probabilités d'atteinte de dynamiques markoviennes élémentaires.

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1. Introduction

We introduced in [1,4] a general method to describe the asymptotic evolution of one-parameter families of continuous-time Markov chains. This method has been succesfully applied in two situations: For zero-range dynamics on a finite set which exhibit condensation [2,18], and for random walks evolving among random traps [16,17]. In the first model the chain admits a finite number of ground sets, while in the second one there is a countable number of ground states. We start in this paper the investigation of a third case, where the limit dynamics is a continuous process.

This article has two purposes. On the one hand, to derive some estimates needed in the proof of the convergence, in the zero-temperature limit, of the two-dimensional Kawasaki dynamics for the Ising model in a large cube to a Brownian motion, presented in [13]. On the other hand, to illustrate the interest of the method introduced in [1,4] by applying it in a simple context. A first step was done in this direction in [3], where we derived the asymptotic behavior of continuous-time Markov chains evolving on a *fixed* and *finite* state space imposing only one simple condition on the jump rates. A second step is performed here, applying the result obtained in [3] to the Kawasaki dynamics for the Ising model on a fixed two-dimensional square with periodic boundary condition.

To present the main result of [3], consider a one-parameter family of irreducible Markov chains $\eta_N(t)$ on a fixed and finite state space E, reversible with respect to a probability measure μ_N. For example, the Glauber or the Kawaski dynamics for the Ising model on a finite space. For $\eta \in E$, denote by $\mathbf{P}^N_\eta$ the distribution of the process $\eta_N(t)$ starting from η. Expectation with respect to $\mathbf{P}^N_\eta$ is represented by $\mathbf{E}^N_\eta$.

Denote by $R_N(\eta,\xi)$ the jump rates of the chain and assume that for all $\eta,\eta',\xi,\xi' \in E$,

$$\lim_{N\to\infty} \frac{R_N(\eta,\eta')}{R_N(\xi,\xi')} \in [0,\infty] \tag{1.1}$$

in the sense that the limit exists with $+\infty$ as a possible value. Note that conditions (2.1) and (2.2) in [3] follow from (1.1). Moreover, since for the Glauber or for the Kawasaki dynamics the jump rates are either 1 or $\mathrm{e}^{-k\beta}$ for some $1 \le k \le 4$, assumption (1.1) is fulfilled.

Under the elementary assumption (1.1) we completely described in [3] the asymptotic evolution of the Markov chain $\eta_N(t)$. More precisely, we proved the existence of a rooted tree whose vertices are subsets of the state space. The tree fulfills the following properties: (a) The root of the tree is the state space; (b) the subsets of each generation form a partition of the state space; and (c) the sucessors of a vertex are subsets of this vertex. To each generation corresponds a tunneling behavior. Let $M+1$, $M \ge 1$, be the number of generations of the tree, let $\kappa_m + 1$ be the number of descendents at generation $m+1$, $1 \le m \le M$, and let $\mathcal{E}^m_1,\dots,\mathcal{E}^m_{\kappa_m}, \Delta_m$ be the vertices of the generation $m+1$. We proved the existence of time scales $\theta^N_1 \gg \cdots \gg \theta^N_M$ such that for each $1 \le m \le M$:

(1) For every $1 \le i \le \kappa_m$ and every state η in $\mathcal{E}^m_i$,

$$\lim_{N\to\infty} \max_{\xi\in\mathcal{E}^m_i} \mathbf{P}^N_\xi [H_{\breve{\mathcal{E}}^m_i} < H_\eta] = 0,$$

where $\breve{\mathcal{E}}^m_i = \bigcup_{j\neq i} \mathcal{E}^m_j$ and where H_A stands for the hitting time of a set $A \subset E$. This means that starting from a set $\mathcal{E}^m_i$ the process visits all the points of $\mathcal{E}^m_i$ before reaching another set $\mathcal{E}^m_j$.

(2) Let $\mathcal{E}^m = \bigcup_i \mathcal{E}^m_i$ and let $\Psi_m : \mathcal{E}^m \to \{1,\dots,\kappa_m\}$ be the index function given by

$$\Psi_m(\eta) = \sum_{i=1}^{\kappa_m} i \mathbf{1}\{\eta \in \mathcal{E}^m_i\}.$$

Denote by $\{\eta^m_N(t)\colon t \ge 0\}$ the trace of the process $\{\eta_N(t)\colon t \ge 0\}$ on $\mathcal{E}^m$. For every $1 \le i \le \kappa_m$, $\eta \in \mathcal{E}^m_i$, under the measure $\mathbf{P}^N_\eta$, the (non-Markovian) index process $X^m_N(t) = \Psi_m(\eta^m_N(t\theta^N_m))$ converges to a Markov process $X^m(t)$ on $\{1,\dots,\kappa_m\}$.

(3) Starting from $\eta \in \mathcal{E}^m$, in the time scale θ^N_m the time spent outside $\mathcal{E}^m$ is negligible: For every $t > 0$,

$$\lim_{N\to\infty} \max_{\eta\in E} \mathbf{E}^N_\eta \left[\int_0^t \mathbf{1}\{\eta_N(s\theta^N_m) \in \Delta_m\} \,\mathrm{d}s \right] = 0.$$

Therefore, in the time scale θ^N_m the process $\eta_N(t)$ behaves as a Markov process on a state space whose κ_m points are the sets $\mathcal{E}^m_1,\dots,\mathcal{E}^m_{\kappa_m}$ and which jumps from $\mathcal{E}^m_i$ to $\mathcal{E}^m_j$ at a rate given by the jump rates of the Markov process $X^m(t)$.

We apply this result to investigate the zero-temperature limit of the Kawasaki dynamics for the Ising model on a two-dimensional square with periodic boundary condition. Here, for a fixed square and a fixed number of particles, we derive the asymptotic behavior of the dynamics among the ground states, configurations whose particles form squares. In [13], we show that the evolution of these square configurations converges to a Brownian motion when the lenght of the square and the number of particles increase with the inverse of the temperature.

The problem of describing the asymptotic behavior of a one-parameter family of Markov chains evolving on a fixed and finite state space has been considered before. Olivieri and Scoppola [22,25] applied the ideas introduced in the pathwise approach to metastability [10] to this context. They supposed that the jump probabilities $P(x,y)$ of a discrete-time chain are given by

$$P(x,y) = q(x,y)\mathrm{e}^{-\beta[H(y)-H(x)]_+}, \tag{1.2}$$

where $[a]_+$ represents the positive part of a, $q(x, y)$ a symmetric function and H an Hamiltonian. A subset A of the state space E is called a cycle if $\max_{x\in A} H(x) < \min_{y\in\partial_+ A} H(y)$, where $\partial_+ A$ stands for the outer boundary of A: $\partial_+ A = \{y \notin A: \exists x \in A, P(x, y) > 0\}$. Under condition (1.2), Olivieri and Scoppola proved that the exit time of a cycle, appropriately renormalized, converges to an exponential random variable, and they obtained estimates, with exponential errors, for the expectation of the exit time. They were also able to describe the exit path from a cycle. These results were generalized by Olivieri and Scoppola [23] to the non-reversible case, and by Manzo et al. [19], who extended the results proved in [22] for the exit time of a cycle to the hitting time of the absolute minima of the Hamiltonian.

Therefore, in the context of a fixed and finite state space, the approach proposed in [1,4] requires weaker assumptions on the jump rates than the pathwise approach, it provides better estimates on the exit times of the wells, and it characterizes the transition probabilities which describe the way the process jumps from one well to another. On the other hand, and in contrast with the pathwise approach to metastability, it does not attempt to characterize the exit paths from a well.

The potential theoretic approach to metastability [5,6] has also been tested [9] in the framework of a fixed and finite state space Markov chain. Bovier and Manzo considered a Hamiltonian H and an irreducible discrete-time jump probability reversible with respect to the Gibbs measure associated to the Hamiltonian H at inverse temperature β. Let $\mathcal{M}$ be a subset of the local minima of the Hamiltonian H and let $\mathcal{M}_x = \mathcal{M} \setminus \{x\}$, $x \in \mathcal{M}$. Under some assumptions on the Hamiltonian, they computed the expectation of the hitting time of $\mathcal{M}_x$ starting from $x \in \mathcal{M}$ and they proved that this hitting time properly renormalized converges to an exponential random variable. They also provided a formula for the probability that starting from $x \in \mathcal{M}$ the process returns to the set $\mathcal{M}$ at a local minima $y \in \mathcal{M}$ in terms of the right eigenvectors of the jump matrix of the chain. This latter formula, although interesting from the theoretical point of view, since it establishes a link between the spectral properties of the generator and the metastable behavior of the process, is of little pratical use because one is usually unable to compute the eigenvectors of the generator.

In our approach, we replace the formula of the jump probabilities written through eigenvectors of the generator by one, [1], Remark 2.9 and Lemma 6.8, expressed only in terms of the capacities, capacities which can be estimated using the Dirichlet and the Thomson variational principles. This latter formula allows us to prove the convergence of the process (in fact, of the trace process in the usual Skorohod topology or of the original process in a weaker topology introduced in [17]) by solving a martingale problem.

Metastability of locally conserved dynamics or of conservative dynamics superposed with non-conservative ones have been considered before. Peixoto [24] examined the metastability of the two dimensional Ising lattice gas at low temperature evolving according to a superposition of the Glauber dynamics with a stirring dynamics. Den Hollander et al. [15] and Gaudillière et al. [11] described the critical droplet, the nucleation time and the typical trajectory followed by the process during the transition from a metastable set to the stable set in a two dimensional Ising lattice gas evolving under the Kawasaki dynamics at very low temperature in a finite square in which particles are created and destroyed at the boundary. This result has been extended to the anistropic case by Nardi et al. [20] and to three dimensions by den Hollander et al. [14]. Using the potential theoretic approach introduced in [5,6], Bovier et al. [7] presented the detailed geometry of the set of critical droplets and provided sharp estimates for the expectation of the nucleation time for this model in dimension two and three.

More recently, Gaudillière et al. [12] proved that the dynamics of particles evolving according to the Kawasaki dynamics at very low temperature and very low density in a two-dimensional torus whose length increases as the temperature decreases can be approximated by the evolution of independent particles. These results were used in [8], together with the potential theoretic approach alluded to above, to obtain sharp estimates for the expectation of the nucleation time for this model.

2. Notation and results

We consider a lattice gas on a torus subjected to a Kawasaki dynamics at inverse temperature β. Let $\Lambda_L = \{1, \dots, L\}^2$, $L \geq 1$, be a square with periodic boundary conditions. Denote by Λ_L^* the set of edges of Λ_L. This is the set of unordered pairs $\{x, y\}$ of Λ_L such that $\|x - y\| = 1$, where $\|\cdot\|$ stands for the Euclidean distance. The configurations are denoted by $\eta = \{\eta(x): x \in \Lambda_L\}$, where $\eta(x) = 1$ if site x is occupied and $\eta(x) = 0$ if site x is vacant. The

Hamiltonian $\mathbb{H}$, defined on the state space $\Omega_L = \{0,1\}^{\Lambda_L}$, is given by

$$-\mathbb{H}(\eta) = \sum_{\{x,y\}\in\Lambda_L^*} \eta(x)\eta(y).$$

The Gibbs measure at inverse temperature β associated to the Hamiltonian $\mathbb{H}$, denoted by μ^β, is given by

$$\mu^\beta(\eta) = \frac{1}{Z_\beta} e^{-\beta\mathbb{H}(\eta)},$$

where Z_β is the normalizing partition function.

We consider the continuous-time Markov chain $\{\eta_t^\beta: t\ge 0\}$ on Ω_L whose generator L_β acts on functions $f:\Omega_L\to\mathbb{R}$ as

$$(L_\beta f)(\eta) = \sum_{\{x,y\}\in\Lambda_L^*} c_{x,y}(\eta)\big[f\big(\sigma^{x,y}\eta\big) - f(\eta)\big],$$

where $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$\big(\sigma^{x,y}\eta\big)(z) = \begin{cases} \eta(z) & \text{if } z\neq x,y,\\ \eta(y) & \text{if } z=x,\\ \eta(x) & \text{if } z=y.\end{cases}$$

The rates $c_{x,y}$ are given by

$$c_{x,y}(\eta) = \exp\big\{-\beta\big[\mathbb{H}\big(\sigma^{x,y}\eta\big) - \mathbb{H}(\eta)\big]_+\big\},$$

where $[a]_+$, $a\in\mathbb{R}$, stands for the positive part of a: $[a]_+ = \max\{a,0\}$. We sometimes represent η_t^β by $\eta^\beta(t)$ and we frequently omit the index β of η_t^β.

A simple computation shows that the Markov process $\{\eta_t: t\ge 0\}$ is reversible with respect to the Gibbs measures μ^β, $\beta>0$, and ergodic on each irreducible component formed by the configurations with a fixed total number of particles. Denote by $|A|$ the cardinality of a finite set A. Let $\Omega_{L,K} = \{\eta\in\Omega_L: \sum_{x\in\Lambda_L}\eta(x) = K\}$, $0\le K\le|\Lambda_L|$, and denote by μ_K^β the Gibbs measure μ^β conditioned on $\Omega_{L,K}$:

$$\mu_K^\beta(\eta) = \frac{1}{Z_{\beta,K}} e^{-\beta\mathbb{H}(\eta)}, \quad \eta\in\Omega_{L,K},$$

where $Z_{\beta,K}$ is the normalizing constant $Z_{\beta,K} = \sum_{\eta\in\Omega_{L,K}}\exp\{-\beta\mathbb{H}(\eta)\}$. We sometimes denote μ_K^β simply by μ_K.

For each configuration $\eta\in\Omega_{L,K}$, denote by $\mathbf{P}_\eta^\beta$ the probability measure on the path space $D([0,\infty),\Omega_{L,K})$ induced by the Markov process $\{\eta_t: t\ge 0\}$ starting from η. Expectation with respect to $\mathbf{P}_\eta^\beta$ is represented by $\mathbf{E}_\eta^\beta$.

Assume from now on that $K=n^2$ for some $4\le n<\sqrt{L}$, and denote by Q the square $\{0,\dots,n-1\}\times\{0,\dots,n-1\}$. For $\mathbf{x}\in\Lambda_L$, let $Q_\mathbf{x} = \mathbf{x}+Q$ and let $\eta^\mathbf{x}$ be the configuration in which all sites of the square $Q_\mathbf{x}$ are occupied. Denote by $\Omega^0 = \Omega_{L,K}^0$ the set of square configurations:

$$\Omega^0 = \big\{\eta^\mathbf{x}: \mathbf{x}\in\Lambda_L\big\}.$$

If $L>2n$ the ground states of the energy $\mathbb{H}$ in $\Omega_{L,K}$ are the square configurations:

$$\mathbb{H}_{\min} := \min_{\eta\in\Omega_{L,K}} \mathbb{H}(\eta) = \mathbb{H}\big(\eta^\mathbf{x}\big) = -2n(n-1), \tag{2.1}$$

and $\mathbb{H}(\eta) > -2n(n-1)$ for all $\eta\in\Omega_{L,K}\setminus\Omega^0$.

To prove this claim, fix a configuration $\eta\in\Omega_{L,K}$. Denote by ξ the configuration obtained from η by moving vertically the particles of the configuration η along the columns of Λ_L in the following way. If there is a particle in the column $C_k = \{\mathbf{x} = (x_1, x_2) \in \Lambda_L\colon x_1 = k\}$, move a particle in this column to the position $(k, 0)$ if this site is empty. Place all the other particles in the contiguous sites above $(k, 0)$. This means that if there are j particles in the column C_k for the configuration η, $\xi(k, i) = 1$ if and only if $0 \le i < j$.

This transformation does not decrease the number of vertical edges connecting particles and maximizes the number of horizontal edges among the configurations with a fixed number of particles per column. Therefore, $\mathbb{H}(\xi) \le \mathbb{H}(\eta)$ and to prove claim (2.1) it is enough to show that $\mathbb{H}(\xi) \ge \mathbb{H}(\eta^{\mathbf{x}})$ and that the equality holds only if ξ is a square configuration. There are two cases which are examined separately. Either all columns have at least one particle, or there is a column with no particle.

In the second case, we may assume that ξ is a configuration of $\{0, 1\}^{\mathbb{Z}^2}$ with n^2 occupied sites. If the set of occupied sites is not a connected subgraph of $\mathbb{Z}^2$, we decrease the energy by moving laterally a cluster of particles until it touches another cluster. We may therefore suppose that the occupied sites form a connected set.

Associate to each particle of ξ a square of lenght 1 centered at the site occupied by the particle. Consider the smallest rectangle in $\mathbb{R}^2$ which contains all squares. By construction, each row and column of the rectangle contains at least one square.

Denote by $m_1 \le m_2$ the lengths of the smallest rectangle which contains all squares. The area of the rectangle, equal to $m_1 m_2$, must be larger than or equal to the number of particles n^2. It follows from this inequality that $m_1 + m_2 \ge 2n$, with an equality if and only if $m_1 = m_2 = n$. Since each row and each column contains at least a square, there exist at least $2(m_1 + m_2)$ edges connecting an occupied site to an empty site.

Since there are n^2 particles, if all 4 bonds of each particle were attached to another particle, the energy would be $-2n^2$. For the configurations ξ, we have seen that $2(m_1 + m_2)$ bonds link a particle to a hole. Hence, the energy of this configuration is at least $-(2n^2 - m_1 - m_2) \ge -2n(n-1)$, with an equality if and only if $m_1 = m_2 = n$, i.e., if ξ' is a square configuration. This proves claim (2.1) if the configuration ξ can be considered as a configuration of $\{0, 1\}^{\mathbb{Z}^2}$.

Assume now that all columns have at least one particle. This means that the configuration ξ has a row of particles forming a ring around the torus Λ_L. The argument presented below to estimate the energy of ξ applies also in the case where a column has no particles. Let h be the maximal height of the columns: $h = \max\{j \ge 1\colon \exists k, \xi(k, j-1) = 1\}$. If all particles at height h have two horizontal neighbors, the configuration ξ forms a strip around the torus Λ_L with $hL = n^2$ particles and its energy is equal to $-(2h-1)L = -(2n^2 - L) > -2n(n-1)$ because $L > 2n$ by assumption.

If there is a particle with maximal height which has one or no horizontal neighbor, we may move this particle to an empty site at minimal height without increasing the energy. We repeat this operation until reaching a configuration formed by a strip of particles surmounted by a row of particles. Denote by h the height of the strip and by $0 \le k < L$ the number of particles forming the top row so that $n^2 = hL + k$. The energy of this configuration is $-[2(hL+k) - (L+1)] = -[2n^2 - (L+1)] > -2n(n-1)$ because $L > 2n$ by assumption. This concludes the proof of claim (2.1).

We examine in this article the asymptotic evolution of the Markov process $\{\eta_t\colon t \ge 0\}$ among the $|\Lambda_L|$ ground states $\{\eta^{\mathbf{x}}\colon \mathbf{x} \in \Lambda_L\}$ in the zero temperature limit. Denote by $\{\xi_t\colon t \ge 0\}$ the trace of the process η_t on the set of ground states Ω^0. We refer to [1] for a precise definition of the trace process. The main theorem of this article reads as follows.

Theorem 2.1. *As $\beta \uparrow \infty$, the speeded up process $\xi(e^{2\beta} t)$ converges to a Markov process on Ω^0 which jumps from $\eta^{\mathbf{x}}$ to $\eta^{\mathbf{y}}$ at a strictly positive rate $r(\mathbf{x}, \mathbf{y})$. Moreover, in the time scale $e^{2\beta}$ the time spent by the original process η_t outside the set of ground states Ω^0 is negligible: for every $\mathbf{x} \in \Lambda_L$, $t > 0$,*

$$\lim_{\beta \to \infty} \mathbf{E}^{\beta}_{\eta^{\mathbf{x}}}\left[\int_0^t \mathbf{1}\{\eta(e^{2\beta} s) \notin \Omega^0\}\, \mathrm{d}s\right] = 0. \tag{2.2}$$

In the terminology introduced in [1], the previous theorem states that the sequence of Markov processes $\{\eta^{\beta}_t\colon t \ge 0\}$ exhibits a tunneling behavior on the time-scale $e^{2\beta}$, with metastable sets $\{\{\eta^{\mathbf{x}}\}\colon \mathbf{x} \in \Lambda_L\}$, metastable points $\eta^{\mathbf{x}}$ and asymptotic Markov dynamics characterized by the strictly positive rates $r(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$.

Remark 2.2. *The asymptotic rates $r(\mathbf{x}, \mathbf{y})$ depend on the parameters L and n. We stress that these rates are strictly positive. The asymptotic behavior is therefore non-local, the limit process being able to jump from a configuration $\eta^{\mathbf{x}}$ to any configuration $\eta^{\mathbf{y}}$ with a positive probability. We present in Corollary* 6.2 *an explicit formula for these rates in terms of the hitting probabilities of simple Markovian dynamics, and we examine in* [13] *the case in which n and*

L increase with β, proving that the trace process ξ_t converges in an appropriate time scale to a two-dimensional Brownian motion.

