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On harmonic functions of symmetric Lévy processes[1]

Ante Mimica

Fakultät für Mathematik, Universität Bielefeld, Universitätsstraße 25, Bielefeld, NRW, 33615, Germany. E-mail: amimica@math.uni-bielefeld.de

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Abstract. We consider some classes of Lévy processes for which the estimate of Krylov and Safonov (as in (*Potential Anal.* **17** (2002) 375–388)) fails and thus it is not possible to use the standard iteration technique to obtain a-priori Hölder continuity estimates of harmonic functions. Despite the failure of this method, we obtain some a-priori regularity estimates of harmonic functions for these processes. Moreover, we extend results from (*Probab. Theory Related Fields* **135** (2006) 547–575) and obtain asymptotic behavior of the Green function and the Lévy density for a large class of subordinate Brownian motions, where the Laplace exponent of the corresponding subordinator is a slowly varying function.

Résumé. On considère des classes de processus de Lévy pour lesquels les estimations de Krylov et Safonov (comme dans (*Potential Anal.* **17** (2002) 375–388)) ne sont pas verifiées donc il n'est pas possible d'utiliser la technique standard d'itération pour obtenir *a priori* des estimations de continuité Hölder pour des fonctions harmoniques. Bien qu'il soit impossible d'appliquer cette méthode, on obtient des estimations *a priori* de régularité de fonctions harmoniques pour ces processus. De plus, on étend les résultats de (*Probab. Theory Related Fields* **135** (2006) 547–575) et on obtient les comportements asymptotiques de la fonction de Green et de la densité de Lévy pour une grande classe de mouvements browniens subordonnés, où l'exposant de Laplace du subordinateur correspondant est une fonction à variation lente.

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1. Introduction

Recently there has been much interest studying the continuity properties of harmonic functions with the respect to various non-local operators. An example of such an operator $\mathcal{L}$ is of the form

$$(\mathcal{L}f)(x) = \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+h) - f(x) - \langle \nabla f(x), h\rangle \mathbb{1}_{|h|\le 1}\big) n(x,h)\,\mathrm{d}h \tag{1.1}$$

for $f \in C^2(\mathbb{R}^d)$ bounded and with bounded first and second derivatives. Here $n : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0,\infty)$ is a measurable function satisfying

$$c_1|h|^{-d-\alpha} \le n(x,h) \le c_2|h|^{-d-\alpha}$$

for some constants $c_1, c_2 > 0$ and $\alpha \in [0,2]$.

It is known that for $\alpha \in (0,2)$ Hölder regularity estimates hold for $\mathcal{L}$-harmonic functions (see [3] for a probabilistic and [19] for an analytic approach).

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For $\alpha = 0$, techniques developed so far are not applicable. One of our aims is to investigate this case using a probabilistic approach.

In many cases the operator of the form (1.1) can be understood as the infinitesimal generator of a Markov jump process. The kernel $n(x,h)$ can be thought of as the measure of intensity of jumps of the process.

Let us describe the stochastic process we are considering. Let $S = (S_t\colon\ t \geq 0)$ be a subordinator such that its Laplace exponent ϕ defined by $\phi(\lambda) = -\log(\mathbb{E}e^{-\lambda S_1})$ satisfies

$$\lim_{\lambda\to+\infty} \frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\alpha/2-1} \quad \text{for any } x > 0 \tag{1.2}$$

for some $\alpha \in [0,2]$. Let $B = (B_t, \mathbb{P}_x)$ be an independent Brownian motion in $\mathbb{R}^d$ and define a new process $X = (X_t, \mathbb{P}_x)$ in $\mathbb{R}^d$ by $X_t = B(S_t)$. It is called a subordinate Brownian motion.

Example 1. *Let S be a subordinator with the Laplace exponent ϕ satisfying*

$$\lim_{\lambda\to+\infty} \frac{\phi(\lambda)}{\lambda^{\alpha/2}\ell(\lambda)} = 1$$

with $\alpha \in (0,2)$ and $\ell\colon (0,\infty) \to (0,\infty)$ that varies slowly at infinity (i.e. for any $x > 0$, $\frac{\ell(\lambda x)}{\ell(\lambda)} \to 1$ as $\lambda \to +\infty$).

Then (1.2) holds (we just use Theorem 1.7.2 in [5] together with the fact that ϕ' is decreasing and $\phi(\lambda) = \int_0^\lambda \phi'(t)\,dt$) and

$$n(x,h) = n(h) \asymp |h|^{-d-\alpha}\ell\big(|h|^{-2}\big), \quad |h| \to 0+,$$

which means that $\frac{n(h)}{|h|^{-d-\alpha}\ell(|h|^{-2})}$ stays between two positive constants as $|h| \to 0+$.

Choosing $\ell \equiv 1$ we see that the rotationally invariant α-stable process (whose infinitesimal generator is the fractional Laplacian $\mathcal{L} = -(-\Delta)^{\alpha/2}$) is included in this class. Other choices of ℓ allow us to consider processes which are not invariant under time–space scaling.

The processes described in Example 1 with $\ell \equiv 1$ belong to the class of Lévy stable or, more generally, stable-like Markov jump processes. The potential theory of such processes is well investigated (see [2,3,6–8,10,11,13,16,20,22]). For example, it is known that harmonic functions of such processes satisfy Hölder regularity estimates and the scale invariant Harnack inequality holds. Also, two-sided heat kernel estimates are obtained for these processes.

Not much is known about harmonic functions in the case when the corresponding subordinators belong to "boundary" cases, i.e. $\alpha \in \{0,2\}$ (see [14,15,18]). For the class of geometric stable processes only the non-scale invariant Harnack inequality was proved and the heat kernel may blow at the diagonal (see [18]).

The following two examples belong to these "boundary" cases and are covered by our approach.

Example 2. *Let $\phi(\lambda) = \log(1+\lambda)$. The corresponding process X is known as the variance gamma process. In (1.2) we have $\alpha = 0$. Moreover, it will be proved (see Theorem 4.1) that*

$$n(x,h) = n(h) \asymp |h|^{-d}, \quad |h| \to 0+. \tag{1.3}$$

This example can be generalized in various ways. For example, we can take $k \in \mathbb{N}$ and consider $\phi_k = \underbrace{\phi \circ \cdots \circ \phi}_{k \text{ times}}$. Then (see Theorem 4.1 or Section 7.1)

$$n_k(x,h) = n_k(h) \asymp |h|^{-d}\left(\underbrace{\log\cdots\log}_{k-1 \text{ times}} \frac{1}{|h|} \cdot \cdots \cdot \log\log\frac{1}{|h|} \cdot \log\frac{1}{|h|}\right)^{-1}, \quad |h| \to 0+.$$

Another generalization is to consider $\phi(\lambda) = \log(1+\lambda^{\beta/2})$ for some $\beta \in (0,2]$. The process X is known as the β-geometric stable process and the behavior of n is given also by (1.3).

Example 3. *Let $\phi(\lambda) = \frac{\lambda}{\log(1+\sqrt{\lambda})}$. Then in (1.2) we have $\alpha = 2$ and (see Theorem 4.1)*

$$n(x,h) = n(h) \asymp |h|^{-d-2}\left(\log \frac{1}{|h|}\right)^{-2}, \quad |h| \to 0+.$$

This behavior shows that small jumps of this process have higher intensity than small jumps of any stable process.

A measurable bounded function $f:\mathbb{R}^d \to \mathbb{R}$ is said to be harmonic in an open set $D \subset \mathbb{R}^d$ if for any relatively compact open set $B \subset \overline{B} \subset D$

$$f(x) = \mathbb{E}_x f(X_{\tau_B}) \quad \text{for any } x \in B,$$

where $\tau_B = \inf\{t > 0\colon X_t \notin B\}$.

The main theorem is the following regularity result, which covers cases $\alpha \in [0,1)$. The novelty of this result rests on the case $\alpha = 0$. By $B_r(x_0)$ we denote the open ball with center $x_0 \in \mathbb{R}^d$ and radius $r > 0$.

Theorem 1.1. *Let S be a subordinator such that its Lévy and potential measures have decreasing densities. Assume that the Laplace exponent of S satisfies (1.2) with $\alpha \in [0,1)$. Let X be the corresponding subordinate Brownian motion.*

There is a constant $c > 0$ such that for any $r \in (0, \frac{1}{4})$ and any bounded function $f:\mathbb{R}^d \to \mathbb{R}$ which is harmonic in $B_{4r}(0)$,

$$\left|f(x) - f(y)\right| \le c\|f\|_\infty \frac{\phi(r^{-2})}{\phi(|x-y|^{-2})} \quad \textit{for all } x, y \in B_{r/4}(0).$$

Applying Theorem 1.1 to Example 1, we obtain expected Hölder regularity estimates. Within this example the result is new when the scaling is lost, e.g.

$$\phi(\lambda) = \lambda^{\alpha/2}\left[\log(1+\lambda)\right]^{1-\alpha/2}.$$

The situation is more interesting in Example 2, e.g. for the geometric stable process. For this process we obtain logarithmic regularity estimates:

$$\left|f(x) - f(y)\right| \le c\|f\|_\infty \log\left(r^{-1}\right) \frac{1}{\log(|x-y|^{-1})}.$$

It is still unknown whether Hölder regularity estimates hold for harmonic functions of this process (or, generally, of the processes belonging to the case $\alpha = 0$).

Let us explain why known analytic and probabilistic techniques do not work in the case $\alpha = 0$. The main idea in the proof of the a priori Hölder estimates of harmonic functions relies on the estimate of Krylov and Safonov.

