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A remarkable σ-finite measure unifying supremum penalisations for a stable Lévy process

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Abstract. The σ-finite measure $\mathcal{P}_{\mathrm{sup}}$ which unifies supremum penalisations for a stable Lévy process is introduced. Silverstein's coinvariant and coharmonic functions for Lévy processes and Chaumont's h-transform processes with respect to these functions are utilized for the construction of $\mathcal{P}_{\mathrm{sup}}$.

Résumé. On introduit la mesure σ-finie $\mathcal{P}_{\mathrm{sup}}$, unifiant les pénalisations selon le supremum pour un processus de Lévy stable. Dans la construction de $\mathcal{P}_{\mathrm{sup}}$ on utilise les fonctions co-invariantes et co-harmoniques de Silverstein pour les processus de Lévy, et les processus h-transformés par rapport à ces fonctions selon l'approche de Chaumont.

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1. Introduction

Roynette, Vallois and Yor ([19] and [20], see also [21] and [22]) have considered the limit laws of Wiener measure weighted by various processes (Γ_t), and they call these studies *Brownian penalisations*. Especially we call the case where the weight process is given by a function of its supremum, i.e., (S) $\Gamma_t = f(S_t)$, *supremum penalisation*. Concerning the Brownian supremum penalisations, the authors [20] have obtained the following result. Let $X = ((X_t), (\mathcal{F}_t), \mathbb{W})$ be the canonical representation of a 1-dimensional standard Brownian motion with $\mathbb{W}(X_0 = 0) = 1$ and let $\mathcal{F}_\infty = \sigma(\bigvee_t \mathcal{F}_t)$. Put $S_t = \sup_{s \le t} X_s$. If f is a non-negative Borel function which satisfies

$$\int_0^\infty f(x)\,\mathrm{d}x = 1, \tag{1.1}$$

then there exists a unique probability law $\mathbb{W}^{(f)}$ on $\mathcal{F}_\infty$ such that

$$\frac{\mathbb{W}[f(S_t)F_s]}{\mathbb{W}[f(S_t)]} \longrightarrow \mathbb{W}^{(f)}[F_s] \quad \text{as } t \to \infty, \tag{1.2}$$

for any fixed $s > 0$ and for any bounded $\mathcal{F}_s$-measurable functional F_s. Moreover the limit measure $\mathbb{W}^{(f)}$ is characterized by

$$\mathbb{W}^{(f)}|_{\mathcal{F}_s} = M_s^{(f)} \cdot \mathbb{W}|_{\mathcal{F}_s}, \tag{1.3}$$

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where $(M_s^{(f)}, s \geq 0)$ is a $((\mathcal{F}_s), \mathbb{W})$-martingale which has the form

$$M_s^{(f)} = f(S_s)(S_s - X_s) + \int_{S_s}^{\infty} f(x)\,\mathrm{d}x. \tag{1.4}$$

We remark that these martingales $(M_s^{(f)})$ which are known as the Azéma–Yor martingales were applied to solve the Skorokhod embedding problem; see [1], [2], also [16] and references therein. In [20] the authors have also obtained the description of the probability measure $\mathbb{W}^{(f)}$ as follows.

Theorem 1.1 (Roynette, Vallois and Yor [20]). *The following holds.*

(i) $\mathbb{W}^{(f)}(S_\infty \in \mathrm{d}x) = f(x)\,\mathrm{d}x$.
(ii) *Let* $g = \sup\{t \geq 0: X_t = S_\infty\}$. *Then* $\mathbb{W}^{(f)}(g < \infty) = 1$ *and, under* $\mathbb{W}^{(f)}$, *we have*
  (a) $(X_u, u \leq g)$ *and* $(X_g - X_{g+u}, u \geq 0)$ *are independent*;
  (b) *conditional on* $S_\infty = x$, *the pre-supremum process* $(X_u, u \leq g)$ *is distributed as a Brownian motion starting from* 0 *and stopped at its first hitting time of* x;
  (c) *the post-supremum process* $(X_g - X_{g+u}, u \geq 0)$ *is distributed as a* 3-*dimensional Bessel process starting from* 0.

Theorem 1.1 implies that, under the limit measure $\mathbb{W}^{(f)}$, the time g when the process attains its overall supremum is finite, so that the supremum penalisation procedure can be interpreted as looking for probabilities on canonical space, which are close to $\mathbb{W}$, and such that $S_\infty < \infty$ a.s.

Roynette, Vallois and Yor considered Brownian penalisations for many other kinds of weighted processes. For instance, (L) $\Gamma_t = f(L_t)$ where L_t denotes the local time of X at the origin, and (K) $\Gamma_t = \exp(-\int L(t,x)V(\mathrm{d}x))$ where $L(t,x)$ denotes the local time of X at x; we call the former case *local time penalisation* and the latter case *Kac killing penalisation*. Meanwhile Najnudel, Roynette and Yor [15] have introduced a certain σ-finite measure $\mathcal{W}$ defined as follows:

$$\mathcal{W} = \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} \bigl(\Pi^{(u)} \bullet P^{3B}\bigr), \tag{1.5}$$

where $\Pi^{(u)}$ denotes the law of Brownian bridge from 0 to 0 of length u and $P^{3B} = (P^{3B,+} + P^{3B,-})/2$ denotes the law of symmetrized 3-dimensional Bessel process; $P^{3B,+}$ is the law of 3-dimensional Bessel process starting from 0, BES(3), whereas $P^{3B,-}$ is the law of $(-\mathrm{BES}(3))$. The authors in [15] have shown that the Brownian penalisations including (S)(L)(K) can be understood in a unified manner, thanks to this measure $\mathcal{W}$. Especially in the supremum penalisation case, they have shown the following absolute continuity relationship between $\mathcal{W}$ and $\mathbb{W}^{(f)}$:

$$f(S_\infty) \cdot \mathcal{W}^- = \mathbb{W}^{(f)} \quad \text{on } \mathcal{F}_\infty, \tag{1.6}$$

where

$$\begin{aligned} \mathcal{W}^- &= \mathbf{1}_{\{S_\infty < \infty\}} \cdot \mathcal{W} \\ &= \int_0^\infty \frac{\mathrm{d}u}{\sqrt{2\pi u}} \left(\Pi^{(u)} \bullet \frac{P^{3B,-}}{2}\right). \end{aligned} \tag{1.7}$$

(See Fig. 1.)

As a generalisation of these studies, Yano, Yano and Yor [27] have considered the two kinds of penalisations (L) and (K) in the case of symmetric α-stable Lévy process with index $\alpha \in (1,2]$. Let us denote by $((X_t), \mathbb{P})$ such a stable Lévy process with $\mathbb{P}(X_0 = 0) = 1$. The authors have introduced a σ-finite measure $\mathcal{P}$ defined as follows, which is the analogue of $\mathcal{W}$:

$$\mathcal{P} = \int_0^\infty \frac{\Gamma(1/\alpha)}{\alpha\pi} \frac{\mathrm{d}u}{u^{1/\alpha}} \bigl(\mathbb{Q}^{(u)} \bullet \mathbb{P}^\times\bigr), \tag{1.8}$$

where $\mathbb{Q}^{(u)}$ denotes the law of the stable bridge from 0 to 0 of length u and $\mathbb{P}^\times$ denotes the h-transform process with respect to the harmonic function $|x|^{\alpha-1}$ of the process killed at the first hitting time of 0. We should remark that the

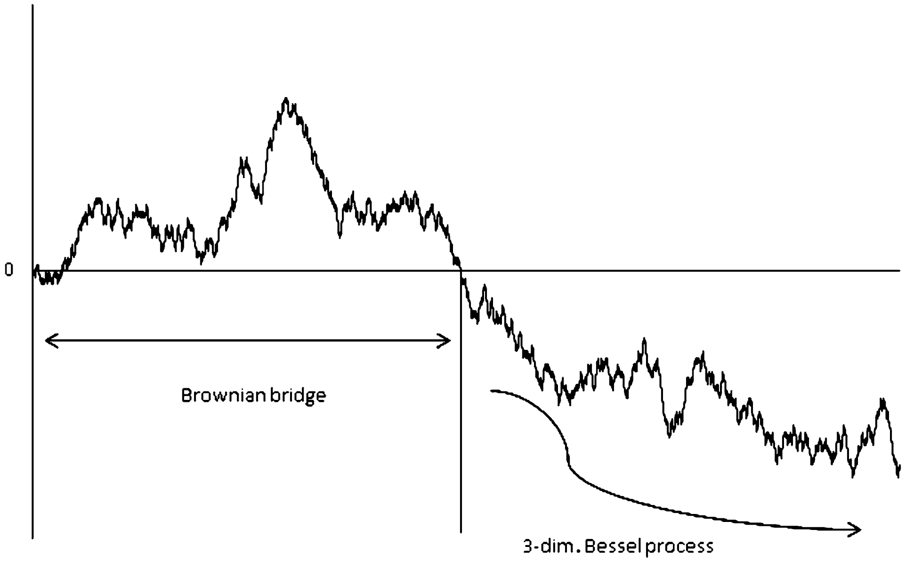

Fig. 1. Sample path of $\Pi^{(u)} \bullet P^{3B,-}$.

process under the measure $\mathbb{P}^{\times}$ is called *conditioned to avoid* 0, because of the following property obtained by K. Yano [25]: if a functional Z is of the form $Z = f(X_{t_1}, \dots, X_{t_n})$ for some $0 < t_1 < \cdots < t_n$ and some continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which vanishes at ∞, then one has

$$\mathbb{P}^{\times}[Z] = \lim_{t\to\infty} \lim_{\varepsilon\to 0+} \mathbb{P}[Z \circ \theta_\varepsilon | \forall u \le t, X_u \circ \theta_\varepsilon \neq 0], \tag{1.9}$$

where $\theta_.$ is the shift operator: $X_u \circ \theta_. = X_{.+u}$. Moreover the following long-time behavior of path under $\mathbb{P}^{\times}$ is also obtained by K. Yano [26]: if $\alpha \in (1,2)$, then

$$\mathbb{P}^{\times}\Big(\limsup_{t\to\infty} X_t = \limsup_{t\to\infty}(-X_t) = \lim_{t\to\infty} |X_t| = \infty\Big) = 1. \tag{1.10}$$

Thus we can see immediately that, under $\mathcal{P}$, $S_\infty = \infty$ a.e. That is, $\mathcal{P}$ cannot unify the supremum penalisations (S) in the stable case.

