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A geometric approach to correlation inequalities in the plane

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Abstract. By elementary geometric arguments, correlation inequalities for radially symmetric probability measures are proved in the plane. Precisely, it is shown that the correlation ratio for pairs of *width-decreasing sets* is minimized within the class of infinite strips. Since open convex sets which are symmetric with respect to the origin turn out to be width-decreasing sets, Pitt's Gaussian correlation inequality (the two-dimensional case of the long-standing Gaussian correlation conjecture) is derived as a corollary, and it is in fact extended to a wide class of radially symmetric measures.

Résumé. En utilisant des arguments géométriques élémentaires, on démontre des inégalités de corrélation pour des mesures de probabilité à symétrie radiale. Plus précisément on montre que, parmi la famille des ensembles *width-decreasing*, le ratio de corrélation est minimisé par des bandes. Comme les ouverts convexes symétriques appartiennent à cette famille, on retrouve comme corollaire le résultat de Pitt sur la validité de la conjecture de corrélation gaussiennne en dimension 2, qui est étendue dans ce papier à une large classe de mesures à symétrie radiale.

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1. Introduction

We address the minimization of the correlation ratio of a radially symmetric probability measure μ on $\mathbb{R}^2$, providing in particular a new and elementary proof of Pitt's correlation inequality for the planar Gaussian measure. Let us say that a set $S\subset\mathbb{R}^2$ is a strip (symmetric with respect to the origin) if there exist $\nu\in\mathbb{S}^1$ and $h>0$ such that

$$S=\{x\in\mathbb{R}^2\colon\ |x\cdot\nu|<h\}.$$

Two strips S and S' are orthogonal if they are associated to orthogonal vectors ν and ν' in $\mathbb{S}^1$. Next we introduce the family of *width-decreasing sets* as the class of those planar, open sets, which are symmetric with respect to the origin, which contain the origin, and whose angular-length (roughly speaking) decreases at least as the angular-length of a strip. Precisely, if we set $B_r=\{x\in\mathbb{R}^2\colon\ |x|<r\}$, then an open set E is a width-decreasing set if $0\in E$, $x\in E$ implies $-x\in E$, and, for every $r>0$ such that $0<\mathscr{H}^1(E\cap\partial B_r)<2\pi r$ we have

$$\mathscr{H}^1(E\cap\partial B_s)\le\mathscr{H}^1(S\cap\partial B_s)\quad\forall s>r,$$

provided S is a strip with $\mathscr{H}^1(S\cap\partial B_r)=\mathscr{H}^1(E\cap\partial B_r)$. We are thus in the position to state our main result.

Theorem 1. *Let μ be a radially symmetric probability measure on $\mathbb{R}^2$. If E and F are two width-decreasing sets in $\mathbb{R}^2$, then there exist two orthogonal strips S_E and S_F such that*

$$\frac{\mu(E\cap F)}{\mu(E)\mu(F)} \geq \frac{\mu(S_E\cap S_F)}{\mu(S_E)\mu(S_F)}. \tag{1.1}$$

In other words, the correlation ratio of μ is minimized, among pairs of width-decreasing sets, over pairs of orthogonal strips.

Remark 1. *Let $\mathcal{K}_n$ denote the family of open, convex sets in $\mathbb{R}^n$, $n\geq 2$, which are symmetric with respect to the origin. Theorem 2, in the Appendix, shows that every set in $\mathcal{K}_2$ is a width-decreasing set. Conversely, non-convex width decreasing sets are easily constructed: for example, if $R_{\pi/2}$ denotes the counter-clockwise rotation by ninety degrees around the origin, $E\in\mathcal{K}_2$ is an ellipse of axes $b>a>0$, and if $c=(a+b)/2$, then $E'=(E\cap B_c)\cup R_{\pi/2}(E\setminus\overline{B_c})$ is a width-decreasing set (since E is a width-decreasing set and $\mathscr{H}^1(E'\cap\partial B_r)=\mathscr{H}^1(E\cap\partial B_r)$ for every $r\neq c$) which, clearly, is not convex.*

Let now $\gamma_n=(2\pi)^{-n/2}\mathrm{e}^{-|x|^2/2}\,\mathrm{d}x$ denote the standard Gaussian measure on $\mathbb{R}^n$. In the case $\mu=\gamma_2$, Fubini's theorem implies that the right-hand side of (1.1) is equal to 1. By combining these two facts with Theorem 1, we provide a new justification of the planar Gaussian correlation inequality (see (1.4) below), that is completely alternative to Pitt's semi-group approach [6], and it is based only on elementary geometric considerations. In fact, the tensorization property of the Gaussian measure is not necessary to obtain non-trivial correlation inequalities in the plane. For example, we can deduce from Theorem 1 the following class of correlation inequalities, which extend Pitt's inequality to a wide class of radially symmetric probability measures.

Corollary 2. *Let $V:[0,\infty)\to\mathbb{R}$ be a Lipschitz function, such that*

$$\frac{V'(r)}{r}\ \textit{is decreasing on}\ (0,\infty), \tag{1.2}$$

and that $\mu=\mathrm{e}^{-V(|z|)}\,\mathrm{d}z$ is a probability measure on $\mathbb{R}^2$. Then

$$\mu(E\cap F)\geq\mu(E)\mu(F) \tag{1.3}$$

for every pair of width-decreasing sets E and F in $\mathbb{R}^2$.

Observe that, by choosing $E=F=\mathbb{R}^2$, we immediately check the sharpness of (1.3).

The paper is divided in three sections. In Section 2 we define the class of width-decreasing sets, prove Theorem 1, and discuss the equality cases in (1.1) under some additional assumption on μ (cf. Remark 3). In Section 3 we prove Corollary 2, and also address the case when $V'(r)/r$ is *increasing* on $(0,\infty)$, see Corollary 7 (in particular, we obtain non-trivial correlation inequalities for all probability measures μ on $\mathbb{R}^2$ of the form $\mu=c_p\mathrm{e}^{-|z|^p}\,\mathrm{d}z$, $p>0$). Finally, in the Appendix we show that every set in $\mathcal{K}_2$ is a width-decreasing set (Theorem 2).