Denote by $\mathbb{H}_j$, $j \ge 0$, the set of configurations with energy equal to $\mathbb{H}_{\min} + j = -2n(n-1) + j$:

$$\mathbb{H}_j = \{\eta \in \Omega_{L,K} \colon \mathbb{H}(\eta) = \mathbb{H}_{\min} + j\}, \qquad \mathbb{H}_{ij} = \bigcup_{k=i}^{j} \mathbb{H}_k,$$

and let

$$\Delta_j = \{\eta \in \Omega_{L,K} \colon \mathbb{H}(\eta) > \mathbb{H}_{\min} + j\}, \tag{2.3}$$

so that $\mathbb{H}_0 = \Omega^0$ and $\{\mathbb{H}_{0j}, \Delta_j\}$ forms a partition of the set $\Omega_{L,K}$.

Remark 2.3. *The proof of Theorem 2.1 requires a precise knowledge of the energy landscape of the Kawasaki dynamics in the graph $\Omega_{L,K}$. This description is carried out in Section 4, where we show that the process η_t visits solely a tiny portion of the state space $\Omega_{L,K}$ during an excursion between two ground states. More precisely, as illustrated in Fig. 1, we show the existence of four disjoint subsets of $\mathbb{H}_1$, denoted by $\Omega^1, \dots, \Omega^4$, and of four subsets of $\mathbb{H}_2$, denoted by $\Gamma_1, \dots, \Gamma_4$, such that, with a probability converging to 1 as the temperature vanishes,*

(1) *After a time of order $e^{2\beta}$, the process jumps from a ground state to a configuration in the set Γ_1;*
(2) *The process spends a time of order 1 in a set Γ_j before reaching a configuration in Ω^{j-1} or in Ω^j;*
(3) *After a time of order e^{β}, the process jumps from a configuration in Ω^j, $1 \le j \le 4$, to a configuration in the set $\Gamma_j \cup \Gamma_{j+1}$;*
(4) *The set Δ_2 is never visited.*

A large portion of the set $\mathbb{H}_1$ can be reached from a ground state only by crossing the set Δ_2. For example, the configurations in which the particles form a $(n-k) \times (n+k)$ rectangle, $3 \le k < \sqrt{n}$, with an extra row or column of particles attached to the longest side of the rectangle. By the previous discussion, these configurations of the set $\mathbb{H}_1$ are never visited during an excursion between two ground states.

The simplicity of the energy landscape emerging from Remark 2.3, and illustrated in Fig. 1, is one of the main by-products of this article. The proof of the convergence of the Kawasaki dynamics to a Brownian motion in [13] relies strongly on this description.

The article is organized as follows. In the next section we present a sketch of the argument and in Section 4 a description of the shallow valleys visited during an excursion between two ground states. In Section 5, we describe the evolution of the Markov process among these shallow valleys in the time scale e^{β}, and in Section 6 the asymptotic behavior of the Kawasaki dynamics among the ground states in the time scale $e^{2\beta}$.

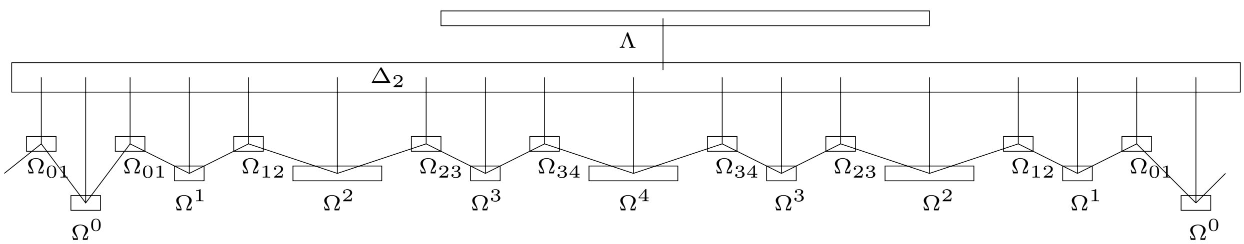

Fig. 1. The energy landscape of the Kawasaki dynamics at low temperature. Ω^0 represents the set of ground states, Ω^j, $1 \le j \le 4$, disjoint subsets of $\mathbb{H}_1$, Γ_j, $1 \le j \le 4$, disjoint subsets of $\mathbb{H}_2$, and $\Lambda = \mathbb{H}_1 \setminus [\bigcup_{1 \le j \le 4} \Omega^j]$. The edges indicate that a configuration from one set may jump to the other. At low temperatures, during an excursion between two ground states the process does not visit the set Δ_2 and all the analysis is reduced to the lower portion of the picture.

3. Sketch of the proof

The proof of Theorem 2.1 relies on the strategy presented in [3] to prove the metastability of reversible Markov processes evolving on finite state spaces. We do not investigate the full tree structure of the chain, presented in the Introduction, but a small portion of it contained in the first and second generation of the tree.

A simple computation shows that assumptions (2.1) and (2.2) of that article are satisfied. Indeed, since $\mathbb{H}(\sigma^{x,y}\eta) - \mathbb{H}(\eta) = (\eta_y - \eta_x)\{\sum_{\|z-y\|=1}\eta_z - \sum_{\|z-x\|=1}\eta_z + \eta_y - \eta_x\}$, the jump rates $c_{x,y}(\eta)$ may only assume the values 1, $\mathrm{e}^{-\beta}$, $\mathrm{e}^{-2\beta}$ and $\mathrm{e}^{-3\beta}$, which proves assumptions (2.1) and (2.2).

Denote by $R_\beta(\eta,\xi)$ the rate at which the process η_t jumps from η to ξ so that $R_\beta(\eta,\xi) = c_{x,y}(\eta)$ if $\xi = \sigma^{x,y}\eta$ for some bond $\{x,y\} \in \Lambda_L^*$, and $R_\beta(\eta,\xi) = 0$, otherwise.

A self-avoiding path γ from $\mathcal{A}$ to $\mathcal{B}$, $\mathcal{A}, \mathcal{B} \subset \Omega_{L,K}$, $\mathcal{A} \cap \mathcal{B} = \varnothing$, is a sequence of configurations $(\xi_0,\dots,\xi_n)$ such that $\xi_0 \in \mathcal{A}$, $\xi_n \in \mathcal{B}$, $\xi_i \neq \xi_j$, $i \neq j$, $R_\beta(\xi_j,\xi_{j+1}) > 0$, $0 \le j < n$. Denote by $\Gamma_{\mathcal{A},\mathcal{B}}$ the set of self-avoiding paths from $\mathcal{A}$ to $\mathcal{B}$ and let

$$G_K(\mathcal{A},\mathcal{B}) := \max_{\gamma\in\Gamma_{\mathcal{A},\mathcal{B}}} G_K(\gamma), \qquad G_K(\gamma) = G_K^\beta(\gamma) := \min_{0\le i<n} \mu_K(\xi_i)R_\beta(\xi_i,\xi_{i+1})$$

if $\gamma = (\xi_0,\dots,\xi_n)$. Since $\mu_K(\xi_i)R_\beta(\xi_i,\xi_{i+1}) = \min\{\mu_K(\xi_i),\mu_K(\xi_{i+1})\}$, $G_K(\gamma) = \min_{0\le i\le n}\mu_K(\xi_i)$ and $G_K(\mathcal{A},\mathcal{B})$ is the measure of the saddle configuration from $\mathcal{A}$ to $\mathcal{B}$.

Denote by $D_K = D_K^\beta$ the Dirichlet form associated to the generator of the Markov process η_t:

$$D_K(f) = \frac{1}{2}\sum_{\{x,y\}\in\Lambda_L^*}\sum_{\xi\in\Omega_{L,K}} \mu_K(\xi)c_{x,y}(\xi)\bigl\{f\bigl(\sigma^{x,y}\xi\bigr) - f(\xi)\bigr\}^2, \quad f:\Omega_{L,K}\to\mathbb{R}.$$

Let $\mathrm{cap}_K(\mathcal{A},\mathcal{B}) = \mathrm{cap}_K^\beta(\mathcal{A},\mathcal{B})$, $\mathcal{A}, \mathcal{B} \subset \Omega_{L,K}$, $\mathcal{A}\cap\mathcal{B} = \varnothing$, be the capacity between $\mathcal{A}$ and $\mathcal{B}$:

$$\mathrm{cap}_K(\mathcal{A},\mathcal{B}) = \inf_f D_K(f),$$

where the infimum is carried over all functions $f:\Omega_{L,K}\to\mathbb{R}$ such that $f(\xi) = 1$ for all $\xi\in\mathcal{A}$, and $f(\xi) = 0$ for all $\xi\in\mathcal{B}$. We proved in [3], Lemma 4.2 and 4.3, that the ratio $\mathrm{cap}_K(\mathcal{A},\mathcal{B})/G_K(\mathcal{A},\mathcal{B})$ converges as $\beta\uparrow\infty$: For every $\mathcal{A}$, $\mathcal{B}\subset\Omega_{L,K}$, $\mathcal{A}\cap\mathcal{B} = \varnothing$,

$$\lim_{\beta\to\infty}\frac{\mathrm{cap}_K(\mathcal{A},\mathcal{B})}{G_K(\mathcal{A},\mathcal{B})} = C(\mathcal{A},\mathcal{B}) \in (0,\infty). \tag{3.1}$$

We claim that $G_K(\{\eta^{\mathbf{x}}\},\{\eta^{\mathbf{y}}\}) = \mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$ for $\mathbf{x}\neq\mathbf{y}$. Denote by e_1, e_2 the canonical basis of $\mathbb{R}^2$. On the one hand, any path γ from $\eta^{\mathbf{x}}$ to a set $\mathcal{A}\not\ni\eta^{\mathbf{x}}$ is such that $G_K(\gamma)\le \mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$. On the other hand, it is easy to construct a self-avoiding path $\gamma = (\eta^{\mathbf{x}} = \xi_0,\dots,\xi_n = \eta^{\mathbf{x}+e_i})$ from $\eta^{\mathbf{x}}$ to $\eta^{\mathbf{x}+e_i}$, and therefore a path from $\eta^{\mathbf{x}}$ to $\eta^{\mathbf{y}}$, such that $\mu_K(\xi_j)\ge \mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$, $0\le j\le n$. This proves the claim.

It follows from the previous claim and from (3.1) that $\mathrm{cap}_K(\{\eta^{\mathbf{x}}\},\{\eta^{\mathbf{y}}\})$ is of order $\mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$. In particular, to examine the evolution of the process η_t among the competing metastable states $\eta^{\mathbf{x}}$ we need only to care of the states whose measure are greater than or equal to $\mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$. Actually, as pointed out in Remark 2.3, only a much smaller class is relevant for the problem.

In the next section we define the sets $\Omega^1,\dots,\Omega^4$ introduced in Remark 2.3. In the following section we show that starting from a configuration in $\bigcup_{0\le j\le 4}\Omega_j$ in the time scale e^β the Kawasaki dynamics evolves as a markov chain whose points are subsets of the sets Ω^j. In this chain the configurations $\eta^{\mathbf{x}}$ are absorbing points and the jump rates are expressed as functions of the hitting probabilities of simple Markovian dynamics.

In Section 6, we deduce from the previous result the tunneling behavior of the process η_t on the longer time scale $\mathrm{e}^{2\beta}$ among the competing metastable states $\eta^{\mathbf{x}}$. The jump rates of this dynamics are expressed in terms of the hitting probabilities of the absorbing states for the Markovian dynamics derived in the previous step.

We conclude this section recalling the definition of a valley presented in [1]. Denote by H_Π, H^+_Π, $\Pi \subset \Omega_{L,K}$, the hitting time and the time of the first return to Π:

$$H_\Pi = \inf\{t > 0:\ \eta^\beta_t \in \Pi\},$$

$$H^+_\Pi = \inf\{t > 0:\ \eta^\beta_t \in \Pi \text{ and } \exists 0 < s < t;\ \eta^\beta_s \notin \Pi\}.$$

We sometimes write $H(\Pi)$, $H^+(\Pi)$ instead of H_Π, H^+_Π.

Consider two subsets $\mathcal{W} \subset \mathcal{B}$ of the state space Ω_K and a configuration $\eta \in \mathcal{W}$. The triple $(\mathcal{W}, \mathcal{B}, \eta)$ is called a valley if:

- Starting from any configuration of $\mathcal{W}$ the process visits η before hitting $\mathcal{B}^c$:

$$\lim_{\beta\to\infty} \max_{\xi\in\mathcal{W}} \mathbf{P}^\beta_\xi [H_{\mathcal{B}^c} < H_\eta] = 0.$$

- There exists a sequence m_β such that, for every $\xi \in \mathcal{W}$, $H_{\mathcal{B}^c}/m_\beta$ converges to a mean 1 exponential random variable under $\mathbf{P}^\beta_\xi$. The sequence m_β is called the depth of the valley.
- The portion of time the process spends in $\mathcal{B} \setminus \mathcal{W}$ before hitting $\mathcal{B}^c$ is negligible: for every $\xi \in \mathcal{W}$ and every $\delta > 0$,

$$\lim_{\beta\to\infty} \mathbf{P}^\beta_\xi \left[\frac{1}{m_\beta} \int_0^{H_{\mathcal{B}^c}} \mathbf{1}\{\eta_s \in \mathcal{B} \setminus \mathcal{W}\} > \delta \right] = 0.$$

4. Some shallow valleys

We examine in this section the evolution of the Markov process $\{\eta^\beta_t:\ t \geq 0\}$ between two consecutive visits to the ground states $\{\eta^{\mathbf{x}}:\ \mathbf{x} \in \Lambda_L\}$. In the next section, we show that at very low temperatures, in the time scale e^β, much smaller than the time scale of an excursion between ground states, the process η_t evolves as a continuous-time Markov chain whose state space consists of subsets of $\mathbb{H}_{01}$. In this asymptotic dynamics the ground states play the role of absorbing states. We present in this section the subsets of $\mathbb{H}_{01}$ which become the points of the asymptotic dynamics and show that these sets are valleys. We also provide explicit formulas for the jump rates of the asymptotic dynamics in terms of four elementary Markov processes.

For a subset B of Λ_L, denote by $\partial_+ B$ the outer boundary of B. This is the set of sites which are at distance one from B:

$$\partial_+ B = \{\mathbf{x} \in \Lambda_L \setminus B:\ \exists \mathbf{y} \in B, \|\mathbf{y} - \mathbf{x}\| = 1\}.$$

4.1. *Elementary Markov processes*

The jump rates among the shallow valleys introduced below are all expressed in terms of the hitting probabilities of four elementary, finite-state, continuous-time Markov chains. We present in this subsection these processes and derive some identities needed later. Let $\{\mathbf{x}_t:\ t \geq 0\}$ be the nearest-neighbor, symmetric random walk on Λ_L which jumps from a site $\mathbf{x}$ to $\mathbf{x} \pm e_i$ at rate 1. Denote by $\mathbb{P}^{\mathbf{x}}_{\mathbf{y}}$, $\mathbf{y} \in \Lambda_L$, the probability measure on $D(\mathbb{R}_+, \Lambda_L)$ induced by $\mathbf{x}_t$ starting from $\mathbf{x}$. We sometimes represent $\mathbf{x}_t$ by $\mathbf{x}(t)$. Denote by $\mathfrak{p}(\mathbf{y}, \mathbf{z}, G)$, $\mathbf{y} \in \Lambda_L$, $\mathbf{z} \in G$, $G \subset \Lambda_L$, the probability that the random walk starting from $\mathbf{y}$ reaches G at $\mathbf{z}$:

$$\mathfrak{p}(\mathbf{y}, \mathbf{z}, G) := \mathbb{P}^{\mathbf{x}}_{\mathbf{y}}[\mathbf{x}(H_G) = \mathbf{z}].$$

By extension, for a subset A of Λ_L, let $\mathfrak{p}(\mathbf{x}, A, G) = \sum_{\mathbf{y}\in A} \mathfrak{p}(\mathbf{x}, \mathbf{y}, G)$. Moreover, when $G = \partial_+ Q$, we omit the set G in the notation: $\mathfrak{p}(\mathbf{x}, A) := \mathfrak{p}(\mathbf{x}, A, \partial_+ Q)$. Let, finally,

$$\mathfrak{p}(A) := \mathfrak{p}(\mathbf{w}_2 + 2e_2, A) + \mathfrak{p}(\mathbf{w}_2 + e_1 + e_2, A). \tag{4.1}$$

Let $\mathbf{y}_t = (\mathbf{y}_t^1, \mathbf{y}_t^2)$ be the continuous-time Markov chain on $D_n = \{(j,k)\colon 0 \le j < k \le n-1\} \cup \{(0,0)\}$ which jumps from a site $\mathbf{y}$ to any of its nearest-neighbor sites $\mathbf{z}$, $\|\mathbf{y} - \mathbf{z}\| = 1$, at rate 1. Let $D_n^+ = \{(j, n-1)\colon 0 \le j < n-1\}$ and let

$$\mathfrak{q}_n = \mathbb{P}^{\mathbf{y}}_{(0,1)}[H_{D_n^+} < H_{(0,0)}], \tag{4.2}$$

where $\mathbb{P}^{\mathbf{y}}_{(0,1)}$ stands for the distribution of $\mathbf{y}_t$ starting from (0, 1).

Let $E_n = \{0, \dots, n-1\}^2$ and let $\mathbf{z}_t = (\mathbf{z}_t^1, \mathbf{z}_t^2)$ be the continuous-time Markov chain on $E_n \cup \{\mathfrak{d}\}$ which jumps from a site $\mathbf{z} \in E_n$ to any of its nearest-neighbor sites $\mathbf{z}' \in E_n$, $\|\mathbf{z}' - \mathbf{z}\| = 1$, at rate 1, and which jumps from (1, 1) (resp. from $\mathfrak{d}$) to $\mathfrak{d}$ (resp. to (1, 1)) at rate 1. Let $E_n^+ = \{(j, n-1)\colon 0 \le j \le n-1\}$, $E_n^- = \{(j, 0)\colon 1 \le j \le n-1\}$, $\partial E_n = E_n^+ \cup E_n^- \cup \{(0,0)\}$ and for $0 \le k \le n-1$, let

$$\begin{aligned} &\mathfrak{r}_n^+ = \mathbb{P}^{\mathbf{z}}_{(0,1)}[H_{\partial E_n} = H_{E_n^+}], \qquad \mathfrak{r}_n^- = \mathbb{P}^{\mathbf{z}}_{(0,1)}[H_{\partial E_n} = H_{E_n^-}], \\ &\mathfrak{r}_n^0(k) = \mathbb{P}^{\mathbf{z}}_{(k,1)}[H_{\partial E_n} = H_{(0,0)}], \qquad \mathfrak{r}_n = \mathfrak{r}_n^+ + \mathfrak{r}_n^-, \end{aligned} \tag{4.3}$$

where $\mathbb{P}^{\mathbf{z}}_{(k,1)}$ stands for the distribution of $\mathbf{z}_t$ starting from $(k, 1)$. Note that the values of $\mathfrak{r}_n^{\pm}$, $\mathfrak{r}_n^0(k)$ are unchanged if we consider the trace of $\mathbf{z}_t$ on the set $\{0, \dots, n-1\}^2$. This latter process is a nearest-neighbor random walk on $\{0, \dots, n-1\}^2$ whose holding time at (1, 1) is longer. The embedded discrete-time chain of the trace process is the symmetric, nearest-neighbor random walk. In particular, $\mathfrak{r}_n^+ = (n-1)^{-1}$.

We claim that

$$\sum_{k=1}^{n-1} \mathfrak{r}_n^0(k) = \mathfrak{r}_n^-. \tag{4.4}$$

Indeed, denote by $\mathbb{Z}_k$ the embedded, discrete-time chain on E_n. Outside of the boundary ∂E_n, $\mathbb{Z}_k$ jumps uniformly to one of its neighbors. At the boundary ∂E_n, it jumps with probability 1 to the unique neighbor in the interior $E_n \setminus \partial E_n$. Therefore,

$$\mathfrak{r}_n^0(k) = \mathbb{P}^{\mathbf{z}}_{(k,1)}[H_{\partial E_n} = H_{(0,0)}] = \sum_{\gamma} p(\gamma) 3^{-1} = \sum_{\gamma} \pi(k,1) p(\gamma) \frac{1}{3\pi(k,1)},$$

where the sum is carried over all paths γ from $(k, 1)$ to (0, 1) which never pass by ∂E_n. The factor 1/3 represents the probability to jump from (0, 1) to (0, 0) and π the reversible stationary measure for the chain $\mathbb{Z}_k$, which is proportional to the degree of the vertices. By reversibility, the previous sum is equal to

$$\sum_{\gamma'} \pi(0,1) p(\gamma') \frac{1}{3\pi(k,1)} = \sum_{\gamma'} p(\gamma') \frac{1}{Z_n \pi(k,1)},$$

where the sum is now carried over all paths γ' from (0, 1) to $(k, 1)$ which never pass by ∂E_n and Z_n is the sum of the degrees of all vertices. Last expression is equal to $\mathbb{P}^{\mathbf{z}}_{(0,1)}[H_{\partial E_n} = H_{(k,0)}]$. Summing over all k yields (4.4).