Yano, Yano and Yor [28] have studied the supremum penalisation for a (α,ρ)-stable Lévy process with index $\alpha \in (0,2]$ and positivity parameter $\rho \in (0,1)$. The authors have introduced a generalised Azéma–Yor martingale $(M_s^{(f)})$ which is defined as

$$M_s^{(f)} = f(S_s)(S_s - X_s)^{\alpha\rho} + \alpha\rho \int_{S_s}^{\infty} f(x)(x - X_s)^{\alpha\rho-1}\,\mathrm{d}x, \tag{1.11}$$

for any non-negative Borel function f satisfying

$$0 < \int_0^\infty f(x)x^{\alpha\rho-1}\,\mathrm{d}x < \infty \tag{1.12}$$

and also introduced the probability measure $\mathbb{P}^{(f)}$ given as

$$\mathbb{P}^{(f)}|_{\mathcal{F}_s} = \frac{M_s^{(f)}}{M_0^{(f)}} \cdot \mathbb{P}\Big|_{\mathcal{F}_s}. \tag{1.13}$$

The authors obtained the following result:

Theorem 1.2 (Yano, Yano and Yor [28]). *Let f be a non-negative function which satisfies either of the following two conditions*:

(i) $f(x) = \mathbf{1}_{\{x \le a\}}$ *for some* $a > 0$;
(ii) f *is absolutely continuous with respect to the Lebesgue measure and satisfies*

$$\lim_{x\to\infty} f(x) = 0 \quad \text{and} \quad 0 < \int_0^\infty \left|f'(x)\right| x^{\alpha\rho}\,\mathrm{d}x < \infty. \tag{1.14}$$

Then it holds that, for any $s > 0$ and any bounded $\mathcal{F}_s$-measurable functional F_s,

$$\frac{\mathbb{P}[f(S_t)F_s]}{\mathbb{P}[f(S_t)]} \longrightarrow \mathbb{P}^{(f)}[F_s] \quad \text{as } t \to \infty. \tag{1.15}$$

We remark that the condition (ii) in Theorem 1.2 is stronger than the condition (1.12) because we have

$$\begin{aligned}\int_0^\infty f'(x)x^{\alpha\rho}\,\mathrm{d}x &= \alpha\rho\int_0^\infty f'(x)\,\mathrm{d}x\int_0^x y^{\alpha\rho-1}\,\mathrm{d}y \\ &= \alpha\rho\int_0^\infty y^{\alpha\rho-1}\,\mathrm{d}y\int_y^\infty f'(x)\,\mathrm{d}x \\ &= k - \alpha\rho\int_0^\infty f(y)y^{\alpha\rho-1}\,\mathrm{d}y.\end{aligned}$$

One may conjecture that the assumption of Theorem 1.2 can be weakened to the condition (1.12) that is sufficient to define the generalised Azéma–Yor martingale and the measure $\mathbb{P}^{(f)}$; however, this is still an open problem.

In the present paper we introduce a certain σ-finite measure $\mathcal{P}_{\sup}$ by using Chaumont's h-transform processes for Lévy processes (cf. Theorem 5.1 below):

$$\mathcal{P}_{\sup} = \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}),$$

where ψ is the function stated below in (2.10), $\mathbb{P}_{0\nearrow x}$ denotes the law of the process starting from 0 and conditioned to hit x continuously (in fact, under $\mathbb{P}_{0\nearrow x}$, the process starting from 0 is killed at the first hitting time at x), and $\mathbb{P}_{x\downarrow x}$ denotes the law of the process starting from x and conditioned to stay below level x. $\mathcal{P}_{\sup}$ is another analogue of $\mathcal{W}$ and $\mathcal{P}$, and it is a generalisation of $\mathcal{W}^-$ given in (1.7). We remark that, in the Brownian case, $\mathcal{P}_{\sup}^{\mathrm{BM}}$ is given by the following:

$$\begin{aligned}\mathcal{P}_{\sup}^{\mathrm{BM}} &= \int_0^\infty \mathrm{d}x\left(\mathbb{W}_{0\nearrow x} \bullet P_x^{3B,-}\right) \\ &= \int_0^\infty \mathrm{d}x\int_0^\infty \mathrm{d}u\frac{x}{\sqrt{2\pi u^3}}\mathrm{e}^{-x^2/(2u)}\left(\mathbb{W}_{0\nearrow x}^{(u)} \bullet P_x^{3B,-}\right),\end{aligned} \tag{1.16}$$

where $\mathbb{W}_{0\nearrow x}$ denotes the law of Brownian motion killed at the first hitting time at x and $\mathbb{W}_{0\nearrow x}^{(u)}(\cdot) = \mathbb{W}_{0\nearrow x}(\cdot|T_{\{x\}} = u)$, and $P_x^{3B,-}$ denotes the law of the translation by x of $(-\mathrm{BES}(3))$. (See Fig. 2.) The latter equality is obtained from the well-known fact (see, e.g., [11]) that

$$\mathbb{W}(T_{\{x\}} \in \mathrm{d}u) = \mathrm{d}u\frac{x}{\sqrt{2\pi u^3}}\mathrm{e}^{-x^2/(2u)}. \tag{1.17}$$

We note that the measure $\mathcal{P}_{\sup}^{\mathrm{BM}}$ equals $\mathcal{W}^-$ by the agreement formula obtained by Pitman and Yor [17].

We then show that the measure $\mathcal{P}_{\sup}$ unifies the supremum penalisations. More precisely, we shall define a probability measure $\mathbb{P}^{(f)}$ as the transformation of the law $\mathbb{P}$ of a Lévy process by the generalised Azéma–Yor martingale defined as (6.2) below. This measure $\mathbb{P}^{(f)}$ is the generalisation of (1.13) for a general Lévy process. We then prove

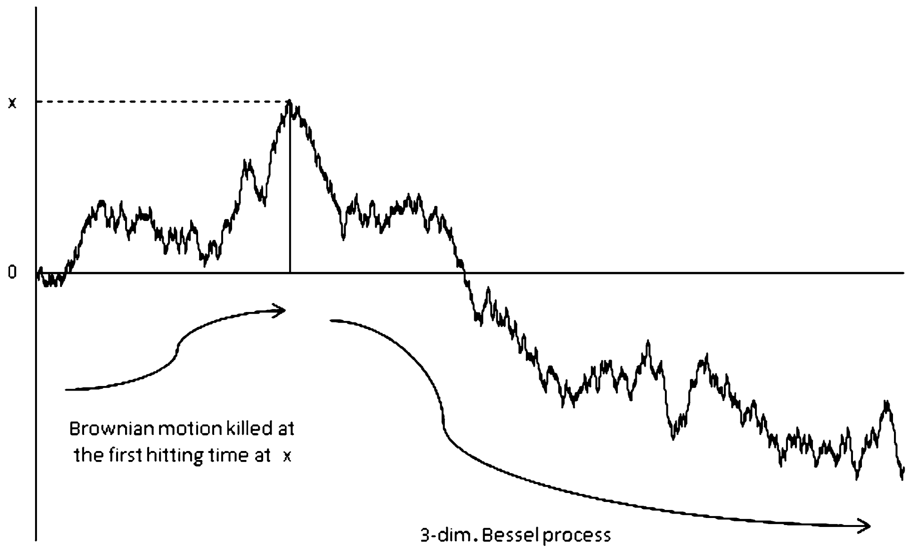

Fig. 2. Sample path of $\mathbb{W}^{(u)}_{0\nearrow x} \bullet P^{3B,-}_x$.

the absolute continuity relationship between $\mathcal{P}_{\sup}$ and $\mathbb{P}^{(f)}$ in the Lévy case, which is the analogue of (1.6) (cf. Theorem 7.3 below):

$$\frac{f(S_\infty) \cdot \mathcal{P}_{\sup}}{\mathcal{P}_{\sup}[f(S_\infty)]} = \mathbb{P}^{(f)} \quad \text{on } \mathcal{F}_\infty.$$

We obtain a detailed description of $\mathbb{P}^{(f)}$ as a consequence of this result (cf. Theorem 7.5 below):

$$\mathbb{P}^{(f)} = \int_0^\infty \mathbb{P}^{(f)}(S_\infty \in \mathrm{d}x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}).$$

To prove the absolute continuity relationship between $\mathcal{P}_{\sup}$ and $\mathbb{P}^{(f)}$, we shall introduce a path decomposition of the law $\mathbb{P}$ of a Lévy process up to a fixed time t with respect to the position and the time where the process attains its supremum before time t.

The organization of the present paper is as follows. In Sections 2 and 3, we recall some preliminary facts about Lévy processes and (α, ρ)-stable Lévy processes, respectively. If a reader needs to see details, he/she may refer to, e.g., [3,10,12,23]. In Section 4, we review Chaumont's two kinds of h-transform processes for a Lévy process. In Section 5, we establish a path decomposition of the law of a Lévy process at the position and the time where the Lévy process attains its supremum up to a fixed time t. In Section 6, we introduce the generalised Azéma–Yor martingale in the general Lévy case, which is the generalisation of (1.4) and (1.11). A certain probability measure which should appear as the limit measure of the supremum penalisation is also introduced in this section. In Section 7, we introduce the σ-finite measure $\mathcal{P}_{\sup}$ which unifies the supremum penalisations and give some properties of the measure $\mathcal{P}_{\sup}$. In Section 8, we compare $\mathcal{P}_{\sup}$ with $\mathcal{P}$ and give some remarks on these measures.