We finally recall that the Gaussian correlation conjecture postulates the validity of the inequality

$$\gamma_n(E\cap F)\geq\gamma_n(E)\gamma_n(F) \tag{1.4}$$

for every pair of sets $E,F\in\mathcal{K}_n$, $n\geq 2$. As pointed out to us by Michael Loss, in [2], Theorem 3.2, Christer Borell proves the n-dimensional Gaussian correlation inequality (1.4) for every pair of sets $E,F\in\mathcal{B}_n$, where, by definition, $E\in\mathcal{B}_n$ if E is open, $0\in E$, $x\in E$ implies $-x\in E$, and, for every $r>0$ such that $0<\mathscr{H}^{n-1}(E\cap\partial B_r)<n\omega_n r^{n-1}$, we have

$$\mathscr{H}^{n-1}(E\cap\partial B_s)\leq\mathscr{H}^{n-1}(C\cap\partial B_s)\quad\forall s>r,$$

provided $C=\{x\in\mathbb{R}^n\colon\ |x-(x\cdot\nu)\nu|<h\}$ ($\nu\in\mathbb{S}^{n-1}$, $h>0$) is such that $\mathscr{H}^{n-1}(E\cap\partial B_r)=\mathscr{H}^{n-1}(C\cap\partial B_r)$. If $n\geq 3$, the classes $\mathcal{B}_n$ and $\mathcal{K}_n$ do not coincide (nor are contained one into the other), although, in some sense, they have a considerably large intersection. Concerning the planar case, it is easily seen that the class $\mathcal{B}_2$ *coincides with the class of width-decreasing sets in* $\mathbb{R}^2$, so that, in particular, Theorem 2 in the Appendix implies $\mathcal{K}_2 \subset \mathcal{B}_2$. In fact, the inclusion $\mathcal{K}_2 \subset \mathcal{B}_2$ is also proved by Borell in Section 4 of his paper. However, for the sake of clarity, we have opted to include an elementary geometric proof of this result in our appendix. We also remark that the argument used by Borell in order to prove (1.4) in the class $\mathcal{B}_n$ makes essential use of the tensorization property of γ_n, and thus, in the planar case, it does not seem suitable to recover the more general Theorem 1.

It is to be noticed that there are many other interesting results about correlation inequalities in various settings, see for instance Khatri [4], Sydak [8], Schechtman, Schlumprecht, Zinn [7], Harge [3], Kolesnikov [5], and the lecture notes [1].

2. Width decreasing sets and planar correlation inequalities

In this section we prove Theorem 1. We begin with some definitions and terminology. A probability measure μ on $\mathbb{R}^2$ is *radially symmetric* if for every Borel set $E \subset \mathbb{R}^2$ and $\theta \in (0, 2\pi)$ we have

$$\mu(E) = \mu\big(R_\theta(E)\big),$$

where $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the counter-clockwise rotation around the origin by the angle θ. By a standard disintegration argument, we see that if μ is a radially symmetric probability measure and E, F are Borel sets in $\mathbb{R}^2$, then

$$\mathscr{H}^1(E \cap \partial B_r) \le \mathscr{H}^1(F \cap \partial B_r) \quad \forall r > 0 \quad \Rightarrow \quad \mu(E) \le \mu(F). \tag{2.1}$$

(Here and in the sequel, $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure of a set.)

Given an open set $E \subset \mathbb{R}^2$ which contains the origin, the (normalized) *angular-length function of* E, $\theta_E : (0, \infty) \to [0, \pi/2]$, is defined as

$$\theta_E(r) = \frac{1}{4}\frac{\mathscr{H}^1(E \cap \partial B_r)}{r}, \quad r > 0.$$

Since E is open and contains the origin, the function θ_E is always lower semicontinuous on $(0, \infty)$, and constantly equal to $\pi/2$ in a neighborhood of 0.

We now reformulate the notion of width-decreasing set defined in the introduction in terms of angular-length functions: we say that E is a *width-decreasing set*, if E is open, symmetric with respect to the origin, contains the origin, and, for every $r > 0$ such that $\theta_E(r) < \pi/2$, the angular-length function of E is bounded from above on (r, ∞) by the angular-length function of a strip S such that $\theta_E(r) = \theta_S(r)$, see Fig. 1. More concisely, we ask that, for any $r \in (0, \infty)$,

$$\theta_E(r) < \frac{\pi}{2} \quad \Rightarrow \quad \theta_E \le \theta_S \quad \text{on } (r, \infty) \tag{2.2}$$

for every strip S with $\theta_E(r) = \theta_S(r)$. It is easily seen that if E is a width-decreasing set, then its angular-length functions θ_E is decreasing.

Proof of Theorem 1. Without loss of generality we can assume that both E and F are different from $\mathbb{R}^2$ (otherwise the result is trivial). Recalling that by assumption E and F are open, we can immediately check that $\theta_E + \theta_F : (0, \infty) \to [0, \pi]$ is a decreasing, lower semicontinuous function satisfying $\theta_E(0^+) + \theta_F(0^+) = \pi$ and $\theta_E(+\infty) + \theta_F(+\infty) = 0$. Hence, if we set

$$r_0 = \inf\left\{r > 0\colon\, \theta_E(r) + \theta_F(r) \le \frac{\pi}{2}\right\},$$

then $r_0 \in (0, \infty)$ and $\theta_E(r_0) + \theta_F(r_0) \le \pi/2$. If $\theta_E(r_0) = \pi/2$ then $\theta_F(r_0) = 0$, and so $F \subset E$. In this case, we set S_E to be any strip such that $E \subset S_E$, and we set S_F to be any strip orthogonal to S_E, so to find

$$\frac{\mu(E \cap F)}{\mu(E)\mu(F)} = \frac{1}{\mu(E)} \ge \frac{1}{\mu(S_E)} \ge \frac{\mu(S_E \cap S_F)}{\mu(S_E)\mu(S_F)}. \tag{2.3}$$