In the probabilistic setting this estimate can be formulated as follows. There is a constant $c > 0$ such that for every closed subset $A \subset B_r(0)$ and $x \in B_{r/2}(0)$

$$\mathbb{P}_x(T_A < \tau_{B(0,r)}) \ge c \frac{|A|}{|B_r(0)|}, \tag{1.4}$$

where $T_A = \tau_{A^c}$ is the first hitting time of A and $|A|$ denotes the Lebesgue measure of the set A.

Performing a computation similar to the one in the proof of Proposition 3.4 in [3] (see also Lemma 3.4 in [20]) we deduce

$$\mathbb{P}_x(T_A < \tau_{B_r(0)}) \ge c \frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})} \frac{|A|}{|B_r(0)|}.$$

If $\alpha \in (0,2)$, it can be seen that $\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})} \asymp 1$ as $r \to 0+$. This gives estimate of the form (1.4) and thus the standard Moser's iteration procedure for obtaining a-priori Hölder regularity estimates of harmonic functions can be applied (see the proof of Theorem 4.1 in [3] for a probabilistic version).

The situation is quite different for $\alpha = 0$. To find a counterexample we will use the following result.

Proposition 1.2. *Let S be a subordinator such that its Lévy and potential measures have decreasing densities and whose Laplace exponent satisfies (1.2) with $\alpha \in [0,1)$. Let X be the corresponding subordinate Brownian motion.*

There is a constant $c > 0$ such that for every $r \in (0, \frac{1}{2})$ and $x \in B_{r/4}(0)$

$$\mathbb{P}_x\big(X_{\tau_{B_{r/2}(0)}} \in B_r(0) \setminus B_{r/2}(0)\big) \le c\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}.$$

For $\alpha = 0$ it will follow that $\lim_{r\to 0+} \frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})} = 0$ (see (2.10)). Therefore in this case (1.4) does not hold, since

$$\lim_{r\to 0+} \mathbb{P}_0(T_{B_r(0)\setminus B_{r/2}(0)} < \tau_{B_r(0)}) \le \lim_{r\to 0+} \mathbb{P}_0\big(X_{\tau_{B_{r/2}(0)}} \in B_r(0) \setminus B_{r/2}(0)\big) = 0.$$

Considering the process X in the setting of metric measure spaces (as in [8] or [1]) some new feature appears. Theorem 4.1 shows that the jumping kernel of the process X is of the form $n(x,h) = j(|h|)$ with

$$j(r) \asymp \frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})} \cdot \frac{\mathbb{E}_0\tau_{B_r(0)}}{|B_r(0)|}, \quad r \to 0+.$$

In the case $\alpha = 0$ the term $\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}$ becomes significant. This has not yet been treated within this framework.

The above discussion shows that the question of the continuity of harmonic functions becomes interesting even in the case when the kernel $n(x,h)$ is space homogeneous, or in other words, in the case of a Lévy process. There is no known technique that covers this situation in the case of a more general jump process.

Our technique relies on asymptotic properties of the underlying subordinator. The potential density can be analyzed using the de Haan theory of slow variation (see [5]).

On the other hand, there is no known Tauberian theorem that can be applied to obtain asymptotic behavior of the Lévy density μ of the subordinator. For this purpose we perform asymptotic inversion of the Laplace transform (see Proposition 3.2) to get

$$\mu(t) \asymp t^{-2}\phi'\big(t^{-2}\big), \quad t \to 0+.$$

These techniques allow us to extend results from [18] to much wider class of subordinators whose Laplace exponents are logarithmic or, more generally, slowly varying functions.

Although we do not obtain regularity estimates of harmonic functions for cases when $\alpha \in [1,2]$, it is possible to say something about the behavior of the jumping kernel and the Green function. In this sense, the case $\alpha = 2$ is also new. For example the Green function of the process corresponding to the Example 3 above has the following behavior:

$$G(x,y) \asymp |x-y|^{2-d}\log\big(|x-y|^{-1}\big), \quad |x-y| \to 0+.$$

We may say that such process X is "between" any stable process and Brownian motion.

Let us briefly comment the technique we are using to prove the regularity result. In Section 2 it will be seen that any bounded function f which is harmonic in $B_{2r}(0)$ can be represented as

$$f(x) = \int_{\overline{B_r(0)}^c} K_{B_r(0)}(x,z) f(z)\,\mathrm{d}z, \quad x \in B_r(0),$$

where $K_{B_r(0)}(x,z)$ is the Poisson kernel of the ball $B_r(0)$.

The following estimate of differences of Poisson kernel is the key to the proof of Theorem 1.1:

$$\left|K_{B_r(0)}(x_1,z)-K_{B_r(0)}(x_2,z)\right| \le \begin{cases} c|z|^{-d}\frac{\phi((|z|-r)^{-2})}{\phi(|x-y|^{-2})}, & r<|z|\le 2r, \\ \frac{j(|z|/2)}{\phi(|x-y|^{-2})}, & |z|>2r \end{cases}$$

for $x_1, x_2 \in B_{r/8}(0)$ (see Proposition 5.3).

Similar type estimate has been obtained in [22] for stable Lévy processes using scaling argument and the explicit behavior of the transition density. In our setting there are many cases where the behavior of the transition density is not known and the scaling argument does not work. Our idea is to establish the following Green function difference estimates:

$$\left|G(x_1,y)-G(x_2,y)\right| \le c\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})^2}r^d\left(1\wedge\frac{|x-y|}{r}\right)$$

for all $y\in\mathbb{R}^d$ and $x_1,x_2\notin B_r(y)$ (see Proposition 5.1).

The paper is organized as follows. In Section 2 we introduce all concepts we need throughout the paper. Section 3 is devoted to the study of subordinators. We obtain asymptotic properties of Lévy and potential densities. In Section 4 asymptotic properties of the Green function and Lévy density of the subordinate Brownian motions are obtained. Difference estimates of the Green function and the Poisson kernel are the main subject of Section 5. This type of estimates are the main ingredient in the proof of the regularity result in Section 6. In Section 7 we apply our results to some new examples.

Notation. *For two functions f and g we write $f(r)\sim g(r), r\to 0+$ if $\lim_{r\to 0+}\frac{f(r)}{g(r)}=1$ and $f(r)\asymp g(r), r\to 0+$ if $f(r)/g(r)$ stays between two positive constants as $r\to 0+$. The nth derivative of f (if exists) is denoted by $f^{(n)}$.*

The logarithm with base e is denoted by log and we introduce the following notation for iterated logarithms: $\log_1=\log$ and $\log_{k+1}=\log\circ\log_k$ for $k\in\mathbb{N}$.

We say that $f:\mathbb{R}\to\mathbb{R}$ is increasing if $s\le t$ implies $f(s)\le f(t)$ and analogously for a decreasing function.

The standard Euclidian norm and the standard inner product in $\mathbb{R}^d$ are denoted by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively. By $B_r(x)=\{y\in\mathbb{R}^d\colon |y-x|<r\}$ we denote the open ball centered at x with radius $r>0$. The Gamma function is defined by $\Gamma(\rho)=\int_0^\infty t^{\rho-1}\mathrm{e}^{-t}\,\mathrm{d}t$ for $\rho>0$.

2. Preliminaries

2.1. *Lévy processes and their potential theory*

A stochastic process $X=(X_t\colon t\ge 0)$ with values in $\mathbb{R}^d$ $(d\ge 1)$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ is said to be a Lévy process if it has independent and stationary increments, its trajectories are $\mathbb{P}$-a.s. right continuous with left limits and $\mathbb{P}(X_0=0)=1$.

The characteristic function of X_t is always of the form

$$\mathbb{E}\exp\left\{\mathrm{i}\langle\xi,X_t\rangle\right\}=\exp\left\{-t\Phi(\xi)\right\},$$

where Φ is called the characteristic (or Lévy) exponent of X. It has the following Lévy–Khintchine representation

$$\Phi(\xi)=\mathrm{i}\langle\gamma,\xi\rangle+\frac{1}{2}\langle A\xi,\xi\rangle+\int_{\mathbb{R}^d}\left(1-\mathrm{e}^{\mathrm{i}\langle x,\xi\rangle}+\mathrm{i}\langle x,\xi\rangle\mathbb{1}_{\{|x|\le 1\}}\right)\Pi(\mathrm{d}x).$$

Here $\gamma\in\mathbb{R}^d$, A is a non-negative definite symmetric $d\times d$ real matrix and Π is a measure on $\mathbb{R}^d$, called the Lévy measure of X, satisfying

$$\Pi\left(\{0\}\right)=0 \quad\text{and}\quad \int_{\mathbb{R}^d}\left(1\wedge|x|^2\right)\Pi(\mathrm{d}x)<\infty.$$

The Brownian motion $B = (B_t \colon t \geq 0)$ in $\mathbb{R}^d$ with transition density $p_0(t,x,y) = (4\pi t)^{-d/2} \exp\{-\frac{|x-y|^2}{4t}\}$ is an example of a Lévy process with the characteristic exponent $\Phi(\xi) = |\xi|^2$.