2. Preliminaries about Lévy processes

Let $\mathcal{D}([0,\infty))$ be the space of càdlàg paths $\omega : [0,\infty) \to \mathbb{R} \cup \{\delta\}$ with lifetime $\zeta(\omega) = \inf\{s\colon \omega(s) = \delta\}$ where δ is a cemetery point. Let (X_t) denote the coordinate process, $X_t(\omega) = \omega_t$, and let $(\mathcal{F}_t)$ denote its natural filtration with $\mathcal{F}_\infty = \bigvee_{t\ge 0} \mathcal{F}_t$. Let $\mathbb{P}$ be the law of a Lévy process $X = (X_t, t \ge 0)$ with $\mathbb{P}(X_0 = 0) = 1$ such that

$$\mathbb{P}\bigl[\exp\{\mathrm{i}\lambda X_t\}\bigr] = \mathrm{e}^{-t\Psi(\lambda)}, \quad t \ge 0, \lambda \in \mathbb{R}, \tag{2.1}$$

where

$$\Psi(\lambda) = \mathrm{i}\gamma\lambda + \frac{\sigma^2\lambda^2}{2} + \int_{\mathbb{R}\setminus\{0\}} \big(1 - \mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{i}\lambda x 1_{\{|x|<1\}}\big)\nu(\mathrm{d}x) \tag{2.2}$$

for some constants γ, σ, and Lévy measure ν on $\mathbb{R}\setminus\{0\}$ which satisfies

$$\int_{\mathbb{R}\setminus\{0\}} \big(x^2 \wedge 1\big)\nu(\mathrm{d}x) < \infty. \tag{2.3}$$

We denote by $\mathbb{P}_x$ the law of $X + x$ under $\mathbb{P}$ for every $x \in \mathbb{R}$. Throughout this paper we assume the following absolute continuity condition (A1):

(A1) For each $\alpha > 0$, there exists an integrable function u_α such that

$$\mathbb{P}_x\left[\int_0^\infty \mathrm{e}^{-\alpha t} f(X_t)\,\mathrm{d}t\right] = \int_{-\infty}^\infty u_\alpha(y) f(x+y)\,\mathrm{d}y, \tag{2.4}$$

for every non-negative Borel function f.

Let S_t and I_t be respectively the supremum and the infimum processes up to time t, that is, for all $t < \zeta(\omega)$,

$$S_t = \sup\{X_s\colon 0 \le s \le t\} \quad \text{and} \quad I_t = \inf\{X_s\colon 0 \le s \le t\}. \tag{2.5}$$

Let T_A denote the first entrance time of a Borel set $A \subset \mathbb{R}$ of X, i.e.,

$$T_A = \inf\{s > 0\colon X_s \in A\}. \tag{2.6}$$

Define

$$R = S - X. \tag{2.7}$$

The process $R = (R_t, t \ge 0)$ is called *the reflected process of X at the supremum*. We recall that R is a strong Markov process (Bingham [5], see also [4]). We consider the following condition (A2):

(A2) 0 is regular for $(0,\infty)$ with respect to X under $\mathbb{P}$, i.e., $\mathbb{P}(T_{(0,\infty)} = 0) = 1$.

Then 0 is regular for itself with respect to R, and hence we can define a local time $L = (L_t, t \ge 0)$ at level 0 of R. We denote by τ the right-continuous inverse of L and let $H = X(\tau) = S(\tau)$. We recall that the pair (τ, H) is a bivariate subordinator, called the (upwards) ladder process, in particular, τ and H are separately also subordinators, called the (upwards) ladder time and the (upwards) ladder height process, respectively. Denote by X^* the dual process of X, i.e., $X^* = -X$. Consider

(A2*) 0 is regular for $(-\infty, 0)$ with respect to X under $\mathbb{P}$.

Then we can define a local time L^* at level 0 of $R^* = S^* - X^* = X - I$, and also get the (downwards) ladder time τ^* and the (downwards) ladder height time H^* of R^*.

We denote by E the set of càdlàg paths $e : [0,\infty) \to \mathbb{R} \cup \{\delta\}$ such that

$$e(t) \begin{cases} \in \mathbb{R}\setminus\{0\}, & 0 < t < \zeta_e; \\ = \delta, & t \ge \zeta_e, \end{cases}$$

where

$$\zeta_e = \inf\big\{t > 0 : e(t) = \delta\big\}. \tag{2.8}$$

We call E the set of excursions and an element $e \in E$ an excursion path. For $e \in E$, we call ζ_e the lifetime of the excursion e. Set $D = \{l : \tau_l - \tau_{l-} > 0\}$. For each $l \in D$, we set

$$e_l(t) = \begin{cases} R_{t+\tau_{l-}}, & 0 \le t < \tau_l - \tau_{l-}; \\ \delta, & t \ge \tau_l - \tau_{l-}. \end{cases}$$

By Itô's theorem, the point process $(e_l, l \in D)$ which takes values on E is a Poisson point process, and its characteristic measure $\mathbf{n}$ is called *the Itô measure of excursions*. Similarly, we can introduce excursions e^* with respect to R^* and denote by $\mathbf{n}^*$ its Itô measure.

We recall the following important formula, see also p. 7 in [4], and Proposition (1.10) in Chapter XII in [18]. Denote by $\mathcal{P}(\mathcal{F}_t)$ the predictable σ-field relative to $(\mathcal{F}_t)$ (cf. p. 47 in [18]), and let $\mathcal{E} = \sigma\{e(t)\}$.

Theorem 2.1 (Compensation formula). *Let $F = F(t, \omega, e)$ be a positive process defined on $[0, \infty) \times \mathcal{D} \times E$, measurable with respect to $\mathcal{P}(\mathcal{F}_t) \otimes \mathcal{E}$ and vanishing at δ. Then one has*

$$\mathbb{P}\left[\sum_{l \in D} F(\tau_{l-}, X, e_l)\right] = \mathbb{P} \otimes \widehat{\mathbf{n}}\left[\int_0^\infty \mathrm{d}L_t F(t, X, \widehat{X})\right], \tag{2.9}$$

where the symbol $\widehat{}$ *means independence.*

Under (A1) and (A2), there exists a unique coexcessive function ψ for the killed process, i.e., $\mathbb{P}_{-x}[\psi(X_t^*) \times \mathbf{1}_{\{t<T_{(0,\infty)}\}}] \le \psi(x)$ for $x \ge 0$, which satisfies

$$\int_0^\infty \psi(y) f(y)\,\mathrm{d}y = \mathbb{P}\left[\int_0^\infty f(S_{\tau_s})\,\mathrm{d}s\right] = \mathbb{P}\left[\int_0^\infty f(S_t)\,\mathrm{d}L_t\right], \tag{2.10}$$

for any non-negative Borel function f on $[0, \infty)$. We remark that ψ is continuous and satisfies that $0 < \psi(x) < \infty$ for $x \in (0, \infty)$. Thanks to Silverstein [24], the function ψ is *coharmonic* on $(0, \infty)$, that is,

$$\mathbb{P}_{-x}\left[\psi\left(X_{T_M}^*\right)\mathbf{1}_{\{T_M<T_{(0,\infty)}\}}\right] = \psi(x), \quad x > 0, \tag{2.11}$$

where M denotes a subinterval of $(-\infty, 0)$ whose complement $(-\infty, 0) \setminus M$ is open and has compact closure. We assume further that

(A3) $\mathbb{P}_x(T_{(-\infty,0)} < \infty) = 1$ for $x > 0$.

Note that (A3) is equivalent to that $I_\infty = -\infty$ $\mathbb{P}$-a.s. Then the function h given by

$$h(x) = \int_0^x \psi(y)\,\mathrm{d}y = \mathbb{P}\left[\int_0^\infty \mathbf{1}_{\{S_t \le x\}}\,\mathrm{d}L_t\right] \tag{2.12}$$

is *coinvariant* by Silverstein [24], that is,

$$\mathbb{P}_{-x}\left[h\left(X_t^*\right)\mathbf{1}_{\{t<T_{(0,\infty)}\}}\right] = h(x), \quad x > 0. \tag{2.13}$$

We remark that the function h is finite, continuous, increasing, and that $h(0) = 0$. We remark that every positive coinvariant function is also coharmonic.

Similarly, under (A1) and (A2*), there exists a version of the potential density of the subordinator $(I_{\tau_s^*})_{s\ge 0}$. That is, there exists a unique coexcessive function ψ^* for the killed process, i.e., $\mathbb{P}_x[\psi^*(X_t)\mathbf{1}_{\{t<T_{(-\infty,0)}\}}] \le \psi^*(x)$ for $x \ge 0$, which satisfies

$$\int_0^\infty \psi^*(y) f(y)\,\mathrm{d}y = \mathbb{P}\left[\int_0^\infty f(I_{\tau_s^*})\,\mathrm{d}s\right] = \mathbb{P}\left[\int_0^\infty f(I_t)\,\mathrm{d}L_t^*\right], \tag{2.14}$$

for any non-negative Borel function f on $(0, \infty)$. Also thanks to Silverstein [24], the function ψ^* is coharmonic on $(0, \infty)$, that is,

$$\mathbb{P}_x\left[\psi^*(X_{T_{M'}})\mathbf{1}_{\{T_{M'}<T_{(-\infty,0)}\}}\right] = \psi^*(x), \quad x > 0, \tag{2.15}$$

where M' denotes a subinterval of $(0, \infty)$ whose complement $(0, \infty) \setminus M'$ is open and has the compact closure. If we assume further that

(A3*) $\mathbb{P}_{-x}(T_{(0,\infty)} < \infty) = 1$ for $x > 0$.

Note that (A3*) is equivalent to that $S_\infty = \infty$ $\mathbb{P}$-a.s. Then the function h^* given by

$$h^*(x) = \int_0^x \psi^*(y)\,\mathrm{d}y = \mathbb{P}\biggl[\int_0^\infty \mathbf{1}_{\{I_t \le x\}}\,\mathrm{d}L_t^*\biggr] \tag{2.16}$$

is coinvariant, that is,

$$\mathbb{P}_x\bigl[h^*(X_t)\mathbf{1}_{\{t<T_{(-\infty,0)}\}}\bigr] = h^*(x), \quad x > 0. \tag{2.17}$$

3. Preliminaries about (α, ρ)-stable Lévy processes

Consider a probability measure $\mathbb{P}$ on $\mathcal{D}([0,\infty))$ with respect to which X is a strictly stable Lévy process of index $\alpha \in (0,2]$ with $\mathbb{P}(X_0 = 0) = 1$. That is,

$$\mathbb{P}\bigl[\mathrm{e}^{i\lambda X_t}\bigr] = \mathrm{e}^{-t\Psi(\lambda)}, \quad t \ge 0, \lambda \in \mathbb{R}, \tag{3.1}$$

where

$$\Psi(\lambda) = \begin{cases} c|\lambda|^\alpha\bigl(1 - \mathrm{i}\beta\,\mathrm{sgn}(\lambda)\tan\frac{\pi\alpha}{2}\bigr), & \alpha \in (0,1)\cup(1,2), \\ c|\lambda| + d\mathrm{i}\lambda, & \alpha = 1, \\ c\lambda^2, & \alpha = 2, \end{cases} \tag{3.2}$$

for some constants $c > 0$, $d \in (-\infty,\infty)$ and $\beta \in [-1,1]$. The Lévy measure ν is given by

$$\nu(\mathrm{d}x) = \begin{cases} (c_+\mathbf{1}_{\{x>0\}} + c_-\mathbf{1}_{\{x<0\}})|x|^{-\alpha-1}\,\mathrm{d}x, & \alpha \in (0,1)\cup(1,2), \\ \widetilde{c}|x|^{-2}\,\mathrm{d}x, & \alpha = 1, \\ 0, & \alpha = 2, \end{cases} \tag{3.3}$$

where $\beta = (c_+ - c_-)/(c_+ + c_-)$, and for some constant $\widetilde{c} > 0$. When $c_{+[-]} = 0$, the process is spectrally negative [positive] (or, has no positive [negative] jumps). We remark that the condition (A1) is also valid in the stable Lévy case because of the scaling property of X.