Finally, consider two independent, nearest-neighbor, continuous-time, random walks $\mathbf{u}_t$, $\mathbf{v}_t$ evolving on an interval $J = \{m, \dots, M\}$, $m < M$, which jump from k (resp. $k+1$) to $k+1$ (resp. k), $m \le k < M$, at rate 1. Let $H_1 = \inf\{t > 0\colon |\mathbf{u}_t - \mathbf{v}_t| = 1\}$. For $a, a+2 \in J$, $b, b+1 \in J$, let

$$\mathfrak{m}(J, a, b) := \mathbb{P}^{\mathbf{uv}}_{(a,a+2)}\big[(\mathbf{u}_{H_1}, \mathbf{v}_{H_1}) = (b, b+1)\big], \tag{4.5}$$

where $\mathbb{P}^{\mathbf{uv}}_{(a,a+2)}$ stands for the distribution of the pair $(\mathbf{u}_t, \mathbf{v}_t)$ starting from $(a, a+2)$.

4.2. *The distribution of $\eta(H^+_{\mathbb{H}_{01}})$ starting from a ground state*

Denote by $\mathcal{N}(\eta^{\mathbf{x}})$ the set of eight configurations which can be obtained from $\eta^{\mathbf{x}}$ by a rate $\mathrm{e}^{-2\beta}$ jump. Two of these configurations deserve a special notation, $\eta_1^\star = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2} \eta^{\mathbf{w}}$ and $\eta_2^\star = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_1} \eta^{\mathbf{w}}$.

Lemma 4.1. *For each $\eta \in \mathcal{N}(\eta^{\mathbf{x}})$, there exists a probability measure $\mathbb{M}(\eta, \cdot)$ on $\mathbb{H}_{01}$ such that*

$$\mathbb{M}(\eta, \mathcal{A}) := \lim_{\beta \to \infty} \mathbf{P}_\eta[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] \tag{4.6}$$

for all $\mathcal{A} \subset \mathbb{H}_{01}$. Set $\mathbb{M}_j(\cdot) = \mathbb{M}(\eta_j^\star, \cdot)$, $j = 1, 2$. Then,

$$\mathbb{M}_1\big(\mathcal{E}_{\mathbf{w}}^{0,0}\big) = \frac{1}{2}\left\{\frac{1 + \mathfrak{r}_n^- + \mathfrak{A}_{1,2}}{4 + \mathfrak{q}_n + \mathfrak{r}_n - \mathfrak{A}} + \frac{1 + \mathfrak{r}_n^- + \mathfrak{A}_2 - \mathfrak{A}_1}{4 + \mathfrak{q}_n + \mathfrak{r}_n + \mathfrak{A}(e_1) - \mathfrak{A}(e_2)}\right\},$$
$$\mathbb{M}_2\big(\mathcal{E}_{\mathbf{w}}^{0,0}\big) = \frac{1}{2}\left\{\frac{1 + \mathfrak{r}_n^- + \mathfrak{A}_{1,2}}{4 + \mathfrak{q}_n + \mathfrak{r}_n - \mathfrak{A}} - \frac{1 + \mathfrak{r}_n^- + \mathfrak{A}_2 - \mathfrak{A}_1}{4 + \mathfrak{q}_n + \mathfrak{r}_n + \mathfrak{A}(e_1) - \mathfrak{A}(e_2)}\right\},$$

where $\mathfrak{A}(e_i) = \mathfrak{p}(\mathbf{w}_2 + e_i)$, $\mathfrak{A}_j = \mathfrak{p}(Q_{\mathbf{w}}^{2,j})$ and

$$\mathfrak{A} = \mathfrak{A}(e_1) + \mathfrak{A}(e_2), \qquad \mathfrak{A}_{1,2} = \mathfrak{A}_1 + \mathfrak{A}_2, \qquad \mathfrak{A}_{0,3} = \mathfrak{A}_0 + \mathfrak{A}_3.$$

Proof. We prove this lemma for $\eta = \eta_1^\star$, $\mathcal{A} = \mathcal{E}_{\mathbf{0}}^{2,2}$ and leave the other cases to the reader. Recall the definition of $\mathfrak{q}_n$, $\mathfrak{r}_n^\pm$ introduced in (4.2), (4.3). We claim that any limit point $\mathbb{M}_j = \mathbb{M}_j(\mathcal{E}_{\mathbf{w}}^{2,2})$ of the sequences $\mathbf{P}_{\eta_j^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}]$ satisfies the equations

$$(4 + \mathfrak{q}_n + \mathfrak{r}_n - \mathfrak{A})(\mathbb{M}_1 + \mathbb{M}_2) = 1 + \mathfrak{A}_{1,2} + \mathfrak{r}_n^-. \tag{4.7}$$

To prove (4.7), assume that $\mathbf{P}_{\eta_j^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}]$ converges and observe that the configuration $\eta_1^\star$ may jump at rate 1 to 6 configurations and at rate $\mathrm{e}^{-\beta}$ or less to $O(n)$ configurations. Among the configurations which can be reached at rate 1 two belong to $\mathbb{H}_{01}$, one of them being $\eta^{\mathbf{w}}$ and the other $\sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2 - e_1}\eta^{\mathbf{w}} \in \mathcal{A}$. Hence, if we denote by $\mathcal{N}_2(\eta_1^\star)$ the set of the remaining four configurations which can be reached from $\eta_1^\star$ by a rate 1 jump, decomposing $\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}]$ according to the first jump we obtain that

$$6\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] = 1 + \sum_{\eta' \in \mathcal{N}_2(\eta_1^\star)} \mathbf{P}_{\eta'}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \varepsilon(\beta), \tag{4.8}$$

where $\varepsilon(\beta)$ is a remainder which vanishes as $\beta \uparrow \infty$.

We examine the four configurations of $\mathcal{N}_2(\eta_1^\star)$ separately. In two configurations, $\sigma^{\mathbf{w}_2, \mathbf{w}_2 + 2e_2}\eta^{\mathbf{w}}$ and $\sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2 + e_1} \times \eta^{\mathbf{w}}$, a particle is detached from the quasi-square $Q_{\mathbf{w}}^2$. The detached particle performs a rate 1 symmetric random walk on Λ_L until it reaches the outer boundary of the square $Q_{\mathbf{0}}$. Denote by H_{hit} the time the detached particle hits the outer boundary of the square $Q_{\mathbf{w}}$. Among the remaining particles, two jumps have rate $\mathrm{e}^{-\beta}$ and the other ones have rate at most $\mathrm{e}^{-2\beta}$. Therefore, by the strong Markov property, for $\eta' = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + 2e_2}\eta^{\mathbf{w}}$, $\sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2 + e_1}\eta^{\mathbf{w}}$,

$$\mathbf{P}_{\eta'}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] = \mathbf{P}_{\eta'}\big[\eta(H_{\text{hit}}) \in \mathcal{E}_{\mathbf{w}}^{2,2}\big] + \sum_{i=1}^{2} \mathbf{P}_{\eta'}\big[\eta(H_{\text{hit}}) = \eta_i^\star\big]\mathbf{P}_{\eta_i^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \varepsilon(\beta).$$

By definition (4.1) of $\mathfrak{p}$, the contribution of the terms $\eta' = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + 2e_2}\eta^{\mathbf{w}}$ and $\eta' = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2 + e_1}\eta^{\mathbf{w}}$ to the sum appearing on the right hand side of (4.8) is

$$\mathfrak{p}(\mathbf{w}_2 + e_2)\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \mathfrak{p}(\mathbf{w}_2 + e_1)\mathbf{P}_{\eta_2^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \mathfrak{p}\big(Q_{\mathbf{w}}^{2,2}\big) + \varepsilon(\beta). \tag{4.9}$$

It remains to analyze the two configurations of $\mathcal{N}_2(\eta)$, $\eta' = \sigma^{\mathbf{w}_2 - e_2, \mathbf{w}_2 + e_2}\eta^{\mathbf{w}}$ and $\eta' = \sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 + e_2}\eta^{\mathbf{w}}$. In the first one, if we denote by $\mathbf{z}_t^1$ the horizontal position of the particle attached to the top side of the square Q and by $\mathbf{z}_t^2$ the vertical position of the hole on the left side of the square, it is not difficult to check that $(\mathbf{z}_t^1, \mathbf{z}_t^2)$ evolves as the Markov chain described just before (4.3) with initial condition $(\mathbf{z}_0^1, \mathbf{z}_0^2) = (0, 1)$.

Denote by H_{hit} the time the hole hits 0 or $n - 1$. Since the hole moves at rate 1, and since all the other $O(n)$ possible jumps have rate at most $\mathrm{e}^{-\beta}$, with probability increasing to 1 as $\beta \uparrow \infty$, H_{hit} occurs before any rate $\mathrm{e}^{-\beta}$ jump takes place. At time H_{hit} three situations can happen. If the process $(\mathbf{z}_t^1, \mathbf{z}_t^2)$ reached $(0,0)$ (resp. E_n^+, E_n^-), the process $\eta(t)$ returned to the configuration $\eta_1^\star$ (resp. hitted a configuration in $\mathcal{E}_{\mathbf{w}}^{1,2}$, $\mathcal{E}_{\mathbf{w}}^{2,2}$). Hence, by definition of $\mathfrak{r}_n^\pm$,

$$\mathbf{P}_{\eta'}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] = \mathfrak{r}_n^- + (1-\mathfrak{r}_n)\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \varepsilon(\beta)$$

for $\eta' = \sigma^{\mathbf{w}_2 - e_2, \mathbf{w}_2 + e_2}\eta^{\mathbf{w}}$.

Assume now that $\eta' = \sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 + e_2}\eta^{\mathbf{0}}$. In this case, if we denote by $\mathbf{y}^1$ the horizontal position of the particle attached to the side of the quasi-square and by $\mathbf{y}^2$ the horizontal position of the hole, the pair $(\mathbf{y}_t^1, \mathbf{y}_t^2)$ evolve according to the Markov process introduced just before (4.2) with initial condition $(\mathbf{y}_0^1, \mathbf{y}_0^2) = (0,1)$. Denote by H_{hit} the time the hole hits 0 or $n-1$. Here again, since the attached particle and the hole move at rate 1 and since all the other $\mathrm{O}(n)$ jumps have rate at most $\mathrm{e}^{-\beta}$, with asymptotic probability equal to 1, H_{hit} occurs before any rate $\mathrm{e}^{-\beta}$ jump takes place. At time H_{hit}, if $\mathbf{y}_{H_{\mathrm{hit}}}^2 = 0$, the process $\eta(t)$ has returned to the configuration $\eta_1^\star$, while if $\mathbf{y}_{H_{\mathrm{hit}}}^2 = n-1$, the process $\eta(t)$ has reached a configuration in $\mathcal{E}_{\mathbf{w}}^{3,2}$. Since the random walk reaches $n-1$ before 0 with probability $\mathfrak{q}_n$, by the strong Markov property

$$\mathbf{P}_{\eta'}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] = (1-\mathfrak{q}_n)\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \varepsilon(\beta)$$

for $\eta' = \sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 + e_2}\eta^{\mathbf{w}}$.

Therefore, the contribution of the last two configurations of $\mathcal{N}_2(\eta_1^\star)$ to the sum on the right hand side of (4.8) is

$$\mathfrak{r}_n^- + (2 - \mathfrak{r}_n - \mathfrak{q}_n)\mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}] + \varepsilon(\beta). \tag{4.10}$$

Equations (4.8), (4.9) and (4.10) yield a linear equation for $\mathbb{M}_1 = \mathbf{P}_{\eta_1^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}]$ in terms of $\mathbb{M}_1$ and $\mathbb{M}_2 = \mathbf{P}_{\eta_2^\star}[H_{\mathbb{H}_{01}} = H_{\mathcal{A}}]$. Analogous arguments provide a similar equation for $\mathbb{M}_2$ in terms of $\mathbb{M}_1$ and $\mathbb{M}_2$. Adding these two equations we obtain (4.7), while subtracting them gives a formula for the difference $\mathbb{M}_1 - \mathbb{M}_2$. The assertion of the lemma follows from these equations for $\mathbb{M}_1 + \mathbb{M}_2$ and $\mathbb{M}_1 - \mathbb{M}_2$. □

Similar computations to the ones carried over in the previous proof permit to derive explicit expressions for $\mathbb{M}_j$. For example, we have that

$$\mathbb{M}_1(\eta^{\mathbf{w}}) = \mathbb{M}_2(\eta^{\mathbf{w}}) = \frac{1}{4 + \mathfrak{q}_n + \mathfrak{r}_n - \mathfrak{A}}.$$

By symmetry, for each $\eta \in \mathcal{N}(\eta^{\mathbf{x}})$, we can represent $\mathbb{M}(\eta, \cdot)$ in terms of $\mathbb{M}_1$ and $\mathbb{M}_2$. Moreover, for all $\eta \in \mathcal{N}(\eta^{\mathbf{x}})$,

$$\mathbb{M}(\eta, \eta^{\mathbf{x}}) + \sum_{0 \le i,j \le 3} \mathbb{M}(\eta, \mathcal{E}_{\mathbf{x}}^{i,j}) = 1.$$

4.3. *The valleys* $\mathcal{E}_{\mathbf{x}}^{i,j}$

Let $Q^i = Q \setminus \{\mathbf{w}_i\}$, $0 \le i \le 3$, where

$$\mathbf{w}_0 = \mathbf{w} = (0,0), \qquad \mathbf{w}_1 = (n-1, 0), \qquad \mathbf{w}_2 = (n-1, n-1), \qquad \mathbf{w}_3 = (0, n-1)$$

are the corners of the square Q. For $\mathbf{x} \in \Lambda_L$, let $Q_{\mathbf{x}}^i = \mathbf{x} + Q^i$, $\mathbf{x}_i = \mathbf{x} + \mathbf{w}_i$.

Denote by $\partial_j Q_{\mathbf{x}}^i$, $0 \le j \le 3$, the jth boundary of $Q_{\mathbf{x}}^i$:

$$\partial_j Q_{\mathbf{x}}^i = \{\mathbf{z} \in \partial_+ Q_{\mathbf{x}}^i \colon \exists \mathbf{y} \in Q_{\mathbf{x}}^i; \mathbf{y} - \mathbf{z} = (1-j)e_2\}, \quad j = 0, 2,$$

$$\partial_j Q_{\mathbf{x}}^i = \{\mathbf{z} \in \partial_+ Q_{\mathbf{x}}^i \colon \exists \mathbf{y} \in Q_{\mathbf{x}}^i; \mathbf{y} - \mathbf{z} = (j-2)e_1\}, \quad j = 1, 3.$$

Let $Q_{\mathbf{x}}^{i,j} = \partial_j Q_{\mathbf{x}}^i \setminus Q_{\mathbf{x}}$, let $\mathcal{E}_{\mathbf{x}}^{i,j}$ be the set of configurations in which all sites of the set $Q_{\mathbf{x}}^i$ are occupied with an extra particle at some location of $Q_{\mathbf{x}}^{i,j}$, and let $\Omega^1 = \Omega_{L,K}^1$ be the union of all such sets:

$$\mathcal{E}_{\mathbf{x}}^{i,j} = \{\sigma^{\mathbf{x}_i, \mathbf{z}}\eta^{\mathbf{x}} \colon \mathbf{z} \in Q_{\mathbf{x}}^{i,j}\}, \qquad \Omega_{\mathbf{x}}^1 = \bigcup_{0 \le i,j \le 3} \mathcal{E}_{\mathbf{x}}^{i,j}, \qquad \Omega^1 = \bigcup_{\mathbf{x} \in \Lambda_L} \Omega_{\mathbf{x}}^1.$$

Fig. 2. Four among the five configurations of the set $\mathcal{E}_{\mathbf{x}}^{0,0}$ for $n=6$. The gray dot indicates the site $\mathbf{x}$. We placed a square $[-1/2,1/2)^2$ around each particle.

Note that $\Omega_1 \subset \mathbb{H}_1$. Figure 2 illustrates typical configurations of the set Ω^1.

The process $\{\eta_t^\beta \colon t \ge 0\}$ can reach any configuration $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$ from any configuration $\eta \in \mathcal{E}_{\mathbf{x}}^{i,j}$ with rate one jumps, while any jump from a configuration in $\mathcal{E}_{\mathbf{x}}^{i,j}$ to a configuration which does not belong to this set has rate at most $\mathrm{e}^{-\beta}$. This means that at low temperatures the process η_t reaches equilibrium in $\mathcal{E}_{\mathbf{x}}^{i,j}$ before exiting this set, which is the first condition for a set to be the well of a valley.

The main result of this subsection states that for any configuration $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$, the triples $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$ are valleys in the terminology of [1]. This means, in particular, that starting from any configuration in $\mathcal{E}_{\mathbf{x}}^{i,j}$, the hitting time of the set $\mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j}$ properly rescaled converges in distribution, as $\beta \uparrow \infty$, to an exponential random variable. We compute in Proposition 4.3 the time scale which turns the limit a mean one exponential distribution, as well as the asymptotic distribution of $\eta(H(\mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j}))$.

Denote by $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$, $\mathcal{N}$ for neighborhood, the configurations which do not belong to $\mathcal{E}_{\mathbf{x}}^{i,j}$, but which can be reached from a configuration in $\mathcal{E}_{\mathbf{x}}^{i,j}$ by performing a jump which has rate $\mathrm{e}^{-\beta}$. The set $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$, for instance, has the following $3n$ elements. There are $n+1$ configurations obtained when the top particle detaches itself from the others: $\sigma^{\mathbf{w}_2,\mathbf{z}}\eta^{\mathbf{w}}$, where $\mathbf{z} = (-1,n)$, $(a,n+1)$, $0 \le a \le n-2$, $(n-1,n)$. There are $n-1$ configurations obtained when the particle at $\mathbf{w}_2 - e_2$ moves upward: $\sigma^{\mathbf{w}_2-e_2,\mathbf{z}}\eta^{\mathbf{w}}$, $\mathbf{z} = (a,n)$, $0 \le a \le n-2$. There are $n-2$ configurations obtained when the particle at $\mathbf{w}_2 - e_1$ moves to the right: $\sigma^{\mathbf{w}_2-e_1,\mathbf{z}}\eta^{\mathbf{w}}$, $\mathbf{z} = (a,n)$, $0 \le a \le n-3$. To complete the description of the set $\mathcal{N}(\mathcal{F}_{\mathbf{w}}^{2,2})$, we have to add the configurations $\sigma^{\mathbf{w}_3,\mathbf{w}_3+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\eta^{\mathbf{w}}$ and $\sigma^{\mathbf{w}_2-e_1,\mathbf{w}_2-e_1+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_2-2e_1+e_2}\eta^{\mathbf{w}}$.

Lemma 4.2. *For $\mathbf{x} \in \Lambda_L$, $0 \le i, j \le 3$, and $\xi \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$, there exists a probability measure $\mathbb{M}(\xi,\cdot)$ defined on $\mathbb{H}_{01}$ such that*

$$\lim_{\beta\to\infty} \mathbf{P}_\xi^\beta\big[\eta(H_{\mathbb{H}_{01}}) \in \mathcal{A}\big] = \mathbb{M}(\xi,\mathcal{A})$$

for all $\mathcal{A} \subset \mathbb{H}_{01}$. Moreover, let Π be one of the sets $\mathcal{E}_{\mathbf{w}}^{i,j}$, $0 \le i, j \le 3$, or one of the singletons $\{\eta^{\mathbf{w}}\}$, $\{\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2} \times \sigma^{\mathbf{w}_0,\mathbf{w}_3+e_2}\eta^{\mathbf{w}}\}$. Then,

(1) *For $\mathbf{z} \in J_1 = \{(-1,n), (n-1,n), (a,n+1) \colon 0 \le a \le n-2\}$,*

$$\mathbb{M}\big(\sigma^{\mathbf{w}_2,\mathbf{z}}\eta^{\mathbf{w}}, \Pi\big) = \sum_{k=0}^{3} \mathfrak{p}\big(\mathbf{z}, Q_{\mathbf{w}}^{2,k}\big)\mathbf{1}\big\{\Pi = \mathcal{E}_{\mathbf{w}}^{2,k}\big\} + \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_2)\mathbb{M}_1(\Pi) + \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_1)\mathbb{M}_2(\Pi).$$

(2) *For $\mathbf{z} \in Q_{\mathbf{w}}^{2,2}$,*

$$\mathbb{M}\big(\sigma^{\mathbf{w}_2-e_2,\mathbf{z}}\eta^{\mathbf{w}}, \Pi\big) = \frac{1}{n-1}\mathbf{1}\big\{\Pi = \mathcal{E}_{\mathbf{w}}^{1,2}\big\} + \mathfrak{r}_n^0(\mathfrak{n}_{\mathbf{z}})\mathbb{M}_1(\Pi) + \left\{\frac{n-2}{n-1} - \mathfrak{r}_n^0(\mathfrak{n}_{\mathbf{z}})\right\}\mathbf{1}\big\{\Pi = \mathcal{E}_{\mathbf{w}}^{2,2}\big\},$$

where $\mathfrak{n}_{\mathbf{z}} = n - 1 - z_1$, $\mathbf{z} = (z_1, z_2)$.

