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Ergodic behaviour of "signed voter models"[1]

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Abstract. We answer some questions raised by Gantert, Löwe and Steif (*Ann. Inst. Henri Poincaré Probab. Stat.* **41** (2005) 767–780) concerning "signed" voter models on locally finite graphs. These are voter model like processes with the difference that the edges are considered to be either positive or negative. If an edge between a site x and a site y is negative (respectively positive) the site y will contribute towards the flip rate of x if and only if the two current spin values are equal (respectively opposed).

Résumé. Nous répondons à des questions soulevées dans le récent papier de Gantert, Löwe et Steif (*Ann. Inst. Henri Poincaré Probab. Stat.* **41** (2005) 767–780) concernant les modèles du votant "signés" sur des graphes localement finis. Ce sont des processus de type modèle du votant à la différence que chaque arête est considérée comme étant positive ou bien négative. Si une arête entre un site x et un site y est négative (respectivement positive), le site y contribura au taux de flip de x si et seulement si les deux valeurs actuelles des spins sont égales (respectivement opposées).

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1. Introduction

This work arises from questions raised in the recent article by Gantert, Löwe and Steif [4]. Following this paper we consider voter model like processes called "signed" voter models. For such a process we suppose that we are given a locally finite, undirected, connected graph $G=(V,E)$ and a function $s: E \to \{-1,1\}$. Our model $(\eta(t)\colon t\ge 0)$ will be a spin system on $\{-1,1\}^V$ with generator

$$\Omega f(\eta)=\sum_{x\in V}\bigl(f(\eta^x)-f(\eta)\bigr)\frac{1}{d(x)}\sum_{y\colon \{x,y\}\in E} I_{\{\eta(x)\eta(y)\neq s(\{x,y\})\}}. \tag{1.1}$$

Here $I_{\cdot}$ denotes the ordinary indicator function. The usual spins, 0 and 1, are replaced by -1 and 1 purely for the resulting notational simplicity. As usual $d(x)$ is the degree of vertex $x\in V$ and configuration η^x is the element of $\{-1,1\}^V$ with spins equal to those of η except at site x. From now on we will abuse notation and write $s(x,y)$ for $s(\{x,y\})$; we will call this the sign of edge $\{x,y\}$. This can be seen as a generalization of the classical voter model (see e.g. [1,6]) in that if the function s is identically 1 (or equivalently if all signs are positive) then the corresponding process is the voter model. Equally, if all the signs are negative, we are faced with the already studied anti-voter model.

As with the voter model, the easiest and most natural way to realize the voter model is via a Harris construction: we introduce for each ordered pair (x,y) of neighbours a Poisson process, $N^{x,y}$, of rate $1/d(x)$ with all Poisson

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processes being independent. The process is built by stipulating that at times $t \in N^{x,y}$, the spin at x becomes equal to $s(x,y)\eta_t(y)$ (which may well represent no change for the process). A.s. no two distinct Poisson processes have common points so the rule is unambiguous. It can easily be checked that with probability one this rule specifies $\eta_t(x)$ for all t and x just as in the classical voter model (see [1]). The Markovian nature is simply inherited from that of the system of Poisson processes. It is then readily seen that this is indeed the process with generator given by (1.1).

As with the voter model, duality plays the dominant role in understanding the "signed" voter model. Before discussing this duality we introduce some notation.

Definition 1.1. *A nearest neighbour path* $(\gamma(r)\colon 0 \le r \le t)$ *having finitely many jumps at times* $0 \le t_1 \le t_2 \le \cdots \le t_n \le t$ *is said to be* even *or* positive *if the number of* $1 \le i \le n$ *so that* $s(\gamma(t_{i}-), \gamma(t_i)) = -1$ *is even. Otherwise the path is said to be* odd *or* negative. *If it is positive we write* $\operatorname{sgn}(\gamma) = 1$ *otherwise* $\operatorname{sgn}(\gamma) = -1$.

Definition 1.2. *For a possibly infinite path* $\gamma = (\gamma(r)\colon 0 \le r \le t)$ *and* $0 \le r_1 \le t_1 \le t$, γ^{r_1,t_1} *signifies the path* $(\gamma(r)\colon r_1 \le r \le t_1)$. *If* $r_1 = 0$ *we write* γ^{t_1} *instead of* γ^{r_1,t_1}.

For fixed $t \ge 0$ and $x \in V$ we define the "dual" random walk on G, $X^{x,t} = (X^{x,t}(r)\colon 0 \le r \le t)$ by the recipe: $X^{x,t}(0) = x$ and the random walk jumps from y to z at time $r \in [0,t]$ if immediately before time r it was at site y and $t - r \in N^{y,z}$. As in [1], we recover $\eta_t(x)$ via the identity

$$\eta_t(x) = \eta_0\bigl(X^{x,t}(t)\bigr)\operatorname{sgn}\bigl(X^{x,t}\bigr). \tag{1.2}$$

It should be noted that the random walks $X^{x,t}(\cdot)$ and $X^{y,t}(\cdot)$ are coalescing. If the two paths meet for the first time at $r_0 \in [0,t]$, then irrespective of η_0 we have

$$\eta_t(x)\eta_t(y) = \operatorname{sgn}\bigl(\gamma^{x,y,r_0}\bigr), \tag{1.3}$$

where $\gamma^{x,y,r_0}\colon [0,2r_0] \to V$ is the concatenation of the path $(X^{x,t}(r)\colon 0 \le r \le r_0)$ with the path $(X^{y,t}(r_0 - r)\colon 0 \le r \le r_0)$. Thus just as with the classical voter model, if the two dual random walks meet then irrespective of the initial configuration, η_0, a relation holds between $\eta_t(x)$ and $\eta_t(y)$. The difference is that this relation is more complicated than for the voter model where the relation is nothing but equality. For more discussion of the dual see the next section.

As stated above, this article is written to address questions raised by [4]; it also follows for instance, the article of [9] which addresses signed voter models on the integer lattice where the signs are assigned to the edges in i.i.d. fashion. See [4] for a more complete bibliography.

A major concern of [4] was *unsatisfied cycles* that are defined as follows.

Definition 1.3. *Unsatisfied cycles are nearest neighbour cycles in G whose sign is negative.*

Such cycles are important since in their absence the vertices can be divided into a "positive" set, V_+ and a "negative" set, V_-: one simply fixes arbitrarily a site $x_0 \in V$ which is designated "positive." A site $y \in V$ is positive if a path (and so, in the absence of unsatisfied cycles, all paths) from x_0 to y is positive; otherwise y is negative. Then the process $(\eta'_t\colon t \ge 0)$ defined by $\eta'_t(x) = \eta_t(x)$ for $x \in V_+$, $\eta'_t(x) = -\eta_t(x)$ for $x \in V_-$ is a classical voter model. Equally, the presence of unsatisfied cycles precludes the existence of fixed configurations η for which the total flip rate is zero (see [4], Section 2 for details). For the classical voter model the configurations $\underline{1}$ of all 1's and $-\underline{1}$ of all -1's are fixed in this manner and so the voter model is never ergodic in the sense of [6], i.e., there exists a unique equilibrium μ and for every initial η_0, η_t converges in distribution to μ as t tends to infinity. In the case of "signed" voter models ergodicity in this sense is a real possibility. It is well known that for the classical voter model one may obtain (possibly nonextremal) equilibria by starting with $\{\eta_0(x)\}_{x\in V}$ i.i.d. with say $\alpha = P(\eta_0(x) = 1)$ and taking the limit distribution of η_t as t tends to infinity. For the signed voter model and general $\alpha \in (0,1)$ the limit may very well not exist, indeed even the limit of $P(\eta_t(x) = 1)$ as t becomes large need not exist. However for the symmetric value $\alpha = 1/2$, it can be shown in much the same way as for the voter model that the limit exists. We denote this special measure by $\nu_{1/2}$.

For the signed voter model it may well be the case that $\nu_{1/2}$ is the unique equilibrium. The question of whether, for such processes, if this equilibrium was indeed unique, the process must necessarily be ergodic was raised in [4]. In fact this holds and can be seen to be a consequence of Matloff's lemma (Lemma 3.1 in [7]), see also Lemma V.1.26 of [6].

Theorem 1.4. *If the "signed" voter model has a unique equilibrium, then the "signed" voter model is ergodic.*

We next consider another raised question ([4], question one). Proposition 1.2 of this work gives a useful robust criterion for there to exist multiple equilibria for a signed voter model: there exists a subset $W \subset V$ such that with positive probability a random walk (starting from an appropriate site) will never leave W and secondly that W, with inherited edge set, has no unsatisfied cycles. The question raised was whether this criterion was in fact necessary as well as sufficient.

Proposition 1.5. *There exist graphs $G = (V, E)$ with sign function s so that the associated signed voter model is not ergodic but such that there does not exist $W \subset V$ with the above property.*

But, under a natural condition, the question can be answered affirmatively.

Proposition 1.6. *If the graph $G = (V, E)$ is of bounded degree and the sign function is such that there are multiple equilibria for the associated signed voter model, then there exists $W \subset V$ on which the inherited graph has no unsatisfied cycle and such that for each $x \in W$, $P^x(T_{W^c} = \infty) > 0$ for $T_{W^c} = \inf\{t\colon X(t) \in W^c\}$.*

Definition 1.7. *For a path $\gamma = (\gamma(s)\colon s \geq 0)$ on V, we say that γ* traverses infinitely many unsatisfied cycles *if there exists sequences $(s_i)_{i\geq 1}$ and $(t_i)_{i\geq 1}$ tending to infinity so that γ^{s_i,t_i} are unsatisfied cycles.*

A simple criterion for ergodicity was the existence of unsatisfied cycles and the recurrence of the associated simple random walk, see Theorem 1.1 of [4]. The following may be seen as a generalization of this result.

Proposition 1.8. *If with probability one a random walk on $G = (V, E)$, $(X(t)\colon t \geq 0)$ traverses infinitely many unsatisfied cycles then the signed voter model is ergodic.*

Another question we are fully able to resolve is the second open question listed in [4]:

Theorem 1.9. *For the graph $\mathbb{Z}^3$ with usual edge set and any sign assignation, s, either the process is not ergodic or a random walk must a.s. traverse infinitely many unsatisfied cycles.*

By Proposition 1.8 above the two statements in Theorem 1.9 are exclusive. The peculiarity of this result is highlighted by the next result

Theorem 1.10. *For the graph $\mathbb{Z}^d$, $d \geq 4$ there are sign functions s on the edge set so that the associated voter model is ergodic but the random walk must a.s. traverse only finitely many unsatisfied cycles.*

Theorem 1.1 of [4] shows that in dimensions 1 and 2, if there is an unsatisfied cycle then necessarily the associated "signed" voter model is ergodic so the above results are in a sense definitive.

An important tool we will use is the existence for two Markov chains on a state space S where the jump rates satisfy

$$\sup_{x\in S} q(x, x) < \infty, \tag{1.4}$$

of a "time shift" coupling. By this we mean that:

Lemma 1.11. *Under condition (1.4), and given $T < \infty$ and $\varepsilon > 0$, there exists a finite t_0 so that for any $r \in [0, T]$ and any $x \in S$, two realizations of the Markov chain starting at x, $(X(t)\colon t \geq 0)$ and $(X'(t)\colon t \geq 0)$ may be coupled so that with probability at least $1 - \varepsilon$*

(a) *for all* $t \geq t_0$, $X(t) = X'(t+r)$ *and*
(b) *the sequence of sites visited* (*allowing repeat visits*) *by the process* $X(\cdot)$ *up to time* t_0 *is equal to that for* $X'(\cdot)$ *up to time* $t_0 + r$.