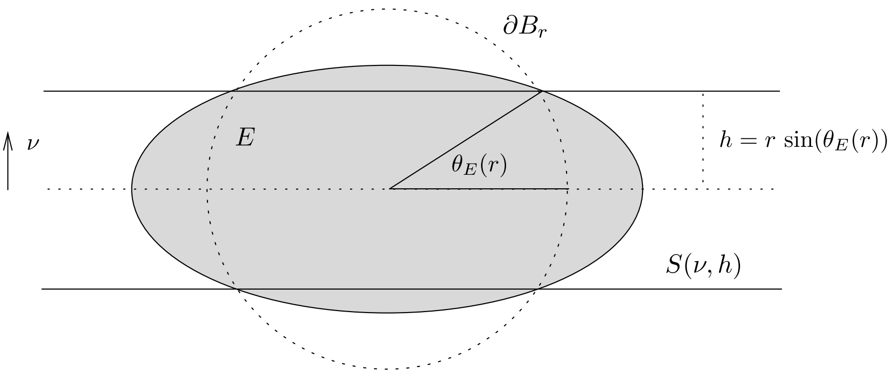

Fig. 1. Width decreasing sets. If $E\cap\partial B_r$ is a proper subset of ∂B_r and S is any strip with $\mathscr{H}^1(E\cap\partial B_r)=\mathscr{H}^1(S\cap\partial B_r)$, then for every $s\geq r$ we have $\mathscr{H}^1(E\cap\partial B_s)\leq\mathscr{H}^1(S\cap\partial B_s)$.

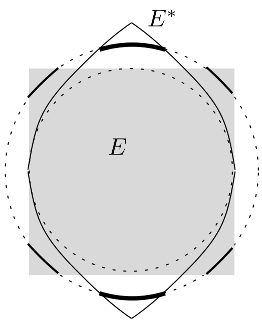

Fig. 2. A square E and a (qualitative picture) of its vertical double-cap symmetrization E^*. E^* is obtained by rearranging the connected components of $E\cap\partial B_r$ into pairs of opposite circular arcs, with center on the vertical axis. The horizontal double-cap symmetrization E_* of E is obtained by a $\pi/2$-rotation of E^*.

The case $\theta_E(r_0)=0$, $\theta_F(r_0)=\pi/2$ is settled by symmetry. Hence we are left to consider the case that

$$\theta_E(r_0)+\theta_F(r_0)\leq\frac{\pi}{2},\qquad 0<\theta_E(r_0)<\frac{\pi}{2},\qquad 0<\theta_F(r_0)<\frac{\pi}{2}. \tag{2.4}$$

In this case we are going to replace E by its *vertical double-cap symmetrization* E^*, defined as (see Fig. 2)

$$E^*=\bigcup_{r>0}\left\{re^{i\theta}\colon\left|\theta-\frac{\pi}{2}\right|<\theta_E(r)\right\}\cup\left\{re^{i\theta}\colon\left|\theta-\frac{3\pi}{2}\right|<\theta_E(r)\right\},$$

and, simultaneously, to replace F by its *horizontal double-cap symmetrization* F_*, defined as

$$F_*=\bigcup_{r>0}\left\{re^{i\theta}\colon|\theta|<\theta_F(r)\right\}\cup\left\{re^{i\theta}\colon|\theta-\pi|<\theta_F(r)\right\},$$

where we write $re^{i\theta}=(r\cos\theta,r\sin\theta)$. By construction, it is clear that $\theta_E=\theta_{E^*}$ and $\theta_F=\theta_{F_*}$. Moreover, by (2.1),

$$\mu(E)=\mu\big(E^*\big),\qquad\mu(F)=\mu(F_*). \tag{2.5}$$

Since

$$\mathscr{H}^1\big(E^*\cap F_*\cap\partial B_r\big)=4\max\left\{0,\theta_E(r)+\theta_F(r)-\frac{\pi}{2}\right\}\leq\mathscr{H}^1(E\cap F\cap\partial B_r), \tag{2.6}$$

again by (2.1) we have

$$\mu\big(E^* \cap F_*\big) \le \mu(E \cap F), \tag{2.7}$$

so that

$$\frac{\mu(E \cap F)}{\mu(E)\mu(F)} \ge \frac{\mu(E^* \cap F_*)}{\mu(E^*)\mu(F_*)}. \tag{2.8}$$

We now observe the crucial fact that the notion of width-decreasing set is invariant under both our symmetrizations, being a property only of the angular-length function. Hence, both E^* and F_* are width-decreasing.

Since the circular slices of E^* are pairs of opposite vertical caps, the fact that E^* is a width-decreasing set and the property $0 < \theta_E(r_0) < \pi/2$ force the inclusions

$$E^* \setminus B_{r_0} \subset S_E, \tag{2.9}$$

$$S_E \cap B_{r_0} \subset E^* \cap B_{r_0}, \tag{2.10}$$

where S_E is the vertical strip such that $\theta_{S_E}(r_0) = \theta_E(r_0)$ (see Fig. 3). More precisely, $S_E = S(\nu, h)$ for

$$\nu = e_1, \qquad h = r_0 \sin\big(\theta_E(r_0)\big).$$

On the other hand, for F we find

$$F_* \setminus B_{r_0} \subset S_F, \tag{2.11}$$

$$S_F \cap B_{r_0} \subset F_* \cap B_{r_0}, \tag{2.12}$$

where S_F is the horizontal strip such that $\theta_{S_F}(r_0) = \pi/2 - \theta_E(r_0)$ (observe that for general width-decreasing sets it can happen that $\pi/2 - \theta_E(r_0) > \theta_F(r_0)$, see Fig. 3). If we now set

$$\widetilde{E} = \big(E^* \cap B_{r_0}\big) \cup (S_E \setminus B_{r_0}), \qquad \widetilde{F} = (F_* \cap B_{r_0}) \cup (S_F \setminus B_{r_0}),$$

then by construction $E^* \cap F_* = \widetilde{E} \cap \widetilde{F}$ (recall that $\theta_E(r_0) + \theta_F(r_0) \le \pi/2$, so $E^* \cap \partial B_{r_0}$ and $F_* \cap \partial B_{r_0}$ are disjoint). Moreover, by (2.9) and (2.11), $E^* \subset \widetilde{E}$ and $F_* \subset \widetilde{F}$. Thus,

$$\frac{\mu(E^* \cap F_*)}{\mu(E^*)\mu(F_*)} \ge \frac{\mu(\widetilde{E} \cap \widetilde{F})}{\mu(\widetilde{E})\mu(\widetilde{F})}. \tag{2.13}$$

