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The right tail exponent of the Tracy–Widom β distribution

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Abstract. The Tracy–Widom β distribution is the large dimensional limit of the top eigenvalue of β random matrix ensembles. We use the stochastic Airy operator representation to show that as $a \to \infty$ the tail of the Tracy–Widom distribution satisfies

$$P(TW_\beta > a) = a^{-(3/4)\beta + \mathrm{o}(1)} \exp\left(-\frac{2}{3}\beta a^{3/2}\right).$$

Résumé. La loi de Tracy–Widom β est la limite de la plus grande valeur propre des ensembles β de matrices aléatoires lorsque leur taille tend vers l'infini. Nous utilisons la représentation par l'opérateur stochastique d'Airy pour montrer que lorsque $a \to \infty$ la queue de la loi de Tracy–Widom vérifie :

$$P(TW_\beta > a) = a^{-(3/4)\beta + \mathrm{o}(1)} \exp\left(-\frac{2}{3}\beta a^{3/2}\right).$$

MSC: 60F10; 60H25

Keywords: Tracy–Widom distribution; Stochastic Airy operator; Beta ensembles

1. Introduction

For $\beta > 0$ fixed, we examine the probability density of $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \mathbb{R}$ given by:

$$\mathbb{P}_\beta(\lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{Z_{n,\beta}} \mathrm{e}^{-\beta \sum_{k=1}^n \lambda_k^2/4} \prod_{j<k} |\lambda_j - \lambda_k|^\beta, \tag{1}$$

in which $Z_{n,\beta}$ is a normalizing constant. This family of distribution is called the β-ensemble. When $\beta = 1, 2$ or 4, this is the joint density of eigenvalues for respectively the Gaussian orthogonal, unitary, or symplectic ensembles of random matrix theory. But the law (1) has a physical sense for all the β as it describes a one-dimensional Coulomb gas at inverse temperature β (see e.g. Chapter 1, Section 1.4 of [7]). Dumitriu and Edelman [5] discovered that (1) is the eigenvalue distribution for the tridiagonal matrix

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ & & & \chi_\beta & g_n \end{bmatrix},$$

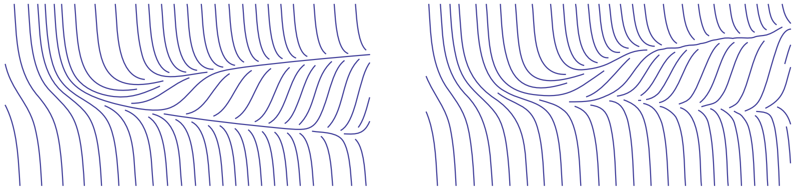

Fig. 1. Flowlines for the ODE of $X - \frac{2}{\sqrt{\beta}} B$.

where the random variables $g_1, g_2, \dots, g_n$ are independent Gaussians with mean 0 and variance 2 and $\chi_\beta, \chi_{2\beta}, \dots, \chi_{(n-1)\beta}$ are independents χ random variables indexed by the shape parameter.

When $n \uparrow \infty$, the largest eigenvalue centered by $2\sqrt{n}$ and scaled by $n^{1/6}$ converges in law to the Tracy–Widom(β) distribution. This was first shown in [9] and [10] for the cases $\beta = 1, 2$ or 4, where exact formulae are available. Ramírez, Rider and Virág [8] extended this result for all the β. They show that the rescaled operator:

$$\tilde{H}_n^\beta := n^{1/6}\bigl(2\sqrt{n} I - H_n^\beta\bigr)$$

converges to the stochastic Airy operator (SAE_β):

$$\mathcal{H}_\beta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x + \frac{2}{\sqrt{\beta}} B_x' \tag{2}$$

in the appropriate sense (here B' is a white noise). In particular, the low-lying eigenvalues of $\tilde{H}_n^\beta$ converge in law to those of $\mathcal{H}_\beta$. Thanks to the Ricatti transform, the eigenvalues of SAE_β can be reinterpreted in terms of the explosion probabilities of a one-dimensional diffusion. In particular, Ramírez, Rider and Virág [8] show that

$$\mathbb{P}(TW_\beta > a) = \mathbb{P}_{+\infty}(X \text{ blows up in a finite time}), \tag{3}$$

where X is the diffusion

$$\begin{cases} \mathrm{d}X(t) = \bigl(t + a - X^2(t)\bigr)\,\mathrm{d}t + \frac{2}{\sqrt{\beta}}\,\mathrm{d}B(t), \\ X(0) = \infty. \end{cases} \tag{4}$$