A subordinator is a stochastic process $S = (S_t \colon t \geq 0)$ which is a Lévy process in $\mathbb{R}$ such that $S_t \in [0,\infty)$ for every $t \geq 0$. In this case it is more convenient to consider the Laplace transform of S_t:

$$\mathbb{E} \exp\{-\lambda S_t\} = \exp\{-t\phi(\lambda)\}, \quad \lambda > 0.$$

The function $\phi \colon (0,\infty) \to (0,\infty)$ is called the Laplace exponent of S and it has the following representation

$$\phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} \big(1 - \mathrm{e}^{-\lambda t}\big)\mu(\mathrm{d}t).$$

Here $\gamma \geq 0$ and μ is also called the Lévy measure of S and it satisfies the following integrability condition: $\int_{(0,\infty)} (1 \wedge t)\mu(\mathrm{d}t) < \infty$.

The potential measure of the subordinator S is defined by

$$U(A) = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{S_t \in A\}}\,\mathrm{d}t\right] \quad \text{for a measurable } A \subset [0,\infty).$$

The Laplace transform of U is then

$$\mathcal{L}U(\lambda) := \int_{(0,\infty)} \mathrm{e}^{-\lambda t} U(\mathrm{d}t) = \frac{1}{\phi(\lambda)}. \tag{2.1}$$

Assume that the processes B and S just described are independent. We define a new stochastic process $X = (X_t \colon t \geq 0)$ by $X_t = B(S_t)$ and call it a subordinate Brownian motion. It is a Lévy process with the characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$ and the Lévy measure of the form $\Pi(\mathrm{d}x) = j(|x|)\,\mathrm{d}x$ with

$$j(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left\{-\frac{r^2}{4t}\right\}\mu(\mathrm{d}t). \tag{2.2}$$

If the subordinator S is not a compound Poisson process, then the process X has a transition density and it is given by

$$p(t,x,y) = \int_{[0,\infty)} (4\pi s)^{-d/2} \exp\left\{-\frac{|x-y|^2}{4s}\right\}\mathbb{P}(S_t \in \mathrm{d}s). \tag{2.3}$$

When X is transient, we can define the Green function of X by

$$G(x,y) = \int_{(0,\infty)} p(t,x,y)\,\mathrm{d}t, \quad x,y \in \mathbb{R}^d, x \neq y.$$

The Green function can be considered as the density of the Green measure defined by

$$G(x,A) = \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{X_t \in A\}}\,\mathrm{d}t\right], \quad A \subset \mathbb{R}^d \text{ measurable},$$

since $G(x,A) = \int_A G(x,y)\,\mathrm{d}y$.

Using (2.3) we can rewrite it as $G(x,y) = g(|y-x|)$ with

$$g(r) = \int_{(0,\infty)} (4\pi t)^{-d/2} \exp\left\{-\frac{r^2}{4t}\right\}U(\mathrm{d}t). \tag{2.4}$$

Let $D \subset \mathbb{R}^d$ be a bounded open set. We define the process killed upon exiting D $X^D = (X_t^D \colon t \geq 0)$ by

$$X_t^D = \begin{cases} X_t, & t < \tau_D, \\ \partial, & t \geq \tau_D, \end{cases}$$

where ∂ is an extra point adjoined to D.

Using the strong Markov property we can see that the Green measure of X^D is

$$\begin{aligned} G_D(x,A) &= \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{X_t^D\in A\}}\,\mathrm{d}t\right] \\ &= \mathbb{E}_x\left[\int_0^\infty \mathbb{1}_{\{X_t\in A\}}\,\mathrm{d}t\right] - \mathbb{E}_x\left[\int_{\tau_D}^\infty \mathbb{1}_{\{X_t\in A\}}\,\mathrm{d}t\right] \\ &= G(x,A) - \mathbb{E}_x\big[G(X_{\tau_D},A);\tau_D<\infty\big]. \end{aligned}$$

Thus in the transient case the Green function of X^D can be written as

$$G_D(x,y) = G(x,y) - \mathbb{E}_x\big[G(X_{\tau_D},y);\tau_D<\infty\big], \quad x,y\in D, x\neq y.$$

Since X is, in particular, an isotropic Lévy process, if the subordinator has no drift (thus there is no Brownian component), it follows from Proposition 4.1 in [12] (see also [21]) that

$$\mathbb{P}_x\big(X_{\tau_{B_r(0)}}\in\partial B_r(0)\big)=0, \quad x\in B_r(0)$$

for any $r>0$ and $x\in B_r(0)$. This allows us to use the Ikeda–Watanabe formula (see Theorem 1 in [9]):

$$\mathbb{P}_x(X_{\tau_{B_r(0)}}\in F) = \int_F\int_{B_r(0)} G_{B_r(0)}(x,y)\,j\big(|z-y|\big)\,\mathrm{d}y\,\mathrm{d}z \tag{2.5}$$

for $x\in B_r(0)$ and $F\subset \overline{B_r(0)}^c$.

Defining a function $K_{B_r(0)}: B_r(0)\times\overline{B_r(0)}^c\to[0,\infty)$ by

$$K_{B_r(0)}(x,z) = \int_{B_r(0)} G_{B_r(0)}(x,y)\,j\big(|z-y|\big)\,\mathrm{d}y \tag{2.6}$$

the Ikeda–Watanabe formula (2.5) reads

$$\mathbb{P}_x(X_{\tau_{B_r(0)}}\in F) = \int_F K_{B_r(0)}(x,z)\,\mathrm{d}z. \tag{2.7}$$

The function $K_{B_r(0)}$ will be called the Poisson kernel for the ball $B_r(0)$.

2.2. *Bernstein functions and subordinators*

A function $\phi:(0,\infty)\to(0,\infty)$ is said to be a Bernstein function if $\phi\in C^\infty(0,\infty)$ and $(-1)^n\phi^{(n)}\le 0$ for all $n\in\mathbb{N}$. Every Bernstein ϕ function has the following representation:

$$\phi(\lambda) = \gamma_1+\gamma_2\lambda+\int_{(0,\infty)}\big(1-\mathrm{e}^{-\lambda t}\big)\mu(\mathrm{d}t), \tag{2.8}$$

where $\gamma_1,\gamma_2\ge 0$ and μ is a measure on $(0,\infty)$ satisfying $\int_{(0,\infty)}(1\wedge t)\mu(\mathrm{d}t)<\infty$.

Using the elementary inequality $y\mathrm{e}^{-y}\le 1-\mathrm{e}^{-y}$, $y>0$ we deduce from (2.8) that every Bernstein function ϕ satisfies:

$$\lambda\phi'(\lambda)\le\phi(\lambda) \quad \text{for any } \lambda>0. \tag{2.9}$$

There is a strong connection between subordinators and Bernstein functions. To be more precise, ϕ is a Bernstein function such that $\phi(0+)=0$ (i.e. $\gamma_1=0$ in (2.8)) if and only if it is the Laplace exponent of some subordinator. If $\phi(0+)>0$, ϕ can be understood as the Laplace exponent of a subordinator killed with rate $\phi(0+)$ (see Chapter 3 in [4]).

A Bernstein function ϕ is a complete Bernstein function if the Lévy measure in (2.8) has a completely monotone density, i.e. $\mu(\mathrm{d}t) = \mu(t)\,\mathrm{d}t$, where $\mu : (0, \infty) \to (0, \infty)$, $\mu \in C^\infty(0, \infty)$ and $(-1)^n \mu^{(n)} \geq 0$ for any $n \in \mathbb{N}$.

Let us mention some properties of complete Bernstein function (see [17]). The composition of two complete Bernstein function is a complete Bernstein function and if ϕ is a complete Bernstein function, then $\phi^\star(\lambda) = \frac{\lambda}{\phi(\lambda)}$ is also a complete Bernstein function.

Assume that S is a subordinator with the Laplace exponent ϕ and infinite Lévy measure. Then $\phi^\star$ is a Bernstein function if and only if the potential measure U has a decreasing density u with respect to the Lebesgue measure. Moreover, if ν denotes the Lévy measure of the subordinator with the Laplace exponent $\phi^\star$, then $u(t) = \nu(t, \infty)$ for any $t > 0$.

2.3. *Regular variation*

A function $f : (0, \infty) \to (0, \infty)$ varies regularly (at infinity) with index $\rho \in \mathbb{R}$ if

$$\lim_{\lambda \to +\infty} \frac{f(\lambda x)}{f(\lambda)} = x^\rho \quad \text{for every } x > 0.$$

If $\rho = 0$, then we say that f is slowly varying. Regular (slow) variation at 0 is defined similarly.

If f varies regularly with index $\rho \in \mathbb{R}$, then there exists a slowly varying function ℓ so that $f(\lambda) = \lambda^\rho \ell(\lambda)$.

Let ℓ be a slowly varying function such that $L(\lambda) = \int_0^\lambda \frac{\ell(t)}{t}\,\mathrm{d}t$ exists for all $\lambda > 0$. Then L is slowly varying,

$$\lim_{\lambda \to +\infty} \frac{L(\lambda)}{\ell(\lambda)} = +\infty \tag{2.10}$$

and

$$\lim_{\lambda \to +\infty} \frac{L(\lambda x) - L(\lambda)}{\ell(\lambda)} = \log x \quad \text{for every } x > 0$$

(see Proposition 1.5.9a and p. 127 in [5]).

3. Subordinators

Let $S = (S_t : t \geq 0)$ be a subordinator with the Laplace exponent ϕ satisfying the following conditions:

(A-1) there is $\alpha \in [0, 2]$ such that ϕ' varies regularly at infinity with index $\frac{\alpha}{2} - 1$, i.e.

$$\lim_{\lambda \to +\infty} \frac{\phi'(\lambda x)}{\phi'(\lambda)} = x^{\alpha/2 - 1} \quad \text{for every } x > 0,$$

(A-2) the Lévy measure is infinite and has a decreasing density μ,
(A-3) the potential measure has a decreasing density u.