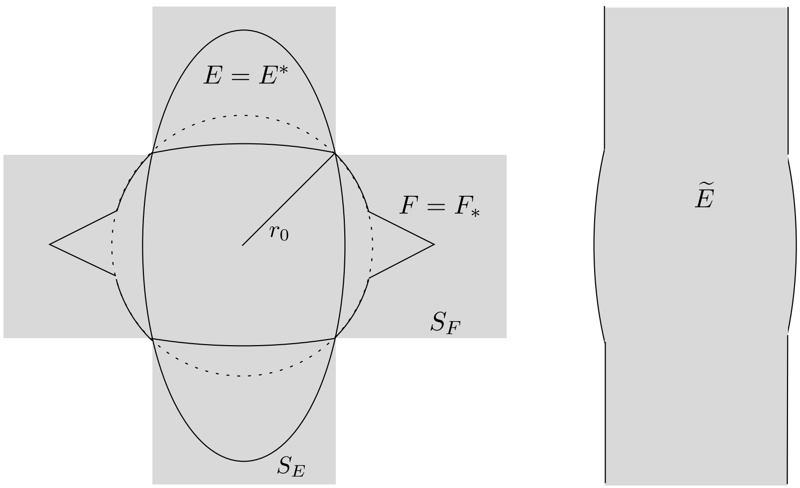

Fig. 3. A pair of sets E and F such that $E = E^*$, $F = F_*$, and $\theta_E(r_0) + \theta_F(r_0) < \pi/2$. On the right, the set $\widetilde{E} = (E^* \cap B_{r_0}) \cup (S_E \setminus B_{r_0})$.

Let us now notice the trivial inequality

$$\frac{b}{a} \geq \frac{b-c}{a-c} \quad \forall 0 < c < b \leq a. \tag{2.14}$$

Since $\theta_E(r_0)+\theta_F(r_0) \leq \pi/2$ we have $B_{r_0} \setminus S_E \subset S_F \cap B_{r_0}$. Hence, recalling that $\widetilde{E} \setminus B_{r_0} = S_E \setminus B_{r_0}$ and $S_F \cap B_{r_0} \subset \widetilde{F}$,

$$\widetilde{E} \setminus S_E = (\widetilde{E} \cap B_{r_0}) \setminus S_E \subset \widetilde{E} \cap S_F \cap B_{r_0} \subset \widetilde{E} \cap \widetilde{F}.$$

Therefore we may apply (2.14) to obtain

$$\frac{\mu(\widetilde{E} \cap \widetilde{F})}{\mu(\widetilde{E})\mu(\widetilde{F})} \geq \frac{\mu(\widetilde{E} \cap \widetilde{F}) - \mu(\widetilde{E} \setminus S_E)}{(\mu(\widetilde{E}) - \mu(\widetilde{E} \setminus S_E))\mu(\widetilde{F})} = \frac{\mu(\widetilde{F} \cap S_E)}{\mu(S_E)\mu(\widetilde{F})}. \tag{2.15}$$

Similarly, from the inclusion $\widetilde{F} \setminus S_F \subset \widetilde{F} \cap S_E$ and by applying again (2.14), we conclude

$$\frac{\mu(\widetilde{F} \cap S_E)}{\mu(S_E)\mu(\widetilde{F})} \geq \frac{\mu(\widetilde{F} \cap S_E) - \mu(\widetilde{F} \setminus S_F)}{\mu(S_E)(\mu(\widetilde{F}) - \mu(\widetilde{F} \setminus S_F))} = \frac{\mu(S_E \cap S_F)}{\mu(S_E)\mu(S_F)}. \tag{2.16}$$

Combining together (2.8), (2.13), (2.15) and (2.16), we finally get (1.1), so the proof is completed. □

Remark 3 (A necessary condition for equality in (1.1)). *Let us now discuss the sharpness of* (1.1) *under the assumption that*

$$\mu(A) > 0 \quad \textit{for every open set } A \subset \mathbb{R}^2.$$

We claim that, in this case, the inequality sign in (1.1) *is strict unless*

$$E^* \textit{ and } F_* \textit{ are orthogonal strips}.$$

To verify this, let us recall that in proving (1.1) *we have considered three separate cases: $F \subset E$, $E \subset F$, or else. In the first case, $F \subset E$, we denoted by S_E any strip containing E, by S_F any strip orthogonal to S_E, and then we deduced* (1.1) *from the chain of inequalities* (2.3),

$$\frac{\mu(E \cap F)}{\mu(E)\mu(F)} = \frac{1}{\mu(E)} \geq \frac{1}{\mu(S_E)} \geq \frac{\mu(S_E \cap S_F)}{\mu(S_E)\mu(S_F)}.$$

The first inequality sign is strict unless $\mu(S_E \setminus E) = 0$; the second inequality is then necessarily strict (unless we are in the trivial case $E = S_E = \mathbb{R}^2$). Therefore, in the case $F \subset E$, (1.1) *is always a strict inequality, and the same holds true in the symmetric case $E \subset F$. Let us now assume that $E \Delta F \neq \varnothing$. In this case, if* (1.1) *holds as an equality, we deduce from* (2.13) *that $\mu(E^*) = \mu(\widetilde{E})$ and $\mu(F_*) = \mu(\widetilde{F})$. By* (2.9), (2.11) *and by our assumptions on μ, this implies that*

$$E^* \setminus B_{r_0} = S_E \setminus B_{r_0}, \qquad F_* \setminus B_{r_0} = S_F \setminus B_{r_0}.$$

At the same time, thanks to (2.15) *and* (2.16), *the equality sign in* (1.1) *implies $\mu(\widetilde{E} \setminus S_E) = \mu(\widetilde{F} \setminus S_F) = 0$, that finally gives*

$$E^* = S_E, \qquad F_* = S_F,$$

using (2.10) *and* (2.12).

3. Extensions of Pitt's correlation inequality

In this section we present some classes of radially symmetric measures such that the right-hand side of (1.1) admits an explicit, non-trivial lower bound, and, in particular, we prove Corollary 2. Precisely, we consider a Borel function $V:[0,\infty)\to\mathbb{R}$, we set

$$f(z)=\mathrm{e}^{-V(|z|)},\quad z\in\mathbb{R}^2, \tag{3.1}$$

so that $f>0$ on $\mathbb{R}^2$, and we work with the measure

$$\mu=f(z)\,\mathrm{d}z. \tag{3.2}$$

We shall assume as usual that μ is a probability measure on $\mathbb{R}^2$, that is, we shall assume that

$$2\pi\int_0^\infty \mathrm{e}^{-V(r)}r\,\mathrm{d}r=\int_{\mathbb{R}^2} f(z)\,\mathrm{d}z=\mu\bigl(\mathbb{R}^2\bigr)=1. \tag{3.3}$$

We begin with the following lemma, that, in combination with Theorem 1, will readily imply Corollary 2. In the following, we will denote the generic point of $\mathbb{R}^2$ as $z=(x,y)$.