Note also that $X - \frac{2}{\sqrt{\beta}} B$ satisfies an ODE, simulated on Fig. 1 with $\beta = \infty$ and 2. The starting time of the separatrix is distributed as $-TW_\beta$.

Asymptotic expansions of beta-ensembles are of active interest in the literature, see for example [3,4,6] and [11].

In this article, we study the diffusion (4) in order to obtain the right tail of the Tracy–Widom law. Our main tool will be the Cameron–Martin–Girsanov theorem: it permits us to change the drift coefficient of the diffusion and evaluate the probability of explosion using the new process.

Using the variational characterization of the eigenvalues of SAE_β (2) and an analysis of the SDE (4) Ramírez, Rider and Virág [8] show that as $a \to \infty$ we have

$$\begin{aligned} &P(TW_\beta < -a) = \exp\biggl(-\frac{1}{24}\beta a^3\bigl(1 + \mathrm{o}(1)\bigr)\biggr) \quad \text{and} \\ &P(TW_\beta > a) = \exp\biggl(-\frac{2}{3}\beta a^{3/2}\bigl(1 + \mathrm{o}(1)\bigr)\biggr). \end{aligned} \tag{5}$$

While we were finishing this article, Borot, Eynard, Majumdar and Nadal [2], in a physics paper, using completely different methods, calculated more precise asymptotics for the *left tail* of the Tracy–Widom distribution.

In this paper we evaluate the exponent of the polynomial factor in the asymptotics of the right tail.

Theorem 1. *When $a \to +\infty$, we have*

$$P(TW_\beta > a) = a^{-3\beta/4} \exp\biggl(-\frac{2}{3}\beta a^{3/2} + \mathrm{O}\bigl(\sqrt{\ln a}\bigr)\biggr). \tag{6}$$

This generalizes, in a less precise form, a result that follows from Painlevé asymptotics for the case $\beta = 2$ (see the slide 3 of the presentation [1]).

$$P(TW_2 > a) = \frac{a^{-3/2}}{16\pi} \exp\left(-\frac{4}{3}a^{3/2} + \mathrm{O}\left(a^{-3/2}\right)\right).$$

The structure of the proof of Theorem 1 is contained in Section 2.

Preliminaries and notation

For every initial condition in $[-\infty, +\infty]$, the SDE (4) admits a unique solution, and this solution is increasing in a for each time t (see Fact 3.1 in [8]). From now on, we denote by $(\Omega, \mathcal{F}, \mathbb{P}_{(t,x)})$ the probability space on which the solution of this SDE X begins at time t with the value $X_t = x$ almost surely, and $\mathbb{E}_{(t,x)}$ its corresponding expectation ($x \in [-\infty, +\infty]$). When the starting time is $t = 0$, we simply write $\mathbb{P}_x$ and $\mathbb{E}_x$.

The first passage time to a level $x \in [-\infty, \infty]$ for the diffusion X will be denoted $T_x := \inf\{s \geq 0, X_s = x\}$.