Put $\rho = \mathbb{P}(X_t \ge 0)$. By the scaling property of X, ρ does not depend on $t > 0$. We call ρ *the positivity parameter*. It is well known that the value of ρ for $\alpha \ne 1, 2$ can be represented in terms of the parameter β as

$$\rho = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\biggl(\beta\tan\frac{\pi\alpha}{2}\biggr). \tag{3.4}$$

See Section 2.6 in [29], and p. 218 in [3]. The range of the value of ρ is classified as follows:

$$\rho \begin{cases} \in [0,1] \quad \text{if } \alpha \in (0,1) \\ \quad \text{(when } \rho = 0 \text{ or } 1, \text{the process is a subordinator or a negative subordinator)}, \\ \in (0,1) \quad \text{if } \alpha = 1, \\ \in [1-1/\alpha, 1/\alpha] \quad \text{if } \alpha \in (1,2) \\ \quad \text{(when } \rho = 1-1/\alpha \text{ or } 1/\alpha, \text{ the process is spectrally positive or spectrally negative)}, \\ = 1/2 \quad \text{if } \alpha = 2. \end{cases}$$

Assume that

(B) $\rho \in (0,1)$.

Note that (B) is equivalent to that $|X|$ is not a subordinator. Then $\alpha\rho \in (0,1]$. We note that the condition (B) for the stable Lévy case implies the conditions (A2) and (A2*), that is, 0 is regular for both $(0,\infty)$ and $(-\infty,0)$ with respect to X. Therefore we can define the local times L, L^*, etc. for the reflected and dual reflected processes in this case. Moreover the condition (B) also implies the conditions (A3) and (A3*): More precisely, when $\alpha \in (1,2]$, (A3) and (A3*) hold since X is strictly stable; when $\alpha \in (0,1]$, they hold because of the condition (B).

Assuming (B), the function h defined in (2.12) is

$$h(x) = Cx^{\alpha\rho}, \quad x > 0 \tag{3.5}$$

for some constant $C > 0$. This is obtained from the fact that the ladder time process τ is a stable subordinator of index ρ and the ladder height process H is a stable process of index $\alpha\rho$ (see Lemma VIII 1 in [4]). Furthermore, in this case, we have

$$\psi(x) = C\alpha\rho x^{\alpha\rho-1}, \quad x > 0. \tag{3.6}$$

Similarly, we have

$$h^*(x) = Dx^{\alpha(1-\rho)} \quad \text{and} \quad \psi^*(x) = D\alpha(1-\rho)x^{\alpha(1-\rho)-1}, \quad x > 0 \tag{3.7}$$

for some constant $D > 0$. These constants C and D may depend upon the choice of the local time L and L^*, respectively.

Example 3.1 (Brownian case). *When $\alpha = 2$ and $\rho = 1/2$, X is a 1-dimensional Brownian motion up to a multiplicative constant. In this case we have*

$$h(x) = x \quad \text{and} \quad \psi(x) = 1, \quad x > 0. \tag{3.8}$$

4. Chaumont's two kinds of conditionings for a Lévy process

In this section we shall review two kinds of conditionings for a Lévy process introduced by Chaumont [6,7], which are obtained by Doob's h-transform.

Let $X = ((X_t), \mathbb{P})$ be a Lévy process with the conditions (A1), (A2) and (A3). The functions ψ and h are stated as (2.10) and (2.12), respectively.

4.1. *The process conditioned to stay negative*

For non-negative $\mathcal{F}_t$-measurable functional F_t, define $(\mathbb{P}_{-x\downarrow 0}, x > 0)$ as

$$\mathbb{P}_{-x\downarrow 0}\big[F_t(X)\big] := \frac{1}{h(x)}\mathbb{P}_{-x}\big[h\big(X_t^*\big)\mathbf{1}_{\{t<T_{(0,\infty)}\}}F_t(X)\big], \quad x > 0. \tag{4.1}$$

The family $(\mathbb{P}_{-x\downarrow 0}|_{\mathcal{F}_t}, t \geq 0)$ is proved to be consistent by the coinvariance of the function h and hence $\mathbb{P}_{-x\downarrow 0}$ is well-defined as a probability measure on $\mathcal{F}_\infty$. The process $(X, \mathbb{P}_{-x\downarrow 0})$ is called *the process starting from $(-x)$ and conditioned to stay negative* since it has the following property:

Theorem 4.1 ([6], Theorem 1). *Let* $\mathbf{e}$ *be an independent exponential random variable with index* 1. *Then, for any $x > 0, t \geq 0$ and any $\mathcal{F}_t$-measurable functional F_t, it holds that*

$$\lim_{\varepsilon\to 0} \mathbb{P}_{-x}[\mathbf{1}_{\{t<\mathbf{e}/\varepsilon\}}F_t|X_s < 0, 0 \leq s \leq \mathbf{e}/\varepsilon] = \mathbb{P}_{-x\downarrow 0}[F_t]. \tag{4.2}$$

It is proved by Chaumont [6] and Chaumont and Doney [9] that $\mathbb{P}_{-x\downarrow 0}$ converges in the Skorokhod sense to $\mathbb{P}_{0\downarrow 0}$ as $x \to 0$. Thus it follows from Theorem 4.1 that, for every $x \geq 0$,

$$\mathbb{P}_{-x\downarrow 0}\Big(X_0 = -x; \zeta = \infty; X_t < 0 \text{ for all } t > 0; \lim_{t\to\infty} X_t = -\infty\Big) = 1. \tag{4.3}$$

Here ζ denotes the lifetime.

Chaumont [6] also showed the absolutely continuity between $\mathbb{P}_{0\downarrow 0}$ and the excursion measure $\mathbf{n}$ of the reflected process $R = S - X$ as follows:

Theorem 4.2 ([6], Theorem 3). *It holds*

$$\mathbb{P}_{0\downarrow 0}\big[F_t(X)\big] = \mathbf{n}\big[h(X_t)\mathbf{1}_{\{t<\zeta_e\}}F_t\big(X^*\big)\big], \tag{4.4}$$

for non-negative $\mathcal{F}_t$-measurable functional F_t.

For $b \le a$, denote by $\mathbb{P}_{b\downarrow a}$ the law of $X + a$ under $\mathbb{P}_{b-a\downarrow 0}$, that is, $(X, \mathbb{P}_{b\downarrow a})$ is the process starting from b and conditioned to stay below level a.

4.2. *The process conditioned to hit* 0 *continuously*

Define $(\mathbb{P}_{-x\nearrow 0}, x > 0)$ as

$$\mathbb{P}_{-x\nearrow 0}\big[\mathbf{1}_{\{t<\zeta\}}F_t(X)\big] := \frac{1}{\psi(x)}\mathbb{P}_{-x}\big[\psi\big(X_t^*\big)\mathbf{1}_{\{t<T_{(0,\infty)}\}}F_t(X)\big], \tag{4.5}$$

for non-negative $\mathcal{F}_t$-measurable functional F_t. The process $(X, \mathbb{P}_{-x\nearrow 0})$ is called *the process starting from* $(-x)$ *and conditioned to hit* 0 *continuously*, or also called *the process conditioned to die at* 0, and has the following property:

Theorem 4.3 ([6], Proposition 2). *For $x > 0$, it holds that*

$$\mathbb{P}_{-x\nearrow 0}(X_0 = -x; \zeta < \infty; X_t < 0 \text{ for all } t < \zeta; X_{\zeta-} = 0) = 1, \tag{4.6}$$

where ζ denotes the lifetime.

The following result is also shown by Chaumont [6]:

Theorem 4.4 ([6], Proposition 3). *For any $x > 0, k > 0, t \ge 0$ and any $\mathcal{F}_t$-measurable functional F_t,*

$$\lim_{\varepsilon\to 0}\mathbb{P}_{-x}[\mathbf{1}_{\{t<T_{(-k,\infty)}\}}F_t \mid S_{T_{(0,\infty)}-} \ge -\varepsilon] = \mathbb{P}_{-x\nearrow 0}[\mathbf{1}_{\{t<T_{(-k,0)}\}}F_t]. \tag{4.7}$$

Denote by $\mathbb{P}_{0\nearrow x}$ the law of $X + x$ under $\mathbb{P}_{-x\nearrow 0}$, that is, $(X, \mathbb{P}_{0\nearrow x})$ is the process starting from 0 and conditioned to hit x continuously. For later use, we rewrite (4.5) by translation to obtain

$$\mathbb{P}_{0\nearrow x}\big[\mathbf{1}_{\{t<\zeta\}}F_t(X)\big] = \frac{1}{\psi(x)}\mathbb{P}\big[\psi(x - X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}}F_t(X)\big], \tag{4.8}$$

since we have

$$\begin{aligned}
\mathbb{P}_{0\nearrow x}\big[\mathbf{1}_{\{t<\zeta\}}F_t(X)\big] &= \mathbb{P}_{-x\nearrow 0}\big[\mathbf{1}_{\{t<\zeta\}}F_t(X + x)\big] \\
&= \frac{1}{\psi(x)}\mathbb{P}_{-x}\big[\psi\big(X_t^*\big)\mathbf{1}_{\{t<T_{(0,\infty)}\}}F_t(X + x)\big] \\
&= \frac{1}{\psi(x)}\mathbb{P}\big[\psi\big(x + X_t^*\big)\mathbf{1}_{\{t<T_{(x,\infty)}\}}F_t(X)\big].
\end{aligned}$$

5. Path decomposition at the position and the time where the Lévy process attains its supremum up to time t

Our aim in this section is to prove Theorem 5.1, which consists of a path decomposition with respect to the position and the time where the Lévy process attains its supremum up to time $t > 0$.