(3) *For $\mathbf{z} = (k,n)$, $0 \le k \le n-3$,*

$$\mathbb{M}\big(\sigma^{\mathbf{w}_2-e_1,\mathbf{z}}\eta^{\mathbf{w}}, \mathcal{E}_{\mathbf{w}}^{2,2}\big) = 1.$$

(4) *Finally, for the last two configurations of* $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$,

$$\mathbb{M}\big(\sigma^{\mathbf{w}_3,\mathbf{w}_3+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\eta^{\mathbf{w}},\Pi\big)=\frac{1}{n}\mathbf{1}\big\{\Pi=\big\{\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\sigma^{\mathbf{w}_0,\mathbf{w}_3+e_2}\eta^{\mathbf{w}}\big\}\big\}$$
$$+\frac{n-1}{n}\mathbf{1}\big\{\Pi=\mathcal{E}_{\mathbf{w}}^{2,2}\big\},$$
$$\mathbb{M}\big(\sigma^{\mathbf{w}_2-e_1,\mathbf{w}_2-e_1+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_2-2e_1+e_2}\eta^{\mathbf{w}},\mathcal{E}_{\mathbf{w}}^{2,2}\big)=1.$$

Proof. We present the proof for $i=j=2$, $\mathbf{x}=\mathbf{w}$, the other cases being analogous. As we have seen, the set $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$ has five different types of configurations. We examine each one separately. Assume first that $\xi=\sigma^{\mathbf{w}_2,\mathbf{z}}\eta^{\mathbf{w}}$ for some $\mathbf{z}\in J_1$, J_1 the set defined in the statement of the lemma. The free particle, initially at $\mathbf{z}$ performs a rate one, nearest-neighbor, symmetric random walk in Λ_L until it reaches the outer boundary of the square Q. All the other possible jumps have rate at most $\mathrm{e}^{-\beta}$ and may therefore be neglected in the argument. When the free particle attains $\partial_+ Q$, the configuration either belongs to one of the sets $\mathcal{E}^{2,k}$, $0\le k\le 3$, or is one of the configurations $\eta_1^\star$ or $\eta_2^\star$ introduced in Lemma 4.1. By definition (4.1) of $\mathfrak{p}$, it belongs to $\mathcal{E}_{\mathbf{w}}^{2,k}$ with probability $\mathfrak{p}(\mathbf{z},Q_{\mathbf{w}}^{2,k})$ and is equal to the configuration $\eta_1^\star$ (resp. $\eta_2^\star$) with probability $\mathfrak{p}(\mathbf{z},\mathbf{w}_2+e_2)$ (resp. $\mathfrak{p}(\mathbf{z},\mathbf{w}_2+e_1)$). This proves the first assertion of the lemma.

Assume now that $\xi=\sigma^{\mathbf{w}_2-e_2,\mathbf{z}}\eta^{\mathbf{w}}$, $\mathbf{z}\in Q_{\mathbf{w}}^{2,2}$. The configuration ξ has a particle attached to the top side of the square Q and a hole on the right side of the square. This pair behaves as the process $\mathbf{z}_t$ introduced in the begining of this section and evolves until the hole reaches the bottom of the square or its original position at the top. There are three cases to be considered. The hole may reach $\mathbf{w}_1$ before $\mathbf{w}_2$. This happens with probability $(n-1)^{-1}$ and the configuration attained belongs to the equivalent class $\mathcal{E}_{\mathbf{w}}^{1,2}$.

The hole may reach $\mathbf{w}_2$ before $\mathbf{w}_1$ when the top particle is not at $\mathbf{w}_2+e_2$. In this case the process reached a configuration in $\mathcal{E}_{\mathbf{w}}^{2,2}$. This event has probability $\mathbb{P}^{\mathbf{z}}_{(j,1)}[H_{\partial E_n}=H_{E_N^-}]$, where $j=n-1-\mathbf{z}\cdot e_1$ and where $\mathbb{P}^{\mathbf{z}}_{(j,1)}$ is the measure introduced in (4.3). By definition of the set ∂E_n,

$$\mathbb{P}^{\mathbf{z}}_{(j,1)}[H_{\partial E_n}=H_{E_N^-}]=1-\mathbb{P}^{\mathbf{z}}_{(j,1)}[H_{\partial E_n}=H_{E_N^+}]-\mathbb{P}^{\mathbf{z}}_{(j,1)}[H_{\partial E_n}=H_{(0,0)}]$$
$$=\frac{n-2}{n-1}-\mathfrak{r}_n^0(j).$$

Finally, the hole may reach $\mathbf{w}_2$ before $\mathbf{w}_1$ at a time where the top particle is at $\mathbf{w}_2+e_2$. In this case the process reached the configuration $\eta_1^\star$ introduced in the previous lemma. This event happens with probability $\mathfrak{r}_n^0(j)$, which concludes the proof of the second assertion of the lemma.

In the case $\xi=\sigma^{\mathbf{w}_2-e_1,\mathbf{z}}\eta^{\mathbf{w}}$, the hole initially at $\mathbf{w}_2-e_1$ performs a horizontal, rate one, symmetric random walk on the interval $\{m+1,\dots,n-1\}$, where m represents the horizontal position of the top particle, which itself performs a horizontal, rate one, symmetric random walk limited on its right by the hole in the row below. This coupled system evolves as the process $\mathbf{y}_t$ introduced in (4.2) until the hole initially at $\mathbf{w}_2-e_1$ reaches its original position at $\mathbf{w}_2$.

Suppose that $\xi=\sigma^{\mathbf{w}_3,\mathbf{w}_3+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\eta^{\mathbf{w}}$. In this situation the hole at $\mathbf{w}_3$ performs a vertical, rate one, symmetric random walk on $\{(0,b)\colon 0\le b\le n\}$. The hole reaches $\mathbf{w}$ before it reaches $\mathbf{w}_3$ with probability n^{-1}.

Finally, if $\xi=\sigma^{\mathbf{w}_2-e_1,\mathbf{w}_2-e_1+e_2}\sigma^{\mathbf{w}_2,\mathbf{w}_2-2e_1+e_2}\eta^{\mathbf{w}}$, there is only one rate one jump which drives the system back to the set $\mathcal{E}_{\mathbf{w}}^{2,2}$. $\square$

By symmetry, the distribution of $\eta(H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,\cdot}))$ can be obtained from the one of $\eta(H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{0,j}))$, $0\le j\le 3$. When the set Π is a singleton $\{\zeta\}$ we represent $\mathbb{M}(\xi,\{\zeta\})$ by $\mathbb{M}(\xi,\zeta)$. This convention is adopted for all functions of sets without further comment. Recall the notation introduced in the beginning of this section and in the statement of Lemma 4.2. Let

$$Z\big(\mathcal{E}_{\mathbf{x}}^{i,j}\big)=\sum_{\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})}\mathbb{M}\big(\xi,\big(\mathcal{E}_{\mathbf{x}}^{i,j}\big)^c\big)=\sum_{\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})}\big\{1-\mathbb{M}\big(\xi,\mathcal{E}_{\mathbf{x}}^{i,j}\big)\big\}. \tag{4.11}$$

Note that $Z(\mathcal{E}_{\mathbf{x}}^{i,j})$ does not depend on $\mathbf{x}$. In this sum, the terms $\mathbb{M}(\xi,\cdot)$ are not multiplied by weights $\omega(\xi)$ because asymptotically the process hits $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$ according to a uniform distribution. In view of the previous lemma and by (4.4),

$$Z\big(\mathcal{E}_{\mathbf{x}}^{2,2}\big) = 1 + \frac{1}{n-1} + \big(1 + \mathfrak{r}_n^-\big)\big[1 - \mathbb{M}_1\big(\mathcal{E}_{\mathbf{w}}^{2,2}\big)\big] \\ + \sum_{\mathbf{z}\in J_1^\star} \big\{\big[1 - \mathfrak{p}\big(\mathbf{z}, Q_{\mathbf{w}}^{2,2}\big)\big] - \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_2)\mathbb{M}_1\big(\mathcal{E}_{\mathbf{w}}^{2,2}\big) - \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_1)\mathbb{M}_2\big(\mathcal{E}_{\mathbf{w}}^{2,2}\big)\big\},$$

where $J_1^\star = J_1 \setminus \{\mathbf{w}_2 + e_2\}$.

Proposition 4.3. *Fix $0 \le i, j \le 3$ and $\mathbf{x} \in \Lambda_L$.*

(1) *For any $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$, the triple $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$ is a valley of depth $\mu_K(\mathcal{E}_{\mathbf{x}}^{i,j})/\operatorname{cap}_K(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)$;*
(2) *For any $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$, under $\mathbf{P}_\xi^\beta$, $H(\mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j})/\mathrm{e}^\beta$ converges in distribution to an exponential random variable of parameter $Z(\mathcal{E}_{\mathbf{x}}^{i,j})/|\mathcal{E}_{\mathbf{x}}^{i,j}|$;*
(3) *For any $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$, $\Pi \subset \mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j}$,*

$$\lim_{\beta\to\infty} \mathbf{P}_\xi^\beta\big[\eta\big(H\big(\mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j}\big)\big) \in \Pi\big] = \frac{1}{Z(\mathcal{E}_{\mathbf{x}}^{i,j})} \sum_{\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbb{M}(\eta, \Pi) =: Q\big(\mathcal{E}_{\mathbf{x}}^{i,j}, \Pi\big).$$

Proof. Recall [1], Theorem 2.6. Condition (2.15) is fulfilled by definition of the set Δ_1. A simple argument shows that $G_K(\xi, \zeta) = \mathrm{e}^{-\beta}\mu_K(\eta^{\mathbf{w}})$ for any pair of configurations $\xi \ne \zeta \in \mathcal{E}_{\mathbf{x}}^{i,j}$, and that $G_K(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c) \le \mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{w}})$. Condition (2.14) follows from these estimates and (3.1). This proves the first assertion of the lemma.

To prove the second assertion of the lemma, we start with a recursive formula for $H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}$. Let τ_1 the time the process leaves the set $\mathcal{E}_{\mathbf{x}}^{i,j}$: $\tau_1 = \inf\{t > 0\colon \eta_t^\beta \notin \mathcal{E}_{\mathbf{x}}^{i,j}\}$. We have that

$$H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}} = \tau_1 + H_{\mathbb{H}_{01}} \circ \theta_{\tau_1} + \mathbf{1}\{H_{\mathbb{H}_{01}} \circ \theta_{\tau_1} = H_{\mathcal{E}_{\mathbf{x}}^{i,j}} \circ \theta_{\tau_1}\} H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}} \circ \theta_{H_{\mathbb{H}_{01}}^+},$$

where $\{\theta_t\colon t \ge 0\}$ stands for the shift operators.

Fix $\lambda > 0$ and let $\lambda_\beta = \lambda\mathrm{e}^{-\beta}$. By the strong Markov property, for any $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$,

$$\mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}}\big] = \mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta\tau_1}\mathbf{E}_{\eta_{\tau_1}}^\beta\big[\mathbf{1}\{H_{\mathbb{H}_{01}} \ne H_{\mathcal{E}_{\mathbf{x}}^{i,j}}\}\mathrm{e}^{-\lambda_\beta H_{\mathbb{H}_{01}}}\big]\big] \\ + \mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta\tau_1}\mathbf{E}_{\eta_{\tau_1}}^\beta\big[\mathbf{1}\{H_{\mathbb{H}_{01}} = H_{\mathcal{E}_{\mathbf{x}}^{i,j}}\}\mathrm{e}^{-\lambda_\beta H_{\mathbb{H}_{01}}} \exp\{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}} \circ \theta_{H_{\mathbb{H}_{01}}}\}\big]\big]. \tag{4.12}$$

Recall the definition of $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$ given just before the statement of Lemma 4.2. With a probability which converges to 1 as $\beta \uparrow \infty$, $\eta_{\tau_1}^\beta$ belongs to $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$. Each configuration in $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$ belongs to an equivalent class which eventually attains $\mathbb{H}_{01}$ after a finite random number of rate one jumps. This proves that

$$\lim_{A\to\infty} \lim_{\beta\to\infty} \max_{\zeta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbf{P}_\zeta^\beta[H_{\mathbb{H}_{01}} > A] = 0.$$

Therefore, we may replace in (4.12) $\exp\{-\lambda_\beta H_{\mathbb{H}_{01}}\}$ by 1 at a cost which vanishes as $\beta \uparrow \infty$.

By the strong Markov property, after the last replacement, the second term on the right hand side of (4.12) can be rewritten as

$$\mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta\tau_1}\mathbf{E}_{\eta_{\tau_1}}^\beta\big[\mathbf{1}\{H_{\mathbb{H}_{01}} = H_{\mathcal{E}_{\mathbf{x}}^{i,j}}\}\mathbf{E}_{\eta_{H_{\mathbb{H}_{01}}}}^\beta\big[\exp\{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}\}\big]\big]\big].$$

Since $\mathcal{E}_{\mathbf{x}}^{i,j}$ is an equivalent class and the process leaves $\mathcal{E}_{\mathbf{x}}^{i,j}$ only after a rate $\mathrm{e}^{-\beta}$ jump, a simple coupling argument shows that

$$\lim_{\beta\to\infty} \max_{\eta,\zeta\in\mathcal{E}_{\mathbf{x}}^{i,j}} \big|\mathbf{E}_\eta^\beta\big[\exp\{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}\}\big] - \mathbf{E}_\zeta^\beta\big[\exp\{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}\}\big]\big| = 0.$$

The previous expectation is thus equal to

$$\mathbf{E}_\xi^\beta\big[\exp\{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}\}\big]\mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta\tau_1}\mathbf{E}_{\eta_{\tau_1}}^\beta\big[\mathbf{1}\{H_{\mathbb{H}_{01}}=H_{\mathcal{E}_{\mathbf{x}}^{i,j}}\}\big]\big]$$

plus an error which vanishes as $\beta\uparrow\infty$.

We claim that $(\mathrm{e}^{-\beta}\tau_1,\eta_{\tau_1}^\beta)$ converges in distribution, as $\beta\uparrow\infty$, to a pair of independent random variables where the first coordinate is an exponential time and the second coordinate has a distribution concentrated on $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$. The proof of this claim relies on [1], Theorem 2.7, and on a coupling argument.

Let $G_{\mathbf{x}}^{i,j}=\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})\cup\mathcal{E}_{\mathbf{x}}^{i,j}$. Consider the Markov process $\{\hat\eta_t^\beta\colon t\ge 0\}$ on $G_{\mathbf{x}}^{i,j}$ whose jump rates $\hat r(\eta,\xi)$ are given by

$$\hat r(\eta,\xi)=\begin{cases} r(\eta,\xi) & \text{if } \eta\in\mathcal{E}_{\mathbf{x}}^{i,j},\ \xi\in G_{\mathbf{x}}^{i,j},\\ r(\xi,\eta) & \text{if } \eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j}),\ \xi\in\mathcal{E}_{\mathbf{x}}^{i,j},\\ 0 & \text{otherwise.}\end{cases}$$

Note that $\hat r(\eta,\xi)=\mathrm{e}^{-\beta}$ or 0 if $\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$, and that we may couple the processes η_t^β and $\hat\eta_t^\beta$ in such a way that the probability of the event $\{\eta_t^\beta=\hat\eta_t^\beta\colon 0\le t\le\tau_1\}$ converges to one as $\beta\uparrow\infty$ if the initial state belongs to $\mathcal{E}_{\mathbf{x}}^{i,j}$.

Let $\{\xi^1,\dots,\xi^m\}$ be an enumeration of the set $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$ and consider the partition $\mathcal{E}_{\mathbf{x}}^{i,j}\cup\{\xi^1\}\cup\cdots\cup\{\xi^m\}$ of the set $G_{\mathbf{x}}^{i,j}$. Assumption (H1) of [1], Theorem 2.7, for the process $\hat\eta_t^\beta$ is empty for the sets $\{\xi^j\}$ and has been checked in the first part of this proof for the set $\mathcal{E}_{\mathbf{x}}^{i,j}$. Assumption (H0) for the process $\hat\eta_t^\beta$ speeded up by e^β can be verified by a direct computation. Therefore, by [1], Theorem 2.7, the pair $(\mathrm{e}^{-\beta}\hat\tau_1,\hat\eta_{\tau_1}^\beta)$ converges in distribution, as $\beta\uparrow\infty$, to a pair of independent random variables in which the first coordinate has an exponential distribution and the second one is concentrated over $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})$. This result can be extended to the original pair $(\mathrm{e}^{-\beta}\tau_1,\eta_{\tau_1}^\beta)$ by the coupling argument alluded to above.

It follows from the claim just proved and the previous estimates that

$$\lim_{\beta\to\infty}\mathbf{E}_\xi^\beta\big[\mathrm{e}^{-\lambda_\beta H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j}}}\big]=\lim_{\beta\to\infty}\frac{\mathbf{E}_\xi^\beta[\mathrm{e}^{-\lambda_\beta\tau_1}]\mathbf{E}_\xi^\beta[\mathbf{P}_{\eta_{\tau_1}}^\beta[H_{\mathbb{H}_{01}}\neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}{1-\mathbf{E}_\xi^\beta[\mathrm{e}^{-\lambda_\beta\tau_1}]\mathbf{E}_\xi^\beta[\mathbf{P}_{\eta_{\tau_1}}^\beta[H_{\mathbb{H}_{01}}=H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}.$$

If τ_1/e^β converges to an exponential random variable of parameter θ, the right hand side becomes

$$\lim_{\beta\to\infty}\frac{\theta\mathbf{E}_\xi^\beta[\mathbf{P}_{\eta_{\tau_1}}^\beta[H_{\mathbb{H}_{01}}\neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]}{\lambda+\theta\mathbf{E}_\xi^\beta[\mathbf{P}_{\eta_{\tau_1}}^\beta[H_{\mathbb{H}_{01}}\neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]]},$$

which means that $H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{i,j})/\mathrm{e}^\beta$ converges to an exponential random variable of parameter

$$\gamma=\theta\lim_{\beta\to\infty}\mathbf{E}_\xi^\beta\big[\mathbf{P}_{\eta_{\tau_1}}^\beta[H_{\mathbb{H}_{01}}\neq H_{\mathcal{E}_{\mathbf{x}}^{i,j}}]\big].$$

We examine the case $i=j=2$, $\mathbf{x}=\mathbf{w}$. Recall the description of the set $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$ presented before Lemma 4.2. By computing the average rates which appear in assumption (H0) of [1], we obtain that under $\mathbf{P}_\xi^\beta$, τ_1/e^β converges in distribution to an exponential random variable of parameter $|\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})|/|\mathcal{E}_{\mathbf{w}}^{2,2}|=3n/(n-1)$, and that $\eta_{\tau_1}^\beta$ converges to a uniform distribution on $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$. Hence, by the conclusions of the previous paragraph and by Lemma 4.2, $H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{2,2})/\mathrm{e}^\beta$ converges to an exponential random variable of parameter $Z(\mathcal{E}_{\mathbf{w}}^{2,2})/|\mathcal{E}_{\mathbf{w}}^{2,2}|$. This proves the second assertion of the proposition.