(Observe in particular that if (a) and (b) are satisfied, $\mathrm{sgn}((X')^{t+r}) = \mathrm{sgn}(X^t)$ $\forall t \geq t_0$.) The lemma is for fixed r given in [8], That the coupling bounds are uniform on compacts follows easily from the proof gven there. The details are left to the reader.

The rest of the paper is organized as follows: Sections 2 is dedicated to the proofs of Theorem 1.4 and Proposition 1.8. Section 3 concerns itself with the proofs of Propositions 1.5 and 1.6. Finally Sections 4 and 5 are respectively devoted to the proofs of Theorems 1.9 and 1.10.

Throughout the paper we will use the following notation:

For a measure ν on a measureable space $(E, \mathcal{E})$ and a measureable function h defined on this space $\langle \nu, h\rangle$ will denote the integral of h with respect to ν (when this exists).

Given a set of vertices B in a graph G, ∂B will be the external boundary of B, that is the set of points in B^c which are neighbours of a point in B.

For a process $(X(t)\colon t \geq 0)$ (typically a random walk on a graph (V, E)) and a set $B \subset V$, $T_B = \inf\{t\colon X(t) \in B\}$.

We write $x \sim y$ to denote $\{x, y\} \in E$.

It is universal practice that Bernoulli random variables denote variables which a.s. take value 0 or 1. Nonetheless, we use Bernoulli in this paper to denote variables taking values -1 or 1. A Bernoulli(α) random variable will equal 1 with probability α and thus -1 with probability $1-\alpha$.

2. Proof of Theorem 1.4 and Proposition 1.8

The following proof for Theorem 1.4 is really just a transcription of Lemma V.1.26 of [6]. It is included for completeness. It rests on a property of the dual for the signed voter model, which we now describe in detail.

We suppose, as usual, a given *Harris system* of Poisson processes for generating signed voter models $(\eta_t\colon t \geq 0)$ from a given initial configuration η_0. That is a collection of independent Poisson processes $N^{x,y}$ of rate $1/d(x)$ for ordered neighbour pairs (x, y). Given an initial configuration η_0, a time $t \geq 0$, an integer h and h points in vertex set V, $x_1, x_2, \ldots, x_h$, the values of $(\eta_t(x_1), \eta_t(x_2), \ldots, \eta_t(x_h))$ are determined by the dual process

$$\underline{X}^t(u) = \big(\big(X_1^{x_1,t}(u), i_1^t(u)\big), \big(X_2^{x_2,t}(u), i_2^t(u)\big), \ldots, \big(X_h^{x_h,t}(u), i_h^t(u)\big)\big), \quad 0 \leq u \leq t, \tag{2.1}$$

at time t, where $X_j^{x_j,t}(u) \in V$, $i_j^j(u) \in \{-1, 1\}$ for all $u \in [0, t]$ and $1 \leq j \leq h$. The process (piecewise constant) evolves as follows: $\underline{X}^t(\cdot)$ jumps at time $u \in [0, t]$ if and only if there exists $j \leq h$ so that $t - u \in N^{X_j^{x_j,t}(u^-),z}$ for some z neighbouring $X_j^{x_j,t}(u^-)$. This being the case

(i) for every index k so that $X_k^{x_k,t}(u^-) \neq X_j^{x_j,t}(u^-)$, there will be no change: $X_k^{x_k,t}(u) = X_k^{x_k,t}(u^-)$ and $i_k^t(u) = i_k^t(u^-)$,
(ii) for every index k so that $X_k^{x_k,t}(u^-) = X_j^{x_j,t}(u^-)$, we will have $X_k^{x_k,t}(u) = z$ and $i_k^t(u) = i_k^t(u^-)s(X_j^{x_j,t}(u^-), z)$ with s the sign function.

Given this dual one recovers the values $\eta_t(x_k)$ by

$$\eta_t(x_k) = \eta_0\big(X_k^{x_k,t}(t)\big) i_k^t(t). \tag{2.2}$$

The key point for the proof is that over the interval $[0, t]$ the process $\underline{X}^t$ will evolve as a Markov chain whose jump rates are bounded (by h) so that Lemma 1.11 may be applied. That is given integer $T, h < \infty$ and $\varepsilon > 0$, uniformly over all $x_1, x_2, \ldots, x_h$ there exists t_0 so that

$$\big\|\underline{X}^t(t) - \underline{X}^{t+u}(t+u)\big\|_{\mathrm{TV}} = \big\|\underline{X}^{t+u}(t) - \underline{X}^{t+u}(t+u)\big\|_{\mathrm{TV}} < \varepsilon \tag{2.3}$$

for all $t \geq t_0$ and $u \in [0, T]$, where by abuse of notation we identify the random variables with their law. Here $\|\cdot\|_{\mathrm{TV}}$ is the usual total variation norm between probability laws.

We may now turn directly to the proof of Theorem 1.4.

Proof of Theorem 1.4. We consider η_0 as fixed. It is sufficient to show that all limit points of the distribution of η_t as t tends to infinity are equilibria. We suppose that for sequence $\{t_n\}_{n\geq 1}$ tending to infinity

$$\eta_{t_n} \to \nu \quad \text{in law.} \tag{2.4}$$

Let h be a cylinder function depending on, say, the spin values at $x_1, x_2, \ldots, x_r$, i.e., $h(\eta) = g(\eta(x_1), \eta(x_2), \ldots, \eta(x_r))$. We have that

$$\langle \nu, h\rangle = \lim_{n\to\infty} E^{\eta_0}\big[h(\eta_{t_n})\big] = \lim_{n\to\infty} E\big[h'\big(\eta_0, \underline{X}^{t_n}(t_n)\big)\big], \tag{2.5}$$

where we have

$$h'\big(\eta_0, \underline{X}^s(s)\big) = g\big(\eta_0\big(X_1^{x_1,s}(s)\big)i_1^s(s), \eta_0\big(X_2^{x_2,s}(s)\big)i_2^s(s), \ldots, \eta_0\big(X_r^{x_r,s}(s)\big)i_r^s(s)\big). \tag{2.6}$$

But equally for any fixed t we have (our signed voter model is easily seen to be a Feller process)

$$\langle \nu, P_t h\rangle = \lim_{n\to\infty} E^{\eta_0}\big[P_t h(\eta_{t_n})\big], \tag{2.7}$$

where as usual $(P_t)_{t\geq 0}$ denotes the Markov semigroup of our signed voter model. The quantity inside the limit in the r.h.s. of (2.7) can be rewritten as $E^{\eta_0}[h(\eta_{t_n+t})]$ which in the notation introduced in (2.6) is equal to

$$E\big[h'\big(\eta_0, \underline{X}^{t_n+t}(t_n+t)\big)\big]. \tag{2.8}$$

But, as already noted, as t_n tends to infinity $\|\underline{X}^{t_n}(t_n) - \underline{X}^{t_n+t}(t_n+t)\|_{\mathrm{TV}}$ tends to zero and so

$$\lim_{n\to\infty}\big(E\big[h'\big(\eta_0, \underline{X}^{t_n+t}(t_n+t)\big)\big] - E\big[h'\big(\eta_0, \underline{X}^{t_n}(t_n)\big)\big]\big) = 0 \tag{2.9}$$

which implies that $\langle \nu, h\rangle = \langle \nu, P_t h\rangle$. By the arbitrariness of t and h we can conclude that measure ν is an equilibrium but, given our hypotheses that there is a unique equilibrium, we have established that any limit point ν must equal this equilibrium $(\nu_{1/2})$. That is we have established ergodicity. □

The following criterion for ergodicity was given as Theorem 6.1 in [4].

Lemma 2.1. *If for every $x \in V$ and every $\eta_0 \in \{-1, 1\}^V$, $P^{\eta_0}(\eta_t(x) = 1) \to 1/2$ as $t \to \infty$, then the signed voter model is ergodic.*

The argument given above permits the following generalization:

Proposition 2.2. *If for every equilibrium μ on $\{-1, 1\}^V$, $\mu(\eta(x) = 1) = 1/2$, then the signed voter model is ergodic.*

Proof. If the system is not ergodic then by Lemma 2.1, there exists $x, \varepsilon > 0, \eta_0$ and sequence t_n increasing to infinity so that $\forall n, |P^{\eta_0}(\eta_{t_n}(x) = 1) - 1/2| > \varepsilon$. But the proof of Theorem 1.4 demonstrated that under P^{η_0} any limit distribution of η_{t_n} must be an equilibrium. This equilibrium must satisfy $\mu(\eta(x) = 1) \neq 1/2$ which contradicts our hypothesis. □

To show Proposition 1.8 we will need the following result. Let, for $x \in V$, $t \geq 0$, the measures $\mu_{x,t,\pm}$ on V be defined by

$$\mu_{x,t,+}(y) = P^x\big(X(t) = y, X^t \text{ is even}\big), \tag{2.10}$$

$$\mu_{x,t,-}(y) = P^x\big(X(t) = y, X^t \text{ is odd}\big). \tag{2.11}$$

Lemma 2.3. *For fixed* $T \in (0, \infty)$, *and* $\varepsilon > 0$, *there exists* $T_0 < \infty$ *so that uniformly over* $r \in [0, T]$, $x \in V$ *and* $t \geq T_0$

$$\|\mu_{x,t,+} - \mu_{x,t-r,+}\|_{\mathrm{TV}} + \|\mu_{x,t,-} - \mu_{x,t-r,-}\|_{\mathrm{TV}} < \varepsilon. \tag{2.12}$$

The lemma is a simple consequence of Lemma 1.11.

Proof of Theorem 1.8. By Lemma 2.1, it is enough to show that for fixed $x \in V$ and $\eta_0 \in \{-1, 1\}^V$, as t tends to infinity $P^{\eta_0}(\eta_t(x) = 1) \to \frac{1}{2}$.

Thus we need to show that $\limsup_{t\to\infty} P^{\eta_0}(\eta_t(x) = 1) \leq \frac{1}{2}$ and $\liminf_{t\to\infty} P^{\eta_0}(\eta_t(x) = 1) \geq \frac{1}{2}$. That is we need to show that for t large $P^{\eta_0}(\eta_t(x) = 1) \in [1 - \alpha, \alpha]$, for any $\alpha > 1/2$.