Let us denote by $X^{(u)}$ the coordinate process considered up to time u, i.e.,

$$X_t^{(u)} = \begin{cases} X_t, & t < u; \\ \delta, & t \ge u \end{cases}$$

and denote by $\mathbb{P}_x^{(u)}$ the law of $X^{(u)}$ under $\mathbb{P}_x$. We denote the concatenation between two independent processes $X^{(u)}$ and $\widehat{X}^{(v)}$ by $X^{(u)} \bullet \widehat{X}^{(v)}$, i.e.,

$$\big(X^{(u)} \bullet \widehat{X}^{(v)}\big)_t = \begin{cases} X_t^{(u)}, & 0 \le t < u; \\ \widehat{X}_{t-u}^{(v)}, & u \le t < u+v; \\ \delta, & t \ge u+v. \end{cases}$$

We define the measure $\mathbb{P}_x^{(u)} \bullet \mathbb{P}_y^{(v)}$ as the law of the concatenation $X^{(u)} \bullet \widehat{X}^{(v)}$ between two independent processes $X^{(u)}$ and $\widehat{X}^{(v)}$ where $(X^{(u)}, \widehat{X}^{(v)})$ is considered under the product measure $\mathbb{P}_x^{(u)} \otimes \widehat{\mathbb{P}}_y^{(v)}$.

For $t > 0$, we denote the last time when the process attains its supremum before t by

$$g_t = \sup\{s \le t\colon\ X_s = S_s\}, \tag{5.1}$$

with the convention $\sup \varnothing = 0$.

Theorem 5.1. *Let $X = ((X_t), \mathbb{P})$ be a Lévy process with $\mathbb{P}(X_0 = 0) = 1$ and assume* (A1), *as well as both* (A2) *and* (A2*). *Let $F_t(X^{(t)}) = F(t, X_{t\wedge \cdot})$. Then it holds that*

$$\mathbb{P}\big[F_t\big(X^{(t)}\big)\big] = \int \rho_t(\mathrm{d}x\,\mathrm{d}u)\big(\mathbb{P}_{0\nearrow x}^{(u)} \bullet \mathbb{M}_x^{(t-u)}\big)\big[F_t\big(X^{(t)}\big)\big], \tag{5.2}$$

where the integral is taken over $[0,\infty) \times [0,t)$ and

$$\rho_t(\mathrm{d}x\,\mathrm{d}u) = \mathrm{d}x\,\psi(x)\mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)\mathbf{n}(\zeta_e > t-u); \tag{5.3}$$

$$\mathbb{P}_{0\nearrow x}^{(u)}(\cdot) = \mathbb{P}_{0\nearrow x}(\cdot \mid \zeta = u) \quad (\zeta \text{ denotes the lifetime}); \tag{5.4}$$

$$\mathbb{M}_x^{(s)}\big[F(X)\big] = \frac{\mathbf{n}[F(x - X^{(s)}); \zeta_e > s]}{\mathbf{n}(\zeta_e > s)} \quad (\zeta_e \text{ denotes the lifetime}). \tag{5.5}$$

In other words, the following statements hold:

(i) *$\rho_t(\mathrm{d}x\,\mathrm{d}u)$ gives the joint distribution of S_t and g_t, i.e.,*

$$\rho_t(\mathrm{d}x\,\mathrm{d}u) = \mathbb{P}(S_t \in \mathrm{d}x, g_t \in \mathrm{d}u); \tag{5.6}$$

(ii) *given $g_t = u$, the pre-supremum process $(X_s, s \le u)$ and the post-supremum process $(X_u - X_{u+s}, 0 \le s \le t-u)$ are independent under $\mathbb{P}$;*

(iii) *given $S_t = x$ and $g_t = u$, $(X_s, s \le u)$ under $\mathbb{P}$ is distributed as $\mathbb{P}_{0\nearrow x}^{(u)}$; the process conditioned to hit x continuously, with duration u;*

(iv) *given $S_t = x$ and $g_t = u$, $(x - X_{u+s}, 0 \le s \le t-u)$ under $\mathbb{P}$ is distributed as the meander $\mathbb{M}^{(t-u)} := \mathbb{M}_0^{(t-u)}$.*

Remark 5.2. *The fact* (ii) *in Theorem* 5.1 *is well-known and can be found in Lemma* VI 6 *in* [4].

Remark 5.3. *We can also see that $X_{g_t} = X_{g_t-}$, that is, the process does not jump at g_t; the last hitting time of its supremum up to time t. This fact is guaranteed by the conditions* (A2) *and* (A2*), *see also* [4], *p.* 160.

Remark 5.4. *Theorem* 5.1 *is obtained independently by Chaumont* [8] *in his recent work for some purpose different from ours.*

Before the proof of Theorem 5.1, we recall the following lemma from Chaumont [7]:

Lemma 5.5 ([7], Lemma 3). *Let* $X = ((X_t), \mathbb{P})$ *be a Lévy process with* $\mathbb{P}(X_0 = 0) = 1$ *satisfying conditions* (A1), (A2) *and* (A2*). *Denote by* L *the local time at* 0 *of the reflected process* $R = S - X$. *Let* H *be a predictable functional. Then it holds that*

$$\mathbb{P}\left[\int_0^\infty H_t(X)\,\mathrm{d}L_t\right] = \int_0^\infty \mathbb{P}_{-x\nearrow 0}\big[H_\zeta(X+x)\big]\psi(x)\,\mathrm{d}x. \tag{5.7}$$

The proof of Lemma 5.5 for the stable Lévy process is given in [7]. Lemma 5.5 for the general Lévy process is proved in the same way, so we omit the proof.

Proof of Theorem 5.1. We have

$$\int_0^\infty \mathrm{d}t\, F_t\big(X^{(t)}\big) = \sum_{l\in D}\int_0^{\zeta(e_l)} F_{\tau_{l-}+r}\big(X^{(\tau_{l-})}\bullet (X_{\tau_{l-}} - e_l)\big)\,\mathrm{d}r. \tag{5.8}$$

Hence we have

$$\begin{aligned}\mathbb{P}\left[\int_0^\infty \mathrm{d}t\, F_t\big(X^{(t)}\big)\right] &= \mathbb{P}\left[\sum_{l\in D}\int_0^{\zeta(e_l)} F_{\tau_{l-}+r}\big(X^{(\tau_{l-})}\bullet (X_{\tau_{l-}} - e_l)\big)\,\mathrm{d}r\right] \\ &= \mathbb{P}\otimes\widehat{\mathbf{n}}\left[\int_0^\infty \mathrm{d}L_s\int_0^{\widehat{\zeta}_e} F_{s+r}\big(X^{(s)}\bullet\big(X_s - \widehat{X}^{(r)}\big)\big)\,\mathrm{d}r\right],\end{aligned} \tag{5.9}$$

by the compensation formula (Theorem 2.1). By Lemma 5.5, we have

$$\begin{aligned}(5.9) &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{-x\nearrow 0}\otimes\widehat{\mathbf{n}})\left[\int_0^\infty F_{\zeta+r}\big(\big(X^{(\zeta)}+x\big)\bullet\big(X_\zeta + x - \widehat{X}^{(r)}\big)\big)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\,\mathrm{d}r\right] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{-x\nearrow 0}\otimes\widehat{\mathbf{n}})\left[\int_0^\infty F_{\zeta+r}\big(\big(X^{(\zeta)}+x\big)\bullet\big(x - \widehat{X}^{(r)}\big)\big)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\,\mathrm{d}r\right].\end{aligned} \tag{5.10}$$

Here we use the fact that $X_\zeta = 0$. By translation by x of $\mathbb{P}_{-x\nearrow 0}$ and then changing of variable $\zeta + r = u$, we have

$$\begin{aligned}(5.10) &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x}\otimes\widehat{\mathbf{n}})\left[\int_0^\infty F_{\zeta+r}\big(X^{(\zeta)}\bullet\big(x - \widehat{X}^{(r)}\big)\big)\mathbf{1}_{\{r<\widehat{\zeta}_e\}}\,\mathrm{d}r\right] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x}\otimes\widehat{\mathbf{n}})\left[\int_0^\infty F_u\big(X^{(\zeta)}\bullet\big(x - \widehat{X}^{(u-\zeta)}\big)\big)\mathbf{1}_{\{u-\zeta<\widehat{\zeta}_e\}}\mathbf{1}_{\{u>\zeta\}}\,\mathrm{d}u\right].\end{aligned} \tag{5.11}$$

This identity holds with F_t replaced by $\mathrm{e}^{-qt}F_t$ for any $q > 0$, and hence, by uniqueness of the Laplace transform, we obtain

$$\mathbb{P}\big[F_t\big(X^{(t)}\big)\big] = \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x}\otimes\widehat{\mathbf{n}})\big[F_t\big(X^{(\zeta)}\bullet\big(x - \widehat{X}^{(t-\zeta)}\big)\big)\mathbf{1}_{\{t-\zeta<\widehat{\zeta}_e\}}\mathbf{1}_{\{t>\zeta\}}\big] \tag{5.12}$$

$$= \int_0^\infty \mathrm{d}x\,\psi(x)\int_0^t \mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\big(\mathbb{P}_{0\nearrow x}^{(u)}\otimes\widehat{\mathbf{n}}\big)\big[F_t\big(X^{(u)}\bullet\big(x - \widehat{X}^{(t-u)}\big)\big)\mathbf{1}_{\{t-u<\widehat{\zeta}_e\}}\big] \tag{5.13}$$

$$= \int_0^\infty \mathrm{d}x\,\psi(x)\int_0^t \mathbb{P}_{0\nearrow x}(\zeta\in\mathrm{d}u)\mathbf{n}(\zeta_e > t-u)\big(\mathbb{P}_{0\nearrow x}^{(u)}\otimes\widehat{\mathbb{M}}_x^{(t-u)}\big)\big[F_t\big(X^{(u)}\bullet\widehat{X}^{(t-u)}\big)\big], \tag{5.14}$$

which completes the proof. □

Remark 5.6.
(i) *In the* (α, ρ)*-stable Lévy case with* $\alpha \in (0, 2]$ *and* $\rho \in (0, 1)$, *it is well known that* (*see Lemma* 3.2 *in* [13])

$$\mathbf{n}(\zeta_e > t) = K \cdot t^{-\rho}, \tag{5.15}$$

where $K > 0$ *is some constant*, *and hence we obtain from* (3.6) *and* (5.3) *that*

$$\mathbb{P}(S_t \in \mathrm{d}x, g_t \in \mathrm{d}u) = \widetilde{K} \cdot \mathrm{d}x\, x^{\alpha\rho-1} \mathbb{P}_{0 \nearrow x}(\zeta \in \mathrm{d}u)(t-u)^{-\rho}, \tag{5.16}$$

where $\widetilde{K} > 0$ *is other constant*. *Furthermore*, *together with the following well-known fact* (*see*, *e.g.*, *Lemma* VIII 13 *in* [4]) *that*

$$\mathbb{P}(g_t \in \mathrm{d}u) = \frac{1}{\Gamma(1-\rho)\Gamma(\rho)} u^{\rho-1}(t-u)^{-\rho}\, \mathrm{d}u, \tag{5.17}$$

then we obtain

$$\mathbb{P}(S_t \in \mathrm{d}x | g_t = u)\, \mathrm{d}u = K \cdot \mathrm{d}x\, x^{\alpha\rho-1} \mathbb{P}_{0 \nearrow x}(\zeta \in \mathrm{d}u) \Gamma(1-\rho)\Gamma(\rho) u^{1-\rho}. \tag{5.18}$$