Lemma 4. *Let f, V, μ be as in (3.1), (3.2), and (3.3). Assume that*

$$f(x,y)f(a,b)\ge f(a,y)f(x,b)\quad \forall 0\le x\le a, 0\le y\le b. \tag{3.4}$$

Then

$$\inf\left\{\frac{\mu((-x,x)\times(-y,y))}{\mu((-x,x)\times\mathbb{R})\mu(\mathbb{R}\times(-y,y))}:\ x,y>0\right\}=1. \tag{3.5}$$

Proof. The fact that the infimum is less than or equal to 1 is easily seen by letting $x,y\to\infty$. Let us now prove the converse inequality.

We define the function

$$F(a,b)=\mu\bigl((0,a)\times(0,b)\bigr)=\frac{1}{4}\mu\bigl((-a,a)\times(-b,b)\bigr),\quad a,b>0.$$

Since $F>0$, if we set $H(a,b)=\log(F(a,b))$, $a,b>0$, thanks to (3.4) we get

$$\begin{aligned}
\frac{\partial H}{\partial a}(a,b)&=\frac{\int_0^b f(a,y)\,\mathrm{d}y}{F(a,b)},\\
\frac{\partial^2 H}{\partial a\,\partial b}(a,b)&=\frac{f(a,b)F(a,b)-\int_0^b f(a,y)\,\mathrm{d}y\int_0^a f(x,b)\,\mathrm{d}x}{F(a,b)^2}\\
&=\frac{\int_0^a \mathrm{d}x\int_0^b[f(a,b)f(x,y)-f(a,y)f(x,b)]\,\mathrm{d}y}{F(a,b)^2}\ge 0.
\end{aligned}$$

In particular, for every $y>0$, $(\partial H/\partial b)(\cdot,y)$ is increasing on $(0,\infty)$, so that

$$H(a,b)-H(a,y)\ge H(x,b)-H(x,y)\quad \forall 0\le x\le a, 0\le y\le b,$$

that is,

$$\frac{F(a,b)F(x,y)}{F(a,y)F(x,b)}\ge 1\quad \forall 0\le x\le a, 0\le y\le b. \tag{3.6}$$

Notice now that F is separately increasing in both its variables and

$$F(x,y)=\frac{1}{4}\mu\big((-x,x)\times(-y,y)\big)\quad\forall 0\le x\le\infty, 0\le y\le\infty.$$

Therefore, by letting $a,b\to\infty$ in (3.6), we get (3.5). $\square$

Proof of Corollary 2. By Lemma 4 we have only to check that f (defined from V by (3.1)) satisfies (3.4), which amounts in proving that

$$V\big(\sqrt{a^2+b^2}\big)-V\big(\sqrt{x^2+b^2}\big)\le V\big(\sqrt{a^2+y^2}\big)-V\big(\sqrt{x^2+y^2}\big)\tag{3.7}$$

for every $0\le x\le a\le\infty$ and $0\le y\le b\le\infty$. In fact, by (1.2) we have that

$$y\mapsto\frac{\partial}{\partial x}V\big(\sqrt{x^2+y^2}\big)=\frac{V'(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}x\quad\text{is decreasing on }(0,\infty),$$

from which we easily deduce (3.7). $\square$

Lemma 5. *Let f, V, μ be as in* (3.1), (3.2), *and* (3.3), *and assume that V is continuous at* 0 *and that*

$$f(x,y)f(a,b)\le f(a,y)f(x,b)\quad\forall 0\le x\le a, 0\le y\le b.\tag{3.8}$$

Then

$$\inf\left\{\frac{\mu((-x,x)\times(-y,y))}{\mu((-x,x)\times\mathbb{R})\mu(\mathbb{R}\times(-y,y))}\colon x,y>0\right\}=\frac{\mathrm{e}^{-V(0)}}{(\int_{\mathbb{R}}\mathrm{e}^{-V(t)}\,\mathrm{d}t)^2}.\tag{3.9}$$

Remark 6. *Notice that, thanks to* (3.3), *the continuity of V at* 0 *ensures that $\int_{\mathbb{R}}\mathrm{e}^{-V(t)}\,\mathrm{d}t<\infty$.*

Proof. The lower bound for the infimum in (3.9) is easily seen by letting $x,y\to 0^+$.

To prove the converse inequality, let us define F and H as in the proof of Lemma 4. Having assumed (3.8) in place of (3.4), instead of (3.6) we now get that

$$\frac{F(a,b)F(x,y)}{F(a,y)F(x,b)}\le 1\quad\forall 0<x\le a, 0<y\le b.\tag{3.10}$$

In particular we find that, whenever $0\le x\le a$, $0\le y\le b$,

$$\begin{aligned}\frac{F(x,b)}{F(x,\infty)F(\infty,b)}&\ge\frac{F(a,b)F(x,y)}{F(x,\infty)F(\infty,b)F(a,y)}\\&=\frac{F(a,b)(y\int_0^x f(t,0)\,\mathrm{d}t+\mathrm{o}(y))}{F(x,\infty)F(\infty,b)(y\int_0^a f(t,0)\,\mathrm{d}t+\mathrm{o}(y))},\end{aligned}$$

that is, letting $y\to 0^+$,

$$\frac{F(x,b)}{F(x,\infty)F(\infty,b)}\ge\frac{F(a,b)\int_0^x f(t,0)\,\mathrm{d}t}{F(x,\infty)F(\infty,b)\int_0^a f(t,0)\,\mathrm{d}t}\quad\forall 0\le x\le a, b>0.$$

We now let $a\to\infty$ to find that

$$\frac{F(x,b)}{F(x,\infty)F(\infty,b)}\ge\frac{\int_0^x f(t,0)\,\mathrm{d}t}{F(x,\infty)\int_0^\infty f(t,0)\,\mathrm{d}t}\quad\forall x,b>0.\tag{3.11}$$