When $\alpha = 2$ we additionaly assume:

(A-4) $\lambda \mapsto (\frac{\lambda}{\phi(\lambda)})'$ varies regularly at infinity with index -1.

Remark 3.1.

(a) *The most important assumption is* (A-1) (*and* (A-4) *when* $\alpha = 2$). *Other assumptions hold for a large class of subordinators* (*e.g. when* ϕ *is a complete Bernstein function with infinite Lévy measure*).
(b) *By Karamata's theorem* (*see Theorem* 1.5.11 *in* [5]) *it follows from* (A-1) *that* ϕ *varies regularly at infinity with index* $\frac{\alpha}{2}$.

Proposition 3.2. *Let* $\alpha \in [0, 2)$ *and let* S *be a subordinator satisfying* (A-1) *and* (A-2). *Then*

$$\mu(t) \asymp t^{-2} \phi'(t^{-1}), \quad t \to 0+.$$

Proof. Let $\varepsilon > 0$. By a change of variable

$$\begin{aligned}\phi(\lambda+\varepsilon)-\phi(\lambda) &= \int_0^\infty \big(\mathrm{e}^{-\lambda t}-\mathrm{e}^{-(\lambda+\varepsilon)t}\big)\mu(t)\,\mathrm{d}t \\ &= \lambda^{-1}\int_0^\infty \mathrm{e}^{-t}\big(1-\mathrm{e}^{-\varepsilon\lambda^{-1}t}\big)\mu\big(\lambda^{-1}t\big)\,\mathrm{d}t. \end{aligned} \tag{3.1}$$

Since μ is decreasing, the following holds

$$\phi(\lambda+\varepsilon)-\phi(\lambda) \ge \lambda^{-1}\mu\big(\lambda^{-1}\big)\int_0^1 \mathrm{e}^{-t}\big(1-\mathrm{e}^{-\varepsilon\lambda^{-1}t}\big)\,\mathrm{d}t.$$

Now we can apply Fatou's lemma to deduce

$$\begin{aligned}\phi'(\lambda) &= \lim_{\varepsilon\to 0+}\frac{\phi(\lambda+\varepsilon)-\phi(\lambda)}{\varepsilon} \ge \lambda^{-2}\mu\big(\lambda^{-1}\big)\int_0^1 t\mathrm{e}^{-t}\,\mathrm{d}t \\ &= \lambda^{-2}\mu\big(\lambda^{-1}\big)\big(1-2\mathrm{e}^{-1}\big).\end{aligned}$$

By setting $\lambda = t^{-1}$ we get the upper bound

$$\mu(t) \le \frac{t^{-2}f'(t^{-1})}{1-2\mathrm{e}^{-1}} \quad \text{for every } t>0. \tag{3.2}$$

Now we prove the lower bound. Using (3.1), for any $r \in (0,1)$ and $\varepsilon > 0$ we can write

$$\phi(\lambda+\varepsilon)-\phi(\lambda) = I_1 + I_2 \tag{3.3}$$

with

$$\begin{aligned} I_1 &= \lambda^{-1}\int_0^r \mathrm{e}^{-t}\big(1-\mathrm{e}^{-\varepsilon\lambda^{-1}t}\big)\mu\big(\lambda^{-1}t\big)\,\mathrm{d}t, \\ I_2 &= \lambda^{-1}\int_r^\infty \mathrm{e}^{-t}\big(1-\mathrm{e}^{-\varepsilon\lambda^{-1}t}\big)\mu\big(\lambda^{-1}t\big)\,\mathrm{d}t.\end{aligned}$$

Since μ is decreasing, the dominated convergence theorem yields

$$\begin{aligned}\limsup_{\varepsilon\to 0+}\frac{I_2}{\varepsilon} &\le \lambda^{-2}\mu\big(\lambda^{-1}r\big)\int_r^\infty t\mathrm{e}^{-t}\,\mathrm{d}t \\ &= (r+1)\mathrm{e}^{-r}\lambda^{-2}\mu\big(\lambda^{-1}r\big).\end{aligned} \tag{3.4}$$

To handle I_1 first we use the theorem of Potter (see Theorem 1.5.6(iii) in [5]) to conclude that there are constants $c_1 > 0$ and $\delta > 0$ such that

$$\frac{\phi'(\lambda t^{-1})}{\phi'(\lambda)} \le c_1 t^{\delta} \quad \text{for all } \lambda \ge 1 \quad \text{and} \quad t \le 1. \tag{3.5}$$

Therefore, by (3.2), (3.5) and the dominated convergence theorem

$$\begin{aligned}\limsup_{\varepsilon\to 0+}\frac{I_1}{\varepsilon} &\le \limsup_{\varepsilon\to 0+}\frac{1}{1-2\mathrm{e}^{-1}}\int_0^r \mathrm{e}^{-t}\frac{1-\mathrm{e}^{-\varepsilon\lambda^{-1}t}}{\varepsilon\lambda^{-1}t}t^{-1}\phi'\big(\lambda t^{-1}\big)\,\mathrm{d}t \\ &\le \phi'(\lambda)\frac{c_1}{1-2\mathrm{e}^{-1}}\int_0^r t^{\delta-1}\mathrm{e}^{-t}\,\mathrm{d}t.\end{aligned} \tag{3.6}$$

Combining (3.3), (3.4) and (3.6) we deduce

$$\phi'(\lambda) \le \phi'(\lambda)\frac{c_1}{1-2\mathrm{e}^{-1}}\int_0^r t^{\delta-1}\mathrm{e}^{-t}\,\mathrm{d}t + (r+1)\mathrm{e}^{-r}\lambda^{-2}\mu\big(\lambda^{-1}r\big).$$

Furthermore, by choosing $r \in (0,1)$ so that $\frac{c_1}{1-2\mathrm{e}^{-1}}\int_0^r t^{\delta-1}\mathrm{e}^{-t}\,\mathrm{d}t \le \frac{1}{2}$ we get

$$\mu\big(\lambda^{-1}r\big) \ge \frac{\mathrm{e}^r}{2(r+1)}\lambda^2\phi'(\lambda) \quad \text{for every } \lambda \ge 1.$$

By (A-1) we can find $t_0 \in (0,r)$ so that $\frac{\phi'(rt^{-1})}{\phi'(t^{-1})} \ge \frac{r^{\alpha/2-1}}{2}$ for any $t \in (0,t_0)$. The lower bound now follows:

$$\mu(t) \ge \frac{r^{1+\alpha/2}\mathrm{e}^r}{4(r+1)}t^{-2}\phi'\big(t^{-1}\big) \quad \text{for every } t \in (0,t_0).$$

□

Remark 3.3. *The precise asymptotic behavior of μ when $\alpha \in (0,2)$ can be obtained by Karamata's Tauberian theorem. This is not the case when $\alpha = 0$.*

Under an additional assumption

$$t \mapsto ta^{at}\mu(t) \text{ is monotone on } (0,T) \text{ for some } a \ge 0 \text{ and } T > 0,$$

it is possible to prove the following precise asymptotics in the case $\alpha = 0$:

$$\mu(t) \sim t^{-2}\phi'\big(t^{-1}\big), \quad t \to 0+.$$

Proposition 3.4. *Let $\alpha \in [0,2)$ and let S be a subordinator satisfying* (A-1) *and* (A-3). *Then*

$$u(t) \sim \frac{1}{\Gamma(1-\alpha/2)}\frac{t^{-2}\phi'(t^{-1})}{\phi(t^{-1})^2}, \quad t \to 0+.$$

Remark 3.5. *It can be proved that*

$$u(t) \asymp \frac{1}{\Gamma(1-\alpha/2)}\frac{t^{-2}\phi'(t^{-1})}{\phi(t^{-1})^2}, \quad t \to 0+$$

similarly as in Proposition 3.2. It is enough to note

$$\psi(\lambda+\varepsilon) - \psi(\lambda) = \int_0^\infty \big(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-(\lambda+\varepsilon)t}\big)u(t)\,\mathrm{d}t$$

with $\psi(\lambda) = -\frac{1}{\phi(\lambda)}$.

The main reason why we need precise asymptotics of u is to be able to handle the case $\alpha = 2$ by duality.

Proof of Proposition 3.4. Let us first consider the case $\alpha = 0$. In this case $\ell(\lambda) = \lambda\phi'(\lambda)$ varies slowly (at infinity) and thus it follows from Section 2.3 that $\phi(\lambda) = \int_0^\lambda \frac{\ell(t)}{t}\,\mathrm{d}t$ also varies slowly and

$$\lim_{\lambda\to+\infty}\frac{\phi(\lambda x) - \phi(\lambda)}{\lambda\phi'(\lambda)} = \log x \quad \text{for every } x > 0.$$

This and (2.1) imply

$$\frac{\mathcal{L}U(1/(\lambda x)) - \mathcal{L}U(1/\lambda)}{(1/\lambda)\phi'(1/\lambda)\phi(1/\lambda)^{-2}} = \frac{\phi(1/\lambda) - \phi(1/(\lambda x))}{(1/\lambda)\phi'(1/\lambda)}\frac{\phi(1/\lambda)}{\phi(1/(\lambda x))} \to \log x, \quad \lambda \to 0+$$

for any $x > 0$. Now we can apply de Haan's Tauberian theorem (see 0-version of [5], Theorem 3.9.1) to deduce

$$\frac{U(\lambda x) - U(\lambda)}{(1/\lambda)\phi'(1/\lambda)\phi(1/\lambda)^{-2}} \to \log x, \quad \lambda \to 0+.$$

If we apply de Haan's monotone density theorem (see Theorem 3.6.8 in [5]) we finally obtain

$$u(t) \sim \frac{t^{-2}\phi'(t^{-1})}{\phi(t^{-1})^2}, \quad t \to 0+.$$

The case $\alpha \in (0,2)$ is already known. We give the proof for the sake of completeness and adapt the result to the formula obtained in the case $\alpha = 0$.