First fix $\alpha > 1/2$ and $\varepsilon > 0$, a small strictly positive constant which will be more fully specified later. We will argue by contradiction and assume that for t large $P^{\eta_0}(\eta_t(x) = 1) > \alpha$; the argument to show for large t that $P^{\eta_0}(\eta_t(x) = 1) \geq 1 - \alpha$ is entirely similar. Fix $T \gg 1$ to be such that for Z^x a random walk beginning at site x.

$$P\big(Z^x(r) \text{ has not traversed an unsatisfied cycle for } 0 \leq r \leq T\big) < \varepsilon/100. \tag{2.13}$$

We suppose that $t \geq T + T_0$ for T_0 given by Lemma 2.3 for this ε and T. Consider the martingale $(M_r\colon 0 \leq r \leq t)$

$$M_r = P\big(\eta_0\big(Z^x(t)\big)\operatorname{sgn}\big(\big(Z^x\big)^t\big) = 1 | Z^x(u), \operatorname{sgn}\big(\big(Z^x\big)^u\big) u \leq r\big) \tag{2.14}$$

$$= P\big(\eta_0\big(Z^x(t)\big)\operatorname{sgn}\big(\big(Z^x\big)^t\big) = 1 | Z^x(r), \operatorname{sgn}\big(\big(Z^x\big)^r\big)\big). \tag{2.15}$$

We may consider Z^x to be the dual random walk. If $P^{\eta_0}(\eta_t(x) = 1) > \alpha$, then $M_0 > \alpha$, and, by the optional sampling theorem see e.g. [2],

$$P\big(\sigma(\alpha) < T\big) \leq \frac{4(1-\alpha)}{3 - 2\alpha} \tag{2.16}$$

for

$$\sigma(\alpha) = \inf\left\{r\colon M_r < \frac{1/2 + \alpha}{2}\right\}. \tag{2.17}$$

(Note that $\alpha > 1/2$ implies that $\frac{4(1-\alpha)}{3-2\alpha} < 1$.) Thus if ε is sufficiently small then with strictly positive probability at least $1 - \frac{4(1-\alpha)}{3-2\alpha} - \frac{\varepsilon}{100}$,

(i) for all $0 \leq r \leq T$, $M_r > \frac{1/2+\alpha}{2}$ and
(ii) there exists $0 \leq r_1 \leq r_2 \leq T$ so that $(Z^x(r)\colon r_1 \leq r \leq r_2)$ traverses an unsatisfied cycle.

But by Lemma 2.3 and our assumption on t we have that

$$\|\mu_{Z^x(r_1),t-r_1,+} - \mu_{Z^x(r_1),t-r_2,+}\|_{\mathrm{TV}} + \|\mu_{Z^x(r_1),t-r_1,-} - \mu_{Z^x(r_1),t-r_2,-}\|_{\mathrm{TV}} < \varepsilon.$$

This and the fact that $M_{r_1} > (1/2 + \alpha)/2$ implies that $M_{r_2} < 1 - (1/2 + \alpha)/2 + 2\varepsilon$. But if ε is chosen sufficiently small then this will contradict (i) above. Thus we have that in fact for $\alpha > 1/2$ the conditional probability that $\eta_0(Z^x(t))\operatorname{sgn}((Z^x)^t) = 1$ given η_0 is less than α for t large. We similarly have that it must also be greater than $1 - \alpha$ and we are done by the arbitrariness of α. □

3. Proof of Propositions 1.5 and 1.6

Proposition 1.2 of [4] stated that if the graph $G = (V, E)$ had the property that there existed $W \subset V$, $x \in V$ so that:

(i) $P^x(T_{W^c} = \infty) > 0$, recall $T_{W^c} = \inf\{t\colon X(t) \in W^c\}$ and
(ii) W with its inherited edge set contained no unsatisfied cycles,

then necessarily the signed voter model could not be ergodic. The question was raised at the end of the paper as to whether a converse existed: can it be that whenever a signed voter model is non ergodic such a W can be found? We first show this is not the case, but then show that with the additional hypothesis that the graph is of bounded degree, it is indeed true. We first state without proof (it follows from [4], Proposition 1.2).

Proposition 3.1. *If a.s. for all random walks $X = (X(t)\colon t \geq 0)$ on the graph G, there exists random T so that on $[T, \infty)$, X does not traverse a negative edge then the signed voter model has multiple equilibria.*

Proof of Theorem 1.5. We will build our counterexample out of a rooted tree, R, with only positive edges by adding a number of negative edges whose density is so small that the property of multiple equilibria is unchanged. Consider a rooted tree, R, so that each ith generation has n_i "children" where n_i increases to infinity as $i \to \infty$ and is always even. We now amend R as follows. We pick strictly increasing $V_n \uparrow \infty$ so that $n_{V_n} \geq 2^n$. At the V_nth generation we pair up the vertices so that each vertex of the V_nth generation is paired with a member having the same father. We add the corresponding edges. For the resulting graph all original edges remain positive but the extra "within generation" edges are fixed as negative. Though this new graph has cycles (indeed unsatisfied cycles), we retain use of the words descendants inherited from the original rooted tree. By the Borel–Cantelli lemma and Proposition 3.1, the signed voter model has multiple equilibria. Let W be a subset of V with the property that, with initial point suitably chosen, the probability of a random walk on G never leaving W is strictly positive. For time t let $T_{r(t)}$ be the first time after t that the random walk is at generation V_n for some generation V_n which is strictly larger than the current generation (that of $X(t)$). Since $X(T_{r(t)})$ is chosen uniformly among all potential descendants of $X(T_{r(t)}-)$, we have that the probability that $X(T_{r(t)})$ is equal to an element of W whose pair does not belong to W is less than the probability that $X(T_{r(t)})$ is not a member of W. But by Lévy's 0–1 law (see e.g. [2]), on the event that the random walk $(X(t)\colon t \geq 0)$ never leaves W, as t becomes large the conditional probability given $\mathcal{F}_t$ (in the natural filtration of X), that the pair of $X(T_{r(t)}) \in W$ tends to one. Thus with probability tending to one (as t tends to infinity) on the event $\{T_{W^c} = \infty\}$ we have

$$\text{both } X(T_{r(t)}) \text{ and its pair belong to } W. \tag{3.1}$$

But this must mean that with probability tending to one as t tends to infinity, the unsatisfied cycle of length 3 involving the point $X(T_{r(t)})$, its pair and their (common) father is in W. □

This counterexample is somewhat cheap, the "real" question is whether the converse to Proposition 1.2 holds for graphs of bounded degree.

Proof of Theorem 1.6. Given Proposition 2.1 it is enough to show the existence of a suitable $W \subset V$ under the existence of an equilibrium μ for which $\mu(\{\eta\colon \eta(x) = 1\})$ is not identically $\frac{1}{2}$ as x varies over V. In the following let $M = \sup_{x \in V} d(x)$, which is supposed finite. We fix equilibrium μ so that for some $x \in V$, $\mu(\{\eta\colon \eta(x) = 1\}) \neq 1/2$. Let

$$\alpha = \sup_{x \in V} \bigl|\mu\bigl(\{\eta\colon \eta(x) = 1\}\bigr) - \mu\bigl(\{\eta\colon \eta(x) = -1\}\bigr)\bigr| > 0. \tag{3.2}$$

Without loss of generality we have

$$\alpha = \sup_{x \in V} \bigl(\mu\bigl(\{\eta\colon \eta(x) = 1\}\bigr) - \mu\bigl(\{\eta\colon \eta(x) = -1\}\bigr)\bigr). \tag{3.3}$$

Now we have (see e.g. [6] or [7]) for any $x \in V$ and $t \geq 0$

$$h(x) := \mu\bigl(\{\eta\colon \eta(x) = 1\}\bigr) - \mu\bigl(\{\eta\colon \eta(x) = -1\}\bigr) = E^x\bigl[h\bigl(X(t)\bigr)\operatorname{sgn}\bigl(X^t\bigr)\bigr]. \tag{3.4}$$

Fix $\varepsilon > 0$ with $\varepsilon \ll 1$ and let $x \in \{y\colon h(y) > \alpha - \varepsilon\}$. For $0 \leq t \leq T$, where T is fixed, let

$$M_t = E\bigl[\eta_T(x) = 1 | \mathcal{G}_t(T)\bigr] - E\bigl[\eta_T(x) = -1 | \mathcal{G}_t(T)\bigr] = h\bigl(X^{x,T}(t)\bigr)\operatorname{sgn}\bigl(\bigl(X^{x,T}\bigr)^t\bigr). \tag{3.5}$$

(Here $\mathcal{G}_t(T) =$ Harris system on interval $[T - t, T]$ and the process is run in equilibrium μ.) Note that $|M_t| \le \alpha$ for all $0 \le t \le T$. Let $\sigma = \inf\{t \ge 0\colon |M_t| \le \alpha - 10\varepsilon\}$ then by optional sampling theorem we have

$$\alpha - \varepsilon < h(x) = E[M_{\sigma\wedge T}] \le (\alpha - 10\varepsilon)P(\sigma \le T) + \alpha P(\sigma > T), \tag{3.6}$$

from which we deduce $P(\sigma \le T) \le 1/10$. Now this (and the arbitrariness of T) implies that if W is the component of $\{y\colon |h(y)| \ge \alpha - 10\varepsilon\}$ containing x, then $P^x(T_{W^c} = \infty) \ge 9/10$. We now show that, provided ε is sufficiently small, W has no unsatisfied cycles. The idea of the proof is that if a site x has close to the maximum value for function h, then as each neighbour has a reasonable chance of being hit by a random walk starting at x, each neighbour, y, must also be extreme and in a way that is in conformity with $s(x, y)$. That is $h(y)$ will have close to the minimum value only if $s(x, y) = -1$.

Suppose for W as above, $x_0, x_1, \dots, x_k$ forms an unsatisfied cycle in W. The point is that for all $i \in \{0, \dots, k\}$

$$h(x_i) = \sum_{y\sim x_i} \frac{h(y)s(x_i, y)}{d(x_i)}, \tag{3.7}$$

thus

$$h(x_i) = \frac{h(x_{i+1})}{M} s(x_i, x_{i+1}) + R_i\left(\frac{M-1}{M}\right), \tag{3.8}$$

where $|R_i| \le \alpha$. From which we have for $h(x_i) > 0$

$$(\alpha - 10\varepsilon) \le \frac{h(x_{i+1})s(x_i, x_{i+1})}{M} + \frac{M-1}{M}\alpha. \tag{3.9}$$

That is $h(x_{i+1})s(x_i, x_{i+1}) \ge \alpha - 10M\varepsilon > 0$ if ε is sufficiently small. Similarly if $h(x_i) < 0$, then $h(x_{i+1})s(x_i, x_{i+1}) < -(\alpha - 10M\varepsilon) < 0$ for ε sufficiently small. This gives a contradiction. □

4. The integer lattice in three dimensions

In this section we consider the signed voter model on $\mathbb{Z}^3$ with simple random walk motion. We address the question of whether the existence of a single equilibrium implies that the simple random walk must a.s. traverse infinitely many unsatisfied cycles. Given the possibility of adapting the example of the following section to three dimensions we interpret the random walk "traversing infinitely many unsatisfied cycles" to mean (recall the definition given in the Introduction (Definition 1.7)): there exist $r_i, t_i \uparrow \infty$ with $r_i < t_i$ for all $i \ge 1$ so that $X(r_i) = X(t_i)$ for all $i \ge 1$ and the path

$$\bigl(X(r)\colon r_i \le r \le t_i\bigr) := X^{r_i,t_i} \quad \text{is odd.} \tag{4.1}$$

We do not require that the path X^{r_i,t_i} visits each site in the range exactly once, with the exception of $X(r_i) = X(t_i)$.

Our approach uses the following simple properties of simple random walks in $\mathbb{Z}^d$ found in e.g. Lawler [5].

(A) There exists $k_{4.1} \in (0, \infty)$ so that for each integer n and for a random walk $(X(t)\colon t \ge 0)$ starting at $X(0) = 0$ and any $x \in \partial B(0, n)$,

$$\frac{1}{k_{4.1}n^{d-1}} \le P\bigl(X(T_{\partial B(0,n)}) = x\bigr) \le \frac{k_{4.1}}{n^{d-1}} \tag{4.2}$$

(see [5], Lemma 1.7.4).