We turn to the third assertion. Denote by $\{H_j\colon j\ge 1\}$ the successive return times to $\mathbb{H}_{01}$:

$$H_1=H^+(\mathbb{H}_{01}),\qquad H_{j+1}=H^+(\mathbb{H}_{01})\circ\theta_{H_j},\quad j\ge 1.$$

With this notation, we may write for every $\xi \in \mathcal{E}_{\mathbf{w}}^{2,2}$,

$$\mathbf{P}_{\xi}^{\beta}\big[\eta(H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{2,2}}) \in \Pi\big] = \sum_{j\ge 1} \mathbf{P}_{\xi}^{\beta}\big[\eta(H_k) \in \mathcal{E}_{\mathbf{w}}^{2,2}, 1 \le k \le j-1, \eta(H_j) \in \Pi\big]. \tag{4.13}$$

By the strong Markov property, for any $\xi' \in \mathcal{E}_{\mathbf{w}}^{2,2}$, $\Pi' \subset \mathbb{H}_{01}$,

$$\mathbf{P}_{\xi'}^{\beta}\big[\eta(H_1) \in \Pi'\big] = \mathbf{E}_{\xi'}^{\beta}\big[\mathbf{P}_{\eta_{\tau_1}}^{\beta}\big[\eta(H_{\mathbb{H}_{01}}) \in \Pi'\big]\big].$$

Under $\mathbf{P}_{\xi}^{\beta}$, the distribution of η_{τ_1} converges to the uniform distribution over $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$ as $\beta \uparrow \infty$. Hence, by Lemma 4.2,

$$\lim_{\beta\to\infty} \mathbf{P}_{\xi'}^{\beta}\big[\eta(H_1) \in \Pi'\big] = \frac{1}{|\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})|} \sum_{\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})} \mathbb{M}\big(\eta, \Pi'\big)$$

for all $\xi' \in \mathcal{E}_{\mathbf{w}}^{2,2}$, $\Pi' \subset \mathbb{H}_{01}$. Denote the right hand side of the previous formula by $q(\Pi')$. It follows from identity (4.13), the strong Markov property and the previous observation that for all $\Pi \subset \mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{w}}^{2,2}$,

$$\lim_{\beta\to\infty} \mathbf{P}_{\xi}^{\beta}\big[\eta(H_{\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{2,2}}) \in \Pi\big] = \frac{q(\Pi)}{1 - q(\mathcal{E}_{\mathbf{w}}^{2,2})},$$

which concludes the proof of the proposition. □

Since $\mathbb{H}_{01} \setminus \mathcal{E}_{\mathbf{x}}^{i,j} = [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c$, it follows from the second assertion of the proposition that the depth of the valley $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$ is $e^{\beta}|\mathcal{E}_{\mathbf{x}}^{i,j}|/Z(\mathcal{E}_{\mathbf{x}}^{i,j})$. In particular,

$$\lim_{\beta\to\infty} \frac{\mu_K(\mathcal{E}_{\mathbf{x}}^{i,j})}{e^{\beta}\,\mathrm{cap}_K(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)} = \frac{|\mathcal{E}_{\mathbf{x}}^{i,j}|}{Z(\mathcal{E}_{\mathbf{x}}^{i,j})}\cdot \tag{4.14}$$

Since $\mu_K(\mathcal{E}_{\mathbf{x}}^{i,j}) = |\mathcal{E}_{\mathbf{x}}^{i,j}|e^{-\beta}\mu_K(\eta^{\mathbf{w}})$,

$$\lim_{\beta\to\infty} \frac{\mathrm{cap}_K(\mathcal{E}_{\mathbf{x}}^{i,j}, [\mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1]^c)}{e^{-2\beta}\mu_K(\eta^{\mathbf{w}})} = Z\big(\mathcal{E}_{\mathbf{x}}^{i,j}\big).$$

In view of Lemma 4.2, we have the following explicit formula for the probability measure $Q(\mathcal{E}_{\mathbf{w}}^{2,2}, \cdot)$ on $\mathbb{H}_{01}$. Let

$$\mathbf{R}\big(\mathcal{E}_{\mathbf{x}}^{i,j}, \Pi\big) = Z\big(\mathcal{E}_{\mathbf{x}}^{i,j}\big)Q\big(\mathcal{E}_{\mathbf{x}}^{i,j}, \Pi\big) = \sum_{\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbb{M}(\eta, \Pi), \quad \Pi \subset \mathbb{H}_{01}. \tag{4.15}$$

Recall the definition of the set $J_1^\star$ introduced one equation below (4.11). Then,

$$\begin{aligned}\mathbf{R}\big(\mathcal{E}_{\mathbf{w}}^{2,2}, \eta^{\mathbf{w}}\big) &= \big(1 + \mathfrak{r}_n^-\big)\mathbb{M}_1\big(\eta^{\mathbf{w}}\big) \\ &\quad + \sum_{\mathbf{z}\in J_1^\star}\big\{\mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_2)\mathbb{M}_1\big(\eta^{\mathbf{w}}\big) + \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_1)\mathbb{M}_2\big(\eta^{\mathbf{w}}\big)\big\},\end{aligned}$$

for $0 \le i, j \le 3$, $(i, j) \ne (2, 2)$;

$$\begin{aligned}\mathbf{R}\big(\mathcal{E}_{\mathbf{w}}^{2,2}, \mathcal{E}_{\mathbf{w}}^{i,j}\big) &= \big(1 + \mathfrak{r}_n^-\big)\mathbb{M}_1\big(\mathcal{E}_{\mathbf{w}}^{i,j}\big) + \mathbf{1}\big\{\mathcal{E}_{\mathbf{w}}^{i,j} = \mathcal{E}_{\mathbf{w}}^{1,2}\big\} + \sum_{\mathbf{z}\in J_1^\star} \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_2)\mathbb{M}_1\big(\mathcal{E}_{\mathbf{w}}^{i,j}\big) \\ &\quad + \sum_{\mathbf{z}\in J_1^\star}\big\{\mathfrak{p}\big(\mathbf{z}, Q_{\mathbf{w}}^{2,j}\big)\mathbf{1}\big\{\Pi = \mathcal{E}_{\mathbf{w}}^{2,j}\big\} + \mathfrak{p}(\mathbf{z}, \mathbf{w}_2 + e_1)\mathbb{M}_2\big(\mathcal{E}_{\mathbf{w}}^{i,j}\big)\big\};\end{aligned}$$

and

$$\mathbf{R}\big(\mathcal{E}^{2,2}_{\mathbf{w}}, \sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\sigma^{\mathbf{w}_0,\mathbf{w}_3+e_2}\eta^{\mathbf{w}}\big) = \frac{1}{n}\cdot$$

The rate $\mathbf{R}(\mathcal{E}^{2,2}_{\mathbf{w}}, \Pi)$ vanishes if Π does not intersect $\{\eta^{\mathbf{w}}, \xi_1\}\bigcup_{(i,j)\neq(2,2)}\mathcal{E}^{i,j}_{\mathbf{w}}$, where ξ_1 is the configuration appearing in the previous displayed formula. Hence, on the time scale e^{β}, starting from the valley $\mathcal{E}^{2,2}_{\mathbf{w}}$ the process may fall in the deep well $\eta^{\mathbf{w}}$, it may reach some valley $\mathcal{E}^{i,j}_{\mathbf{w}}$, $(i,j)\neq(2,2)$, which are similar to $\mathcal{E}^{2,2}_{\mathbf{w}}$, or attain the configuration $\sigma^{\mathbf{w}_2,\mathbf{w}_3+e_1+e_2}\sigma^{\mathbf{w}_0,\mathbf{w}_3+e_2}\eta^{\mathbf{w}}$. In the next subsection we show that this configuration is the well of a valley, a property shared by a class of configurations.

4.4. *The valleys* $\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}$

Let $R^{\mathfrak{l}}$, $R^{\mathfrak{s}}$ be the rectangles $R^{\mathfrak{l}}=\{1,\dots,n-1\}\times\{1,\dots,n-2\}$, $R^{\mathfrak{s}}=\{1,\dots,n-2\}\times\{1,\dots,n-1\}$, where $\mathfrak{l}$ stands for lying and $\mathfrak{s}$ for standing. Let $n^{\mathfrak{s}}_0=n^{\mathfrak{s}}_2=n-2$, $n^{\mathfrak{s}}_1=n^{\mathfrak{s}}_3=n-1$ be the length of the sides of the standing rectangle $R^{\mathfrak{s}}$. Similarly, denote by $n^{\mathfrak{l}}_i$, $0\le i\le 3$, the length of the sides of the lying rectangle $R^{\mathfrak{l}}$: $n^{\mathfrak{l}}_i=n^{\mathfrak{s}}_{i+1}$, where the sum over the index i is performed modulo 4.

Denote by $\mathbb{I}_{\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})=(k_0,\ell_0;k_1,\ell_1;k_2,\ell_2;k_3,\ell_3)$ such that

- $0\le k_i\le \ell_i\le n^{\mathfrak{a}}_i$,
- If $k_j=0$, then $\ell_{j-1}=n^{\mathfrak{a}}_{j-1}$.

For $(\mathbf{k},\boldsymbol{\ell})\in\mathbb{I}_{\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, let $R^{\mathfrak{l}}(\mathbf{k},\boldsymbol{\ell})$, $R^{\mathfrak{s}}(\mathbf{k},\boldsymbol{\ell})$ be the sets

$$\begin{aligned}
R^{\mathfrak{l}}(\mathbf{k},\boldsymbol{\ell}) &= R^{\mathfrak{l}}\cup\big\{(a,0)\colon k_0\le a\le\ell_0\big\}\cup\big\{(n,b)\colon k_1\le b\le\ell_1\big\}\\
&\quad\cup\big\{(n-a,n-1)\colon k_2\le a\le\ell_2\big\}\cup\big\{(0,n-1-b)\colon k_3\le b\le\ell_3\big\},\\
R^{\mathfrak{s}}(\mathbf{k},\boldsymbol{\ell}) &= R^{\mathfrak{s}}\cup\big\{(a,0)\colon k_0\le a\le\ell_0\big\}\cup\big\{(n-1,b)\colon k_1\le b\le\ell_1\big\}\\
&\quad\cup\big\{(n-1-a,n)\colon k_2\le a\le\ell_2\big\}\cup\big\{(0,n-b)\colon k_3\le b\le\ell_3\big\}.
\end{aligned}$$

Note that a hole between particles on the side of a rectangle is not allowed in the sets $R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, $R^{\mathfrak{s}}(\mathbf{k},\boldsymbol{\ell})$.

Denote by $I_{\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})\in\mathbb{I}_{\mathfrak{a}}$ such that $|R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})|=n^2$. For $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}$, denote by $M_i(\mathbf{k},\boldsymbol{\ell})$ the number of particles attached to the side i of the rectangle $R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$:

$$M_i(\mathbf{k},\boldsymbol{\ell})=\begin{cases}\ell_i-k_i+1 & \text{if } k_{i+1}\ge 1,\\ \ell_i-k_i+2 & \text{if } k_{i+1}=0.\end{cases}$$

Clearly, for $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}$, $\sum_{0\le i\le 3}M_i(\mathbf{k},\boldsymbol{\ell})=3n-2+A$, where A is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by $I^*_{\mathfrak{a}}\subset I_{\mathfrak{a}}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}$ whose rectangles $R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$ have at least two particles on each side: $M_i(\mathbf{k},\boldsymbol{\ell})\ge 2$, $0\le i\le 3$. Note that if $(\mathbf{k},\boldsymbol{\ell})$ belongs to $I^*_{\mathfrak{a}}$, for all $\mathbf{x}\in R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, there exist $\mathbf{y},\mathbf{z}\in R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, $\mathbf{y}\neq\mathbf{z}$, with the property $\|\mathbf{x}-\mathbf{y}\|=\|\mathbf{x}-\mathbf{z}\|=1$.

For $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $\mathbf{x}\in\Lambda_L$, let $R^{\mathfrak{a}}_{\mathbf{x}}(\mathbf{k},\boldsymbol{\ell})=\mathbf{x}+R^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, and let $\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}$ represent the configurations defined by

$$\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}(a,b)=1 \quad\text{if and only if}\quad (a,b)\in R^{\mathfrak{a}}_{\mathbf{x}}(\mathbf{k},\boldsymbol{\ell}).$$

The configurations $\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}$, $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}\setminus I^*_{\mathfrak{a}}$, belong to Ω^1 or form a $(n-1)\times(n+1)$ rectangle of particles with one extra particle attached to a side of length $n+1$. Let $\Omega^2=\Omega^2_{L,K}$, be the set of configurations associated to the pairs $(\mathbf{k},\boldsymbol{\ell})$ in $I^*_{\mathfrak{a}}$:

$$\Omega^2_{\mathbf{x}}=\big\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\colon \mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\},(\mathbf{k},\boldsymbol{\ell})\in I^*_{\mathfrak{a}}\big\},\qquad \Omega^2=\bigcup_{\mathbf{x}\in\Lambda_L}\Omega^2_{\mathbf{x}}.$$

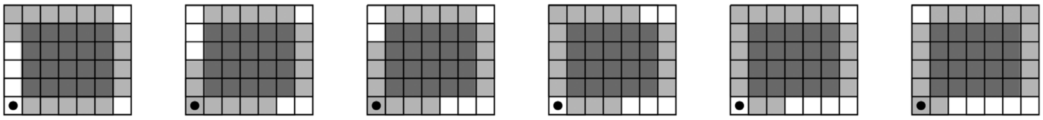

Fig. 3. Some configurations $\eta_{\mathbf{x}}^{\mathfrak{l},(\mathbf{k},\boldsymbol{\ell})}$ for $n=6$. The first one corresponds to the vector $(\mathbf{k},\boldsymbol{\ell})=((1,5);(1,4);(1,6);(0,1))$ and the last one to the vector $(\mathbf{k},\boldsymbol{\ell})=((0,1);(1,5);(0,5);(1,5))$. The inner gray rectangle represents the set $\mathbf{x}+R^{\mathfrak{l}}$ and the black dot the site $\mathbf{x}$.

Figure 3 illustrates typical configurations of the set Ω^2.

To describe the valleys which can be attained from $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ we have to define a map from Ω^2 to $\mathbb{H}_{01}$ which translates by one unit all particles in an external row or column of a rectangle $R_{\mathbf{x}}^{\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$. This must be done carefully because the translation of one row may produce a configuration which does not belong to $\mathbb{H}_{01}$, or a configuration $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k}',\boldsymbol{\ell}')}$, where the vector $(\mathbf{k}',\boldsymbol{\ell}')$ differs from $(\mathbf{k},\boldsymbol{\ell})$ in more than one coordinate.

Denote by $I_{\mathfrak{a},i}^{-}$ (resp. $I_{\mathfrak{a},i}^{+}$), $0\le i\le 3$, the pairs $(\mathbf{k},\boldsymbol{\ell})$ in $I_{\mathfrak{a}}^{*}$ for which the particle sitting at k_i (resp. ℓ_i) jumps to k_i-1 (resp. ℓ_i+1) at rate $\mathrm{e}^{-\beta}$. The abuse of notation is clear. For instance, by site k_0 we mean the site $(k_0,0)$, or, if $\mathfrak{a}=\mathfrak{s}$, by site ℓ_2 we mean site $(n-1-\ell_2,n)$. The subsets $I_{\mathfrak{a},i}^{\pm}$ of $I_{\mathfrak{a}}^{*}$ are given by

$$I_{\mathfrak{a},i}^{-}=\bigl\{(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}^{*}\colon\ k_i\ge 2 \text{ or } k_i=1,\ \ell_{i-1}=n_{i-1}^{\mathfrak{a}}\bigr\},$$

$$I_{\mathfrak{a},i}^{+}=\bigl\{(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}^{*}\colon\ \ell_i\le n_i^{\mathfrak{a}}-1 \text{ or } \ell_i=n_i^{\mathfrak{a}},\ k_{i+1}=1\bigr\}.$$

For $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a},i}^{-}$, denote by $\hat{T}_{\mathfrak{a},i}^{-}\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ the configuration obtained from $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ by moving the particle sitting at k_i to k_i-1, with the same abuse of notation alluded to before. Similarly, for $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a},i}^{+}$, denote by $\hat{T}_{\mathfrak{a},i}^{+}\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ the configuration obtained from $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ by moving the particle sitting at ℓ_i to ℓ_i+1.

Define the map $T_{\mathfrak{a},i}^{-}:I_{\mathfrak{a},i}^{-}\to I_{\mathfrak{a}}$ by

$$T_{\mathfrak{a},i}^{-}(\mathbf{k},\boldsymbol{\ell})=\begin{cases}(\mathbf{k}-\mathfrak{e}_i,\boldsymbol{\ell}-\mathfrak{e}_i) & \text{if } k_{i+1}\ge 1,\\ (\mathbf{k}-\mathfrak{e}_i+\mathfrak{e}_{i+1},\boldsymbol{\ell}) & \text{if } k_{i+1}=0,\end{cases}$$

where $\{\mathfrak{e}_1,\dots,\mathfrak{e}_4\}$ stands for the canonical basis of $\mathbb{R}^4$. The map $T_{\mathfrak{a},i}^{+}:I_{\mathfrak{a},i}^{+}\to I_{\mathfrak{a}}$ is defined in an analogous way. Hence, the map $T_{\mathfrak{s},2}^{+}$ translate to the *left* all particles on the top row of the rectangle $R^{\mathfrak{s}}$ and the map $T_{\mathfrak{l},3}^{-}$ translate in the *upward* direction all particles on the leftmost column of $R^{\mathfrak{l}}$.

The vector $T_{\mathfrak{a},i}^{\pm}(\mathbf{k},\boldsymbol{\ell})$ may not belong to $I_{\mathfrak{a}}^{*}$ when there are only two particles on one side of a rectangle $R^{\mathfrak{a}}$ and one of them is translated along another side. For example, suppose that $k_0=1$, $\ell_0=n-2$, $k_1=0$, $\ell_1=1$ for a vector $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{s}}^{*}$. In this case, necessarily $k_2=1$, $\ell_2=n-2$, $k_3=0$, $\ell_3=n-1$, and $T_{\mathfrak{s},0}^{-}(\mathbf{k},\boldsymbol{\ell})\notin I_{\mathfrak{s}}^{*}$. In fact, the configuration $\eta_{\mathbf{x}}^{\mathfrak{s},T_{\mathfrak{s},0}^{-}(\mathbf{k},\boldsymbol{\ell})}$ belongs to the set Ω^3 to be introduced in the next subsection. Similarly, if $k_1=2$, $\ell_1=n-1$, $k_2=0$, $\ell_2=1$ for a vector $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{s}}^{*}$, $T_{\mathfrak{s},1}^{-}(\mathbf{k},\boldsymbol{\ell})\notin I_{\mathfrak{s}}^{*}$, and $\eta_{\mathbf{x}}^{\mathfrak{s},T_{\mathfrak{s},1}^{-}(\mathbf{k},\boldsymbol{\ell})}\in\Omega^1$.

Fix a vector $(\mathbf{k},\boldsymbol{\ell})\in I_{\mathfrak{a}}^{*}$ such that $M_i(\mathbf{k},\boldsymbol{\ell})=2$ for some $0\le i\le 3$. Denote by $J_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ the interval over which the particles on side i may move:

$$J_{\mathfrak{a},i}=J_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})=\bigl\{1-\mathbf{1}\bigl\{\ell_{i-1}=n_{i-1}^{\mathfrak{a}}\bigr\},\dots,n_i^{\mathfrak{a}}+\mathbf{1}\{k_{i+1}\le 1\}\bigr\},$$

and by $T_{\mathfrak{a},i}^{b}(\mathbf{k},\boldsymbol{\ell})$, $b,b+1\in J_{\mathfrak{a},i}$, the vector obtained from $(\mathbf{k},\boldsymbol{\ell})$ by replacing the occupied sites k_i, k_i+1 by the sites b, $b+1$. Note that $T_{\mathfrak{a},i}^{b}(\mathbf{k},\boldsymbol{\ell})$ belongs to $I_{\mathfrak{a}}^{*}$ because we assumed $n>3$. Note also that we did not excluded the possibility that $b=k_i$ in which case $T_{\mathfrak{a},i}^{b}(\mathbf{k},\boldsymbol{\ell})=(\mathbf{k},\boldsymbol{\ell})$.

Denote by $\mathcal{N}(\eta)$ the set of all configurations which can be attained from $\eta\in\Omega^2$ by a rate $\mathrm{e}^{-\beta}$ jump. Note that the set $\mathcal{N}(\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})})$ may have more than 8 configurations. For example, if $\mathfrak{a}=\mathfrak{s}$, $\mathbf{x}=\mathbf{w}$, $\ell_0=n-3$ and $k_1\ge 2$, the particle at $(n-2,1)$ jumps at rate $\mathrm{e}^{-\beta}$ to $(n-2,0)$. However, starting from this configuration, the probability of the event $H(\mathbb{H}_{01})\ne H(\eta_{\mathbf{w}}^{\mathfrak{s},(\mathbf{k},\boldsymbol{\ell})})$ converges to 0 since the unique rate one jump from this configuration is the return to $\eta_{\mathbf{w}}^{\mathfrak{s},(\mathbf{k},\boldsymbol{\ell})}$.

The proof of the next result is straightforward and left to the reader. One just needs to identify all configurations which can be reached by rate 1 jumps from a configuration in $\mathcal{N}(\eta)$.

Lemma 4.4. *Fix $\eta \in \Omega^2$. Then, for all $\xi \in \mathcal{N}(\eta)$, there exists a probability measure $\mathbb{M}(\xi, \cdot)$ defined on $\mathbb{H}_{01}$ such that*

$$\lim_{\beta\to\infty} \mathbf{P}^{\beta}_{\xi}\big[\eta\big(H(\mathbb{H}_{01})\big) \in \Pi\big] = \mathbb{M}(\xi, \Pi), \quad \Pi \subset \mathbb{H}_{01}.$$

Moreover, for $0 \le i \le 3$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $(\mathbf{k}, \boldsymbol{\ell}) \in I^{\pm}_{\mathfrak{a},i}$

$$\mathbb{M}\big(\hat{T}^{\pm}_{\mathfrak{a},i}\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \eta^{\mathfrak{a},T^{\pm}_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \frac{1}{M_i(\mathbf{k}, \boldsymbol{\ell})},$$
$$\mathbb{M}\big(\hat{T}^{\pm}_{\mathfrak{a},i}\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \frac{M_i(\mathbf{k}, \boldsymbol{\ell}) - 1}{M_i(\mathbf{k}, \boldsymbol{\ell})}$$

if $M_i(\mathbf{k}, \boldsymbol{\ell}) \ge 3$; and

$$\mathbb{M}\big(\hat{T}^{-}_{\mathfrak{a},i}\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \eta^{\mathfrak{a},T^{b}_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \mathfrak{m}\big(J_{\mathfrak{a},i}(\mathbf{k}, \boldsymbol{\ell}), k_i - 1, b\big), \quad b, b+1 \in J_{\mathfrak{a},i}(\mathbf{k}, \boldsymbol{\ell}),$$
$$\mathbb{M}\big(\hat{T}^{+}_{\mathfrak{a},i}\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \eta^{\mathfrak{a},T^{b}_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \mathfrak{m}\big(J_{\mathfrak{a},i}(\mathbf{k}, \boldsymbol{\ell}), k_i, b\big), \quad b, b+1 \in J_{\mathfrak{a},i}(\mathbf{k}, \boldsymbol{\ell}),$$

if $M_i(\mathbf{k}, \boldsymbol{\ell}) = 2$, where the probability $\mathfrak{m}(J, a, c)$ has been introduced in (4.5).