Since

$$\mathcal{L}U(\lambda) = \frac{1}{\phi(\lambda)}$$

varies regularly at infinity with index $-\frac{\alpha}{2}$, Karamata's Tauberian theorem (see Theorem 1.7.1 in [5]) implies

$$U\big([0,t]\big) \sim \frac{1}{\Gamma(1-\alpha/2)}\frac{1}{\phi(t^{-1})}, \quad t \to 0+.$$

Then by Karamata's monotone density theorem (see Theorem 1.7.2 in [5]) we deduce

$$u(t) \sim \frac{\alpha}{2\Gamma(1-\alpha/2)}\frac{1}{t\phi(t^{-1})} \sim \frac{1}{\Gamma(1-\alpha/2)}\frac{t^{-2}\phi'(t^{-1})}{\phi(t^{-1})^2}, \quad t \to 0+.$$

□

Now we consider the case $\alpha = 2$.

Proposition 3.6. *Let $\alpha = 2$ and let S be a subordinator satisfying* (A-1)–(A-4). *Then*

$$\mu(t) \sim t^{-2}\big(t\phi\big(t^{-1}\big) - \phi'\big(t^{-1}\big)\big), \quad t \to 0+.$$

Proof. Since potential density exists by (A-3) we see that ϕ it follows that $\phi^\star(\lambda) = \frac{\lambda}{\phi(\lambda)}$ defines the Laplace exponent of a (possibly killed) subordinator, which we denote by T (see Section 2).

Note that the subordinator T corresponds to the case of $\alpha = 0$. If we denote potential density of T by v, then

$$v(t) = \mu(t,\infty), \quad t > 0.$$

Proposition 3.4 yields

$$\int_t^\infty \mu(s)\,\mathrm{d}s \sim \frac{t^{-2}(\phi^\star)'(t^{-1})}{\phi^\star(t^{-1})^2}, \quad t \to 0+. \tag{3.7}$$

By assumption (A-4) we know that $t \mapsto (\phi^\star)'(t^{-1})$ varies regularly at 0 with index 1 and thus $t \mapsto \frac{t^{-2}(\phi^\star)'(t^{-1})}{\phi^\star(t^{-1})^2}$ varies regularly at 0 with index -1.

Now we change variable in the integral on the left-hand side in (3.7) and conclude

$$\int_0^{t^{-1}} \mu\big(s^{-1}\big)\frac{\mathrm{d}s}{s^2} \sim \frac{t^{-2}(\phi^\star)'(t^{-1})}{(\phi^\star(t^{-1}))^2}, \quad t \to 0+.$$

This gives ($r = t^{-1}$):

$$\int_0^r \mu\big(s^{-1}\big)\frac{\mathrm{d}s}{s^2} \sim \frac{r^2(\phi^\star)'(r)}{\phi^\star(r)^2}, \quad r \to \infty.$$

Note that the right-hand side is now regularly varying at infinity with index 1 and thus by Karamata's monotone density theorem (see Theorem 1.7.2 in [5]) we deduce

$$\frac{\mu(r^{-1})}{r^2} \sim \frac{r(\phi^\star)'(r)}{\phi^\star(r)^2}, \quad r \to \infty.$$

Going back ($t = r^{-1}$) we conclude

$$\mu(t) \sim \frac{t^{-3}(\phi^\star)'(t^{-1})}{\phi^\star(t^{-1})^2}, \quad t \to 0+.$$

□

Proposition 3.7. *Let $\alpha = 2$ and let S be a subordinator satisfying* (A-1) *and* (A-3). *Then the following is true*

$$u(t) \sim \frac{1}{\phi'(t^{-1})} \sim \frac{1}{t\phi(t^{-1})}, \quad t \to 0+.$$

Proof. By (2.1) we get

$$\mathcal{L}U(\lambda) = \frac{1}{\phi(\lambda)} \sim \frac{1}{\lambda\phi'(\lambda)}, \quad \lambda \to +\infty$$

and thus by Karamata's Tauberian theorem (see Theorem 1.7.1 in [5]) it follows that

$$U\big([0,t]\big) \sim \frac{1}{\Gamma(2)} \frac{t}{\phi'(t^{-1})}, \quad t \to 0+$$

since $\lambda \mapsto \lambda\phi'(\lambda)$ varies regularly at infinity with index 1. By applying Karamata's monotone density (see Theorem 1.7.2 in [5]) theorem we deduce

$$u(t) \sim \frac{1}{\phi'(t^{-1})}, \quad t \to 0+.$$

□

4. Lévy density and Green function

Let S be a subordinator as in Section 3 and let X be the corresponding subordinate Brownian motion in $\mathbb{R}^d$. Our aim is to establish asymptotic behavior of the Lévy density and Green function of X.

Recall that the Lévy density of X is of the form $j(|x|)$, where j is given by (2.2).

Theorem 4.1. *Assume that S satisfies* (A-1) *and* (A-2). *If $\alpha \in [0,2)$, then*

$$j(r) \asymp r^{-d-2}\phi'\big(r^{-2}\big), \quad r \to 0+.$$

If $\alpha = 2$ and (A-4) *holds, then*

$$j(r) \asymp r^{-d-2}\big(r^2\phi\big(r^{-2}\big) - \phi'\big(r^{-2}\big)\big), \quad r \to 0+.$$

Proof. This result follows directly from Proposition 3.2 and Proposition 3.6 together with Lemma A.1, where $a = \frac{1}{4}$, $b = 1 + \frac{\alpha}{2}$, $p = -\frac{d}{2}$ and $w = \mu$.

More precisely, for $\alpha \in [0,2)$ the slowly varying function is given by $\ell(t) = t^{\alpha/2-1}\phi'(t^{-1})$. When $\alpha = 2$ we take

$$\ell(t) = t\phi\big(t^{-1}\big) - \phi'\big(t^{-1}\big) = \frac{t(\phi^\star(t^{-1}))'}{\phi^\star(t^{-1})^2},$$

which varies slowly, since $\phi^\star(\lambda) = \frac{\lambda}{\phi(\lambda)}$ is slowly varying by (A-1) and Karamata's theorem (see Theorem 1.5.11 in [5]) and has a derivative that varies regularly with index 1 by (A-4). Thus, condition (a) of the Lemma A.1 holds.

Since μ is a Lévy measure, condition (b) also holds:

$$\int_1^\infty t^{-d/2}\mu(t)\,\mathrm{d}t \le \mu(1,\infty) < \infty. \qquad \square$$

The Green function of X is of the form $G(x,y) = g(|y-x|)$, where g is given by (2.4).

Theorem 4.2. *Assume that S satisfies* (A-1) *and* (A-3). *Let $d > \alpha$ and assume that X is transient. If $\alpha \in [0,2)$, then*

$$g(r) \asymp r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2}, \quad r \to 0+.$$

If $\alpha = 2$, then

$$g(r) \asymp r^{-d+2}\frac{1}{\phi'(r^{-2})} \asymp r^{-d}\frac{1}{\phi(r^{-2})}, \quad r \to 0+.$$

Proof. We use Lemma A.1 with $a = \frac{1}{4}$, $b = 1 - \frac{\alpha}{2}$, $p = \frac{d}{2}$, $w = u$, Proposition 3.4 and Proposition 3.7.

First we check condition (a) of Lemma A.1. When $\alpha \in (0,2]$ we define $\ell(t) = \frac{t^{-1}\phi'(t^{-1})}{\phi(t^{-1})}$ which varies slowly at 0. In the case $\alpha = 2$, we let $\ell(t) = \frac{1}{\phi'(t^{-1})}$ or $\ell(t) = \frac{1}{t\phi(t^{-1})}$ both of which vary slowly at 0.

To check condition (b), note that

$$\infty > g(1) = \int_0^\infty t^{-d/2}\mathrm{e}^{-1/(4t)}u(t)\,\mathrm{d}t \ge \mathrm{e}^{-1/4}\int_1^\infty t^{-d/2}u(t)\,\mathrm{d}t.$$

$\square$

Also, using $d > \alpha$ it follows that $p > 1 - b$.

Using asymptotic results from this section we can now prove the proposition that gives a counterexample for the estimate of Krylov and Safonov.