We finally notice that, again by (3.8),

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{\int_0^x f(t,0)\,\mathrm{d}t}{F(x,\infty)}=\frac{f(x,0)\int_0^x \mathrm{d}t\int_0^\infty f(t,s)\,\mathrm{d}s-\int_0^x f(t,0)\,\mathrm{d}t\int_0^\infty f(x,s)\,\mathrm{d}s}{F(x,\infty)^2}$$
$$=\frac{\int_0^x \mathrm{d}t\int_0^\infty [f(x,0)f(t,s)-f(t,0)f(x,s)]\,\mathrm{d}s}{F(x,\infty)^2}\geq 0,$$

so that, in particular,

$$\inf_{x>0}\frac{\int_0^x f(t,0)\,\mathrm{d}t}{F(x,\infty)}=\lim_{x\to 0^+}\frac{\int_0^x f(t,0)\,\mathrm{d}t}{F(x,\infty)}=\frac{f(0,0)}{\int_0^\infty f(0,s)\,\mathrm{d}s}.$$

Combining (3.11) with

$$f(0,0)=\mathrm{e}^{-V(0)},\qquad \int_0^\infty f(t,0)\,\mathrm{d}t=\int_0^\infty f(0,s)\,\mathrm{d}s=\int_{\mathbb{R}}\mathrm{e}^{-V(t)}\,\mathrm{d}t,$$

we conclude the proof of (3.9). □

Corollary 7. *Let $V:[0,\infty)\to\mathbb{R}$ be a Lipschitz function such that*

$$\frac{V'(r)}{r}\ \textit{is increasing on}\ (0,\infty),\tag{3.12}$$

and $\mu=\mathrm{e}^{-V(|z|)}\,\mathrm{d}z$ is a probability measure on $\mathbb{R}^2$. Then

$$\mu(E\cap F)\geq\frac{\mathrm{e}^{-V(0)}}{(\int_{\mathbb{R}}\mathrm{e}^{-V(t)}\,\mathrm{d}t)^2}\mu(E)\mu(F)\tag{3.13}$$

for every pair of width-decreasing sets E and F in $\mathbb{R}^2$.

Proof. Arguing as in the proof of Corollary 2, we now see that (3.12) implies the validity of (3.8). In particular, combining Lemma 5 with Theorem 1, we immediately deduce (3.13). □

Appendix: Planar symmetric convex sets are width-decreasing sets

In this section we prove that $\mathcal{K}_2$ is a subset of the family of width-decreasing sets. This result was first proved by Borell [2]. For the sake of clarity, we include here a new elementary geometric proof of this fact.

Theorem 2. *A planar convex set, symmetric with respect to the origin, is a width-decreasing set.*

We first present two simple lemmas, that shall be used in the proof of the theorem. Let us first note that any strip S of width h (i.e., $S=\{x\in\mathbb{R}^2:\ |x\cdot\nu|<h\}$ for some $\nu\in\mathbb{S}^1$), satisfies

$$\theta_S(s)=\begin{cases}\pi/2 & \text{if } s\in(0,h),\\ \arcsin(h/s) & \text{if } s\geq h\ ,\end{cases}\tag{A.1}$$

where $\arcsin:[0,1]\to[0,\pi/2]$ is the inverse function of the sinus on $[0,\pi/2]$. In particular (2.2) is equivalent to check that, if $\theta_E(r)<\pi/2$, then

$$\theta_E(s)\leq\arcsin\left(\frac{r\sin(\theta_E(r))}{s}\right)\quad\forall s\geq r,\tag{A.2}$$

since any strip S with $\theta_E(r) = \theta_S(r)$ has width $h = r\sin(\theta_E(r))$. From these remarks we easily obtain the following criterion for a set to be width-decreasing.

Lemma 8. *Let E be an open set in $\mathbb{R}^2$, symmetric with respect to the origin, containing the origin, and such that*

$$\theta_E(r) < \frac{\pi}{2} \quad \Rightarrow \quad \text{there exists } \delta > 0 \text{ such that } \theta_E \le \theta_S \text{ on } [r, r+\delta) \tag{A.3}$$

for every strip S with $\theta_E(r) = \theta_S(r)$. Then E is a width-decreasing set.

Proof. It is clear from (A.3) that θ_E is a decreasing function. In fact, with the help of (A.2), we easily deduce from (A.3) that, if we set,

$$D^+\theta_E(r) = \limsup_{\varepsilon\to 0^+} \frac{\theta_E(r+\varepsilon) - \theta_E(r)}{\varepsilon}, \tag{A.4}$$

then, for every $r > 0$ such that $\theta_E(r) < \pi/2$, we have

$$D^+\theta_E(r) \le -\frac{\tan(\theta_E(r))}{r} \le 0. \tag{A.5}$$

Let now $r > 0$ be such that $\theta_E(r) < \pi/2$, let S be a strip such that $\theta_E(r) = \theta_S(r)$, and set

$$r_0 = \sup\big\{s \ge r\colon\ \theta_E \le \theta_S \text{ on } (r,s)\big\}.$$

We have to prove that $r_0 = \infty$. Assume on the contrary that $r_0 < \infty$. Since θ_E is decreasing, $\theta_E(r_0) \le \theta_E(r) < \pi/2$. By (A.3) (applied with r_0 in place of r), there exists $\delta > 0$ such that

$$\theta_E \le \theta_{S'}, \quad \text{on } (r_0, r_0+\delta), \tag{A.6}$$

where S' is any strip such that $\theta_{S'}(r_0) = \theta_E(r_0)$. The strip S has width $h = r\sin(\theta_E(r))$, while the strip S' has width $h' = r_0\sin(\theta_E(r_0))$. By (A.5), the map $s \mapsto s\sin(\theta_E(s))$ is decreasing, therefore

$$\theta_{S'} \le \theta_S, \quad \text{on } (0,\infty). \tag{A.7}$$

By (A.6) and (A.7) we conclude that $\theta_E \le \theta_S$ on $(r, r_0+\delta)$, against the maximality of r_0. Thus $r_0 = \infty$ and the lemma is proved. □

Given three points P, Q, R in the plane, we denote by (PQR) the angle at Q defined by the points P and R.