(ii) *In the Brownian case*, *i.e.*, $\alpha = 2$ *and* $\rho = 1/2$ *in* (3.2), *we note that* $(X_t) \overset{law}{=} (W_{2ct})$ *for a* 1*-dimensional standard Brownian motion* (W_t), *and we have the following*:

$$\mathbb{P}(S_t \in \mathrm{d}x, g_t \in \mathrm{d}u) = \mathrm{d}x\, \mathrm{d}u \frac{x}{2c\pi\sqrt{u^3(t-u)}} \mathrm{e}^{-x^2/(4u)}; \tag{5.19}$$

$$\mathbb{P}_{0 \nearrow x}(\zeta \in \mathrm{d}u) = \mathbb{P}(T_{\{x\}} \in \mathrm{d}u) = \mathrm{d}u \frac{x}{2c\sqrt{\pi u^3}} \mathrm{e}^{-x^2/(4u)}, \tag{5.20}$$

because of the following well-known facts (*see*, *e.g.*, *p*. 102 *and p*. 80 *in* [11], *respectively*):

$$\mathbb{P}(\widetilde{S}_t \in \mathrm{d}x, \widetilde{g}_t \in \mathrm{d}u) = \mathrm{d}x\, \mathrm{d}u \frac{x}{\pi\sqrt{u^3(t-u)}} \mathrm{e}^{-x^2/(2u)}; \tag{5.21}$$

$$\mathbb{P}(\widetilde{T}_{\{x\}} \in \mathrm{d}u) = \mathrm{d}u \frac{x}{\sqrt{2\pi u^3}} \mathrm{e}^{-x^2/(2u)}, \tag{5.22}$$

where $\widetilde{S}_t = \sup_{s \le t} W_s$, $\widetilde{g}_t = \sup\{s \le t\colon\ W_s = \widetilde{S}_t\}$, *and* $\widetilde{T}_A = \inf\{s > 0\colon\ W_s \in A\}$ *for a Borel set* $A \subset \mathbb{R}$. *Thus we can easily check that the equality* (5.16) *is valid.*

Remark 5.7. *Assume moreover* (A3). *Then*, *thanks to Bertoin's result*; *Corollary* 3.2 *in* [3], *it holds that*

$$\lim_{t\to\infty} \mathbb{M}^{(t)}\big[F(X)\big] = \mathbb{P}_{0\downarrow 0}\big[F(X)\big], \tag{5.23}$$

where

$$\mathbb{M}^{(t)}\big[F(X)\big] = \mathbb{M}_0^{(t)}\big[F(X)\big] = \frac{\mathbf{n}[F(-X^{(t)}); \zeta_e > t]}{\mathbf{n}(\zeta_e > t)}. \tag{5.24}$$

6. Generalised Azéma–Yor martingales and definition of a probability measure $\mathbb{P}^{(f)}$

Let us introduce a generalisation of (1.4) and (1.11). Let $X = ((X_t), \mathbb{P})$ be a Lévy process with notation given in Section 2 and assume (A1), (A2) and (A3). Let ψ and h be the functions given by (2.10) and (2.12), respectively. Let f be a non-negative Borel function on $[0, \infty)$ satisfying

$$(0 <) \int_0^\infty f(x)\psi(x)\, \mathrm{d}x < \infty. \tag{6.1}$$

We introduce the process $(M_t^{(f)}, t \geq 0)$ by

$$M_t^{(f)} = f(S_t)h(S_t - X_t) + \int_{S_t}^{\infty} f(x)\psi(x - X_t)\,\mathrm{d}x. \tag{6.2}$$

Theorem 6.1. *$(M_t^{(f)}, t \geq 0)$ is a $((\mathcal{F}_t), \mathbb{P})$-martingale.*

The proof of Theorem 6.1 is done in the same way as in [28] in the stable Lévy case; the coinvariance of the function h plays a key role. Thus we omit it.

We introduce the probability measure $\mathbb{P}^{(f)}$ on $\mathcal{F}_\infty$ as follows:

$$\mathbb{P}^{(f)}|_{\mathcal{F}_t} = \frac{M_t^{(f)}}{M_0^{(f)}} \cdot \mathbb{P}\bigg|_{\mathcal{F}_t}. \tag{6.3}$$

Since $(M_t^{(f)})$ is a martingale, the consistency holds, and hence $\mathbb{P}^{(f)}$ is well defined.

7. The σ-finite measure which unifies the supremum penalisations

Let us consider a Lévy process $X = ((X_t), \mathbb{P})$ with $\mathbb{P}(X_0 = 0) = 1$. In this section we assume:

(A1), i.e., absolute continuity condition for the resolvent;
(A2) & (A2*), i.e., 0 is regular for both $(0, \infty)$ and $(-\infty, 0)$ with respect to X;
(A3) & (A3*), i.e., $I_\infty = -\infty$ and $S_\infty = \infty$ $\mathbb{P}$-a.s.,

where I_∞ and S_∞ are the overall infimum and supremum of X_t, respectively, i.e., $I_\infty = \inf\{X_t\colon t \geq 0\}$ and $S_\infty = \sup\{X_t\colon t \geq 0\}$. Remark again that the condition (B) in the (α, ρ)-stable Lévy case implies all the above conditions.

We introduce $\mathcal{P}_{\mathrm{sup}}$ as follows.

Definition 7.1. *Define*

$$\mathcal{P}_{\mathrm{sup}} = \int_0^{\infty} \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}), \tag{7.1}$$

where $\mathbb{P}_{0\nearrow x}$ denotes the law of $X + x$ under $\mathbb{P}_{-x\nearrow 0}$, i.e., $\mathbb{P}_{0\nearrow x}$ denotes the law of the process starting from 0 and conditioned to hit x continuously, and $\mathbb{P}_{x\downarrow x}$ denotes the law of $X + x$ under $\mathbb{P}_{0\downarrow 0}$, i.e., $\mathbb{P}_{x\downarrow x}$ denotes the law of the process starting from x and conditioned to stay below level x.

Denote

$$g = \sup\{t \geq 0\colon X_t = S_\infty\}. \tag{7.2}$$

Theorem 7.2. *The following statements hold*:

(i) $\mathcal{P}_{\mathrm{sup}}(S_\infty \in \mathrm{d}x, g \in \mathrm{d}u) = \mathrm{d}x\,\psi(x)\mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)$, *in particular*, $\mathcal{P}_{\mathrm{sup}}(S_\infty \in \mathrm{d}x) = \mathrm{d}x\,\psi(x)$;
(ii) $\mathcal{P}_{\mathrm{sup}}$ *is a σ-finite measure on* $\mathcal{F}_\infty$;
(iii) $\mathcal{P}_{\mathrm{sup}}$ *is singular to* $\mathbb{P}$ *on* $\mathcal{F}_\infty$;
(iv) *For each $t > 0$ and $A \in \mathcal{F}_t$, it holds that*

$$\mathcal{P}_{\mathrm{sup}}(A) = \begin{cases} 0, & \text{if } \mathbb{P}(A) = 0; \\ \infty, & \text{if } \mathbb{P}(A) > 0. \end{cases} \tag{7.3}$$

Consequently, $\mathcal{P}_{\mathrm{sup}}$ is not σ-finite on $\mathcal{F}_t$ for $t < \infty$.

Proof.

(i) We have

$$\mathcal{P}_{\sup} = \int_0^\infty \mathrm{d}x\,\psi(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)\big(\mathbb{P}^{(u)}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\big), \tag{7.4}$$

and hence

$$\begin{aligned}\mathcal{P}_{\sup}\big[F(S_\infty)G(g)\big] &= \int_0^\infty \mathrm{d}x\,\psi(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)\big(\mathbb{P}^{(u)}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x}\big)\big[F(S_\infty)G(g)\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)F(x) \int_0^\infty \mathbb{P}_{0\nearrow x}(\zeta \in \mathrm{d}u)G(u),\end{aligned}$$

for any test functions F and G. Thus we obtain the desired result.

(ii) For each $x > 0$, $\mathcal{P}_{\sup}(S_\infty < x) = \int_0^x \psi(y)\,\mathrm{d}y$ is finite, which shows the desired conclusion.

(iii) We have $\mathcal{P}_{\sup}(S_\infty = \infty) = 0$ by definition. On the other hand, we have $\mathbb{P}(S_\infty < \infty) = 0$ by our assumption (A3*). This implies that $\mathcal{P}_{\sup}$ is singular to $\mathbb{P}$ on $\mathcal{F}_\infty$.