Throughout this paper, we study many solutions of stochastic differential equations by comparing them to expressions involving Brownian motion. The letter B will denote a standard Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. We will use the following easy estimates. For the upper bounds, these inequalities hold for every $x \geq 0$:

$$\begin{cases} P(B_1 > x) \leq \mathrm{e}^{-(1/2)x^2}, \\ P\left(\sup_{t\in[0,1]} |B_t| > x\right) \leq 4\mathrm{e}^{-(1/2)x^2}. \end{cases} \tag{7}$$

For the lower bounds, there exists $c_{bm} > 0$ such that for every $\varepsilon \in (0, 1)$:

$$P\Big(\sup_{t\in[0,1]} |B_t| < \varepsilon\Big) \geq \exp\left(-c_{bm}\frac{1}{\varepsilon^2}\right). \tag{8}$$

In the sequel, asymptotic notation always refers to $a \to \infty$ unless stated otherwise. Inequalities are meant to hold for all large enough a.

2. Proof of Theorem 1

This section gives the structure of proof of the main theorem. The proof of technical points will be treated in the following sections in chronological order.

We rely on the characterization (3), and separate our study of the diffusion (4) into three distinguished parts demarcated by the *critical parabola*

$$\left\{(t, x)\colon\ t + a - x^2 = 0\right\}$$

where the drift vanishes (see Fig. 2). The exponential leading term of the asymptotic (6) comes from the part inside the parabola (Stretch II): the drift is positive and makes it difficult for the particle to go down. One part of the logarithmic term comes from the time it takes to reach the upper part of the parabola (Stretch I): the t-term of the drift adds this cost.

2.1. *Upper bound*, *above the parabola*

At first, let us approximate the critical parabola by the two horizontal lines $\sqrt{a}$ and $-\sqrt{a}$ (as the blow-down times will be typically very small). Moreover, the part below the parabola gives no contribution for the upper bound, and we use

$$\mathbb{P}_\infty(T_{-\infty} < +\infty) \leq \mathbb{P}_\infty(T_{-\sqrt{a}} < +\infty).$$

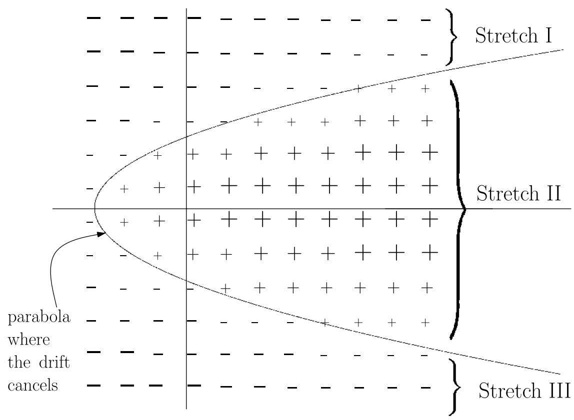

Fig. 2. The critical parabola $\{(t,x)\colon t+a-x^2=0\}$.

The first step is to control the time it takes to reach $\sqrt{a}$. Indeed, as the cost for crossing the interval $[-\sqrt{a},\sqrt{a}]$ increases with time, we need to find a good lower bound for this time. A comparison with the solution of an ODE linked to our SDE enables us to have a quite precise information: its typical value is $3/8\ln a/\sqrt{a}$, which does not depend on the factor β. It is very unlikely to happen in time faster than

$$\tau_- = \bigl(3/8 - 1/\sqrt{\ln a}\bigr)\ln a/\sqrt{a}.$$

This is the content of Proposition 2. Therefore, using this proposition, the decreasing property in t and the Markov property, we can write:

$$\mathbb{P}_\infty(T_{\sqrt{a}} < \tau_-, T_{-\sqrt{a}} < \infty) \le \exp\biggl(-\frac{4}{3}\beta \mathrm{e}^{2\sqrt{\ln a}}\biggr)\mathbb{P}_{\sqrt{a}}(T_{-\sqrt{a}} < \infty). \tag{9}$$

The asymptotic formula (15) given by Lemma 6 will highlight the fact that even if the process is considered to start immediately at $\sqrt{a}$ in line (9), the award is small (of the order $\exp(\mathrm{O}(\ln a))$) compared to the cost it takes to go down quickly. Consequently, with a much more significant probability, it will take a longer time than the one considered in (9) to reach $\sqrt{a}$. Let us find an upper bound for this case:

$$\mathbb{P}_\infty(T_{-\sqrt{a}} < \infty, T_{\sqrt{a}} \ge \tau_-) \le \mathbb{P}_{\tau_-,\sqrt{a}}(T_{-\sqrt{a}} < \infty).$$