Let

$$Z\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \sum_{\xi\in\mathcal{N}(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \sum_{\zeta\neq\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}} \mathbb{M}(\xi, \zeta) = \sum_{\xi\in\mathcal{N}(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \big\{1 - \mathbb{M}\big(\xi, \eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big)\big\}.$$

Note that $Z(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$ does not depend on $\mathbf{x}$ and that

$$\begin{aligned} Z\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{w}}\big) &= \sum_{i=0}^{3} \frac{\mathbf{1}\{M_i(\mathbf{k}, \boldsymbol{\ell}) > 2\}}{M_i(\mathbf{k}, \boldsymbol{\ell})} \big\{\mathbf{1}\big\{(\mathbf{k}, \boldsymbol{\ell}) \in I^{-}_{\mathfrak{a},i}\big\} + \mathbf{1}\big\{(\mathbf{k}, \boldsymbol{\ell}) \in I^{+}_{\mathfrak{a},i}\big\}\big\} \\ &\quad + \sum_{i=0}^{3} \mathbf{1}\big\{M_i(\mathbf{k}, \boldsymbol{\ell}) = 2\big\}\mathbf{1}\big\{(\mathbf{k}, \boldsymbol{\ell}) \in I^{-}_{\mathfrak{a},i}\big\}\big[1 - \mathfrak{m}(J_{\mathfrak{a},i}, k_i - 1, k_i)\big] \\ &\quad + \sum_{i=0}^{3} \mathbf{1}\big\{M_i(\mathbf{k}, \boldsymbol{\ell}) = 2\big\}\mathbf{1}\big\{(\mathbf{k}, \boldsymbol{\ell}) \in I^{+}_{\mathfrak{a},i}\big\}\big[1 - \mathfrak{m}(J_{\mathfrak{a},i}, k_i, k_i)\big]. \end{aligned}$$

Proposition 4.5. *Fix $\mathbf{x} \in \Lambda_L$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $(\mathbf{k}, \boldsymbol{\ell}) \in I^{*}_{\mathfrak{a}}$. Then,*

(1) *The triple $(\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}, \{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\} \cup \Delta_1, \eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$ is a valley of depth given by $\mu_K(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})/\mathrm{cap}_K(\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}, [\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\} \cup \Delta_1]^c)$;*
(2) *Under $\mathbf{P}^{\beta}_{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}}$, $H(\mathbb{H}_{01} \setminus \{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\})/\mathrm{e}^{\beta}$ converges in distribution to an exponential random variable of parameter $Z(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$;*
(3) *For any $\Pi \subset \mathbb{H}_{01} \setminus \{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}$,*

$$\begin{aligned} &\lim_{\beta\to\infty} \mathbf{P}^{\beta}_{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}}\big[\eta\big(H\big(\mathbb{H}_{01} \setminus \big\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big\}\big)\big) \in \Pi\big] \\ &\quad = \frac{1}{Z(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \sum_{\xi\in\mathcal{N}(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \mathbb{M}(\xi, \Pi) =: Q\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \Pi\big). \end{aligned}$$

Proof. Recall [1], Theorem 2.6. Assumption (2.14) is fulfilled by default and assumption (2.15) follows from the definition of the set Δ_1. This proves the first assertion of the proposition.

The proof of the second claim is simpler than the one of the second assertion of Proposition 4.3 if we take τ_1 as the time of the first jump. With this definition, τ_1 and $\eta^\beta_{\tau_1}$ are independent random variables by the Markov property, τ_1/e^β converges to an exponential random variable of parameter $|\mathcal{N}|$, where $\mathcal{N}=\mathcal{N}(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$, and $\eta^\beta_{\tau_1}$ converges to a random variable which is uniformly distributed over $\mathcal{N}$.

By the arguments of Proposition 4.3, starting from $\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}$, $H(\mathbb{H}_{01}\setminus\{\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\})/e^\beta$ converges in distribution to an exponential random variable of parameter

$$\gamma=\lim_{\beta\to\infty}\sum_{\xi\in\mathcal{N}}\mathbf{P}^\beta_\xi\big[H(\mathbb{H}_{01})\neq H\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big)\big].$$

To conclude the proof, it remains to recall the statement of Lemma 4.4, and the definition of $Z(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$.

The proof of the third assertion of the proposition is identical to the one of the third claim of Proposition 4.3. □

As in (4.14), the second assertion of this proposition gives an explicit expression for the depth of the valley presented in the first statement. On the other hand, following (4.15), for $\eta\in\Omega_2$, let

$$\mathbf{R}(\eta,\Pi)=Z(\eta)Q(\eta,\Pi)=\sum_{\xi\in\mathcal{N}(\eta)}\mathbb{M}(\xi,\Pi),\qquad \Pi\subset\mathbb{H}_{01}.$$

By Lemma 4.4, if $M_i(\mathbf{k},\boldsymbol{\ell})>2$ for some $0\le i\le 3$,

$$\mathbf{R}\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}},\eta^{\mathfrak{a},T^\pm_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big)=\frac{\mathbf{1}\{(\mathbf{k},\boldsymbol{\ell})\in I^\pm_{\mathfrak{a},i}\}}{M_i(\mathbf{k},\boldsymbol{\ell})}$$

and if $M_i(\mathbf{k},\boldsymbol{\ell})=2$,

$$\mathbf{R}\big(\eta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}},\eta^{\mathfrak{a},T^b_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big)=p_{\mathfrak{a}}(\boldsymbol{k},\boldsymbol{\ell},i,b),\qquad b,b+1\in J_{\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell}),$$

where

$$p_{\mathfrak{a}}(\boldsymbol{k},\boldsymbol{\ell},i,b)=\mathbf{1}\big\{(\mathbf{k},\boldsymbol{\ell})\in I^-_{\mathfrak{a},i}\big\}\mathfrak{m}(J_{\mathfrak{a},i},k_i-1,b)+\mathbf{1}\big\{(\mathbf{k},\boldsymbol{\ell})\in I^+_{\mathfrak{a},i}\big\}\mathfrak{m}(J_{\mathfrak{a},i},k_i,b).$$

It follows from Proposition 4.5 and Lemma 4.4 that starting from a configuration $\zeta\in\Omega^2$ the process η^β_t reaches $\mathbb{H}_{01}$ only in a configuration of $\Omega^1\cup\Omega^2$ or in a configuration in which all sites of a $(n-1)\times(n+1)$ rectangle are occupied and an extra particle is attached to a side of length $n+1$. To pursue our analysis, we have to investigate this new set of configurations.

4.5. *The valleys* $\mathcal{E}^{\mathfrak{a},i}_{\mathbf{x}}$

The arguments of this subsection are similar to the ones of Section 4.3. Let $T^{\mathfrak{l}}$, $T^{\mathfrak{s}}$ be the rectangles $T^{\mathfrak{l}}=\{0,\dots,n\}\times\{0,\dots,n-2\}$, $T^{\mathfrak{s}}=\{0,\dots,n-2\}\times\{0,\dots,n\}$. Denote by $T^{\mathfrak{a}}_{\mathbf{x}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $\mathbf{x}\in\Lambda_L$, the rectangle $T^{\mathfrak{a}}$ translated by $\mathbf{x}$: $T^{\mathfrak{a}}_{\mathbf{x}}=\mathbf{x}+T^{\mathfrak{a}}$, and by $\eta_{\mathbf{x},\mathfrak{a}}$ the configuration in which all sites of $T^{\mathfrak{a}}_{\mathbf{x}}$ are occupied. Note that $\eta_{\mathbf{x},\mathfrak{a}}$ belongs to $\Omega_{L,K-1}$ and not to $\Omega_{L,K}$.

For $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $\mathbf{x}\in\Lambda_L$, $\mathbf{z}\in\partial_+T^{\mathfrak{a}}_{\mathbf{x}}$, denote by $\eta^{\mathbf{z}}_{\mathbf{x},\mathfrak{a}}$ the configuration in which all sites of the rectangle $T^{\mathfrak{a}}_{\mathbf{x}}$ and the site $\mathbf{z}$ are occupied: $\eta^{\mathbf{z}}_{\mathbf{x},\mathfrak{a}}=\eta_{\mathbf{x},\mathfrak{a}}+\mathfrak{d}_z$, where $\mathfrak{d}_y$, $y\in\Lambda_L$, is the configuration with a unique particle at y and summation of configurations is performed componentwise. Denote by $\partial_j T^{\mathfrak{a}}_{\mathbf{x}}$, $0\le j\le 3$, the jth boundary of $T^{\mathfrak{a}}_{\mathbf{x}}$:

$$\partial_j T^{\mathfrak{a}}_{\mathbf{x}}=\big\{\mathbf{z}\in\partial_+T^{\mathfrak{a}}_{\mathbf{x}}:\ \exists\mathbf{y}\in T^{\mathfrak{a}}_{\mathbf{x}};\ \mathbf{y}-\mathbf{z}=(1-j)e_2\big\},\qquad j=0,2,$$
$$\partial_j T^{\mathfrak{a}}_{\mathbf{x}}=\big\{\mathbf{z}\in\partial_+T^{\mathfrak{a}}_{\mathbf{x}}:\ \exists\mathbf{y}\in T^{\mathfrak{a}}_{\mathbf{x}};\ \mathbf{y}-\mathbf{z}=(j-2)e_1\big\},\qquad j=1,3.$$

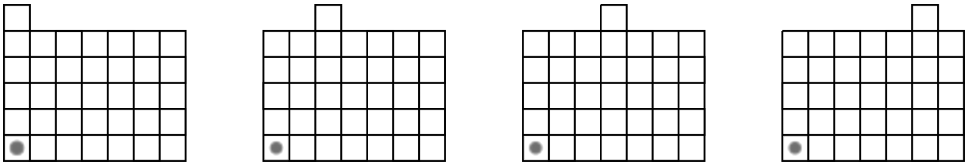

Fig. 4. Four among the seven configurations of $\mathcal{E}_{\mathbf{x}}^{\mathfrak{l},2}$ for $n=6$. The gray dot represents the site $\mathbf{x}$.

Figure 4 illustrates typical configurations of the set Ω^3.

Let

$$\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j} = \big\{\eta_{\mathbf{x},\mathfrak{a}}^{\mathbf{z}}\colon \mathbf{z}\in\partial_j T_{\mathbf{x}}^{\mathfrak{a}}\big\},$$

and let $\Omega^3=\Omega^3_{L,K}$ be the set of all such configurations:

$$\Omega^3_{\mathbf{x}} = \bigcup_{j=0}^{3}\bigcup_{\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}}\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j},\qquad \Omega^3=\bigcup_{\mathbf{x}\in\Lambda_L}\Omega^3_{\mathbf{x}}.$$

The process $\{\eta_t^\beta\colon t\ge 0\}$ can reach any configuration $\xi\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$ from any configuration $\eta\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$ with rate one jumps. The main result of this subsection states that for any configuration $\xi\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$, the triples $(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j},\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\cup\Delta_1,\xi)$ are valleys.

Denote by $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})$, the configurations which do not belong to $\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$, but which can be reached from a configuration in $\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$ by performing a jump of rate $\mathrm{e}^{-\beta}$. The set $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0})$ has the following $n+3$ elements. There are $n+1$ configurations obtained when the bottom particle detaches itself from the others: $\eta_{\mathbf{w},\mathfrak{s}}+\mathfrak{d}_z$, where $\mathbf{z}\in J_2=\{(-1,-1),(a,-2),(n-1,-1)\colon 0\le a\le n-2\}$. There is a configuration in $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0})$ which is obtained when the bottom particle is at $(1,-1)$ and the particle at $\mathbf{w}$ moves to $\mathbf{w}-e_2$: $\sigma^{\mathbf{w},\mathbf{w}-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(1,-1)}$. The last configuration of $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0})$ is obtained when the bottom particle is at $(n-3,-1)$ and the particle at $\mathbf{w}_1-e_1$ moves to $\mathbf{w}_1-e_1-e_2$: $\sigma^{\mathbf{w}_1-e_1,\mathbf{w}_1-e_1-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(n-3,-1)}$.

Lemma 4.6. *Fix $\mathbf{x}\in\Lambda_L$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$ and $0\le j\le 3$. For each $\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})$, there exists a probability measure $\mathbb{M}(\xi,\cdot)$ defined on $\mathbb{H}_{01}$ such that*

$$\lim_{\beta\to\infty}\mathbf{P}_\xi^\beta\big[\eta(H_{\mathbb{H}_{01}})\in\Pi\big]=\mathbb{M}(\xi,\Pi),\qquad \Pi\subset\mathbb{H}_{01}.$$

Moreover, if $\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0})$,

$$\mathbb{M}\big(\eta_{\mathbf{w},\mathfrak{s}}+\mathfrak{d}_z,\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},j}\big)=\mathfrak{p}\big(\mathbf{z},\partial_j T_{\mathbf{w}}^{\mathfrak{s}},\partial_+T_{\mathbf{w}}^{\mathfrak{s}}\big),\qquad 0\le j\le 3,\mathbf{z}\in J_2,$$

$$\mathbb{M}\big(\sigma^{\mathbf{w},\mathbf{w}-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(1,-1)},\Pi\big)=\begin{cases}\frac{1}{n+1} & \textit{if } \Pi=\big\{\sigma^{\mathbf{w}_3+e_2,\mathbf{w}-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(1,-1)}\big\},\\ \frac{n}{n+1} & \textit{if } \Pi=\big\{\eta_{\mathbf{w},\mathfrak{s}}^{(1,-1)}\big\},\end{cases}$$

$$\mathbb{M}(\xi,\Pi)=\begin{cases}\frac{1}{n+1} & \textit{if } \Pi=\big\{\sigma^{\mathbf{w}_2-e_1+e_2,\mathbf{w}_1-e_1-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(n-3,-1)}\big\},\\ \frac{n}{n+1} & \textit{if } \Pi=\big\{\eta_{\mathbf{w},\mathfrak{s}}^{(n-3,-1)}\big\},\end{cases}$$

if $\xi=\sigma^{\mathbf{w}_1-e_1,\mathbf{w}_1-e_1-e_2}\eta_{\mathbf{w},\mathfrak{s}}^{(n-3,-1)}$.

The proof of the previous lemma is simpler than the one of Lemma 4.2 and left to the reader. By symmetry, the distribution of $\eta(H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{\cdot,\cdot}))$ can be obtained from the one of $\eta(H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0}))$ or from the one of $\eta(H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},1}))$.

Define

$$Z\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\big)=\sum_{\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})}\mathbb{M}\big(\xi,\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\big)^c\big)=\sum_{\xi\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})}\big\{1-\mathbb{M}\big(\xi,\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\big)\big\}.$$

By the previous lemma,

$$Z\big(\mathcal{E}_{\mathbf{w}}^{\mathfrak{s},0}\big) = \frac{2}{n+1} + \sum_{\mathbf{y}\in J_2}\big[1-\mathfrak{p}\big(\mathbf{y}, \partial_0 T_{\mathbf{w}}^{\mathfrak{s}}, \partial_+ T_{\mathbf{w}}^{\mathfrak{s}}\big)\big].$$

Proposition 4.7. *Fix* $0\le j\le 3$, $\mathfrak{a}\in\{\mathfrak{l},\mathfrak{s}\}$, $\mathbf{x}\in\Lambda_L$.

(1) *For every* $\xi\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$, *the triple* $(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}, \mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\cup\Delta_1, \xi)$ *is a valley of depth* $\mu_K(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})/\operatorname{cap}_K(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}, [\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\cup\Delta_1]^c)$;

(2) *For any* $\xi\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$, *under* $\mathbf{P}_\xi^\beta$, $H(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})/\mathrm{e}^\beta$ *converges in distribution to an exponential random variable of parameter* $Z(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})/|\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}|$;

(3) *For any* $\xi\in\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$, $\Pi\subset\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$,

$$\lim_{\beta\to\infty}\mathbf{P}_\xi^\beta\big[\eta\big(H\big(\mathbb{H}_{01}\setminus\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\big)\big)\in\Pi\big] = \frac{1}{Z(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})}\sum_{\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})}\mathbb{M}(\eta,\Pi) =: Q\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j},\Pi\big).$$

The proof of this proposition is similar to the one of Proposition 4.3, with τ_1 defined as the first time the process leaves the set $\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$. Remark (4.14) concerning the explicit formula for the depth of the valley appearing in the first statement of Proposition 4.7 also holds.

For $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $\mathbf{x}\in\Lambda_L$, $0\le j\le 3$, let $\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}$

$$\mathbf{R}\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j},\Pi\big) = Z\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j}\big)Q\big(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j},\Pi\big) = \sum_{\eta\in\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},j})}\mathbb{M}(\eta,\Pi), \quad \Pi\subset\mathbb{H}_{01}.$$

It follows from the previous two results that starting from a configuration $\zeta\in\Omega^3$ the process η_t^β reaches $\mathbb{H}_{01}$ only in a configuration of $\Omega^2\cup\Omega^3$ or in a configuration in which all sites of a $(n-3)\times n$ rectangle are occupied with $3n$ extra particles attached to the boundary. This is the last set of configurations which needs to be examined.

4.6. *The valleys* $\{\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}\}$

The arguments of this subsection are similar to the ones of Section 4.4. Let $R^{2,\mathfrak{l}}$, $R^{2,\mathfrak{s}}$ be the rectangles $R^{2,\mathfrak{l}} = \{1,\dots,n\}\times\{1,\dots,n-3\}$, $R^{2,\mathfrak{s}} = \{1,\dots,n-3\}\times\{1,\dots,n\}$. Let $n_0^{2,\mathfrak{s}} = n_2^{2,\mathfrak{s}} = n-3$, $n_1^{2,\mathfrak{s}} = n_3^{2,\mathfrak{s}} = n$ be the length of the sides of the standing rectangle $R^{2,\mathfrak{s}}$. Similarly, denote by $n_i^{2,\mathfrak{l}}$, $0\le i\le 3$, the length of the sides of the lying rectangle $R^{2,\mathfrak{l}}$: $n_i^{2,\mathfrak{l}} = n_{i+1}^{2,\mathfrak{s}}$, where the sum over the index i is performed modulo 4.

Denote by $\mathbb{I}_{2,\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})$ such that

- $0\le k_i\le\ell_i\le n_i^{2,\mathfrak{a}}$,
- If $k_j=0$, then $\ell_{j-1} = n_{j-1}^{2,\mathfrak{a}}$.

For $(\mathbf{k},\boldsymbol{\ell})\in\mathbb{I}_{2,\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, let $R^{2,\mathfrak{l}}(\mathbf{k},\boldsymbol{\ell})$, $R^{2,\mathfrak{s}}(\mathbf{k},\boldsymbol{\ell})$ be the sets

$$\begin{aligned} R^{2,\mathfrak{l}}(\mathbf{k},\boldsymbol{\ell}) = R^{2,\mathfrak{l}} &\cup \big\{(a,0)\colon k_0\le a\le\ell_0\big\}\cup\big\{(n+1,b)\colon k_1\le b\le\ell_1\big\} \\ &\cup\big\{(n+1-a,n-2)\colon k_2\le a\le\ell_2\big\}\cup\big\{(0,n-2-b)\colon k_3\le b\le\ell_3\big\}, \\ R^{2,\mathfrak{s}}(\mathbf{k},\boldsymbol{\ell}) = R^{2,\mathfrak{s}} &\cup\big\{(a,0)\colon k_0\le a\le\ell_0\big\}\cup\big\{(n-2,b)\colon k_1\le b\le\ell_1\big\} \\ &\cup\big\{(n-2-a,n+1)\colon k_2\le a\le\ell_2\big\}\cup\big\{(0,n+1-b)\colon k_3\le b\le\ell_3\big\}. \end{aligned}$$

Denote by $I_{2,\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})\in\mathbb{I}_{2,\mathfrak{a}}$ such that $|R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})| = n^2$. For $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}$, denote by $M_i^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$ the number of particles attached to the side i of the rectangle $R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$:

$$M_i^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell}) = \begin{cases}\ell_i-k_i+1 & \text{if } k_{i+1}\ge 1,\\ \ell_i-k_i+2 & \text{if } k_{i+1}=0.\end{cases}$$

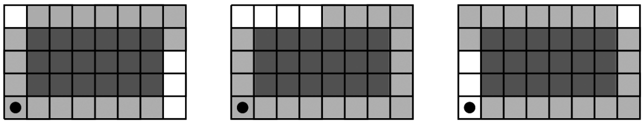

Fig. 5. Examples of configurations in $\Omega^4_{\mathbf{x}}$ for $n=6$. In general $3n$ particles (or $n-2$ holes) have to be placed around the rectangle, respecting the constraints introduced above. The black dot represents the site $\mathbf{x}$.

Clearly, for $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}$, $\sum_{0\le i\le 3} M_i^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})=3n+A$, where A is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by $I^*_{2,\mathfrak{a}}\subset I_{2,\mathfrak{a}}$, the set of pairs $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}$ whose rectangles $R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$ have at least two particles on each side: $M_i^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})\ge 2$, $0\le i\le 3$. Note that if $(\mathbf{k},\boldsymbol{\ell})$ belongs to $I^*_{2,\mathfrak{a}}$, for all $\mathbf{x}\in R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, there exist $\mathbf{y},\mathbf{z}\in R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, $\mathbf{y}\ne\mathbf{z}$, with the property $\|\mathbf{x}-\mathbf{y}\|=\|\mathbf{x}-\mathbf{z}\|=1$.