(iv) Suppose that $\mathbb{P}(A) = 0$ for $A \in \mathcal{F}_t$. We have

$$\begin{aligned}\mathcal{P}_{\sup}(A) &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x})(A) \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x})[\mathbf{1}_A; t < \zeta] + \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x})[\mathbf{1}_A; t \ge \zeta] =: I_1 + I_2.\end{aligned}$$

On one hand, we have

$$\begin{aligned}I_1 &= \int_0^\infty \mathrm{d}x\,\psi(x)\mathbb{P}_{0\nearrow x}[\mathbf{1}_A; t < \zeta] \\ &= \int_0^\infty \mathrm{d}x\mathbb{P}\big[\psi(x - X_t)\mathbf{1}_{\{t < T_{(x,\infty)}\}}\mathbf{1}_A\big] \quad \big(\text{by (4.8)}\big) \\ &= 0.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}I_2 &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \bullet \mathbb{P}_{x\downarrow x})\big[\mathbf{1}_A(X); t \ge \zeta\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \otimes \widehat{\mathbb{P}}_{0\downarrow 0})\big[\mathbf{1}_A\big(X^{(\zeta)} \bullet \big(x + \widehat{X}^{(t-\zeta)}\big)\big)\mathbf{1}_{\{t \ge \zeta\}}\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \otimes \widehat{\mathbf{n}})\big[h(\widehat{X}_{t-\zeta})\mathbf{1}_{\{t-\zeta < \widehat{\zeta}_e\}}\mathbf{1}_A\big(X^{(\zeta)} \bullet \big(x - \widehat{X}^{(t-\zeta)}\big)\big)\mathbf{1}_{\{t \ge \zeta\}}\big],\end{aligned} \tag{7.5}$$

by the definition of $\mathbb{P}_{0\downarrow 0}$. Then

$$\begin{aligned}(7.5) &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \otimes \widehat{\mathbf{n}})\big[h\big(x - (x - \widehat{X}_{t-\zeta})\big)\mathbf{1}_A(X)\mathbf{1}_{\{0 \le t-\zeta < \widehat{\zeta}_e\}}\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x} \otimes \widehat{\mathbf{n}})\big[h\big(x - \big(X^{(\zeta)} \bullet \big(x - \widehat{X}^{(t-\zeta)}\big)\big)_t\big)\mathbf{1}_A(X)\mathbf{1}_{\{0 \le t-\zeta < \widehat{\zeta}_e\}}\big] \\ &= \mathbb{P}\big[h(S_t - X_t)\mathbf{1}_A\big] \quad (\text{by Theorem 5.1}) \\ &= 0.\end{aligned}$$

Thus we obtain $\mathcal{P}_{\sup}(A) = 0$.

Conversely, suppose that $\mathbb{P}(A) > 0$ for $A \in \mathcal{F}_t$. Then we see that

$$\begin{aligned}\mathcal{P}_{\sup}(A) &\geq \int_0^\infty \mathrm{d}x\,\psi(x)\mathbb{P}_{0\nearrow x}[\mathbf{1}_A; t < \zeta] \\ &= \int_0^\infty \mathrm{d}x\,\mathbb{P}\big[\psi(x - X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}}\mathbf{1}_A\big] \\ &\geq \int_1^\infty \mathrm{d}x\,\mathbb{P}\big[\psi(x - X_t)\mathbf{1}_{\{t<T_{(1,\infty)}\}}\mathbf{1}_A\big] \\ &= \mathbb{P}\big[\big\{h(\infty) - h(1 - X_t)\big\}\mathbf{1}_{\{t<T_{(1,\infty)}\}}\mathbf{1}_A\big].\end{aligned}$$

Since we have

$$h(\infty) = \lim_{x\to\infty} h(x) = \lim_{x\to\infty} \mathbb{P}\bigg[\int_0^\infty \mathbf{1}_{\{S_t\leq x\}}\,\mathrm{d}L_t\bigg] = \mathbb{P}\bigg[\int_0^\infty \mathrm{d}L_t\bigg] = \mathbb{P}[L_\infty] = \infty,$$

thus $\mathcal{P}_{\sup}(A) = \infty$. Therefore the proof is completed. □

We shall give some relationships between the measures $\mathcal{P}_{\sup}$, $\mathbb{P}$ and $\mathbb{P}^{(f)}$.

Theorem 7.3. *It holds that*

$$\mathcal{P}_{\sup}\big[f(S_\infty)F_t(X)\big] = \mathbb{P}\big[M_t^{(f)}F_t(X)\big], \tag{7.6}$$

for any $\mathcal{F}_t$-measurable functional $F_t(X)$. Consequently, one has

$$\frac{\mathcal{P}_{\sup}[f(S_\infty)F_t(X)]}{\mathcal{P}_{\sup}[f(S_\infty)]} = \mathbb{P}\bigg[\frac{M_t^{(f)}}{M_0^{(f)}}F_t(X)\bigg] = \mathbb{P}^{(f)}\big[F_t(X)\big] \tag{7.7}$$

and

$$\frac{f(S_\infty)\cdot\mathcal{P}_{\sup}}{\mathcal{P}_{\sup}[f(S_\infty)]} = \mathbb{P}^{(f)} \quad \text{on } \mathcal{F}_\infty. \tag{7.8}$$

Proof. Recall the computation in the proof of Theorem 7.2(iv). We have

$$\begin{aligned}\mathcal{P}_{\sup}\big[f(S_\infty)F_t(X)\big] &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\big[f(S_\infty)F_t(X)\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\big[F_t(X)\big],\end{aligned} \tag{7.9}$$

since $S_\infty = x$ under the measure $\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x}$. Then

$$\begin{aligned}(7.9) &= \int_0^\infty \mathrm{d}x\,\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\big[F_t(X); t < \zeta\big] \\ &\quad + \int_0^\infty \mathrm{d}x\,\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\big[F_t(X); t \geq \zeta\big] \\ &=: I_1 + I_2.\end{aligned}$$

On one hand, we have

$$\begin{aligned}I_1 &= \int_0^\infty \mathrm{d}x\,\psi(x)f(x)\mathbb{P}_{0\nearrow x}\big[F_t(X); t < \zeta\big] = \int_0^\infty \mathrm{d}x\,f(x)\mathbb{P}\big[\psi(x - X_t)\mathbf{1}_{\{t<T_{(x,\infty)}\}}F_t(X)\big] \\ &= \mathbb{P}\bigg[F_t(X)\int_0^\infty \mathrm{d}x\,f(x)\psi(x - X_t)\mathbf{1}_{\{S_t\leq x\}}\bigg].\end{aligned} \tag{7.10}$$

On the other hand, we obtain from the same computation in the proof of (iv) in the previous theorem that

$$\begin{aligned} I_2 &= \int_0^\infty \mathrm{d}x\,\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x})\big[F_t(X); t\ge\zeta\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)f(x)(\mathbb{P}_{0\nearrow x}\otimes\widehat{\mathbf{n}})\big[h(x-X_t)F_t(X)\mathbf{1}_{\{0\le t-\zeta<\widehat{\zeta}_e\}}\big] \\ &= \int_0^\infty \mathrm{d}x\,\psi(x)(\mathbb{P}_{0\nearrow x}\otimes\widehat{\mathbf{n}})\big[f(S_t)h(S_t-X_t)\mathbf{1}_{\{t-\zeta<\widehat{\zeta}_e\}}F_t(X)\mathbf{1}_{\{t\ge\zeta\}}\big]. \end{aligned} \tag{7.11}$$

By Theorem 5.1, we get

$$(7.11)=\mathbb{P}\big[f(S_t)h(S_t-X_t)F_t(X)\big]. \tag{7.12}$$

Combining (7.10) and (7.12), we obtain

$$\begin{aligned} \mathcal{P}_{\sup}\big[f(S_\infty)F_t(X)\big] &= \mathbb{P}\bigg[F_t(X)\int_{S_t}^\infty \mathrm{d}x f(x)\psi(x-X_t)\bigg]+\mathbb{P}\big[F_t(X)f(S_t)h(S_t-X_t)\big] \\ &= \mathbb{P}\bigg[F_t(X)\bigg\{\int_{S_t}^\infty \mathrm{d}x f(x)\psi(x-X_t)+f(S_t)h(S_t-X_t)\bigg\}\bigg], \end{aligned} \tag{7.13}$$

that is,

$$\mathcal{P}_{\sup}\big[f(S_\infty)F_t(X)\big]=\mathbb{P}\big[M_t^{(f)}F_t(X)\big]. \tag{7.14}$$

Especially, when $t=0$, we have

$$\mathcal{P}_{\sup}\big[f(S_\infty)\big]=\int_0^\infty \mathrm{d}x f(x)\psi(x). \tag{7.15}$$

Therefore we obtain

$$\frac{\mathcal{P}_{\sup}[f(S_\infty)F_t(X)]}{\mathcal{P}_{\sup}[f(S_\infty)]}=\mathbb{P}\bigg[\frac{M_t^{(f)}}{M_0^{(f)}}F_t(X)\bigg]=\mathbb{P}^{(f)}\big[F_t(X)\big]. \tag{7.16}$$

This completes the proof. □

Remark 7.4. *Recently Najnudel and Nikeghbali [14] gave a generalization of $\mathcal{W}$. A non-negative submartingale $(X_t, t\ge 0)$ is said to be of the class (Σ) if it can be decomposed as $X_t=N_t+A_t$ where $(N_t,t\ge 0)$ and $(A_t,t\ge 0)$ are $\mathcal{F}_t$-adapted process, (N_t) is a càdlàg martingale, and (A_t) is a continuous increasing process which is carried by the set of zeros with $A_0=0$. Then they proved that there exists a σ-finite measure $\mathcal{Q}$ such that*

$$\mathcal{Q}[F_t; g\le t]=\mathbb{P}[F_tX_t], \tag{7.17}$$

for all non-negative $\mathcal{F}_t$-measurable functional F_t. Here g is the last exit time from 0. It may be quite natural to ask now whether the process $(h(S_t-X_t), t\ge 0)$ is of the class (Σ) or not. However, we have not succeeded in answering the question.

The measure $\mathcal{P}_{\sup}$ does not depend upon f. Recall that $\mathbb{P}^{(f)}$ is the limit measure of supremum penalisation. The measure $\mathcal{P}_{\sup}$ implies the following fact that gives the detailed description of $\mathbb{P}^{(f)}$.