(B) Harnack principle: for all $\alpha < 1$ there exists $k_{4.2} = k_{4.2}(\alpha) < \infty$ so that

$$\frac{1}{k_{4.2}} \le \frac{P^z(X(T_{\partial B(0,n)}) = x)}{P^0(X(T_{\partial B(0,n)}) = x)} \le k_{4.2} \tag{4.3}$$

uniformly over $z \in B(0, \alpha n)$, $x \in \partial B(0, n)$ and n (see [5], Theorem 1.7.6).

In this and the following section we will employ the following notation: $C_n = \partial B(0, 2^n)$ and $B_n = B(0, 2^n)$. The picture to be conveyed by the above two results is that, essentially, the points $X(T_{C_n}), n \geq 1$ are uniformly distributed on C_n and that they are close to being independent. Consider the quantity

$$H(z) = \sum_{n=1}^{\infty} \sum_{\substack{x \in C_n \\ y \in C_{n+1}}} P\big(X(T_{C_n}) = x | X(0) = z\big) P\big(X(T_{C_{n+1}}) = y | X(0) = x\big) N_n^{x,y}, \tag{4.4}$$

where for all $x \in C_n$ and $y \in C_{n+1}$

$$\begin{aligned} N_n^{x,y} = \min\big\{ & P^x\big(\text{path } X^{T_{C_{n+1}}} \text{ is even} | X(T_{C_{n+1}}) = y\big), \\ & P^x\big(\text{path } X^{T_{C_{n+1}}} \text{ is odd} | X(T_{C_{n+1}}) = y\big)\big\}. \end{aligned} \tag{4.5}$$

Then, by (4.2) and (4.3), the following are clear:

(i) $H(\cdot) \equiv \infty$ or $H(z) < \infty \ \forall z$;
(ii) $H(z) < \infty$ if and only if $I < \infty$ with

$$I = \sum_{n=1}^{\infty} \sum_{\substack{x \in C_n \\ y \in C_{n+1}}} \frac{1}{2^{4n+2}} N_n^{x,y}. \tag{4.6}$$

Furthermore:

Lemma 4.1. *$I = \infty$ implies that a.s. $\forall z$,*

$$P^z\big(\textit{path } X^{T_{C_n}} \textit{ is even} | X(T_{C_n})\big) \to 1/2 \tag{4.7}$$

and the signed voter model is ergodic.

The lemma follows from the following elementary 0–1 law whose proof (which rests on (4.2) and (4.3)) is left to the reader.

Lemma 4.2. *Consider sequences $\{n_i\}$ and $\{m_i\}$ satisfying $n_i + 1 < m_i < n_{i+1} - 1$ and positive uniformly bounded random variables V_i measureable with respect to $\sigma\{X^{T_{C_{n_i}}, T_{C_{m_i}}}\}$. Then $\sum_i V_i < \infty$ if and only if $\sum_i E[V_i] < \infty$. Equivalently if and only if*

$$\sum_i \frac{1}{2^{2n_i}} \frac{1}{2^{2m_i}} \sum_{x \in C_{n_i}} \sum_{y \in C_{m_i}} E^{x,y}[V_i] < \infty,$$

where $E^{x,y}$ denotes the expectation for a random walk starting at x and hitting C_{m_i} at y.

Proof of Lemma 4.1. We first note that the condition $I = \infty$ implies that for some $j \in \{0, 1, \dots, 5\}$,

$$\sum_{n=j \bmod 6} \sum_{x \in C_n} \sum_{y \in C_{n+1}} \frac{1}{2^{4n+2}} N_n^{x,y} = \infty. \tag{4.8}$$

Without loss of generality we suppose that this holds for $j = 0$. Consider the quantity

$$\prod_{i=1}^{N} \big(1 - N_{6i}^{X_{T_{C_{6i}}}, X_{T_{C_{6i+1}}}}\big). \tag{4.9}$$

We have via our hypothesis and Lemma 4.2 that a.s. this tends to zero as N tends to infinity. Hence

$$\prod_{i=1}^{N}\bigl(1-N_i^{X_{T_{C_i}},X_{T_{C_{i+1}}}}\bigr) \tag{4.10}$$

tends to zero as N tends to infinity. But

$$\begin{aligned}
&P\bigl(X^{T_{C_N}} \text{ is odd}|X(T_{C_i})\, i=1,2,\dots,N\bigr)-\frac{1}{2}\\
&\quad=\left(P\bigl(X^{T_{C_{N-1}}} \text{ is odd}|X(T_{C_i})\, i=1,2,\dots,N-1\bigr)-\frac{1}{2}\right)\\
&\qquad\times P^{X_{T_{C_{N-1}}},X_{T_{C_N}}}\bigl(X^{X_{T_{C_{N-1}}},X_{T_{C_N}}} \text{ is even}\bigr)\\
&\qquad+\left(P\bigl(X^{T_{C_{N-1}}} \text{ is even}|X(T_{C_i})\, i=1,2,\dots,N-1\bigr)-\frac{1}{2}\right)\\
&\qquad\times P^{X_{T_{C_{N-1}}},X_{T_{C_N}}}\bigl(X^{X_{T_{C_{N-1}}},X_{T_{C_N}}} \text{ is odd}\bigr).
\end{aligned} \tag{4.11}$$

Thus

$$\left|P\bigl(X^{T_{C_N}} \text{ is odd}|X(T_{C_i})\, i=1,2,\dots,N\bigr)-\frac{1}{2}\right|\le\prod_{i=1}^{N}\bigl(1-N_i^{X_{T_{C_{i-1}}},X_{T_{C_i}}}\bigr) \tag{4.12}$$

and we are done. □

Theorem 1.9 will follow from the two following results:

Proposition 4.3. *If $I=\infty$ then a.s. the random walk traverses infinitely many unsatisfied cycles.*

Proposition 4.4. *If $I<\infty$ then a.s. the signed voter model has multiple equilibria.*

Given Proposition 1.8, it is immediate that Proposition 4.3 implies that we have ergodicity when $I=\infty$.

4.1. *Proof of Proposition 4.3*

If $I=\infty$ then again as in the proof of Lemma 4.2, we may assume without loss of generality that

$$\sum_{n=0 \bmod 6}\ \sum_{\substack{x\in C_n\\ y\in C_{n+1}}}\frac{1}{2^{4n+2}}N_n^{x,y}=\infty. \tag{4.13}$$

Lemma 4.2 ensures that a.s.

$$\sum_{n=0 \bmod 6}N_n^{X(T_{C_n}),X(T_{C_{n+1}})}=\infty \tag{4.14}$$

for random walk $(X(r)\colon r\ge 0)$.

Definition 4.5. *For positive integer n and subset $A\subset\mathbb{Z}^3$, let function $H_n(A)$ denote the probability that a simple random walk $(X'(m))_{m\ge 0}$ starting at point $(2^n,0,0)$ hits set A before hitting C_{n+2}.*

The fixing of $(2^n,0,0)$ as the initial point is somewhat arbitrary but is not so important if the set A in question is of distance of order 2^n from $(2^n,0,0)$. In the following we will be interested in $H_n(A)$ for A random, indeed we will take A to be a segment of the path of a random walk X. We must emphasize that the random walk X' invoked in the definition of $H_n(A)$ is always taken to be independent of any randomness producing random set A including random walk X.

It is well know that in three dimensional lattice space, the paths of two independent random walks starting at points in a ball of radius R centred at the origin and killed on leaving the ball of radius $2R$, will, with positive probability not depending on R, meet. The following is simply a concretization of this.

Lemma 4.6. *There exists a constant $k_{4.3} > 0$ so that for all n, uniformly over "initial point" $x \in C_{n-2}$ of a random walk X,*

$$P^x\big(H_n\big(X^{T_{C_{n-1}}}\big) > k_{4.3}\big) > k_{4.3}. \tag{4.15}$$

Proof. It is only necessary to show this for n large. Our approach is simply to use the two moment argument twice. We use the limiting identities of Lawler [5], Proposition 1.5.9, to show that there exist $0 < k_1 < k_2 < \infty$ not depending on n, such that for $|x - y| \in (2^{n-2}/10, 2^{n-2}/5)$,

$$2^{n-2} P^x(T_y < T_{C_{n+2}}) \in (k_1, k_2), \qquad 2^{n-2} P^x(T_y < T_{C_{n-1}}) \in (k_1, k_2). \tag{4.16}$$

Consider A, the set of points y satisfying:

(i) $|x - y| \in (2^{n-2}/10, 2^{n-2}/5)$,
(ii) $y \in X^{T_{C_{n-1}}}$.

We associate two variables with A:

- $|A| = \sum_y I_A(y)$. The above bounds on $P^x(T_y < T_{C_{n-1}})$ immediately give that $E[|A|] \geq k_1' 2^{2n}$ for some strictly positive k_1' and for all n,
- $S_A = \sum_{z,y\in A} 2^n/(1 + |z - y|)$. Equally we have from the bounds of Lawler [5], Proposition 1.5.9 that $E[S_A] \leq k_2' 2^{4n}$ for some universal, finite k_2' and all n.

From this and the usual two moment argument (see e.g. [2]), we have the existence of universal, nontrivial k so that

$$P\left(A > \frac{1}{k} 2^{2n}, S_A < k 2^{4n}\right) \geq 1/2. \tag{4.17}$$

We now condition on process $X(\cdot)$ and in particular on set A and thus on random variables $|A|$ and S_A. Let us define random variable for X' a random walk starting at $(2^n, 0, 0)$ and hitting C_{n+2} at time T'

$$W = \sum_{y\in A} I_{\{\exists r \leq T':\ X'(r)=y\}}. \tag{4.18}$$

Using the inequalities from Lawler, we have that

$$E[W|X] \geq k_1''|A|/2^n, \qquad E\big[W^2|X\big] \leq k_2'' S_A/2^{2n}, \tag{4.19}$$

for universal nontrivial k_i''. Profiting once more from the two moment argument we have that

$$P\big(W > E[W|X]/2|X\big) \geq \frac{1}{4}\frac{(k_1'')^2}{k_2''}\frac{|A|^2}{S_A}. \tag{4.20}$$

In particular we have, by (4.17), with probability at least $\frac{1}{2}$,

$$P(W > 0|X) \geq \frac{1}{4}\frac{(k_1'')^2}{k_2''}\frac{1}{k^3}. \tag{4.21}$$

□

Define the events

$$D_n = \{X^{T_{C_{n+1}},T_{C_{n+2}}} \cap X^{T_{C_{n-2}},T_{C_{n-1}}} \neq \varnothing\} \tag{4.22}$$

and

$$\mathcal{H}_n = \{H_n(X^{T_{C_{n-2}},T_{C_{n-1}}}) > k_{4.3}\}. \tag{4.23}$$

Corollary 4.7. *There exists some universal $k_{4.4}$ not depending on n so that on event $\mathcal{H}_n$,*

$$P(D_n|\mathcal{F}_{T_{C_{n+1}}}) > k_{4.4}, \tag{4.24}$$

where $\{\mathcal{F}_t\}_{t\geq 0}$ is the natural filtration associated to the random walk X.