Proof of Proposition 1.2. By (2.5),

$$\begin{aligned}\mathbb{P}_x\big(X_{\tau_{B_{r/2}(0)}} \in B_r(0) \setminus B_{r/2}(0)\big) &= \int_{B_r(0)\setminus\overline{B_{r/2}(0)}}\int_{B_{r/2}(0)} G_{B_{r/2}(0)}(x,y)\,j\big(|z-y|\big)\,\mathrm{d}y \\ &= I_1 + I_2.\end{aligned}$$

Using Theorems 4.1 and 4.2 (note that $d \ge 1 > \alpha$) it follows

$$\begin{aligned}I_1 &= \int_{B_r(0)\setminus\overline{B_{r/2}(0)}}\int_{B_{5r/16}(0)} G_{B_{r/2}(0)}(x,y)\,j\big(|z-y|\big)\,\mathrm{d}y\,\mathrm{d}z \\ &\le j\left(\frac{r}{8}\right)\big|B_r(0)\setminus B_{r/2}(0)\big|\int_{B_r(0)} g\big(|y|\big)\,\mathrm{d}y \\ &\le c_1 j(r) r^d \int_0^r \frac{\phi'(s^{-2})}{\phi(s^{-2})^2}\frac{\mathrm{d}s}{s^3} \le c_2\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}.\end{aligned}$$

On the other hand, since $x \in B_{r/4}(0)$,

$$\begin{aligned}I_2 &= \int_{B_r(0)\setminus\overline{B_{r/2}(0)}}\int_{B_{r/2}(0)\setminus B_{5r/16}(0)} G_{B_{r/2}(0)}(x,y)\,j\big(|z-y|\big)\,\mathrm{d}y\,\mathrm{d}z \\ &\le g\left(\frac{r}{16}\right)\int_{B_r(0)\setminus\overline{B_{r/2}(0)}}\int_{B_{r/2}(z)} j\big(|y|\big)\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$

To estimate the inner integral, note that $B_{r/2}(z) \subset B_1(0) \setminus B_{|z|-r/2}(0)$ for any $z \in B_r(0) \setminus \overline{B_{r/2}(0)}$ and so, by Theorem 4.1,

$$\int_{B_{r/2}(z)} j\big(|y|\big)\,\mathrm{d}y \le c_3 \int_{|z|-r/2}^{1} s^{-3}\phi'\big(s^{-2}\big)\,\mathrm{d}s \le c_4\phi\left(\left(|z| - \frac{r}{2}\right)^{-2}\right). \tag{4.1}$$

Therefore, by Theorem 4.2 we conclude

$$\begin{aligned} I_2 &\le c_5 r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2}\int_{r/2}^{r}\phi\left(\left(|z| - \frac{r}{2}\right)^{-2}\right)\mathrm{d}z \\ &\le c_5 r^{-3}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2}\int_0^{r/2}\phi\big(s^{-2}\big)\,\mathrm{d}s \\ &\le c_6\frac{r^{-2}\phi'(r^{-2})}{\phi(r^{-2})}. \end{aligned}$$

In the last equality we have used Karamata's theorem (see Theorem 1.5.11 in [5]) and the fact that $\alpha \in [0, 1)$. □

5. Difference estimates

Let X be the stochastic process in $\mathbb{R}^d$ as in Section 4. Assume that X is transient and $d > \alpha$.

In this section we prove the difference estimates of the Green function and the Poisson kernel.

Although we are slightly abusing notation, we set $G(x) := G(0, x) = g(|x|)$.

Proposition 5.1. *There is a constant $c > 0$ such that for every $r \in (0, 1)$*

$$\big|G(x) - G(y)\big| \le cg(r)\left(1 \wedge \frac{|x-y|}{r}\right) \quad \textit{for all } x, y \notin B_r(0).$$

Proof. Assume first that $|x - y| < \frac{r}{2}$. By the mean value theorem it follows that for any $t > 0$ there exists $\vartheta = \vartheta(x, y, t) \in [0, 1]$ such that

$$\begin{aligned} \big|\mathrm{e}^{-|x|^2/(4t)} - \mathrm{e}^{-|y|^2/(4t)}\big| &\le \frac{|x + \vartheta(y-x)|}{2t}\mathrm{e}^{-|x+\vartheta(y-x)|^2/(4t)}|x-y| \\ &\le 2\frac{|x-y|}{\sqrt{t}}\mathrm{e}^{-|x+\vartheta(y-x)|^2/(8t)}, \end{aligned}$$

where in the last line the following elementary inequality was used

$$s\mathrm{e}^{-s^2} < 2\mathrm{e}^{-s^2/2}, \quad s > 0.$$

Then $|x + \vartheta(y - x)| \ge |x| - \vartheta|y - x| \ge \frac{r}{2}$ implies

$$\big|\mathrm{e}^{-|x|^2/(4t)} - \mathrm{e}^{-|y|^2/(4t)}\big| \le 2\frac{|x-y|}{\sqrt{t}}\mathrm{e}^{-r^2/(32t)}. \tag{5.1}$$

By (5.1)

$$\begin{aligned} \big|G(x) - G(y)\big| &\le (4\pi)^{-d/2}\int_0^\infty t^{-d/2}\big|\mathrm{e}^{-|x|^2/(4t)} - \mathrm{e}^{-|y|^2/(4t)}\big|u(t)\,\mathrm{d}t \\ &\le 2(4\pi)^{-d/2}|x-y|\int_0^\infty t^{-d/2-1/2}\mathrm{e}^{-r^2/(32t)}u(t)\,\mathrm{d}t. \end{aligned}$$

Since u is non-increasing and varies regularly at 0 with index $\frac{\alpha}{2}-1$, by Lemma A.1 we see that there is a constant $c_1>0$ so that

$$\int_0^\infty t^{-d/2-1/2}\mathrm{e}^{-r^2/(32t)}u(t)\,\mathrm{d}t\le c_1 r^{-d+1}u\bigl(r^2\bigr)\quad \text{for every } r\in(0,1).$$

Theorem 4.2 yields

$$\bigl|G(x)-G(y)\bigr|\le c_2 g(r)\frac{|x-y|}{r}.$$

When $|x-y|\ge\frac{r}{2}$,

$$\bigl|G(x)-G(y)\bigr|\le G(x)+G(y)\le 2g(r)$$

since $|x|,|y|\ge r$. □

Proposition 5.2. *There is a constant $c>0$ such that for all $R\in(0,1)$, $r\in(0,\frac{R}{2}]$, $y\in B_R(0)$ and $x_1,x_2\in B_{R/2}(0)\setminus B_r(y)$*

$$\bigl|G_{B_R(0)}(x_1,y)-G_{B_R(0)}(x_2,y)\bigr|\le cg(r)\biggl(1\wedge\frac{|x_1-x_2|}{r}\biggr).$$

Proof. By symmetry of the Green function,

$$\begin{aligned}G_{B_R(0)}(x_i,y)=G_{B_R(0)}(y,x_i)&=G(x_i-y)-\mathbb{E}_y\bigl[G(X_{\tau_{B_R(0)}}-x_i)\bigr]\\&=G(x_i-y)-\mathbb{E}_y\bigl[G(X_{\tau_{B_R(0)}}-x_i)\bigr]\end{aligned}$$

for $i\in\{1,2\}$. Now the result follows from Proposition 5.1. □

Proposition 5.3. *There is a constant $c>0$ such that for any $r\in(0,1)$ and $x,y\in B_{r/8}(0)$:*

(i) *if $z\in B_{2r}(0)\setminus B_r(0)$, then*

$$\bigl|K_{B_r(0)}(x,z)-K_{B_r(0)}(y,z)\bigr|\le c|z|^{-d}\frac{\phi(((|z|-r)^{-2})}{\phi(|x-y|^{-2})};$$

(ii) *if $z\notin B_{2r}(0)$, then*

$$\bigl|K_{B_r(0)}(x,z)-K_{B_r(0)}(y,z)\bigr|\le c\frac{j(|z|/2)}{\phi(|x-y|^{-2})}.$$

Proof. In the estimate

$$\bigl|K_{B_r(0)}(x,z)-K_{B_r(0)}(y,z)\bigr|\le\int_{B_r(0)}\bigl|G_{B_r(0)}(x,v)-G_{B_r(0)}(y,v)\bigr|j\bigl(|z-v|\bigr)\,\mathrm{d}v$$

we split the integral into three parts:

$$\begin{aligned}I_1&=\int_{B_{2|x-y|}(x)}\bigl|G_{B_r(0)}(x,v)-G_{B_r(0)}(y,v)\bigr|j\bigl(|z-v|\bigr)\,\mathrm{d}v,\\I_2&=\int_{B_{r/4}(x)\setminus B_{2|x-y|}(x)}\bigl|G_{B_r(0)}(x,v)-G_{B_r(0)}(y,v)\bigr|j\bigl(|z-v|\bigr)\,\mathrm{d}v,\\I_3&=\int_{B_r(0)\setminus B_{r/4}(x)}\bigl|G_{B_r(0)}(x,v)-G_{B_r(0)}(y,v)\bigr|j\bigl(|z-v|\bigr)\,\mathrm{d}v.\end{aligned}$$

For the first part we obtain

$$
\begin{aligned}
I_1 &\le \int_{B_{2|x-y|}(x)} G_{B_r(0)}(x,v)\, j\big(|z-v|\big)\,\mathrm{d}v + \int_{B_{3|x-y|}(y)} G_{B_r(0)}(y,v)\, j\big(|z-v|\big)\,\mathrm{d}v \\
&\le 2j\left(\frac{|z|}{2}\right)\int_{B_{3|x-y|}(0)} G(v)\,\mathrm{d}v \le c_1 \frac{j(|z|/2)}{\phi(|x-y|^{-2})}
\end{aligned}
\tag{5.2}
$$

for any $z \notin B_r(0)$. We have used Theorem 4.2 to get the last inequality in (5.2).