For $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $\mathbf{x}\in\Lambda_L$, let $R^{2,\mathfrak{a}}_{\mathbf{x}}(\mathbf{k},\boldsymbol{\ell})=\mathbf{x}+R^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})$, and let $\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ represent the configurations defined by

$$\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}(a,b)=1\quad\text{if and only if}\quad(a,b)\in R^{2,\mathfrak{a}}_{\mathbf{x}}(\mathbf{k},\boldsymbol{\ell}).$$

The configurations $\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$, $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}$, have at least four particles attached to the longer side, and the configurations $\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$, $(\mathbf{k},\boldsymbol{\ell})\in I_{2,\mathfrak{a}}\setminus I^*_{2,\mathfrak{a}}$, belong to Ω^3, forming a $(n-1)\times(n+1)$ rectangle of particles with one extra particle attached to a side of length $n-1$. Let $\Omega^4=\Omega^4_{L,K}$, be the set of configurations associated to the pairs $(\mathbf{k},\boldsymbol{\ell})$ in $I^*_{2,\mathfrak{a}}$:

$$\Omega^4_{\mathbf{x}}=\big\{\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}\colon\ \mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\},(\mathbf{k},\boldsymbol{\ell})\in I^*_{2,\mathfrak{a}}\big\},\qquad \Omega^4=\bigcup_{\mathbf{x}\in\Lambda_L}\Omega^4_{\mathbf{x}}.$$

Figure 5 illustrates typical configurations of the set Ω^4.

We now describe the configurations which can be attained from a configuration in Ω^4. Denote by $I^{\pm}_{2,\mathfrak{a},i}$, $\mathfrak{a}\in\{\mathfrak{s},\mathfrak{l}\}$, $0\le i\le 3$, the subset of $I^*_{2,\mathfrak{a}}$ defined by

$$I^-_{2,\mathfrak{a},i}=\big\{(\mathbf{k},\boldsymbol{\ell})\in I^*_{2,\mathfrak{a}}\colon\ k_i\ge 2\text{ or }k_i=1,\ \ell_{i-1}=n^{2,\mathfrak{a}}_{i-1}\big\},$$
$$I^+_{2,\mathfrak{a},i}=\big\{(\mathbf{k},\boldsymbol{\ell})\in I^*_{2,\mathfrak{a}}\colon\ \ell_i\le n^{2,\mathfrak{a}}_i-1\text{ or }\ell_i=n^{2,\mathfrak{a}}_i,\ k_{i+1}=1\big\}.$$

For $(\mathbf{k},\boldsymbol{\ell})\in I^-_{2,\mathfrak{a},i}$, denote by $\hat T^-_{2,\mathfrak{a},i}\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ the configuration obtained from $\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ by moving the particle sitting at k_i to k_i-1. As in Section 4.4, the abuse of notation is clear. Similarly, for $(\mathbf{k},\boldsymbol{\ell})\in I^+_{2,\mathfrak{a},i}$, denote by $\hat T^+_{2,\mathfrak{a},i}\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ the configuration obtained from $\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}$ by moving the particle sitting at ℓ_i to ℓ_i+1.

Define the map $T^-_{2,\mathfrak{a},i}:I^-_{2,\mathfrak{a},i}\to I_{2,\mathfrak{a}}$ by

$$T^-_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})=\begin{cases}(\mathbf{k}-\mathfrak{e}_i,\boldsymbol{\ell}-\mathfrak{e}_i)&\text{if }k_{i+1}\ge 1,\\(\mathbf{k}-\mathfrak{e}_i+\mathfrak{e}_{i+1},\boldsymbol{\ell})&\text{if }k_{i+1}=0.\end{cases}$$

The map $T^+_{2,\mathfrak{a},i}:I^+_{2,\mathfrak{a},i}\to I_{2,\mathfrak{a}}$ is defined in an analogous way.

The vector $T^{\pm}_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ may not belong to $I^*_{2,\mathfrak{a}}$ when there are only two particles on one side of a rectangle $R^{2,\mathfrak{a}}$ and one of them is translated along another side. Since there are at least four particles attached to the longer sides of the rectangle, this may happen only in the shorter sides of the rectangles. In this case the configuration associated to the vector $T^{\pm}_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ belongs to Ω^3.

Fix a vector $(\mathbf{k},\boldsymbol{\ell})\in I^*_{2,\mathfrak{a}}$ such that $M_i^{2,\mathfrak{a}}(\mathbf{k},\boldsymbol{\ell})=2$ for some $0\le i\le 3$. Denote by $J_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ the interval over which the particles on side i may move:

$$J_{2,\mathfrak{a},i}=J_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})=\big\{1-\mathbf{1}\big\{\ell_{i-1}=n^{2,\mathfrak{a}}_{i-1}\big\},\dots,n^{2,\mathfrak{a}}_i+\mathbf{1}\{k_{i+1}\le 1\}\big\},$$

and by $T^b_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$, $b, b+1 \in J_{2,\mathfrak{a},i}$, the vector obtained from $(\mathbf{k},\boldsymbol{\ell})$ by replacing the occupied sites (k_i, k_i+1) by $(b, b+1)$. Note that $T^b_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ always belongs to $I^*_{2,\mathfrak{a}}$, and that we did not excluded the possibility that $b = k_i$ in which case $T^b_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell}) = (\mathbf{k},\boldsymbol{\ell})$.

Denote by $\mathcal{N}(\eta)$ the set of all configurations which can be reached from $\eta \in \Omega^4$ by a rate $e^{-\beta}$ jump.

Lemma 4.8. *Fix $\eta \in \Omega^4$. For each $\xi \in \mathcal{N}(\eta)$, there exists a probability measure $\mathbb{M}(\xi,\cdot)$ defined on $\mathbb{H}_{01}$ such that*

$$\lim_{\beta\to\infty} \mathbf{P}^\beta_\xi\big[\eta\big(H(\mathbb{H}_{01})\big) \in \Pi\big] = \mathbb{M}(\xi,\Pi), \quad \Pi \subset \mathbb{H}_{01}.$$

Moreover, for $0 \le i \le 3$, $\mathfrak{a} \in \{\mathfrak{s},\mathfrak{l}\}$ and $(\mathbf{k},\boldsymbol{\ell}) \in I^\pm_{2,\mathfrak{a},i}$,

$$\mathbb{M}\big(\hat{T}^\pm_{2,\mathfrak{a},i}\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \zeta^{\mathfrak{a},T^\pm_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \frac{1}{M^{2,\mathfrak{a}}_i(\mathbf{k},\boldsymbol{\ell})},$$

$$\mathbb{M}\big(\hat{T}^\pm_{2,\mathfrak{a},i}\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \frac{M^{2,\mathfrak{a}}_i(\mathbf{k},\boldsymbol{\ell})-1}{M^{2,\mathfrak{a}}_i(\mathbf{k},\boldsymbol{\ell})}$$

if $M^{2,\mathfrak{a}}_i(\mathbf{k},\boldsymbol{\ell}) \ge 3$; and

$$\mathbb{M}\big(\hat{T}^-_{2,\mathfrak{a},i}\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \zeta^{\mathfrak{a},T^b_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \mathfrak{m}\big(J_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell}), k_i-1, b\big),$$

$$\mathbb{M}\big(\hat{T}^+_{2,\mathfrak{a},i}\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \zeta^{\mathfrak{a},T^b_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \mathfrak{m}\big(J_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell}), k_i, b\big),$$

for $b, b+1 \in J_{2,\mathfrak{a},i}(\mathbf{k},\boldsymbol{\ell})$ if $M^{2,\mathfrak{a}}_i(\mathbf{k},\boldsymbol{\ell}) = 2$. The probability measure $\mathfrak{m}(J,a,c)$ has been introduced in (4.5).

Let

$$Z\big(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big) = \sum_{\xi\in\mathcal{N}(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \mathbb{M}\big(\xi, \big\{\zeta^{\mathfrak{s},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big\}^c\big) = \sum_{\xi\in\mathcal{N}(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \big\{1 - \mathbb{M}\big(\xi, \zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big)\big\}.$$

Proposition 4.9. *Fix $\mathbf{x} \in \Lambda_L$, $\mathfrak{a} \in \{\mathfrak{l},\mathfrak{s}\}$, $(\mathbf{k},\boldsymbol{\ell}) \in I^*_{2,\mathfrak{a}}$. Then,*

(1) *The triple $(\{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}, \{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\} \cup \Delta_1, \zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$ is a valley of depth $\mu_K(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})/\operatorname{cap}_K(\{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}, [\{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\} \cup \Delta_1]^c)$;*
(2) *Under $\mathbf{P}^\beta_{\zeta^{\mathfrak{a},(\mathbf{k},\ell)}_{\mathbf{x}}}$, $H(\mathbb{H}_{01} \setminus \{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\})/e^\beta$ converges in distribution to an exponential random variable of parameter $Z(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})$;*
(3) *For any $\Pi \subset \mathbb{H}_{01} \setminus \{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\}$,*

$$\lim_{\beta\to\infty} \mathbf{P}^\beta_{\zeta^{\mathfrak{a},(\mathbf{k},\ell)}_{\mathbf{x}}}\big[\eta\big(H\big(\mathbb{H}_{01} \setminus \big\{\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}\big\}\big)\big) \in \Pi\big]$$
$$= \frac{1}{Z(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \sum_{\xi\in\mathcal{N}(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}})} \mathbb{M}(\xi,\Pi) =: Q\big(\zeta^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}_{\mathbf{x}}, \Pi\big).$$

For $\eta \in \Omega_4$, let

$$\mathbf{R}(\eta,\Pi) = Z(\eta)Q(\eta,\Pi) = \sum_{\xi\in\mathcal{N}(\eta)} \mathbb{M}(\xi,\Pi), \quad \Pi \subset \mathbb{H}_{01}.$$

5. Tunneling behavior among shallow valleys

We examine in this section the evolution of the Markov process $\{\eta_t^\beta\colon t \ge 0\}$ in the time scale e^β among the shallow valleys introduced in the previous section. We first introduce a family of deep valleys or traps.

Lemma 5.1. *Fix* $\mathbf{x} \in \Lambda_L$. *The triple* $(\{\eta^{\mathbf{x}}\}, \{\eta^{\mathbf{x}}\} \cup \Delta_1, \eta^{\mathbf{x}})$ *is a valley of depth* $\mu_K(\eta^{\mathbf{x}})/\operatorname{cap}_K(\{\eta^{\mathbf{x}}\}, [\{\eta^{\mathbf{x}}\} \cup \Delta_1]^c)$.

This result follows from [1], Theorem 2.6. Up to this point, we introduced five types of disjoint subsets of $\Omega_{L,K}$:

- $\{\eta^{\mathbf{x}}\}$, $\mathbf{x} \in \Lambda_L$;
- $\mathcal{E}_{\mathbf{x}}^{i,j}$, $0 \le i, j \le 3$, $\mathbf{x} \in \Lambda_L$;
- $\{\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}\}$, $\mathbf{x} \in \Lambda_L$, $\mathfrak{a} \in \{\mathfrak{l}, \mathfrak{s}\}$, $(\mathbf{k}, \boldsymbol{\ell}) \in I_{\mathfrak{a}}^*$;
- $\mathcal{E}_{\mathbf{x}}^{\mathfrak{a},i}$, $\mathfrak{a} \in \{\mathfrak{l}, \mathfrak{s}\}$, $0 \le i \le 3$, $\mathbf{x} \in \Lambda_L$;
- $\{\zeta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\boldsymbol{\ell})}\}$, $\mathbf{x} \in \Lambda_L$, $\mathfrak{a} \in \{\mathfrak{l}, \mathfrak{s}\}$, $(\mathbf{k}, \boldsymbol{\ell}) \in I_{2,\mathfrak{a}}^*$.

Denote by $\mathcal{E}_1, \dots, \mathcal{E}_\kappa$ an enumeration of these sets. In this enumeration we shall assume that $\mathcal{E}_1 = \{\eta^{\mathbf{w}}\}$ and that the first $|\Lambda_L|$ sets correspond to the square configurations: for $1 \le i \le |\Lambda_L|$, $\mathcal{E}_i = \{\eta^{\mathbf{x}_i}\}$ for some $\mathbf{x}_i \in \Lambda_L$. Some sets $\mathcal{E}_j$ are singletons, as the first $|\Lambda_L|$ sets, and some are not, as the set $\mathcal{E}_{|\Lambda_L|+1} = \mathcal{E}_{\mathbf{w}}^{0,0}$. Let $\mathcal{E} = \bigcup_{1 \le j \le \kappa} \mathcal{E}_j$ be the union of all subsets and let $\check{\mathcal{E}}_j = \bigcup_{i \ne j} \mathcal{E}_i$. For $1 \le i \le |\Lambda_L|$, we sometimes denote $\mathcal{E}_i = \{\eta^{\mathbf{x}}\}$ by $\mathcal{E}_{\mathbf{x}}$.

Let $\Delta_1^* = \Delta_1 \cup [\mathbb{H}_1 \setminus \mathcal{E}]$. Fix a configuration ξ_i in each set $\mathcal{E}_i$, $1 \le i \le \kappa$. We proved above and in the previous section that the triples $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1, \xi_i)$ are valleys. The next result states that we may increase Δ_1 to Δ_1^*.

Lemma 5.2. *The triples* $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1^*, \xi_i)$, $|\Lambda_L| < i \le \kappa$, *are valleys of depth* $e^\beta|\mathcal{E}_i|/Z(\mathcal{E}_i)$. *Moreover, for every* $|\Lambda_L| < i \le \kappa$, $1 \le j \ne i \le \kappa$, $\xi \in \mathcal{E}_i$,

$$\lim_{\beta\to\infty} \mathbf{P}_\xi^\beta\big[H(\check{\mathcal{E}}_i) = H(\mathcal{E}_j)\big] = Q(\mathcal{E}_i, \mathcal{E}_j).$$

Proof. As already remarked in (4.14), it follows from the second assertion of the propositions stated in the previous section that the depth of the valleys $(\mathcal{E}_i, \mathcal{E}_i \cup \Delta_1, \xi_i)$, $|\Lambda_L| < i \le \kappa$, is $e^\beta|\mathcal{E}_i|/Z(\mathcal{E}_i)$. The first assertion of the lemma follows from Lemma 7.1 below and from the fact, proved in the previous section, that for $|\Lambda_L| < i \le \kappa$,

$$\lim_{\beta\to\infty} \min_{\xi\in\mathcal{E}_i} \mathbf{P}_\xi^\beta\big[H(\mathbb{H}_{01} \setminus \mathcal{E}_i) = H(\mathcal{E})\big] = 1.$$

The second statement of the lemma follows from the definition of the probability measure $Q(\mathcal{E}_i, \cdot)$ introduced in the previous section. □

Denote by $\{\eta_t^{\mathcal{E}}\colon t \ge 0\}$ the trace of the process η_t^β on $\mathcal{E}$. The jumps rates of the Markov process $\eta_t^{\mathcal{E}}$ are represented by $R_\beta^{\mathcal{E}}(\eta, \xi)$. Recall that ξ_i is a fixed configuration in the set $\mathcal{E}_i$.

Proposition 5.3. *The sequence of Markov processes* $\{\eta_t^\beta\colon t \ge 0\}$ *exhibits a tunneling behavior on the time-scale* e^β, *with metastable states* $\{\mathcal{E}_j\colon 1 \le j \le \kappa\}$, *metastable points* ξ_j, $1 \le j \le \kappa$, *and asymptotic Markov dynamics characterized by the rates*

$$r(\mathcal{E}_i, \mathcal{E}_j) = 0, \quad 1 \le i \le |\Lambda_L|, 1 \le j \ne i \le \kappa,$$
$$r(\mathcal{E}_i, \mathcal{E}_j) = \mathbf{R}(\mathcal{E}_i, \mathcal{E}_j), \quad |\Lambda_L| < i \le \kappa, 1 \le j \ne i \le \kappa.$$

Proof. We check that the first two assumptions of [1], Theorem 2.7, are fulfilled. We start with assumption (H1). For the valleys $\mathcal{E}_j$ which are singletons, there is nothing to prove. For the other ones, as $\check{\mathcal{E}}_j \subset [\mathcal{E}_j \cup \Delta_1]^c$, assumption (H1) follows from the proofs of Propositions 4.3 and 4.7.

We turn to assumption (H0). Denote by $r_\beta(\mathcal{E}_i,\mathcal{E}_j)$ the average rates of the trace process:

$$r_\beta(\mathcal{E}_i,\mathcal{E}_j)=\frac{1}{\mu_K(\mathcal{E}_i)}\sum_{\eta\in\mathcal{E}_i}\mu_K(\eta)\sum_{\xi\in\mathcal{E}_j}R^{\mathcal{E}}_\beta(\eta,\xi).$$

We claim that $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\mathcal{E}_j)$, $1\le i\ne j\le\kappa$, converges to a limit denoted by $r(i,j)$, and that $\sum_{j\ne i}r(i,j)=0$, $1\le i\le|\Lambda_L|$, $\sum_{j\ne i}r(i,j)\in(0,\infty)$, $i>|\Lambda_L|$.

Consider first the case $i>|\Lambda_L|$. We may rewrite $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\mathcal{E}_j)$ as $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)\times[r_\beta(\mathcal{E}_i,\mathcal{E}_j)/r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)]$. By [3], Corollary 4.4, $r_\beta(\mathcal{E}_i,\mathcal{E}_j)/r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)$ converges to a number $p(\mathcal{E}_i,\mathcal{E}_j)\in[0,1]$.

On the other hand, by [1], Lemma 6.7, $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)=\mathrm{e}^\beta\operatorname{cap}_K(\mathcal{E}_i,\check{\mathcal{E}}_i)/\mu_K(\mathcal{E}_i)$. From the results stated in the previous section, it is easy to construct a path γ from $\mathcal{E}_i$ to $\check{\mathcal{E}}_i$ such that $G_K(\gamma)=\mathrm{e}^{-\beta}\mu_K(\eta)$, $\eta\in\mathcal{E}_i$. It is also easy to see that any path γ' from $\mathcal{E}_i$, to $\check{\mathcal{E}}_i$ is such that $G_K(\gamma)\le\mathrm{e}^{-\beta}\mu_K(\eta)$, $\eta\in\mathcal{E}_i$. Hence, $G_K(\mathcal{E}_i,\check{\mathcal{E}}_i)=\mathrm{e}^{-\beta}\mu_K(\eta)$, $\eta\in\mathcal{E}_i$. Assumption (H0) for $i>|\Lambda_L|$ follows from this identity and (3.1).

Fix now $i\le|\Lambda_L|$. Since $r_\beta(\mathcal{E}_i,\mathcal{E}_j)\le r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)$, we have to show that the rescaled rate $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\check{\mathcal{E}}_i)=\mathrm{e}^\beta\operatorname{cap}_K(\mathcal{E}_i,\check{\mathcal{E}}_i)/\mu_K(\mathcal{E}_i)$ vanishes as $\beta\uparrow\infty$. Since $G_K(\mathcal{E}_i,\check{\mathcal{E}}_i)\le\mathrm{e}^{-2\beta}\mu_K(\eta)$, $\eta\in\mathcal{E}_i$, the result follows from (3.1).

In view of the proof of [3], Lemma 10.2, and Lemma 5.2, $\mathrm{e}^\beta r_\beta(\mathcal{E}_i,\mathcal{E}_j)$, $1\le i\ne j\le\kappa$, converges to $Z(\mathcal{E}_i)Q(\mathcal{E}_i,\mathcal{E}_j)=\mathbf{R}(\mathcal{E}_i,\mathcal{E}_j)$.

It remains to show property (M3) of tunneling, which states that the time spent outside $\mathcal{E}$ is negligible. Fix $1\le i\le\kappa$ and $\xi\in\mathcal{E}_i$. Denote by $\{H_j\colon j\ge1\}$ the times of the successive returns to $\mathcal{E}:H_1=H^+(\mathcal{E})$, $H_{j+1}=H^+(\mathcal{E})\circ\theta_{H_j}$, $j\ge1$. To prove (M3), it is enough to show that for all $t>0$

$$\begin{aligned}&\lim_{k\to\infty}\lim_{\beta\to\infty}\mathbf{P}^\beta_\xi\bigl[H_k\le t\mathrm{e}^\beta\bigr]=0\quad\text{and}\\&\lim_{\beta\to\infty}\mathbf{E}^\beta_\xi\biggl[\mathrm{e}^{-\beta}\int_0^{H_k\wedge t\mathrm{e}^\beta}\mathbf{1}\bigl\{\eta^\beta_s\in\Delta^*_1\bigr\}\,\mathrm{d}s\biggr]=0\quad\text{for all }k\ge1.\end{aligned}\tag{5.1}$$

Since $H_1=H^+(\mathcal{E})$ is greater than the time of the first jump, there exists a positive constant c_0, independent of β, which turns $H_1=H^+(\mathcal{E})$ bounded below by a mean $c_0\mathrm{e}^\beta$ exponential time, $\mathbf{P}^\beta_\eta$ almost surely for all $\eta\in\mathcal{E}$. The first result of (5.1) follows from this observation and of the strong Markov property.