Proof. To prove this it suffices to see that, because of the invariance principle for the random walk X', uniformly on $y \in C_{n+1}$ for some nontrivial k not depending on n,

$$P^y\big(X' \text{ hits } B\big((2^n,0,0),2^{n-2}/40\big)\big) > k > 0. \tag{4.25}$$

Then combining Lemma 4.6 and (4.3) and using the Markov property, we get on $\mathcal{H}_n$,

$$P^{X(T_{C_{n+1}})}(D_n|\mathcal{F}_{T_{C_{n+1}}}) > \frac{4kk_{4.3}}{k_{4.2}(1/2)}. \tag{4.26}$$

□

Now Corollary 4.7 and Lemma 4.2 ensure that

$$\sum_{n=0 \bmod 6} I_{D_n} N_n^{X(T_{C_n}),X(T_{C_{n+1}})} = \infty \quad \text{a.s.} \tag{4.27}$$

under the conditions given. We now introduce the discrete filtration

$$\mathcal{J}_n = \mathcal{F}_{T_{C_{6n}}} \quad \text{and} \quad \mathcal{G}_{n+1} = \sigma\big(\mathcal{J}_n, X^{T_{C_{6n+1}},T_{C_{6n+2}}}\big) \tag{4.28}$$

and consider the filtration

$$\mathcal{J}_1, \mathcal{G}_2, \ldots, \mathcal{J}_n, \mathcal{G}_{n+1}, \ldots. \tag{4.29}$$

Note that on $D_{6n} \in \mathcal{G}_{n+1}$ we can define measurably $t_n \in [T_{C_{6n-2}}, T_{C_{6n-1}}]$, $r_n \in [T_{C_{6n+1}}, T_{C_{6n+2}}]$ so that $X(t_n) = X(r_n)$. Note that on D_{6n}

$$P\big(X^{t_n,r_n} \text{ is odd}|\mathcal{G}_{n+1}\big) \geq N_{6n}^{X(T_{C_{6n}}),X(T_{C_{6n+1}})}. \tag{4.30}$$

So by (4.30) and Lévy 0–1 law (see e.g. [2]) we have a.s. infinitely many unsatisfied cycles. This complete the proof of of Proposition 4.3.

4.2. *Proof of Proposition 4.4*

One way to show Proposition 4.4 would be to find a suitable subset W of $\mathbb{Z}^3$ satisfying the conditions of Proposition 1.2 of [4]. In fact, since the graph $\mathbb{Z}^3$, the degree is bounded (by 6), Proposition 1.6 applies and so proving Proposition 4.4 shows the existence of a suitable subset W of $\mathbb{Z}^3$. Actually we do not explicitly find such a W but our approach comes down to finding a way of assigning signs to "most" points in $\mathbb{Z}^3$ which will asymptotically (as the points becomes large) be respected by a random walk. As we will shortly see, we begin by talking of the sign "between" certain points in $\mathbb{Z}^3$ and the greater part of the work consists of "sewing" these signs together to give a sign function which "asymptotically" works.

As before, we denote by C_n the external boundary of $B(0, 2^n)$, the Euclidean ball centered at the origin of radius 2^n. For $u \in B_{n+1} = B(0, 2^{n+1})$, $z \in C_{n+1}$, the law $P^{u,z,n}$ is the law of the random walk started at u conditioned to exit $B(0, 2^{n+1})$ at z. Define

$$\operatorname{sgn}(u, z, n) = \begin{cases} 1, & \text{if } P^{u,z,n}\bigl(X^{T_{C_{n+1}}} \text{ is even}\bigr) > 3/4, \\ -1, & \text{if } P^{u,z,n}\bigl(X^{T_{C_{n+1}}} \text{ is odd}\bigr) > 3/4, \\ 0, & \text{otherwise.} \end{cases} \tag{4.31}$$

In this definition the choice of $\frac{3}{4}$ is not very important beyond it being strictly larger than $\frac{1}{2}$. If $\operatorname{sgn}(u, z, n) = 1$, we say z is positive for or with respect to u. Now for $\alpha < 1$ such that $1 - \alpha \ll 1$ (and certainly $\le 1/4$) and for $x \in C_n$ define two complementary sets:

$$S(x, n) = \bigl\{v \in C_{n+1}\colon N_n^{x,v} < 1 - \alpha\bigr\} \quad \text{and} \quad U(x, n) = \bigl\{v \in C_{n+1}\colon N_n^{x,v} \ge 1 - \alpha\bigr\}. \tag{4.32}$$

We first have from the Borel–Cantelli lemma and (4.2) and (4.3):

Lemma 4.8. *For a random walk $(X(t)\colon t \ge 0)$ on $\mathbb{Z}^3$ and for any $\alpha < 1$, under condition $I < \infty$ a.s.*

$$X(T_{C_{n+1}}) \in S\bigl(X(T_{C_n}), n\bigr) \tag{4.33}$$

for all n sufficiently large.

Lemma 4.9. *For any $x \in C_n$, $w \in C_{n+1}$ with $w \in S(x, n)$, the $P^{x,w,n}$ probability that the path $X(\cdot)$ satisfies for all $t \le T_{C_{n+1}}$*

$$\operatorname{sgn}\bigl(X^t\bigr) \operatorname{sgn}\bigl(X(t), w, n\bigr) = \operatorname{sgn}(x, w, n) \tag{4.34}$$

is at least $1 - 4(1 - \alpha)$.

Proof. Suppose without loss of generality that $\operatorname{sgn}(x, w, n) = 1$. Then the $P^{x,w,n}$ probability of event

$$A = \bigl\{\text{path } X^{T_{C_{n+1}}} \text{ is odd}\bigr\} \tag{4.35}$$

is less than $1 - \alpha$. Consider, with respect to the natural filtration, the càdlàg martingale $M_t = E(1_A|\mathcal{F}_t)$. By Doob's optional sampling theorem (see e.g. [2]) the probability that this value ever gets above $1/4$ is bounded above by $4(1 - \alpha)$. This gives the result. □

The following lemma follows in the same way as (4.24) with some elementary conditioning arguments.

Lemma 4.10. *There exists a universal $k_{4.5} > 0$ so that for any $x, y \in C_n$ and $w, v \in C_{n+1}$, if X is a random walk under law $P^{x,w,r}$ and X' an independent random walk under law $P^{y,v,n}$,*

$$P\bigl(P^{y,v,n}\bigl(\bigl(X'\bigr)^{\tau_n',\sigma_n'} \cap X^{\tau_n,\sigma_n} \ne \varnothing | \bigl(X(t)\colon t \ge 0\bigr)\bigr) > k_{4.5}\bigr) > 2k_{4.5}, \tag{4.36}$$

where

$$\tau_n = \inf\bigl\{t\colon \bigl|X(t)\bigr| \ge 3 \times 2^{n-1}\bigr\}, \qquad \sigma_n = \inf\bigl\{t > \tau_n\colon \bigl|X(t)\bigr| \ge 7 \times 2^{n-2} \text{ or } \le 5 \times 2^{n-2}\bigr\} \tag{4.37}$$

and τ_n', σ_n' are analogous stopping times for X'.

Remark. *We assume that n be large enough for the relevant sets to be nonempty.*

Definition 4.11. *We say $\{x, y, v, w\}$ with $x, y \in C_n$ and $v, w \in C_{n+1}$ are* 1-compatible *if*

$$\operatorname{sgn}(x, v, n) \operatorname{sgn}(x, w, n) \operatorname{sgn}(y, v, n) \operatorname{sgn}(y, w, n) = 1. \tag{4.38}$$

Lemma 4.12. *There exists some universal constant $k_{4.6} > 0$ so that (provided $1 - \alpha$ has been fixed sufficiently small) for all n large and $\{x, y, v, w\}$ not 1-compatible with $x, y \in C_n$ and $v, w \in C_{n+1}$ if, for each $u \in \{v, w\}$, $N_n^{x,u} < 1 - \alpha$ then for at least one $u \in \{v, w\}$ $N_n^{y,u} > k_{4.5}/k_{4.6}$, where $k_{4.5}$ is the constant defined in Lemma 4.10.*

Proof. We suppose without loss of generality that v and w are both positive with respect to x but that while w is positive with respect to y, v is not. By our assumption on the largeness of α we have by Lemmas 4.9 and 4.10, that there exists a nearest neighbour path $\gamma(\cdot)$ from x to w on which for all times s,

$$\operatorname{sgn}(\gamma^s)\operatorname{sgn}(\gamma(s), w, r) = 1 \tag{4.39}$$

and for which τ_n', σ_n' are defined for path γ as in Lemma 4.10, we have

$$P^{z,u,n}\big(X^{\tau_n,\sigma_n} \text{ hits } \gamma^{\tau_n',\sigma_n'}\big) > k_{4.5} \tag{4.40}$$

for each $(z, u) \in \{(x, v), (x, w), (y, v), (y, w)\}$. We consider two processes, $(Z^x(t)\colon t \geq 0)$ and $(Z^y(t)\colon t \geq 0)$ starting respectively in x and y, running until C_{n+1} is hit and so that for $u \in \{x, y\}$ the process $(Z^u(t)\colon t \geq 0)$ has law $1/2P^{u,v,n} + 1/2P^{u,w,n}$. Then we define the measures $\mu^u(z)$ by

$$\mu^u\big(\{z\}\big) = P\big(Z^u\big(T_\gamma^u\big) = z, T_\gamma^u < \sigma_n^u\big) \quad \forall u \in \{x, y\}, z \in \gamma^{\tau_n',\sigma_n'}, \tag{4.41}$$

where T_γ^u is the first hitting time for path $\gamma^{\tau_n',\sigma_n'}$ for process Z^u and σ_n^u is the hitting time for this process analagous to the stopping time of Lemma 4.10. From facts (4.2)–(4.3), we have that there exists universal K so that

$$\frac{1}{K}\mu^y\big(\{z\}\big) \leq \mu^x\big(\{z\}\big) \leq K\mu^y\big(\{z\}\big) \quad \forall z \in \gamma^{\tau_n',\sigma_n'} \tag{4.42}$$

and for either u,

$$P\big(Z^u\big(T_{C_{n+1}}^u\big) = v | Z^u\big(T_\gamma^u\big) = z, T_\gamma^u < \sigma_n^u\big) \in (1/K, 1 - 1/K) \quad \forall z \in \gamma^{\tau_n',\sigma_n'}. \tag{4.43}$$

We classify the points in $\gamma^{\tau_n',\sigma_n'}$ into five sets:

$$\begin{aligned} A_{++} &= \big\{z\colon\ P^{z,w,n}\big(X^{T_{C_{n+1}}} \text{ is even}\big) \geq 3/4, P^{z,v,n}\big(X^{T_{C_{n+1}}} \text{ is even}\big) \geq 3/4\big\} \\ &= \big\{z\colon\ \operatorname{sgn}(z, w, n) = 1, \operatorname{sgn}(z, v, n) = 1\big\}, \\ A_{+-} &= \big\{z\colon\ \operatorname{sgn}(z, w, n) = 1, \operatorname{sgn}(z, v, n) = -1\big\}, \\ A_{-+} &= \big\{z\colon\ \operatorname{sgn}(z, w, n) = -1, \operatorname{sgn}(z, v, n) = 1\big\}, \\ A_{--} &= \big\{z\colon\ \operatorname{sgn}(z, w, n) = -1, \operatorname{sgn}(z, v, n) = -1\big\}, \\ D &= \big\{z\colon\ \operatorname{sgn}(z, w, n)\operatorname{sgn}(z, v, n) = 0\big\}. \end{aligned} \tag{4.44}$$