In order to estimate I_2 we split the integral in the following way. We let $N = \lfloor \frac{\log(r/(4|x-y|))}{\log 2} \rfloor$ and write

$$
I_2 \le \sum_{n=1}^{N} \int_{B_{2^{n+1}|x-y|}(x)\setminus B_{2^n|x-y|}(x)} \big|G_{B_r(0)}(x,v) - G_{B_r(0)}(y,v)\big|\, j\big(|z-v|\big)\,\mathrm{d}v.
$$

Now, for each $n \in \{1,\dots,N\}$ we can apply Proposition 5.2 (with the corresponding radii $(2^n-1)|x-y|$ and r) to get

$$
\begin{aligned}
&\int_{B_{2^{n+1}|x-y|}(x)\setminus B_{2^n|x-y|}(x)} \big|G_{B_r(0)}(x,v) - G_{B_r(0)}(y,v)\big|\, j\big(|z-v|\big)\,\mathrm{d}v \\
&\quad\le c_3 \frac{g((2^n-1)|x-y|)}{2^n-1} \int_{B_{2^{n+1}|x-y|}(x)} j\big(|z-v|\big)\,\mathrm{d}v.
\end{aligned}
$$

By Theorem 4.2

$$
\frac{g((2^n-1)|x-y|)}{g(|x-y|)} \le c_4 \frac{\eta((2^n-1)|x-y|)}{\eta(|x-y|)} \quad \text{for all } n \in \{1,2,\dots,N\}
$$

with $\eta(r) = r^{-d-2}\frac{\phi'(r^{-2})}{\phi(r^{-2})^2}$.

Noting that η varies regularly at zero with index $\alpha - d < 0$, the uniform convergence theorem for regularly varying functions (see Theorem 1.5.2 in [5]) gives

$$
\frac{\eta((2^n-1)|x-y|)}{\eta(|x-y|)} \le c_5 \big(2^n-1\big)^{\alpha-d} \quad \text{for all } n \in \mathbb{N} \text{ and } |x-y| \le \tfrac{1}{2}.
$$

By Theorem 4.2 and (2.9) $g(|x-y|) \le \frac{c_5|x-y|^{-d}}{\phi(|x-y|^{-2})}$ and so

$$
\begin{aligned}
I_2 &\le c_6 \sum_{n=1}^{N} \big(2^n-1\big)^{\alpha-d-1} g\big(|x-y|\big)\big(2^{n+1}|x-y|\big)^d j\left(\frac{|z|}{2}\right) \\
&\le c_7 \frac{j(|z|/2)}{\phi(|x-y|^{-2})} \sum_{n=1}^{N} 2^{(\alpha-1)n} \\
&\le \frac{c_7}{1-2^{\alpha-1}} \frac{j(|z|/2)}{\phi(|x-y|^{-2})} \quad \text{for every } z \notin B_r(0).
\end{aligned}
$$

It remains to estimate I_3. Applying Theorem 5.2 we get

$$
\begin{aligned}
I_3 &\le c_8 g(r)\frac{|x-y|}{r} \int_{B_r(z)} j\big(|v|\big)\,\mathrm{d}v \\
&\le c_9 \frac{|x-y|\phi(|x-y|^{-2})}{r\phi(r^{-2})} \frac{r^{-d}}{\phi(|x-y|^{-2})} \int_{B_r(z)} j\big(|v|\big)\,\mathrm{d}v \\
&\le c_{10} \frac{r^{-d}}{\phi(|x-y|^{-2})} \int_{B_r(z)} j\big(|v|\big)\,\mathrm{d}v.
\end{aligned}
\tag{5.3}
$$

In the last inequality we have used the theorem of Potter (cf. [5], Theorem 1.5.6(iii)) to conclude that for $\delta < 1-\alpha$ there is a constant $A_\delta > 0$ such that

$$\frac{|x-y|\phi(|x-y|^{-2})}{r\phi(r^{-2})} \le A_\delta \left(\frac{|x-y|}{r}\right)^{1-\alpha-\delta} \le A_\delta,$$

since $r \mapsto r\phi(r^{-2})$ varies regularly at zero with index $1-\alpha$.

Since

$$j\big(|v|\big) \ge j\left(\frac{|z|}{2}\right) \quad \text{for all } v \in B_r(z) \text{ and } z \in B_{2r}(0)^c$$

it follows from (5.3) that

$$I_3 \le c_{11} \frac{j(|z|/2)}{\phi(|x-y|^{-2})}.$$

On the other hand, for $z \in B_{2r}(0) \setminus B_r(0)$ we deduce from

$$B_r(z) \subset B_3(0) \setminus B_{|z|-r}(0)$$

(similarly as in (4.1)) that

$$\int_{B_r(z)} j\big(|v|\big)\,\mathrm{d}v \le c_{12}\phi\big(\big(|z|-r\big)^{-2}\big).$$

By (5.3)

$$I_3 \le c_{13}|z|^{-d} \frac{\phi((|z|-r)^{-2})}{\phi(|x-y|^{-2})} \quad \text{for all } z \in B_{2r}(0) \setminus B_r(0).$$

□

6. Regularity of harmonic functions

Recall that (2.7) gives the representation for any bounded function $f:\mathbb{R}^d \to \mathbb{R}$ that is harmonic in $B_{2r}(x_0)$:

$$f(x) = \mathbb{E}_x\big[f(X_{\tau_{B_r(x_0)}})\big] = \int_{\overline{B_r(x_0)}^c} K_{B_r(x_0)}(x,z) f(z)\,\mathrm{d}z, \quad x \in B_r(x_0). \tag{6.1}$$

Proof of Theorem 1.1. Assume first that $d \ge 3$. In particular $d > \alpha$ and X is transient. Thus, in this case we can use results from Section 5.

By (6.1)

$$\big|f(x) - f(y)\big| \le \|f\|_\infty \int_{\overline{B_{2r}(0)}^c} \big|K_{B_{2r}(0)}(x,z) - K_{B_{2r}(0)}(y,z)\big|\,\mathrm{d}z. \tag{6.2}$$

It remains to estimate the integral in (6.2), which we split in the following way

$$I_1 = \int_{B_{4r}(0)\setminus\overline{B_{2r}(0)}} \big|K_{B_{2r}(0)}(x,z) - K_{B_{2r}(0)}(y,z)\big|\,\mathrm{d}z,$$

$$I_2 = \int_{B_1(0)\setminus B_{4r}(0)} \big|K_{B_{2r}(0)}(x,z) - K_{B_{2r}(0)}(y,z)\big|\,\mathrm{d}z,$$

$$I_3 = \int_{B_1(0)^c} \big|K_{B_{2r}(0)}(x,z) - K_{B_{2r}(0)}(y,z)\big|\,\mathrm{d}z.$$

In order to estimate I_1 we use Proposition 5.3(i). More precisely,

$$\begin{aligned} I_1 &\le \frac{c_1}{\phi(|x-y|^{-2})} \int_{B_{4r}(0)\setminus\overline{B_{2r}(0)}} |z|^{-d}\phi\big(\big(|z|-2r\big)^{-2}\big)\,\mathrm{d}z \\ &= \frac{c_2}{\phi(|x-y|^{-2})} \int_{2r}^{4r} t^{-1}\phi\big((t-2r)^{-2}\big)\,\mathrm{d}t \\ &\le \frac{c_2}{\phi(|x-y|^{-2})}(2r)^{-1} \int_0^{2r} \phi\big(t^{-2}\big)\,\mathrm{d}t \\ &\le \frac{c_3}{\phi(|x-y|^{-2})}\phi\big(r^{-2}\big), \end{aligned}$$

where in the last inequality we have used Karamata's theorem (see the 0-version of Theorem 1.5.11 in [5]).

We estimate I_2 and I_3 with the help of Proposition 5.3(ii). Since the Lévy measure is finite away from the origin,

$$I_3 \le \frac{c_4}{\phi(|x-y|^{-2})} \int_{B_1(0)^c} j\left(\frac{|z|}{2}\right)\mathrm{d}z \le \frac{c_5}{\phi(|x-y|^{-2})}.$$

Also,

$$I_2 \le \frac{c_6}{\phi(|x-y|^{-2})} \int_{B_1(0)\setminus B_{4r}(0)} j\left(\frac{|z|}{2}\right)\mathrm{d}z \le \frac{c_7\phi(r^{-2})}{\phi(|x-y|^{-2})},$$

where in the last inequality we have used Theorem 4.1. This proves theorem in the case $d \ge 3$.

We are going to use what we have proved to deduce that the theorem is also true for $d = 1, 2$.

Let us introduce the following notation. If $x = (x^1, \dots, x^{d-1}, x^d) \in \mathbb{R}^d$, then we set $\widetilde{x} = (x^1, \dots, x^{d-1})$. Let X be a subordinate Brownian motion in $\mathbb{R}^d$ with the characteristic exponent $\Phi(\xi) = \phi(|\xi|^2)$.

In particular, for $\xi = (\widetilde{\xi}, 0)$ we get

$$\mathbb{E}_0\big[\mathrm{e}^{i\langle\widetilde{\xi},\widetilde{X}_t\rangle}\big] = \mathrm{e}^{-t\phi(|\widetilde{\xi}|^2)}.$$

This shows that $\widetilde{X}$ is a $(d-1)$-dimensional subordinate Brownian motion with the characteristic exponent $\widetilde{\Phi}(\widetilde{\xi}) = \phi(|\widetilde{\xi}|^2)$.