Lemma 9. *Given $r > 0$, $\theta \in (0, \pi/2)$ and $\alpha \in (0, \pi/2 - \theta)$, let $P = r\mathrm{e}^{\mathrm{i}\theta}$, $Q = r\mathrm{e}^{\mathrm{i}(\pi-\theta-\alpha)}$. The angle β between the lines L_{PQ} and L_{OP} depicted in Fig. 4, satisfies*

$$\beta = \theta + \frac{\alpha}{2}. \tag{A.8}$$

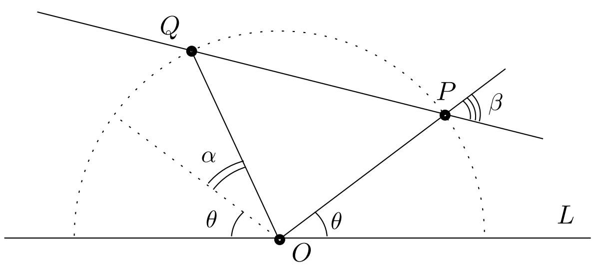

Fig. 4. The situation in Lemma 9.

Proof. Since $|P|=|Q|$, the angles (OPQ) and (OQP) are both equal to β. This implies that

$$2\beta+(QOP)=\pi,$$

from which the formula for β follows immediately observing that

$$(QOP)=\pi-\theta-(\alpha+\theta),$$

see Fig. 4. □

We are now in position to show Theorem 2.

Proof of Theorem 2. Let $E\in\mathcal{K}_2$, let $r>0$ be such that $\theta_E(r)<\pi/2$, and let $S=S(h,\nu)$ be any strip of width $h=r\sin(\theta_E(r))$, so that $\theta_E(r)=\theta_S(r)$. We consider two cases.

Case I. *If $E\cap\partial B_r$ consists of a pair of disjoint open circular arcs.*

In this case, by convexity, we easily find that, for a suitable $\nu\in\mathbb{S}^1$, the strip $S=S(h,\nu)$ satisfies

$$S\cap B_r\subset E\cap B_r.$$

Then, again by convexity, $E\setminus B_r\subset S\setminus B_r$, and thus

$$\mathscr{H}^1(E\cap\partial B_s)\le\mathscr{H}^1(S\cap\partial B_s)\quad\forall s\ge r.$$

In particular, $\theta_E(s)\le\theta_S(s)$ for every $s\ge r$, as required.

Case II. *If $E\cap\partial B_r$ consists of $N\ge2$ pairs of disjoint open circular arcs.*

Thanks to Lemma 8 (see (A.5)), we are left to prove that

$$D^+\theta_E(r)<-\frac{\tan(\theta_E(r))}{r},\tag{A.9}$$

where $D^+\theta_E(r)$ is defined as in (A.4) (note that we need the strict sign in (A.9) to obtain the validity of (A.3) for some $\delta>0$). By assumption, we know that

$$E\cap\partial B_r=\bigcup_{h=1}^N I_h\cup J_h,\quad N\ge2,$$

where I_h and J_h are open circular arcs in ∂B_r, opposite to each other (i.e. $J_h=\{-x\colon\ x\in I_h\}$). Since E is open, convex, and symmetric with respect to the origin, we find that, for every $\varepsilon>0$ sufficiently small

$$E\cap\partial B_{r+\varepsilon}=\bigcup_{h=1}^N I_h^\varepsilon\cup J_h^\varepsilon,\tag{A.10}$$

where I_h^ε and J_h^ε are opposite open circular arcs in $\partial B_{r+\varepsilon}$, with

$$I_h^\varepsilon\subseteq\frac{r+\varepsilon}{r}I_h,\qquad J_h^\varepsilon\subseteq\frac{r+\varepsilon}{r}J_h.$$

We are going to prove (A.9) from the following upper bound for $\mathscr{H}^1(I_h^\varepsilon)$ in terms of $\mathscr{H}^1(I_h)$: whenever $1\le h\le N$,

$$\mathscr{H}^1\big(I_h^\varepsilon\big)\le\mathscr{H}^1(I_h)+\frac{\varepsilon}{r}\Big(\mathscr{H}^1(I_h)-2r\tan\big(\theta_E(r)\big)\Big)+\mathrm{o}(\varepsilon),\tag{A.11}$$

as $\varepsilon \to 0$. Before coming to the proof of (A.11), let us see why it does imply (A.9). From (A.10) we find that

$$\begin{aligned}
\theta_E(r+\varepsilon) - \theta_E(r) &= \frac{1}{4}\left(\frac{\mathscr{H}^1(E \cap \partial B_{r+\varepsilon})}{r+\varepsilon} - \frac{\mathscr{H}^1(E \cap \partial B_r)}{r}\right) \\
&= \frac{1}{2r}\sum_{h=1}^{N}\left(\mathscr{H}^1\big(I_h^\varepsilon\big)\left(1-\frac{\varepsilon}{r}\right) - \mathscr{H}^1(I_h)\right) + \mathrm{o}(\varepsilon) \\
&\le \frac{1}{2r}\sum_{h=1}^{N}\big(-2\varepsilon \tan\big(\theta_E(r)\big)\big) + \mathrm{o}(\varepsilon) = -\frac{\varepsilon}{r} N \tan\big(\theta_E(r)\big) + \mathrm{o}(\varepsilon).
\end{aligned}$$

Dividing by ε and letting $\varepsilon \to 0$, we immediately find (A.9) (note that we can achieve the strict sign thanks to the fact that $N \ge 2$).

We are thus left with proving (A.11). Without loss of generality, we can argue on I_1 and I_1^ε. Up to a rotation we can assume that

$$I_1 = \big\{r\mathrm{e}^{\mathrm{i}t}\colon |t| < \theta\big\}, \qquad J_1 = \big\{-r\mathrm{e}^{\mathrm{i}t}\colon |t| < \theta\big\},$$

where θ satisfies

$$0 < \theta < \theta_E(r) \tag{A.12}$$

(in particular, $\theta < \pi/2$, and the point $P = r\mathrm{e}^{\mathrm{i}\theta}$ belongs to the relative boundary of I_1 in ∂B_r). Let now $\varphi \in (0, \pi)$ be such that the point $Q = r\mathrm{e}^{\mathrm{i}\varphi}$ satisfies

$$Q \in \big(E \cap \partial B_r^+\big) \setminus (I_1 \cup J_1),$$

where $\partial B_r^+ = \partial B_r \cap \{x_2 > 0\}$. It is clear that φ cannot be too close to 0 or to π. More precisely, if we define α so that

$$\varphi = \pi - \theta - \alpha,$$

the fact that

$$\mathscr{H}^1\big(\big(E \cap \partial B_r^+\big) \setminus (I_1 \cup J_1)\big) = r\big(2\theta_E(r) - 2\theta\big),$$

gives us the estimate

$$\alpha > 2\theta_E(r) - 2\theta, \tag{A.13}$$

see Fig. 5. Now, let L_{PQ} denote the line passing through P and Q, and let

$$P_\varepsilon = (r+\varepsilon)\mathrm{e}^{\mathrm{i}\theta(\varepsilon,\varphi)}, \tag{A.14}$$

be the point satisfying

$$P_\varepsilon \in L_{PQ} \cap \partial B_{r+\varepsilon}, \qquad \lim_{\varepsilon \to 0^+} P_\varepsilon = P,$$