We have by the optimal stopping time reasoning of proof of Lemma 4.9 and our assumptions on x and v that

$$\mu^x(D) < 4(1 - \alpha). \tag{4.45}$$

By (4.42), this implies that

$$\mu^y(D) < 4(1 - \alpha)K. \tag{4.46}$$

That is D is a small set for the conditioned random walks started at x or y alike. We claim that

$$\mu^x(A_{+-}) < \varepsilon = 4K(1 - \alpha). \tag{4.47}$$

To see this suppose the contrary, then we must have either

$$P\big(Z^x(T_\gamma) \in A_{+-}, T_\gamma < \sigma_n^u, \big(Z^x\big)^{T_\gamma} \text{ is even}\big) \ge \varepsilon/2 \tag{4.48}$$

or

$$P\big(Z^x(T_\gamma) \in A_{+-}, < T_\gamma < \sigma_n^u, \big(Z^x\big)^{T_\gamma} \text{ is odd}\big) \ge \varepsilon/2. \tag{4.49}$$

In the former case we have via (4.43) that

$$P^{x,v,n}\big(X(T_\gamma) \in A_{+-}, T_\gamma < \sigma_n, X^{T_\gamma} \text{ is even}\big) \ge \varepsilon/(2K) \tag{4.50}$$

for σ_n the analagous stopping time to σ_n^u for X and so by the Markov property

$$P^{x,v,n}\big(X^{T_{C_{n+1}}} \text{ is odd}\big) > \varepsilon/(4K) = 1-\alpha \tag{4.51}$$

(provided $1-\alpha$ has been fixed sufficiently small) which contradicts our hypothesis on x and v. Similarly in the other case we are forced to conclude that $P^{x,v,n}(X^{T_{C_{n+1}}} \text{ is odd}) > 1-\alpha$. Arguing similarly with set A_{+-} replaced by A_{-+}, we are able to deduce that

$$\mu^x(A_{-+}) < \varepsilon. \tag{4.52}$$

From (4.42) we can conclude that $\mu^y(A_{-+} \cup A_{+-}) < 2K\varepsilon$. We thus have that either $\mu^y(A_{++}) \ge (k_{4.5} - 2K\varepsilon)/2$ or $\mu^y(A_{--}) \ge (k_{4.5} - 2K\varepsilon)/2$. Without loss of generality we suppose the former. Note that assuming, as we may, that our assumption on the closeness of α to 1 was suitably stringent, we have that $(k_{4.5} - 2K\varepsilon)/2 > k_{4.5}/3$. Then for identical reasons, either

$$P\big(Z^y\big(T_\gamma^y\big) \in A_{++}, T_\gamma < \sigma_n^y, \big(Z^x\big)^{T_\gamma^x} \text{ is even}\big) \ge (k_{4.5} - 2K\varepsilon)/4 > k_{4.5}/6 \tag{4.53}$$

or

$$P\big(Z^y\big(T_\gamma^y\big) \in A_{++}, T_\gamma^y < \sigma_n^y, \big(Z^x\big)^{T_\gamma^x} \text{ is odd}\big) \ge (k_{4.5} - 2K\varepsilon)/4 > k_{4.5}/6. \tag{4.54}$$

Again without loss of generality we suppose the former. In this case we have

$$P^{y,w}\big(X\big(T_\gamma^y\big) \in A_{++}, T_\gamma^y < \sigma_n^y, X^{T_\gamma^y} \text{ is even}\big) \ge k_{4.5}/(12K) \tag{4.55}$$

and so

$$P^{y,w}\big(X^{T_\gamma^y} \text{ is even}\big) \ge k_{4.5}/(16K). \tag{4.56}$$

□

In the following we assume that α has been fixed so large that $240k_{4.6}^2(1-\alpha) < k_{4.5}$. The condition that $I < \infty$, (4.2), (4.3) and Borel–Cantelli immediately yield:

Corollary 4.13. *Let $(X(t)\colon t \ge 0)$ and $(Y(t)\colon t \ge 0)$ be two independent random walks. For any $\alpha < 1$ a.s., for n sufficiently large*

$$N_n^{X(T_{C_n}),X(T_{C_{n+1}})} < 1-\alpha \quad \textit{and} \quad N_n^{Y(T_{C_n}),Y(T_{C_{n+1}})} < 1-\alpha \tag{4.57}$$

and so a.s. for n sufficiently large $\{X(T_{C_n}^X), Y(T_{C_n}^Y), X(T_{C_{n+1}}^X), Y(T_{C_{n+1}}^Y)\}$ are 1*-compatible.*

Definition 4.14. *We say $\{x, y, z, w\}$ with $x \in C_{n-1}$, $y, z \in C_n$ and $w \in C_{n+1}$ are* 2-compatible *if*

$$\operatorname{sgn}(x, y, n-1)\operatorname{sgn}(x, z, n-1)\operatorname{sgn}(y, w, n)\operatorname{sgn}(z, w, n) = 1. \tag{4.58}$$

Definition 4.15. *For all $x \in C_{n-1}$ and $y \in C_{n+1}$*

$$N_{n+}^{x,y} = \min\big\{P^x\big(\textit{path } X^{T_{C_{n+1}}} \textit{ is even}|X(T_{C_{n+1}}) = y\big),\\ P^x\big(\textit{path } X^{T_{C_{n+1}}} \textit{ is odd}|X(T_{C_{n+1}}) = y\big)\big\}. \tag{4.59}$$

Given this definition we define analagously:

Definition 4.16. $I_+ = \sum_n^\infty \sum_{\substack{x\in C_{n-1}\\ y \in C_{n+1}}} \frac{1}{2^{4n+2}} N_{n+}^{x,y}$.

Lemma 4.17. *The condition $I < \infty$ implies that $I_+ < \infty$.*

Proof. For n sufficiently large to ensure that C_{n-5} is nontrivial and has zero intersection with C_{n-4}, we fix $x \in C_{n-1}$ and $y \in C_n$. By the invariance principle for random walks, we know that there exists universal $k > 0$ so that for any choice of n and $x \in C_{n-1}$, $P^x(T_{C_{n-4}} < T_{C_n}) > k$.

Furthermore by (4.3), we have $\forall x, x' \in C_{n-1}$,

$$\frac{P^x(T_{C_n} = y)}{P^{x'}(T_{C_n} = y)} \in \left(\frac{1}{(k_{4.2}(1/2))^2}, \left(k_{4.2}\left(\frac{1}{2}\right)\right)^2\right). \tag{4.60}$$

Thus from the Markov property we obtain

$$N_n^{x,y} \geq \frac{k}{k_{4.2}(1/2)^4(k_{4.1})^2} \sum_{\substack{u\in C_{n-3}\\ u\in C_{n-1}}} N_{n-2+}^{u,v} \frac{1}{2^{2(n-3)}} \frac{1}{2^{2(n-1)}} \tag{4.61}$$

from which the desired conclusion is immediate. □

This lemma is necessary for:

Lemma 4.18. *Under the hypothesis that $I < \infty$, for any two independent random walks $(X(t){:}\ t \geq 0)$ and $(Y(t){:}\ t \geq 0)$ with probability one $\{X(T_{C_{n-1}}), X(T_{C_n}), Y(T_{C_n}), X(T_{C_{n+1}})\}$ are 2-compatible for all n large.*

Here, as before T_{C_n} as an argument denotes the stopping time appropriate to the process.

Proof. Given $x \in C_{n-1}$, $y \in C_{n+1}$, we define for $N_{n+}^{x,y} < \frac{1}{100}$, the set $H(x, y, +)$ to be the subset of elements $w \in C_n$ such that

$$\mathrm{sgn}(x, w, n-1)\,\mathrm{sgn}(w, y, n) = \mathrm{sgn}(x, y, n+),$$

where $\mathrm{sgn}(x, y, n+)$ is defined analagously to (4.31). We take $H(x, y, -)$ to be simply the complement in C_n of $H(x, y, +)$, $C_n \backslash H(x, y, +)$. We note that given the condition $I < \infty$ (and thus $I_+ < \infty$), by Corollary 4.13 for a random walk $X(\cdot)$ a.s. we have eventually $N_{n+}^{X(T_{C_{n-1}}),X(T_{C_{n-1}})} < \frac{1}{100}$. So in this case we need not define $H(x, y, -)$ for the remaining cases where $N_{n+}^{X(T_{C_{n-1}}),X(T_{C_{n-1}})} \geq \frac{1}{100}$.

If $w \in H(x, y, -)$, then either $\mathrm{sgn}(x, w, n-1)\,\mathrm{sgn}(w, y, n) = 0$, in which case

$$P^x\big(\mathrm{sgn}\big(X^{T_{C_{n+1}}}\big) \neq \mathrm{sgn}(x, y, n+)|X(T_{C_n}) = w, X(T_{C_{n+1}}) = y\big) \geq \frac{1}{4}$$

or $\mathrm{sgn}(x, w, n-1)\,\mathrm{sgn}(w, y, n) = -\,\mathrm{sgn}(x, y, n+)$, in which case

$$P^x\big(\mathrm{sgn}\big(X^{T_{C_{n+1}}}\big) \neq \mathrm{sgn}(x, y, n+)|X(T_{C_n}) = w, X(T_{C_{n+1}}) = y\big) \geq \left(\frac{3}{4}\right)^2.$$

In either we have the lower bound $\frac{1}{4}$, so

$$N_{n+}^{x,y} \geq \frac{1}{4} \sum_{w \in H(x,y,-)} P^x\big(X(T_{C_n}) = w | X(T_{C_{n+1}}) = y\big).$$

An application of (4.2) gives that for random walks $X(\cdot)$ and $Y(\cdot)$, $\sum_n P(A(n)^c)$ and $\sum_n P(A'(n)^c)$ are both majorized by a multiple of I for events

$$\begin{aligned} A(n) = \big\{&\operatorname{sgn}\big(X(T_{C_{n-1}}), X(T_{C_n}), n-1\big) \operatorname{sgn}\big(X(T_{C_n}), X(T_{C_{n+1}}), n\big) \\ &= \operatorname{sgn}\big(X(T_{C_{n-1}}), X(T_{C_{n+1}}), n+\big) \cdot \big\} \\ &\cap \big\{N_+^{X(T_{C_{n-1}}), X(T_{C_{n+1}})} < 1/100\big\} \end{aligned}$$

and

$$\begin{aligned} A'(n) = \big\{&\operatorname{sgn}\big(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\big) \operatorname{sgn}\big(Y(T_{C_n}), X(T_{C_{n+1}}), n\big) \\ &= \operatorname{sgn}\big(X(T_{C_{n-1}}), X(T_{C_{n+1}}), n+\big) \cdot \big\} \\ &\cap \big\{N_+^{X(T_{C_{n-1}}), X(T_{C_{n+1}})} < 1/100\big\}, \end{aligned}$$

so our result follows from the first Borel–Cantelli lemma. □

Given Lemmas 4.12 and 4.18, (4.3) and Fubini's theorem, we can find a path realization $X = X(\cdot, \omega)$ so that for a.s. every independant random walk path Y the conclusion of the lemmas hold a.s. (here we use the notation $X(\cdot, \omega)$ to underline the fact that we consider a fixed path of the random walk $(X(s)\colon s \geq 0)$ at time t). Let us pick and fix a "good" path X so that for a.s. path Y we have that for n large, $\{X(T_{C_{n-1}}), X(T_{C_n}), Y(T_{C_n}), X(T_{C_{n+1}})\}$ are 2-compatible and $\{X(T_{C_n}), Y(T_{C_n}), Y(T_{C_{n+1}}), X(T_{C_{n+1}})\}$ are 1-compatible and also such that for any $\alpha < 1$ eventually $N_n^{X(T_{C_n}), X(T_{C_{n+1}})} < 1 - \alpha$. We will use this path to designate sites in C_n as positive or negative (at least for n large): we first choose r_0 so that for all $n \geq r_0$, $\operatorname{sgn}(X(T_{C_{n-1}}), X(T_{C_n}), n)$ is nonzero. We say that $X_{C_{r_0}}$ is a positive site, i.e., $\operatorname{sgn}(X_{C_{r_0}}) = 1$. Subsequently, we define recursively

$$\operatorname{sgn}\big(X(T_{C_n})\big) = \operatorname{sgn}\big(X(T_{C_{n-1}}), X(T_{C_n}), n\big) \operatorname{sgn}\big(X(T_{C_{n-1}})\big). \tag{4.62}$$

Given this assignation we now assign signs to arbitrary $y \in C_n$ by

$$\operatorname{sgn}(y) = \operatorname{sgn}\big(X(T_{C_{n-1}}), y, n\big) \operatorname{sgn}\big(X(T_{C_{n-1}})\big). \tag{4.63}$$

It may help the reader to note that the objective in assigning a sign to sites is really to divide up the sites into two classes so that this choice is asymptotically respected by random walks. Thus initial arbitrariness in choosing the sign is not problematic.