To estimate the second term of (5.1), fix $k\ge1$ and rewrite the time integral as $\sum_{0\le j<k}\int_{H_j\wedge t\mathrm{e}^\beta}^{H_{j+1}\wedge t\mathrm{e}^\beta}$. For a fixed j, the integral vanishes unless $H_j<t\mathrm{e}^\beta$. In this case, we may apply the strong Markov property to estimate the expectation by

$$k\sup_{\xi\in\mathcal{E}}\mathbf{E}^\beta_\xi\biggl[\mathrm{e}^{-\beta}\int_0^{H_1\wedge t\mathrm{e}^\beta}\mathbf{1}\bigl\{\eta^\beta_s\in\Delta^*_1\bigr\}\,\mathrm{d}s\biggr].$$

If ξ belongs to $\mathcal{E}_i$, $1\le i\le|\Lambda_L|$, the expectation is bounded above by $t\mathbf{P}^\beta_\xi[\tau_1\le t\mathrm{e}^\beta]$, where τ_1 is the time of the first jump. This expression vanishes because τ_1 is an exponential time whose mean is of order $\mathrm{e}^{2\beta}$. For $i>|\Lambda_L|$, we have seen in the proofs of the propositions of the previous section that the time spent between two visits to $\mathcal{E}$ can be estimated by the time a rate 1, finite state, irreducible Markov process needs to visit a specific set. This concludes the proof of the proposition. □

Let $\Psi:\mathcal{E}\to\{1,\dots,\kappa\}$ be the index function $\Psi(\eta)=\sum_{1\le j\le\kappa}j\mathbf{1}\{\eta\in\mathcal{E}_j\}$. It follows from the previous result that the non-Markovian process $X^\beta_t=\Psi(\eta^{\mathcal{E}}_{t\mathrm{e}^\beta})$ converges to the Markov process on $\{1,\dots,\kappa\}$ with jump rates $r(i,j)=\mathbf{R}(\mathcal{E}_i,\mathcal{E}_j)$. The states $\{1,\dots,|\Lambda_L|\}$ are absorbing, while the states $\{|\Lambda_L|+1,\dots,\kappa\}$ are transient for the asymptotic dynamics.

Let $q(i,j)$, $1\le i\le\kappa$, $1\le j\le|\Lambda_L|$, be the probability that starting from i the asymptotic process eventually reaches the absorbing point j:

$$q(i,j)=\mathbb{P}_i[X_t=j\text{ for some }t>0],\tag{5.2}$$

where $\mathbb{P}_i$ stands for the probability on the path space $D([0,\infty),\{1,\dots,\kappa\})$ induced by the Markov process with rates $r(j,k)$ starting from i. We sometimes denote $q(i,j)$ by $q(\mathcal{E}_i,\mathcal{E}_j)$.

6. Tunneling among the deep valleys

We prove in this section the main result of this article. Recall that we denoted by $\mathcal{E}_{\mathbf{x}}$, $\mathbf{x}\in\Lambda_L$, the singletons $\{\eta^{\mathbf{x}}\}$, and that we denoted by $\mathcal{N}(\eta^{\mathbf{x}})$ the set of configurations which can be reached from $\eta^{\mathbf{x}}$ by a jump of rate $\mathrm{e}^{-2\beta}$. By Lemma 4.1, for each $\xi\in\mathcal{N}(\eta^{\mathbf{x}})$ there exists a probability measure $\mathbb{M}(\xi,\cdot)$ defined on $\mathbb{H}_{01}$ such that

$$\lim_{\beta\to\infty}\mathbf{P}^{\beta}_{\xi}\big[\eta(H_{\mathbb{H}_{01}})\in\Pi\big]=\mathbb{M}(\xi,\Pi),\qquad \Pi\subset\mathbb{H}_{01}.$$

Recall from (5.2) the definition of the probability $q(\mathcal{E}_j,\cdot)$. Let

$$Z=\sum_{\xi\in\mathcal{N}(\eta^{\mathbf{x}})}\sum_{j=1}^{\kappa}\mathbb{M}(\xi,\mathcal{E}_j)q(\mathcal{E}_j,\breve{\mathcal{F}}_{\mathbf{x}})=\sum_{\xi\in\mathcal{N}(\eta^{\mathbf{x}})}\sum_{j=1}^{\kappa}\mathbb{M}(\xi,\mathcal{E}_j)\big[1-q(\mathcal{E}_j,\mathcal{E}_{\mathbf{x}})\big],$$

where $\breve{\mathcal{F}}_{\mathbf{x}}=\bigcup_{\mathbf{y}\neq\mathbf{x}}\mathcal{E}_{\mathbf{y}}$, the union being carried over $\mathbf{y}\in\Lambda_L$. Recall that we denote by Δ_0 the configurations which are not ground states: $\Delta_0=\Omega_{L,K}\setminus\Omega^0$, and let $\mathcal{F}=\bigcup_{\mathbf{y}\in\Lambda_L}\mathcal{E}_{\mathbf{y}}$.

Proposition 6.1. *Fix* $\mathbf{x}\in\Lambda_L$.

(1) *The triple* $(\mathcal{E}_{\mathbf{x}},\mathcal{E}_{\mathbf{x}}\cup\Delta_0,\eta^{\mathbf{x}})$ *is a valley of depth* $\mu_K(\eta^{\mathbf{x}})/\operatorname{cap}_K(\mathcal{E}_{\mathbf{x}},\breve{\mathcal{F}}_{\mathbf{x}})$;
(2) *Under* $\mathbf{P}^{\beta}_{\eta^{\mathbf{x}}}$, $H(\breve{\mathcal{F}}_{\mathbf{x}})/\mathrm{e}^{2\beta}$ *converges in distribution to an exponential random variable of parameter* Z;
(3) *For any* $\mathbf{y}\neq\mathbf{x}$,

$$\lim_{\beta\to\infty}\mathbf{P}^{\beta}_{\eta^{\mathbf{x}}}\big[\eta\big(H(\breve{\mathcal{F}}_{\mathbf{x}})\big)=\eta^{\mathbf{y}}\big]=\frac{1}{Z}\sum_{\xi\in\mathcal{N}(\eta^{\mathbf{x}})}\sum_{j=1}^{\kappa}\mathbb{M}(\xi,\mathcal{E}_j)q(\mathcal{E}_j,\mathcal{E}_{\mathbf{y}})=:\mathbb{Q}(\mathbf{x},\mathbf{y}).$$

Proof. Recall [1], Theorem 2.4. By definition of the set Δ_0, $\mu_K(\Delta_0)/\mu_K(\mathcal{E}_{\mathbf{x}})$ is of order $\mathrm{e}^{-\beta}$. Condition (2.15) is therefore fulfilled. Since $\mathcal{E}_i$ is a singleton, condition (2.14) holds automatically and the result follows.

The proof of the second assertion is similar to the one of the second claim in Proposition 4.3 with the following modifications. We first need to replace the normalization e^{β} by $\mathrm{e}^{2\beta}$ and to define τ_1 as the time of the first jump, to write

$$H(\breve{\mathcal{F}}_{\mathbf{x}})=\tau_1+H(\mathcal{F})\circ\theta_{\tau_1}+\mathbf{1}\big\{H(\mathcal{F})\circ\theta_{\tau_1}=H(\mathcal{E}_{\mathbf{x}})\circ\theta_{\tau_1}\big\}H(\breve{\mathcal{F}}_{\mathbf{x}})\circ\theta_{H^+(\mathcal{F})}.$$

At this point, we repeat the arguments presented in the proof of Proposition 4.3. In the present context, τ_1 and η_{τ_1} are independent by the Markov property, and $\eta_{H(\mathcal{E}_{\mathbf{x}})}=\eta^{\mathbf{x}}$. We may therefore skip the coupling arguments of Proposition 4.3.

In contrast, we need to show that

$$\lim_{A\to\infty}\lim_{\beta\to\infty}\max_{\zeta\in\mathcal{N}(\eta^{\mathbf{x}})}\mathbf{P}^{\beta}_{\zeta}\big[H(\mathcal{F})>A\mathrm{e}^{\beta}\big]=0. \tag{6.1}$$

Starting from $\zeta\in\mathcal{N}(\eta^{\mathbf{x}})$, in a time of order one the process reaches $\mathcal{E}$. It follows from Proposition 5.3 that once at $\mathcal{E}$ in a time of order e^{β} the process reaches one of the absorbing point $\{\eta^{\mathbf{x}}\colon \mathbf{x}\in\Lambda_L\}$ of the asymptotic Markovian dynamics characterized by the rates $r(\cdot,\cdot)$. This proves (6.1).

It follows from this result and the proof of Proposition 4.3 that to prove the second assertion of the proposition it is enough to show that

$$\lim_{\beta\to\infty}\sum_{\zeta\in\mathcal{N}(\eta^{\mathbf{x}})}\mathbf{P}^{\beta}_{\zeta}\big[H(\mathcal{F})\neq H(\mathcal{E}_{\mathbf{x}})\big]=Z.$$

Since $H(\mathcal{E}) \le H(\mathcal{F}) \le H(\mathcal{E}_{\mathbf{x}})$, by the strong Markov property we may rewrite the previous probability as

$$\mathbf{E}^\beta_\zeta\big[\mathbf{P}^\beta_{\eta(H(\mathcal{E}))}\big[H(\mathcal{F}) \neq H(\mathcal{E}_{\mathbf{x}})\big]\big].$$

We computed in Lemma 4.1 the asymptotic distribution of $\eta(H(\mathcal{E}))$ and we represented by $q(\mathcal{E}_j, \mathcal{E}_{\mathbf{y}})$ the probability that the asymptotic process starting from a set $\mathcal{E}_j$, $1 \le j \le \kappa$, eventually reaches the absorbing state $\mathcal{E}_{\mathbf{y}}$, $\mathbf{y} \in \Lambda_L$. The second assertion of the proposition follows from these two results.

We now turn to the third assertion of the proposition. Fix $\mathbf{y} \neq \mathbf{x}$. This argument is also similar to the one of Proposition 4.3. Denote by $\{H_j\colon j \ge 1\}$ the successive return times to $\mathcal{F}$:

$$H_1 = H^+(\mathcal{F}), \qquad H_{j+1} = H^+(\mathcal{F}) \circ \theta_{H_j}, \quad j \ge 1.$$

With this notation,

$$\mathbf{P}^\beta_{\eta^{\mathbf{x}}}\big[\eta\big(H(\check{\mathcal{F}}_{\mathbf{x}})\big) = \eta^{\mathbf{y}}\big] = \sum_{j \ge 1} \mathbf{P}^\beta_{\eta^{\mathbf{x}}}\big[\eta(H_k) = \eta^{\mathbf{x}}, 1 \le k \le j-1, \eta(H_j) = \eta^{\mathbf{y}}\big]. \tag{6.2}$$

By the strong Markov property, if τ_1 stands for the time of the first jump, for any $\mathbf{z} \in \Lambda_L$,

$$\mathbf{P}^\beta_{\eta^{\mathbf{x}}}\big[\eta(H_1) = \eta^{\mathbf{z}}\big] = \mathbf{E}^\beta_{\eta^{\mathbf{x}}}\big[\mathbf{E}^\beta_{\eta_{\tau_1}}\big[\mathbf{P}^\beta_{\eta_{H(\mathcal{E})}}\big[\eta(H_{\mathcal{F}}) = \eta^{\mathbf{z}}\big]\big]\big].$$

As $\beta \uparrow \infty$, this expression converges to

$$\frac{1}{8} \sum_{\xi \in \mathcal{N}(\eta^{\mathbf{x}})} \sum_{j=1}^{\kappa} \mathbb{M}(\xi, \mathcal{E}_j) q(\mathcal{E}_j, \mathcal{E}_{\mathbf{z}}).$$

The third assertion of the proposition follows from (6.2), this identity and the strong Markov property. □

It follows from (1) and (2) that the triple

$$\big(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{x}} \cup \Delta_0, \eta^{\mathbf{x}}\big) \text{ is in fact a valley of depth } \mathrm{e}^{2\beta}/Z. \tag{6.3}$$

Corollary 6.2. *The sequence of Markov processes $\{\eta^\beta_t\colon t \ge 0\}$ exhibits a tunneling behavior on the time-scale $\mathrm{e}^{2\beta}$, with metastable states $\{\mathcal{E}_{\mathbf{x}}\colon \mathbf{x} \in \Lambda_L\}$, metastable points $\{\eta^{\mathbf{x}}\}$ and asymptotic Markov dynamics characterized by the rates*

$$r(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}}) = Z\mathbb{Q}(\mathbf{x}, \mathbf{y}), \quad \mathbf{x} \neq \mathbf{y} \in \Lambda_L.$$

Proof. The proof is similar to the one of Proposition 5.3. We first check that assumptions (H0) and (H1) of [1], Theorem 2.7, are fulfilled. Hypothesis (H1) is trivially satisfied since the sets $\mathcal{E}_{\mathbf{x}}$ are singletons.

To prove assumption (H0), denote $\{\eta^{\mathcal{F}}_t\colon t \ge 0\}$ the trace of the process η^β_t on $\mathcal{F}$, and by $R^{\mathcal{F}}_\beta$ the jump rates of the trace process. Note that in this case of singleton valleys, the average rates coincide with the rates. We claim that $\mathrm{e}^{2\beta} R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}})$, $\mathbf{x} \neq \mathbf{y} \in \Lambda_L$, converges to a limit denoted by $R(\mathbf{x}, \mathbf{y})$.

We may rewrite $\mathrm{e}^{2\beta} R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}})$ as $\mathrm{e}^{2\beta} R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}}) \times [R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}})/R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}})]$. By [3], Corollary 4.4, $R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}})/R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}})$ converges to a number $p(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}}) \in [0, 1]$. On the other hand, by [1], Lemma 6.7, $\mathrm{e}^{2\beta} R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}}) = \mathrm{e}^{2\beta} \operatorname{cap}_K(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}})/\mu_K(\mathcal{E}_{\mathbf{x}})$. Clearly, $G_K(\mathcal{E}_{\mathbf{x}}, \check{\mathcal{F}}_{\mathbf{x}}) = \mathrm{e}^{-2\beta}\mu_K(\eta^{\mathbf{x}})$. Hence, assumption (H0) follows from (3.1).

In view of [3], Lemma 10.2, Proposition 6.1 and (6.3), $\mathrm{e}^{2\beta} R^{\mathcal{F}}_\beta(\mathcal{E}_{\mathbf{x}}, \mathcal{E}_{\mathbf{y}})$, $\mathbf{x} \neq \mathbf{y} \in \Lambda_L$, converges to $Z\mathbb{Q}(\mathbf{x}, \mathbf{y})$.

It remains to show property (M3) of tunneling, which states that the time spent outside $\mathcal{F}$ is negligible. Fix $\mathbf{x} \in \Lambda_L$. Denote by $\{H_j\colon j \ge 1\}$ the times of the successive returns to $\mathcal{F}$: $H_1 = H^+(\mathcal{F})$, $H_{j+1} = H^+(\mathcal{F}) \circ \theta_{H_j}$, $j \ge 1$. To prove (M3), it is enough to show that for all $t > 0$

$$\begin{aligned} &\lim_{k\to\infty} \lim_{\beta\to\infty} \mathbf{P}^\beta_{\eta^{\mathbf{x}}}\big[H_k \le t\mathrm{e}^{2\beta}\big] = 0 \quad \text{and} \\ &\lim_{\beta\to\infty} \mathbf{E}^\beta_{\eta^{\mathbf{x}}}\left[\mathrm{e}^{-2\beta} \int_0^{H_k \wedge t\mathrm{e}^{2\beta}} \mathbf{1}\{\eta^\beta_s \in \Delta_0\}\,\mathrm{d}s\right] = 0 \quad \text{for all } k \ge 1. \end{aligned} \tag{6.4}$$

Since $H_1 = H^+(\mathcal{F})$ is greater than the time of the first jump, H_1 is bounded below by an exponential time of parameter $8\mathrm{e}^{-2\beta}$, $\mathbf{P}^\beta_\eta$ almost surely for all $\eta \in \mathcal{E}$. The first line of (6.4) follows from this observation and from the strong Markov property.

To estimate the second term of (6.4), fix $k \geq 1$ and rewrite the time integral as $\sum_{0\leq j<k} \int_{H_j \wedge t\mathrm{e}^{2\beta}}^{H_{j+1}\wedge t\mathrm{e}^{2\beta}}$. For a fixed j, the integral vanishes unless $H_j < t\mathrm{e}^{2\beta}$. Hence, by the strong Markov property, the expectation is less than or equal to

$$k \max_{\mathbf{y}\in\Lambda_L} \mathbf{E}^\beta_{\eta^\mathbf{y}} \left[\mathrm{e}^{-2\beta} \int_0^{H_1 \wedge t\mathrm{e}^{2\beta}} \mathbf{1}\{\eta^\beta_s \in \Delta_0\} \, \mathrm{d}s \right].$$

Recall that we denoted by $\mathcal{F}(\eta^\mathbf{y})$ the set of configurations which can be reached from $\eta^\mathbf{y}$ by a jump of rate $\mathrm{e}^{-2\beta}$. By the strong Markov property, this expression is bounded by

$$k \max_{\mathbf{y}\in\Lambda_L} \max_{\xi\in\mathcal{F}(\eta^\mathbf{y})} \mathbf{E}^\beta_\xi \left[\mathrm{e}^{-2\beta} H(\mathcal{F}) \wedge t \right].$$

By (6.1) this expression vanishes as $\beta \uparrow \infty$. □

7. General results

We prove in this section an useful general result. Fix a sequence $(E_N\colon N \geq 1)$ of countable state spaces. The elements of E_N are denoted by the Greek letters η, ξ. For each $N \geq 1$ consider a matrix $R_N\colon E_N \times E_N \to \mathbb{R}$ such that $R_N(\eta,\xi) \geq 0$ for $\eta \neq \xi$, $-\infty < R_N(\eta,\eta) \leq 0$ and $\sum_{\xi\in E_N} R_N(\eta,\xi) = 0$ for all $\eta \in E_N$.

Let $\{\eta^N_t\colon t \geq 0\}$ be the *minimal* right-continuous Markov process associated to the jump rates $R_N(\eta,\xi)$ [21]. It is well known that $\{\eta^N_t\colon t \geq 0\}$ is a strong Markov process with respect to the filtration $\{\mathcal{F}^N_t\colon t \geq 0\}$ given by $\mathcal{F}^N_t = \sigma(\eta^N_s\colon s \leq t)$. Let $\mathbf{P}_\eta$, $\eta \in E_N$, be the probability measure on $D(\mathbb{R}_+, E_N)$ induced by the Markov process $\{\eta^N_t\colon t \geq 0\}$ starting from η.

Consider two sequences $\mathcal{W} = (W_N \subseteq E_N\colon N \geq 1)$, $\mathcal{B} = (B_N \subseteq E_N\colon N \geq 1)$ of subsets of E_N, the second one containing the first and being properly contained in E_N : $W_N \subseteq B_N \subsetneqq E_N$. Fix a point $\boldsymbol{\xi} = (\xi_N \in W_N\colon N \geq 1)$ in $\mathcal{W}$ and a sequence of positive numbers $\boldsymbol{\theta} = (\theta_N\colon N \geq 1)$.

Next result states an obvious fact. We may add to the basin $\mathcal{B}$ of a valley $(\mathcal{W}, \mathcal{B}, \boldsymbol{\xi})$ a set $\mathcal{C}$ never visited by the process without modifying the properties of the valley.

Lemma 7.1. *Assume that the triple $(\mathcal{W}, \mathcal{B}, \boldsymbol{\xi})$ is a valley of depth $\boldsymbol{\theta}$ and attractor $\boldsymbol{\xi}$. Let $\mathcal{C} = (C_N \subset E_N\colon N \geq 1)$ be a sequence of sets such that B^c_N is attained before C_N when starting from W_N*:

$$\lim_{N\to\infty} \inf_{\eta\in W_N} \mathbf{P}_\eta[H_{B^c_N} < H_{C_N}] = 1. \tag{7.1}$$

Then, the triple $(\mathcal{W}, \mathcal{B} \cup \mathcal{C}, \boldsymbol{\xi})$ is a valley of depth $\boldsymbol{\theta}$ and attractor $\boldsymbol{\xi}$.

Proof. We have to check the three conditions of [1], Definition 2.1. The first one is obvious because $B^c_N \supset (B_N \cup C_N)^c$. On the event $\{H_{B^c_N} < H_{C_N}\}$, $H_{B^c_N} = H_{(B_N\cup C_N)^c}$. Hence, the convergence in distribution of $H_{(B_N\cup C_N)^c}/\theta_N$ to a mean one exponential variable follows from (7.1) and from the one of $H_{B^c_N}/\theta_N$. For the same reasons, on the set $\{H_{B^c_N} < H_{C_N}\}$, $\int_0^{H_{B^c_N}} \mathbf{1}\{\eta^N_s \in A\}\,\mathrm{d}s = \int_0^{H_{(B_N\cup C_N)^c}} \mathbf{1}\{\eta^N_s \in A\}\,\mathrm{d}s$. In particular, property (V3) for the triple $(\mathcal{W}, \mathcal{B} \cup \mathcal{C}, \boldsymbol{\xi})$ follows from (7.1) and (V3) for the valley $(\mathcal{W}, \mathcal{B}, \boldsymbol{\xi})$. □

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