Theorem 7.5. *One has*

$$\mathbb{P}^{(f)}=\int_0^\infty \mathbb{P}^{(f)}(S_\infty\in \mathrm{d}x)(\mathbb{P}_{0\nearrow x}\bullet\mathbb{P}_{x\downarrow x}). \tag{7.18}$$

That is, it holds that, under $\mathbb{P}^{(f)}$,

(i) $\mathbb{P}^{(f)}(S_\infty \in \mathrm{d}x) = \frac{1}{M_0^{(f)}}\psi(x)f(x)\,\mathrm{d}x$ *where* $M_0^{(f)} = \int_0^\infty \psi(x)f(x)\,\mathrm{d}x$;

(ii) *given* $g = u$, $(X_s, s \le u)$ *and* $(X_u - X_{u+s}, s \ge 0)$ *are independent*;

(iii) *given* $S_\infty = x$ *and* $g = u$, $(X_s, s \le u)$ *is distributed as the process conditioned to hit* x *continuously with duration* u;

(iv) *given* $S_\infty = x$ *and* $g = u$, $(-(x - X_{u+s}), s \ge 0)$ *is distributed as the process conditioned to stay negative*.

Under our assumption in this section, the following result for the martingale $(M_t^{(f)})$ can be proved.

Theorem 7.6. *Let* $X = ((X_t), \mathbb{P})$ *be a Lévy process with* (A1), (A2), (A2*), (A3) *and* (A3*), *and let* $M_t^{(f)}$ *be the process given in* (6.2). *Then* $M_t^{(f)}$ *converges to* 0 $\mathbb{P}$*-a.s. as* $t \to \infty$.

Proof. We show that $M_t^{(f)} \to 0\,\mathbb{P}$-a.s. through the measure $\mathcal{P}_{\sup}$. Since $(M_t^{(f)})$ is a non-negative $\mathbb{P}$-martingale as proved before, there exists a $\mathcal{F}_\infty$-measurable functional $M_\infty^{(f)}$ such that $M_t^{(f)} \to M_\infty^{(f)}$ $\mathbb{P}$-a.s. by the martingale convergence theorem. For $a > 0$,

$$\begin{aligned}\mathbb{P}\big[M_\infty^{(f)}\big] &= \mathbb{P}\big[M_\infty^{(f)}\mathbf{1}_{\{S_\infty \ge a\}}\big] \quad \big(\text{by the fact that } \mathbb{P}(S_\infty = \infty) = 1\big)\\ &\le \liminf_{t\to\infty} \mathbb{P}\big[M_t^{(f)}\mathbf{1}_{\{S_t \ge a\}}\big] \quad (\text{by Fatou's lemma})\\ &= \liminf_{t\to\infty} \mathcal{P}_{\sup}\big[f(S_\infty)\mathbf{1}_{\{S_t \ge a\}}\big] \quad \big(\text{by (7.7)}\big)\\ &= \mathcal{P}_{\sup}\big[f(S_\infty)\mathbf{1}_{\{S_\infty \ge a\}}\big] \quad (\text{by the dominated convergence theorem}).\end{aligned}$$

Letting $a \to \infty$, then $\mathcal{P}_{\sup}[f(S_\infty)\mathbf{1}_{\{S_\infty \ge a\}}] \to 0$. Thus $\mathbb{P}[M_\infty^{(f)}] = 0$, and therefore we obtain $\mathbb{P}(M_\infty^{(f)} = 0) = 1$. □

8. Some remarks on $\mathcal{P}$ and $\mathcal{P}_{\sup}$

Let us consider a symmetric (i.e., $\rho = 1/2$) stable Lévy process X with index $\alpha \in (1, 2]$, and recall the σ-finite measure $\mathcal{P}$ which is given in [27] (see also [25]):

$$\mathcal{P} = \int_0^\infty \mathbb{P}\big[\mathrm{d}L_u^X\big]\big(\mathbb{Q}^{(u)} \bullet \mathbb{P}^\times\big), \tag{8.1}$$

where L_t^X denotes the local time at 0 of X itself, $\mathbb{Q}^{(u)}$ denotes the law of the stable bridge from 0 to 0 with length u and $\mathbb{P}^\times$ denotes the h-transform process with respect to the harmonic function $|x|^{\alpha-1}$ of the process killed at the first hitting time of 0. On comparison, it becomes clear that the two σ-finite measures $\mathcal{P}_{\sup}$ and $\mathcal{P}$ are quite different: $\mathcal{P}_{\sup}$ is based on the excursion theory for the reflected process of a Lévy process, whereas $\mathcal{P}$ comes from the excursion theory for a Lévy process itself. We stress that this difference cannot appear in the Brownian case because of the fact that $(S_t, S_t - X_t)_{t\ge0} \overset{law}{=} (L_t^X, |X_t|)_{t\ge0}$ which is known as Lévy's theorem.

Finally, we would like to emphasize the following fact again:

$$\mathcal{P}(S_\infty < \infty) = 0 \quad \text{and} \quad \mathcal{P}_{\sup}(S_\infty = \infty) = 0. \tag{8.2}$$

We mention the relationship between $\mathcal{P}_{\sup}$ and $\mathcal{P}$ as follows:

(i) $\mathcal{P} \perp \mathcal{P}_{\sup}$ on $\mathcal{F}_\infty$;

(ii) if $A \in \mathcal{F}_t$, then

$$\mathcal{P}(A) > 0 \quad \Longleftrightarrow \quad \mathcal{P}_{\sup}(A) > 0, \tag{8.3}$$

and both are infinite.

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References

[1] J. Azéma and M. Yor. Une solution simple au problème de Skorokhod. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)* 90–115. *Lecture Notes in Math*. **721**. Springer, Berlin, 1979. MR0544782

[2] J. Azéma and M. Yor. Le problème de Skorokhod: compléments à "Une solution simple au problème de Skorokhod." In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)* 625–633. *Lecture Notes in Math*. **721**. Springer, Berlin, 1979. MR0544832

[3] J. Bertoin. Splitting at the infimum and excursions in half-lines for random walks and Lévy processes. *Stochastic Process. Appl.* **47** (1993) 17–35. MR1232850

[4] J. Bertoin. *Lévy Processes*. Cambridge Univ. Press, Cambridge, 1996. MR1406564

[5] N. Bingham. Maxima of sums of random variables and suprema of stable processes. *Z. Wahrsch. Verw. Gebiete* **26** (1973) 273–296. MR0415780

[6] L. Chaumont. Conditionings and path decompositions for Lévy processes. *Stochastic Process. Appl.* **64** (1996) 39–54. MR1419491

[7] L. Chaumont. Excursion normalisée, méandre et pont pour des processus stables. *Bull. Sci. Math.* **121** (1997) 377–403. MR1465814

[8] L. Chaumont. On the law of the supremum of Lévy processes. *Ann. Probab.* **41** (2013) 1191–1217.

[9] L. Chaumont and R. A. Doney. On Lévy processes conditioned to stay positive. *Electron. J. Probab.* **10** (2005) 948–961 (electronic); corrections in **13** (2008) 1–4 (electronic). MR2164035

[10] R. A. Doney. *Fluctuation Theory for Lévy Processes*. *Lectures from the 35th Summer School on Probability Theory held in Saint-Flour, July 6–23, 2005. Lecture Notes in Math.* **1897**. Springer, Berlin, 2007. MR2320889

[11] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*, 2nd edition. Springer, New York, 1991. MR1121940

[12] A. E. Kyprianou. *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. *Universitext*. Springer, Berlin, 2006. MR2250061

[13] D. Monrad and M. L. Silverstein. Stable processes: Sample function growth at a local minimum. *Z. Wahrsch. Verw. Gebiete* **49** (1979) 177–210. MR0543993

[14] J. Najnudel and A. Nikeghbali. On some properties of a universal sigma-finite measure associated with a remarkable class of submartingales. *Publ. Res. Inst. Math. Sci.* **47** (2011) 911–936. MR2880381

[15] J. Najnudel, B. Roynette and M. Yor. *A Global View of Brownian Penalisations*. *MSJ Memoirs* **19**. Mathematical Society of Japan, Tokyo, 2009. MR2528440

[16] J. Obłój. The Skorokhod embedding problem and its offsprings. *Probability Surveys* **1** (2004) 321–390. MR2068476

[17] J. Pitman and M. Yor. Decomposition at the maximum for excursions and bridges of one-dimensional diffusions. In *Itô's Stochastic Calculus and Probability Theory* 293–310. Springer, Tokyo, 1996. MR1439532

[18] D. Revuz and M. Yor. *Continuous Martingales and Brownian Motion*, 3rd edition. Springer, Berlin, 1999. MR1725357

[19] B. Roynette, P. Vallois and M. Yor. Limiting laws associated with Brownian motion perturbed by normalized exponential weights, I. *Studia Sci. Math. Hungar.* **43** (2006) 171–246. MR2229621

[20] B. Roynette, P. Vallois and M. Yor. Limiting laws associated with Brownian motion perturbed by its maximum, minimum and local time, II. *Studia Sci. Math. Hungar.* **43** (2006) 295–360. MR2253307

[21] B. Roynette, P. Vallois and M. Yor. Some penalisations of the Wiener measure. *Jpn. J. Math.* **1** (2006) 263–290. MR2261065

[22] B. Roynette and M. Yor. *Penalising Brownian Paths*. *Lecture Notes in Math.* **1969**. Springer, Berlin, 2009. MR2504013

[23] K. Sato. *Lévy Processes and Infinitely Divisible Distributions*. Translated from the 1990 Japanese original, Revised by the author. *Cambridge Studies in Advanced Mathematics* **68**. Cambridge University Press, Cambridge, 1999. MR1739520

[24] M. L. Silverstein. Classification of coharmonic and coinvariant functions for Lévy processes. *Ann. Probab.* **8** (1980) 539–575. MR0573292

[25] K. Yano. Two kinds of conditionings for stable Lévy processes. In *Proceedings of the 1st MSJ-SI, "Probabilistic Approach to Geometry,"* 493–503. *Adv. Stud. Pure Math.* **57**. Math. Soc. Japan, Tokyo. MR2648275

[26] K. Yano. Excursions away from a regular point for one-dimensional symmetric Lévy processes without Gaussian part. *Potential Anal.* **32** (2010) 305–341. MR2603019

[27] K. Yano, Y. Yano and M. Yor. Penalising symmetric stable Lévy paths. *J. Math. Soc. Japan* **61** (2009) 757–798. MR2552915

[28] K. Yano, Y. Yano and M. Yor. Penalisation of a stable Lévy process involving its one-sided supremum. *Ann. Inst. H. Poincaré Probab. Statist.* **46** (2010) 1042–1054.

[29] V. M. Zolotarev. *One-Dimensional Stable Distributions*. *Translations of Mathematical Monographs* **65**. Amer. Math. Soc., Providence, RI, 1986. MR0854867