Lemma 4.19. *With probability one there exists a finite random r_0 so that either*

$$\forall n \geq n_0, \quad \operatorname{sgn}\big(Y^{T_{C_n}}\big) \operatorname{sgn}\big(Y(T_{C_n})\big) = 1 \tag{4.64}$$

or

$$\forall n \geq n_0, \quad \operatorname{sgn}\big(Y^{T_{C_n}}\big) \operatorname{sgn}\big(Y(T_{C_n})\big) = -1. \tag{4.65}$$

Proof. We first observe that for n large enough all the terms

$$N_{n-1}^{X(T_{C_{n-1}}), X(T_{C_n})} \quad \text{and} \quad N_{n-1}^{X(T_{C_{n-1}}), Y(T_{C_n})} \tag{4.66}$$

are less than $1-\alpha \ll 1$. Furthermore by Lemmas 4.12 and 4.18 for n large, 1- and 2-compatibility give

$$\begin{aligned}&\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_n}), X(T_{C_{n+1}}), n\bigr)\operatorname{sgn}\bigl(Y(T_{C_n}), X(T_{C_{n+1}}), n\bigr)=1\end{aligned}\tag{4.67}$$

and

$$\begin{aligned}&\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\operatorname{sgn}\bigl(Y(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_n}), X(T_{C_{n+1}}), n\bigr)\operatorname{sgn}\bigl(Y(T_{C_n}), X(T_{C_{n+1}}), n\bigr)=1.\end{aligned}\tag{4.68}$$

Therefore their product

$$\begin{aligned}&\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\operatorname{sgn}\bigl(Y(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)=1.\end{aligned}\tag{4.69}$$

Using our assumptions, we have

$$\begin{aligned}\operatorname{sgn}\bigl(Y(T_{C_{n+1}})\bigr)&=\operatorname{sgn}\bigl(X(T_{C_n})\bigr)\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\\&=\operatorname{sgn}\bigl(X(T_{C_{n-1}})\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\\&=\operatorname{sgn}\bigl(X(T_{C_{n-1}})\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr)^2\\&=\operatorname{sgn}\bigl(X(T_{C_{n-1}})\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr)\\&=\operatorname{sgn}\bigl(Y(T_{C_n})\bigr)\operatorname{sgn}\bigl(X(T_{C_{n-1}}), X(T_{C_n}), n-1\bigr)\operatorname{sgn}\bigl(X(T_{C_n}), Y(T_{C_{n+1}}), n\bigr)\\&\quad\times\operatorname{sgn}\bigl(X(T_{C_{n-1}}), Y(T_{C_n}), n-1\bigr).\end{aligned}\tag{4.70}$$

Therefore, combining (4.69) and (4.70), we get for all n large

$$\operatorname{sgn}\bigl(Y(T_{C_{n+1}})\bigr)=\operatorname{sgn}\bigl(Y(T_{C_n})\bigr)\operatorname{sgn}\bigl(Y(T_{C_n}), Y(T_{C_{n+1}}), n\bigr).\tag{4.71}$$

Now, conditional upon $Y(T_{C_n}), Y(T_{C_{n+1}})$, the probability that

$$\operatorname{sgn}\bigl(Y^{T_{C_n}}\bigr)\operatorname{sgn}\bigl(Y(T_{C_n})\bigr)\neq\operatorname{sgn}\bigl(Y^{T_{C_n}}\bigr)\operatorname{sgn}\bigl(Y(T_{C_{n+1}})\bigr)\tag{4.72}$$

is simply $N_n^{Y(T_{C_n}),Y(T_{C_{n+1}})}$. Hence the result follows by Lemma 4.8. □

Thus we have defined the sign for points in $\bigcup C_n$ in a way that is a.s. asymptotically respected by random walks. Define the function

$$\begin{aligned}h(x)&=P^x\bigl(\text{for all } n \text{ large } \operatorname{sgn}\bigl(Y^{T_{C_n}}\bigr)\operatorname{sgn}\bigl(Y(T_{C_n})\bigr)=1\bigr)\\&\quad-P^x\bigl(\text{for all } n \text{ large } \operatorname{sgn}\bigl(Y^{T_{C_n}}\bigr)\operatorname{sgn}\bigl(Y(T_{C_n})\bigr)=-1\bigr)\end{aligned}\tag{4.73}$$

and the product measures $\mu_\pm$ by $\mu_+(\{\eta\colon \eta(x)=1\})=(1+h(x))/2$, $\mu_-(\{\eta\colon \eta(x)=1\})=(1-h(x))/2$. We have by Lévy's 0–1 law and the Markov property that with probability 1 $\lim_{t\to\infty}|h(Y(t))|$ exists and equals 1. So there exists $x\in\mathbb{Z}^3$ for which $|h(x)|$ is arbitrarily close to 1 and in particular for which $h(x)\neq 0$. But in this case we have for all t by duality and the Markov property that

$$P_t\mu_\pm\bigl(\{\eta\colon\eta(x)=1\}\bigr)=\bigl(1\pm h(x)\bigr)/2.\tag{4.74}$$

Then using a similar argument as in [4] (Section 7, Proof of Proposition 1.2), this implies non-uniqueness of equilibria. This complete the proof of Proposition 4.4.

5. The integer lattice in dimensions four and higher

We show Theorem 1.10 in this section. For notational convenience we give the proof for four dimensions but the proof is easily seen to hold in all higher dimensions.

To begin, we introduce the basic building block for our counterexample. Consider the following choice of sign function s: $s(e) = 1$ for all edges in $\mathbb{Z}^4$ except those of the form $\{(R, y, z, w), (R+1, y, z, w)\}$. For this choice if random walk $X(\cdot)$ begins at $X(0)$ with the first coordinate $X_1(0) \le R$, then for any $t \ge 0$, we have

$$\operatorname{sgn}(X^t) = 1 \tag{5.1}$$

if and only if $X_1(t) \le R$. Similar considerations enable us to see that there are no unsatisfied cycles for s. We now regard a modification. Define $s(e) = 1$ for all edges in $\mathbb{Z}^4$ except those of the form $\{(R, y, z, w), (R+1, y, z, w)\}$ for $|y|, |z|, |w| \le R$. The two above properties have disappeared. There now exist unsatisfied cycles and one cannot identify $\operatorname{sgn}(X^t)$ merely from $X(t)$. Indeed, if say, $R \gg 1$ and $X(0) \in [-R/2, R/2]^4$, we have that $P(\operatorname{sgn}(X^{R^2}) = 1 | X(R^2))$ is bounded away from 0 and 1 uniformly over R and $X(R^2) \in [-2R, 2R]^4$.

Thus uncertainty is introduced into the sign of the random walk path. Our purpose is to choose a sequence of integer scales R_n so that R_{n+1}/R_n tends to infinity sufficiently rapidly. Then we will give sign $+1$ to all edges except those of the form $(x, x+e_1)$ for $e_1 = (1, 0, 0, 0)$ and $x \in \{R_n\} \times [-R_n, R_n]^3$ for some n. The basic idea is that if the R_n are sufficiently separated, then "infinite uncertainty" is introduced into the sign of a random walk path but that this can be done in such a way that a.s. only a finite number of unsatisfied cycles are traversed.

In the first part of this section we argue from invariance principle considerations that if $R_{n+1}/R_n \ge 2(n+1)^2/K_{n+2}$ for constants $(K_n)_{n\ge 2}$ small then almost surely a random walk does not traverse infinitely many unsatisfied cycles. Then we argue that we have ergodicty.

We now undertake the first part of the program. This involves a more precise discussion of the preceding sketch. Consider a Brownian motion in 4 dimensions, $(B(t)\colon t \ge 0)$. Let V_r^i, $r \ge 0$ and $i = 3, 4$ be the cube $[-r, r]^i$ (r will typically but not always be an integer) and given a process $(Y(t)\colon t \ge 0)$, $T(n) = \inf\{t\colon Y(t) \text{ leaves } V_n^4\}$. It follows from the a.s. nonexistence of double points for 4-dim Brownian motion (see e.g. [3]) (and the fact that two dimensional subspaces are polar) that, with probability 1, there does not exist $t_1, t_2 \le T(n)$ so that $t_1 < t_2$ and

$$\left.\begin{matrix} (B(t_1), B(t_2)) \\ \text{or} \\ (B(t_2), B(t_1)) \end{matrix}\right\} \in (\{1\} \times V_1^3) \times (\partial V_1^4 \setminus (\{1\} \times V_1^3)) \tag{5.2}$$

and

$$B(t_3) = B(t_4) \quad \text{for } t_3 \le t_1 \le t_2 \le t_4. \tag{5.3}$$

Bearing in mind the transcience of the Brownian motion and that B does not hit the intersections of the faces of ∂V_1^4, there exists $K_n > 0$ (which we can and will take to be less than $1/4$ and decreasing in n) so that with probability strictly greater than $1 - 1/2n^2$

$$K_n \le \inf |B(t_3) - B(t_4)| \tag{5.4}$$

for t_1, t_2, t_3, t_4 as above.

Now (possibly reducing K_n) we can also have that this is so for Brownian motion starting in $V_{K_n}^4$ uniformly over the initial point.

Now we profit from the invariance principle, see e.g. [2], to conclude that for all R sufficiently large, with probability at least $1 - 1/2n^2$ for a random walk starting in $V_{RK_n}^4$ for $t_3 \le t_1 \le t_2 \le t_4 \le T(Rn)$,

$$\left.\begin{matrix} (X(t_1), X(t_2)) \\ \text{or} \\ (X(t_2), X(t_1)) \end{matrix}\right\} \in (\{R\} \times V_R^3) \times (\partial V_R^4 \setminus (\{R\} \times V_R^3)), \tag{5.5}$$

$$K_n R \le \inf |X(t_3) - X(t_4)|. \tag{5.6}$$

In particular the probability that there exist t_1, t_2 as above, so that X^{t_1} intersects $X^{t_2,T(nR)}$ is less than $1/2n^2$.