Thanks to the Markov property, the process X under the probability measure $\mathbb{P}_{\tau_-,\sqrt{a}}$ is identically distributed with $\tilde{X}$ defined with the same SDE (4) where the variable a is replaced by $\tilde{a} := a + \tau_-$ with the initial condition $\tilde{X}(0) = \sqrt{a}$. Observe now $\sqrt{a} < \sqrt{\tilde{a}}$, but it does not matter as we will be allowed to reduce the interval $[-\sqrt{a},\sqrt{a}]$ a bit without affecting the relevant terms in our asymptotics. More precisely, the interval we will study for the middle is $[-\sqrt{a}+\delta, \sqrt{a}-\delta]$, where $\delta := \sqrt[4]{\ln a/a}$.

Let $\tilde{T}_x$ denote the first passage times of $\tilde{X}$. The inequality $\sqrt{\tilde{a}} - \tilde{\delta} \le \sqrt{a}$ gives:

$$\mathbb{P}_{\tau_-,\sqrt{a}}(T_{-\sqrt{a}} < \infty) = \mathbb{P}_{\sqrt{a}}(\tilde{T}_{-\sqrt{a}} < \infty) \le \mathbb{P}_{\sqrt{\tilde{a}}-\tilde{\delta}}(\tilde{T}_{-\sqrt{\tilde{a}}+\tilde{\delta}} < \infty). \tag{10}$$

2.2. *Preliminary upper bound inside the parabola*

Recall $\delta := \sqrt[4]{\ln a/a}$ we would like to find the asymptotic of:

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta} < \infty) \ge \mathbb{P}_{\sqrt{a}}(T_{-\sqrt{a}} < \infty). \tag{11}$$

The key is Girsanov formula.

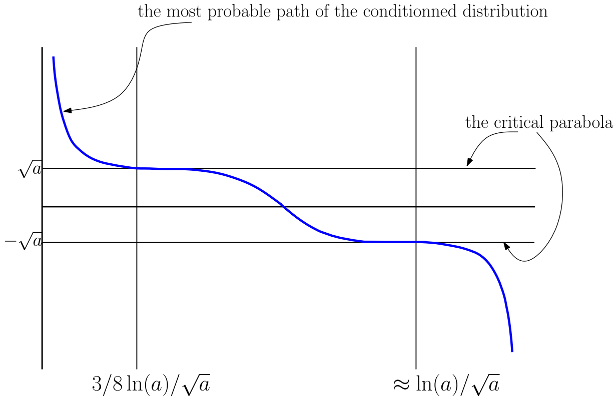

Fig. 3. The most probable path of the conditioned diffusion for $a = 100$.

Girsanov formula

To evaluate the cost inside the parabola, we use the Cameron–Martin–Girsanov formula which allows us to change the drift coefficient of X and evaluate the relevant probability by analyzing the new process. The issue is to find a suitable new drift. The best would be to have the one corresponding to the conditional distribution of the diffusion X under the event it crosses the critical parabola in a finite time: this would lead to an exact formula. As we are not able to do that, we use an approximation of the conditional diffusion; see Fig. 3. In this direction, we introduce a new SDE in which the drift of X is reversed with a correction term given by a function φ:

$$\mathrm{d}Y_t = \big(-a + Y_t^2 - t + \varphi(Y_t)\big)\,\mathrm{d}t + \frac{2}{\sqrt{\beta}}\,\mathrm{d}B_t.$$

Let T'_x denote the first passage time of the process Y to the level x.