Assume that theorem holds in dimension $d \ge 2$. If $f : \mathbb{R}^{d-1} \to \mathbb{R}$ is a function that is harmonic in $B_r(\widetilde{0}) \subset \mathbb{R}^{d-1}$ with respect to $\widetilde{X}$, then $F : \mathbb{R}^d \to \mathbb{R}$ defined by $F(\widetilde{x}, x^d) = f(\widetilde{x})$ is harmonic in $B_r(\widetilde{0}) \times \mathbb{R}$ with respect to X. This follows from the strong Markov property and the fact that for any open set $B \subset B_r(\widetilde{0})$

$$\tau_{B\times\mathbb{R}} = \inf\{t > 0\colon\ \widetilde{X}_t \notin B\}.$$

Since theorem is true in dimension d, we obtain

$$\big|f(\widetilde{x}) - f(\widetilde{y})\big| = \big|F(\widetilde{x}, 0) - F(\widetilde{y}, 0)\big| \le c\|f\|_\infty \frac{\phi(r^{-2})}{\phi(|\widetilde{x}-\widetilde{y}|^{-2})}.$$

If we use this procedure first for $d = 3$ and then for $d = 2$, we obtain the claim in the case of $\mathbb{R}^2$ and $\mathbb{R}$. □

7. Examples

In this section is to illustrate our results by some examples.

7.1. (*Iterated*) *geometric stable processes*

This class of examples belongs to the case of $\alpha = 0$.

Let $\beta \in (0,2]$. We define a family of functions $\{\phi_n : (0,\infty) \to (0,\infty) \colon n \in \mathbb{N}\}$ recursively by

$$\phi_1(\lambda) = \log\bigl(1+\lambda^{\beta/2}\bigr), \quad \lambda > 0,$$
$$\phi_{n+1} = \phi_1 \circ \phi_n, \quad n \in \mathbb{N}.$$

The function ϕ_1 is a complete Bernstein function. Since complete Bernstein functions are closed under operation of composition, ϕ_n belongs to this class for every $n \in \mathbb{N}$.

Let S^n be a subordinator with the Laplace exponent ϕ_n. S^1 is known as the geometric $\frac{\beta}{2}$-stable subordinator. We call S^n the iterated geometric $\frac{\beta}{2}$-stable subordinator. The corresponding subordinate Brownian motions X^n will be called (iterated) geometric β-stable processes.

As already remarked in [18], these processes show quite different behavior compared to the one of stable processes. Our contribution to this class of examples is that now we can obtain behavior of the Lévy density as a special case of Theorem 4.1 (even for iterated geometric stable processes).

The Lévy density of X^n is comparable to

$$\frac{1}{|x|^d} \cdot \prod_{k=1}^{n-1} \frac{1}{\log_k(|x|^{-1})} \quad \text{as } |x| \to 0+,$$

which is almost integrable. We can say that (intially) this process jumps slower than any stable processes.

This can be also seen from the behavior of the Green function:

$$G(x,y) \asymp \frac{1}{|x-y|^d \log_n^2(|x-y|^{-1})} \cdot \prod_{k=1}^{n-1} \frac{1}{\log_k(|x-y|^{-1})} \quad \text{as } |x-y| \to 0+.$$

As a consequence, $\mathbb{E}_0 \tau_{B_r(0)} \asymp \frac{1}{\log_n(r^{-1})}$ as $r \to 0+$. Therefore X^n needs (on average) more time to exit ball $B_r(0)$ than any stable process or Brownian motion.

Theorem 1.1 implies the following a-priori local regularity estimates of harmonic functions:

$$\bigl|f(x) - f(y)\bigr| \le c\|f\|_\infty \log_n\bigl(r^{-1}\bigr) \frac{1}{\log_n(|x-y|^{-1})} \quad \text{for all } x, y \in B_{r/2}(0)$$

and any bounded function f which is harmonic in $B_r(0)$.

This tells us that the modulus of continuity is bounded by a logarithmic term. It is still an open problem whether these harmonic functions satisfy a-priori local Hölder continuity estimates.

7.2. *Conjugates of* (*iterated*) *geometric stable processes*

This class of examples corresponds to the case $\alpha = 2$.

Let $\psi_n(\lambda) = \frac{\lambda}{\phi_n(\lambda)}$, where ϕ_n are as in Section 7.1.

Since ϕ_n are complete Bernstein functions, ψ_n are also complete Bernstein functions. Therefore, there exist (killed) subordinators T^n with the Laplace exponent ψ_n. Killing will not affect the behavior of the Lévy and potential densities of T^n near zero.

In this case the Lévy density of the corresponding subordinate Brownian motion Y^n behaves near the origin as

$$\frac{1}{|x|^{d+2} \log_n^2(|x|^{-1})} \cdot \prod_{k=1}^{n-1} \frac{1}{\log_k(|x|^{-1})} \quad \text{as } |x| \to 0+.$$

Note that the integrability conditions of the Lévy measure are barely satisfied in this case.

Comparing this behavior to the behavior of the small jumps of the α-stable process, we see that small jumps of Y^n are more intensive.

Another interesting feature of this process is the following behavior of the Green function:

$$G(x,y) \asymp |x-y|^{2-d} \log_n\bigl(|x-y|^{-1}\bigr) \quad \text{as } |x-y| \to 0+.$$

In this sense the process Y^n is "between" stable processes and Brownian motion, since their Green functions are given by

$$G^{(\alpha)} = c_\alpha |x-y|^{\alpha-d} \quad \text{and} \quad G^{(2)} = c_\alpha |x-y|^{2-d}.$$

Appendix: Asymptotic properties

In the Appendix we prove a technical lemma which is used throughout the paper.

Lemma A.1. *Let $w:(0,\infty)\to(0,\infty)$ be a decreasing function satisfying the following assumptions*:

(a) *there exist $b \ge 0$ and a function $\ell:(0,\infty)\to(0,\infty)$ that varies slowly at 0 such that*

$$w(t) \asymp t^{-b}\ell(t), \quad t \to 0+;$$

(b) *there exists $p > 1-b$ such that*

$$\int_1^\infty t^{-p} w(t)\,\mathrm{d}t < \infty.$$

For $a > 0$ we define $I_a:(0,1)\to[0,\infty)$ by

$$I_a(r) = \int_0^\infty t^{-p} \mathrm{e}^{-(ar)/t} w(t)\,\mathrm{d}t.$$

Then

$$I_a(r) \asymp r^{-p+1} w(r), \quad r \to 0+.$$

Remark A.2. *Since w is decreasing*, *condition* (*b*) *is satisfied for $p > 1$.*

Proof of Lemma A.1. Write $I_a(r) = A(r) + B(r)$ with

$$A(r) = \int_0^{ar} t^{-p}\mathrm{e}^{-ar/t} w(t)\,\mathrm{d}t \quad \text{and} \quad B(r) = \int_{ar}^\infty t^{-p}\mathrm{e}^{-ar/t} w(t)\,\mathrm{d}t.$$

First we estimate $A(r)$. Changing variables ($s = \frac{ar}{t}$) it follows

$$A(r) = (ar)^{-p+1} \int_1^\infty \mathrm{e}^{-s} s^{p-2} w\left(\frac{ar}{s}\right)\mathrm{d}s. \tag{A.1}$$

By assumption (a), there are constants $c_1, c_2 > 0$ such that for $r > 0$ small enough,

$$c_1 t^{-b}\frac{\ell(tr)}{\ell(r)} \le \frac{w(tr)}{w(r)} \le c_2 t^{-b}\frac{\ell(tr)}{\ell(r)} \quad \text{for } t > 0 \tag{A.2}$$

and so

$$\frac{c_1}{c_2}\int_1^\infty \mathrm{e}^{-s} s^{p-2+b}\frac{\ell(sr)}{\ell(r)}\,\mathrm{d}s \le \frac{A(r)}{(ar)^{-p+1}a^{-b}w(r)} \le \frac{c_2}{c_1}\int_1^\infty \mathrm{e}^{-s} s^{p-2+b}\frac{\ell(sr)}{\ell(r)}\,\mathrm{d}s.$$

Since ℓ varies slowly at zero, dominated convergence theorem implies

$$\lim_{r\to 0+}\int_1^\infty \mathrm{e}^{-s}s^{p-2+b}\frac{\ell(sr)}{\ell(r)}\,\mathrm{d}s=\int_1^\infty \mathrm{e}^{-s}s^{p-2+b}\,\mathrm{d}s<\infty.$$

Therefore, there are constants $c_3, c_4>0$ such that for $r>0$ small enough

$$c_3\le \frac{A(r)}{(ar)^{-p+1}a^{-b}w(r)}\le c_4. \tag{A.3}$$

It is left to estimate $B(r)$ from above.

By (A.2), the fact that w is decreasing and condition (b) we have, for $r>0$ small enough,

$$\begin{aligned} B(r)&\le \int_{ar}^1 t^{-p}w(t)\,\mathrm{d}t+\int_1^\infty t^{-p}w(t)\,\mathrm{d}t\\ &\le w(ar)\int_{ar}^1 t^{-p}\,\mathrm{d}t+c_5\\ &\le c_6\bigl(r^{-p+1}w(r)+1\bigr)\le c_7 r^{-p+1}w(r). \end{aligned}\tag{A.4}$$

The last inequality follows from $p+b-1>0$,

$$r^{-p+1}w(r)\ge c' r^{-p-b+1}\ell(r)$$

and the fact that for a slowly varying function ℓ

$$\lim_{r\to 0+} r^{-\varepsilon}\ell(r)=+\infty\quad\text{for any }\varepsilon>0$$

(see Proposition 1.3.6(v) in [5]).

Now it follows from (A.3) and (A.4) that, for $r>0$ small enough,

$$c_3\le\frac{I_a(r)}{r^{-p+1}w(r)}\le c_4+c_7,$$

which proves the lemma. □

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