Let us inductively define R_n as follows: R_1 is such that for a 4-dimensional random walk (starting at 0) X, the probability that there exist $t_1 < t_2 \le T(2R_1)$ (recall that $T(2R_1)$ is the leaving time of $V^4_{2R_1}$) so that

(i)

$$\left.\begin{matrix}\big(X(t_1), X(t_2)\big)\\ \text{or}\\ \big(X(t_2), X(t_1)\big)\end{matrix}\right\} \in \big(\{R_1\} \times V^3_{R_1}\big) \times \big(\partial V^4_{R_1} \setminus \big(\{R_1\} \times V^3_{R_1}\big)\big);$$

(ii) there exists $t_3 \le t_1 \le t_2 \le t_4$ so that $t_4 \le T(2R_1)$ and $|X(t_3) - X(t_4)| \le K_2 R_1$

is less than $1/8$. Such an R_1 exists by the preceding. Now, given R_{j-1} take $R_j \ge 2j^2 R_{j-1}/K_{j+1}$, so that for any random walk $X(\cdot)$ starting in $V^4_{K_{j+1}R_j}$, the probability that there exists $t_1 < t_2 < T(2(j+1)R_j)$ so that

(i)

$$\left.\begin{matrix}\big(X(t_2), X(t_1)\big)\\ \text{or}\\ \big(X(t_1), X(t_2)\big)\end{matrix}\right\} \in \big(\{R_j\} \times V^3_{R_j}\big) \times \big(\partial V^4_{R_j} \setminus \big(\{R_j\} \times V^3_{R_j}\big)\big);$$

and

(ii) there exists $t_3 \le t_1 \le t_2 \le t_4$ so that $t_4 \le T(2(j+1)R_j)$ and $|X(t_3) - X(t_4)| \le K_{j+1} R_j$

is bounded by $1/4(j+1)^2$. Now take the configuration of ± 1 bonds on $\mathbb{Z}^4$ as follows: all bonds are +1 except bonds

$$(x, x + e_1) \quad \text{for } x \in \{R_j\} \times V^3_{R_j} \text{ for some } j. \tag{5.7}$$

Define $A(n)$ to be the event that after stopping time $T(2(n+1)R_n)$, the random walk returns to $V^4_{R_n}$ and $B(n)$ to be the event

$$\exists t > T\big((n+1)^2 R_n\big): \quad X(t) \in V^4_{2(n+1)R_n}. \tag{5.8}$$

Elementary potential theory and Borel–Cantelli immediately yield:

Lemma 5.1. *With probability one there exists $j_0 < \infty$ such that for all $j \ge j_0$, events $A(j)$ and $B(j)$ do not occur.*

Define event $D(n)$ to be be that there exists $T(R_n) \le t_3 < t_1 < t_2 \le t_4 \le T(2(n+1)R_n)$ so that

$$\left.\begin{matrix}\big(X(t_2), X(t_1)\big)\\ \text{or}\\ \big(X(t_1), X(t_2)\big)\end{matrix}\right\} \in \big(\{R_j\} \times V^3_{R_j}\big) \times \big(\partial V^4_{R_j} \setminus \big(\{R_j\} \times V^3_{R_j}\big)\big);$$

and $X(t_3) = X(t_4)$. By our choice of the $\{R_j\}_{j\ge 1}$ and again Borel–Cantelli, we have

Lemma 5.2. *With probability one there exists $j_0 < \infty$ such that for all $j \ge j_0$, events $D(j)$ does not occur.*

This yields:

Proposition 5.3. *With the above choice of sign, the random walk a.s. traverses through only finitely many unsatisfied cycles.*

Proof. By the two preceding lemmas we have a.s. there exists j_0 so that $A(j)$, $B(j)$ and $D(j)$ do not happen for $j \ge j_0$. We will show, by contradiction, that if $s < t$ are both greater than $T(R_{j_0})$, then $X^{s,t}$ cannot be an unsatisfied cycle.

Let j be the largest integer such that $s \geq T(R_j)$. We either have $s \in [T(R_j), T(2(j+1)R_j)), [T(2(j+1)R_j), T((j+1)^2 R_j)]$ or $[T((j+1)^2 R_j), T(R_{j+1}))$. We consider each of these possibilities in turn. For the first case since by assumption $j \geq j_0$, (and so $A(j)$ does not occur) $X^{s,t}$ being an unsatisfied cycle must imply that $D(j)$ has occurred, a contradiction.

For the second case, given that $B(j)$ does not happen, all edges in $X^{s,T(R_{j+1})}$ are positive and so we must have $t \geq T(R_{j+1})$. It is immediate that $X^{s,t}$ being an unsatisfied cycle must entail forbidden event $B(j+1)$.

Finally, the third case immediately implies forbidden event $D(j+1)$. The proposition is proven. □

We now begin the second step, to show that with the above choice of sign function the signed voter model is ergodic.

It is easily seen that the Harnack principle (4.3) yields:

Lemma 5.4. *Let $\pi_r(w,\cdot)$ be the harmonic measure for a random walk starting at w, at the boundary of the ball $B(0,r)$. Then*

$$\lim_{m\to\infty} \limsup_{r\to\infty} \sup_{\substack{x,y\in B(0,r)\\ z\in\partial B(0,mr)}} \frac{\pi_{mr}(x,z)}{\pi_{mr}(y,z)} = 1. \tag{5.9}$$

Let $(R_n)_{n\geq 1}$ be a sequence satisfying the above conditions and consider $C_n = \partial B(0,n)$ (note change of notation!) and $S_n = nR_n$.

Lemma 5.5. *There exists $k_{5.1} \in (0,1/2)$ so that for all n large enough and all $x \in C_{S_n}$, $y \in C_{S_{n+1}}$*

$$P^x\big(X^{T_{C_{S_{n+1}}}} \text{ is odd} | X(T_{C_{S_{n+1}}}) = y\big) > k_{5.1} \tag{5.10}$$

and

$$P^x\big(X^{T_{C_{S_{n+1}}}} \text{ is even} | X(T_{C_{S_{n+1}}}) = y\big) > k_{5.1}. \tag{5.11}$$

Proof. By the invariance principle we have that if n is large, uniformly for each $x \in C_{S_n}$ the probability of leaving the box $V^4_{R_{n+1}}$ for the first time through $\{R_{n+1}\} \times V^3_{R_{n+1}/2}$, then passing to $\partial V^4_{2R_{n+1}}$ without leaving $[2R_{n+1}/3, +\infty) \times V^3_{2R_{n+1}/3}$ is greater than $k_2 \in (0,1)$ for some universal k_2. From here, uniformly over the random hitting point of $\partial V^4_{2R_{n+1}}$, the conditional probability of hitting $\partial B(0,(n+1)R_{n+1}/2)$ before hitting $V^4_{R_{n+1}}$ will be greater than $k_3 \in (0,1)$ provided n is large. This follows from the invariance principle and the classical hitting estimates of Lawler (see (4.2) and (4.3)). From (4.2) we have the existence of a constant k_4 so that for all $w \in \partial B(0,(n+1)R_{n+1}/2)$

$$\frac{1}{k_4 S^3_{n+1}} \leq P^w\big(X(T_{C_{S_{n+1}}}) = z\big) \leq \frac{k_4}{S^3_{n+1}}. \tag{5.12}$$

So using

$$\begin{aligned} &P^w\big(X(T_{C_{S_{n+1}}}) = z, T_{V^4_{R_{n+1}}} > T_{C_{S_{n+1}}}\big) \\ &\quad \geq P^w\big(X(T_{C_{S_{n+1}}}) = z\big) - \sup_{v\in V^4_{R_{n+1}}} P^v\big(X(T_{C_{S_{n+1}}}) = z\big) P^w(T_{C_{S_{n+1}}} > T_{V^4_{R_{n+1}}}), \end{aligned} \tag{5.13}$$

we obtain

$$P^w\big(X(T_{C_{S_{n+1}}}) = z, T_{V^4_{R_{n+1}}} > T_{C_{S_{n+1}}}\big) \geq \frac{1}{2k_4 S^3_{n+1}}, \tag{5.14}$$

for n large, uniformly over $w \in \partial B(0,(n+1)R_{n+1}/2)$. Hence for all $x \in C_{S_n}$ and $y \in C_{S_{n+1}}$

$$P^x\big(X^{T_{C_{S_{n+1}}}} \text{ is odd}, X(T_{C_{S_{n+1}}}) = y\big) \ge \frac{k_2 k_3}{2k_4 S_{n+1}^3}. \tag{5.15}$$

This, given

$$P^x\big(X(T_{C_{S_{n+1}}}) = y\big) \le \frac{k_4}{S_{n+1}^3} \tag{5.16}$$

gives

$$P^x\big(X^{T_{C_{S_{n+1}}}} \text{ is odd} | X(T_{C_{S_{n+1}}}) = y\big) \ge \frac{k_2 k_3}{2k_4^2}. \tag{5.17}$$

We argue similarly for the second part. □

The following is a simple consequence of Lemmas 5.4 and 5.5.

Corollary 5.6. *For m fixed, there exists N_0 so that for all $n \in \mathbb{Z}_+$ and all $x \in B(0,S_m)$, $y \in C_{S_{N_0+n}}$*

$$\big|2P^x\big(X^{T_{C_{S_{N_0+n}}}} \textit{ is odd} | X(T_{C_{S_{N_0+n}}}) = y\big) - 1\big| \le 2(1-k_{5.1})^n. \tag{5.18}$$

We are now ready to complete the proof of Theorem 1.10.

Proposition 5.7. *For R_n and sign functions as previously described, the signed voter model is ergodic.*

Proof. By Theorem 2.1, it is enough to show that for any η_0 and any $x \in V$, $P(\eta_t(x) = 1) \to \frac{1}{2}$ as t tends to infinity. We fix $x \in B(0,S_m)$ and choose N_0 of the preceding corollary. We also fix positive integer r which we will ultimately let tend to infinity. From the preceding corollary we have that for t sufficiently large and for all $y \in C_{N+r}$,

$$\big|2P^x\big(X^{T_{C_{S_{N+r}}}} \text{ is odd} | X(T_{C_{S_{N+r}}}) = y, T_{C_{S_{N+r}}} < t\big) - 1\big| \le 4(1-k_{5.1})^n \tag{5.19}$$

and so $|P^x(X^{T_{C_{S_{N+r}}}} \text{ is odd} | X(T_{C_{S_{N+r}}}) = y, T_{C_{S_{N+r}}} < t) - \frac{1}{2}|$ is less than or equal to $2(1-k_{5.1})^n$.

We now introduce the sigma-field $\mathbb{G}$, the algebra generated by the random element $X(T_{C_{S_{N+r}}})$ and the event $\{T_{C_{S_{N+r}}} < t\}$. Of course

$$P\big(\eta_t(x) = 1\big) = E\big(P\big(\eta_t(x) = 1 | \mathbb{G}\big)\big). \tag{5.20}$$

But for t so large that (5.19) holds, we have that on event $\{T_{C_{S_{N+r}}} < t\}$,

$$\left|P\big(\eta_t(x) = 1 | \mathbb{G}\big) - \frac{1}{2}\right| \le 2(1-k_{5.1})^r.$$

Thus integrating (5.20), we have

$$\left|P\big(\eta_t(x) = 1\big) - \frac{1}{2}\right| \le 2(1-k_{5.1})^r + P(T_{C_{S_{N+r}}} \ge t).$$

Our result now follows by, first, letting t tend to infinity to obtain $\limsup_{t\to\infty} |P(\eta_t(x) = 1) - \frac{1}{2}| \le 2(1-k_{5.1})^r$ and then letting r become large. □

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