(see Fig. 5). Since E is convex and open, we have the inclusion

$$I_1^\varepsilon \subset \big\{(r+\varepsilon)\mathrm{e}^{\mathrm{i}t}\colon |t| < \theta(\varepsilon, \varphi)\big\},$$

from which we derive the following upper bound for $\mathscr{H}^1(I_1^\varepsilon)$:

$$\mathscr{H}^1\big(I_1^\varepsilon\big) \le 2(r+\varepsilon)\theta(\varepsilon, \varphi).$$

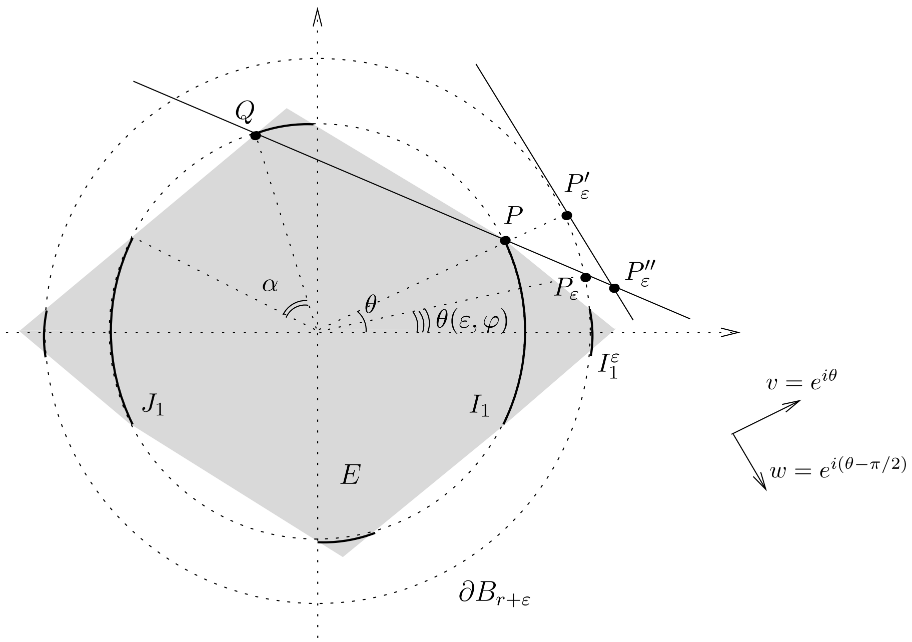

Fig. 5. To prove (A.11) we use the bound $\mathscr{H}^1(I_1^\varepsilon) \le 2(r+\varepsilon)\theta(\varepsilon,\varphi)$, and we show that $\theta(\varepsilon,\varphi) = \theta - \varepsilon\gamma + \mathrm{O}(\varepsilon^2)$, with $\gamma \ge \tan(\theta_E(r))/r$.

Let us now estimate $\theta(\varepsilon,\varphi)$.

Set $v = \mathrm{e}^{\mathrm{i}\theta}$, $w = \mathrm{e}^{\mathrm{i}(\theta-\pi/2)}$, and define the points $P'_\varepsilon = P + \varepsilon v$, $P''_\varepsilon = P'_\varepsilon + \varepsilon\gamma r w$, where $\gamma > 0$ is such that $P''_\varepsilon \in L_{PQ}$. Let us observe that $|P_\varepsilon - P''_\varepsilon| \le C\varepsilon^2$, or equivalently

$$(r+\varepsilon)\mathrm{e}^{\mathrm{i}\theta(\varepsilon,\varphi)} = P_\varepsilon = P + \varepsilon v + \varepsilon\gamma r w + \mathrm{O}\big(\varepsilon^2\big),$$

from which we get

$$(r+\varepsilon)\tan\big(\theta - \theta(\varepsilon,\varphi)\big) = \varepsilon\gamma r + \mathrm{O}\big(\varepsilon^2\big),$$

see Fig. 5. Hence, using that $\tan(\delta) = \delta + \mathrm{O}(\delta^2)$ for δ small, we easily obtain

$$\theta(\varepsilon,\varphi) = \theta - \varepsilon\gamma + \mathrm{O}\big(\varepsilon^2\big).$$

Now, if we define $\beta = (P'_\varepsilon P P''_\varepsilon)$, then we have $\gamma r = \tan(\beta)$, and by Lemma 9

$$\beta = \theta + \frac{\alpha}{2}.$$

By (A.13), this implies the crucial estimate

$$\gamma \ge \frac{\tan(\theta_E(r))}{r}. \tag{A.15}$$

Collecting all together and recalling that $\mathscr{H}^1(I_1) = 2r\theta$, we finally obtain

$$\begin{aligned}
\mathscr{H}^1\big(I_1^\varepsilon\big) &\le 2(r+\varepsilon)\theta(\varepsilon,\varphi) = 2(r+\varepsilon)(\theta - \varepsilon\gamma) + \mathrm{O}\big(\varepsilon^2\big) \\
&= 2r\theta + \varepsilon(2\theta - 2r\gamma) + \mathrm{O}\big(\varepsilon^2\big) \\
&= \mathscr{H}^1(I_1) + \frac{\varepsilon}{r}\big(\mathscr{H}^1(I_1) - 2r^2\gamma\big) + \mathrm{O}\big(\varepsilon^2\big),
\end{aligned}$$

which combined with (A.15) leads to (A.11), and hence to the proof of the theorem. ☐

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