With this diffusion and under some mild assumptions, the Cameron–Martin–Girsanov formula gives for every non-negative measurable function f and every fixed time $t > 0$ and level $l \in [0, 1]$:

$$\mathbb{E}_{\sqrt{a}-l}\big(f(X_u, u \le T_{-\sqrt{a}+l} \wedge t)\big) = \mathbb{E}_{\sqrt{a}-l}\big(f\big(Y_u, u \le T'_{-\sqrt{a}+l} \wedge t\big)\exp\big(G_{T'_{-\sqrt{a}+l}\wedge t}(Y)\big)\big). \tag{12}$$

More details about this and the application of the Girsanov formula can be found in Section 4.2. The exact expression of $G_{T'_{-\sqrt{a}+l}}(Y)$ contains $\beta/4$ times

$$-\frac{8}{3}a^{3/2} - \frac{4}{3}l^3 + 4\sqrt{a}l^2 + 2lT'_{-\sqrt{a}+l} - 2\sqrt{a}T'_{-\sqrt{a}+l} + \left(\frac{8}{\beta} - 2\right)\int_0^{T'_{-\sqrt{a}+l}} Y_t\,\mathrm{d}t \tag{13}$$

plus terms involving the function φ.

Notice that we can already see the correct coefficient in front of the main term. We are now confronted with an expectation over the paths of the diffusion Y. To find a good estimate of the exponential martingale, we need to control the first passage time to the level $-\sqrt{a} + l$ and check that the diffusion do not go far above $\sqrt{a}$ when this time is finite (in order to control the last integral in (13)). We will at first focus on a preliminary bound, for which we do not need any result about the first passage time. The price of this approach is that it uses a finer control of the paths which go down.

Control of the paths

To have a good control of the paths, we examine at first a smaller interval than $[\sqrt{a} - \delta, -\sqrt{a} + \delta]$. On this new interval, the diffusion will go down without hitting $\sqrt{a}$ with a sufficiently large probability. Indeed, we show that for $\varepsilon := \frac{4}{\sqrt{\beta}}\sqrt{\ln a}/\sqrt[4]{a}$ we have

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_{-\sqrt{a}+\varepsilon} < \infty) = \big(1 - \mathrm{o}(1)\big)\mathbb{P}_{\sqrt{a}-\varepsilon}(T_{-\sqrt{a}+\varepsilon} < \infty, T_{-\sqrt{a}+\varepsilon} < T_{\sqrt{a}-\varepsilon/2}). \tag{14}$$

This is accomplished by two applications of the strong Markov property. From now on, we denote $T_+ = T_{\sqrt{a}-\varepsilon/2}$, $T_- = T_{-\sqrt{a}+\varepsilon}$ and $\mathcal{A} = \{T_+ > T_-\}$, and have:

$$\mathbb{P}_{\sqrt{a}-\varepsilon}\bigl(T_- < \infty, \mathcal{A}^c\bigr) \le \mathbb{P}_{\sqrt{a}-\varepsilon/2}(T_- < \infty) \le \mathbb{P}_{\sqrt{a}-\varepsilon/2}(T_{\sqrt{a}-\varepsilon} < \infty)\mathbb{P}_{\sqrt{a}-\varepsilon}(T_- < \infty).$$

Both inequalities use the fact that the hitting probability of any level below the starting place is decreasing in the starting time of the diffusion X. Rearranging this formula we get

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_- < \infty) \le \frac{\mathbb{P}_{\sqrt{a}-\varepsilon}(T_- < \infty, \mathcal{A})}{\mathbb{P}_{\sqrt{a}-\varepsilon/2}(T_{\sqrt{a}-\varepsilon} = \infty)}.$$

Lemma 4 shows that as $a \to \infty$ the denominator converges to 1.

Application of the Girsanov formula
We will now study the right-hand side of (14) $\mathbb{P}_{\sqrt{a}-\varepsilon}(T_- < \infty, \mathcal{A})$ which is the probability of an event under which the absolute value of the diffusion is bounded by $\sqrt{a}$.

In order to find an upper bound for the term (13) with $l = \varepsilon$, it would be useful to have a bound on the time T_-. As we do not have any information about that yet, one idea is to choose the function φ such that the coefficient appearing in front of the $\sqrt{a}T_-$ term becomes negative. The function φ_1 of Section 4.3 works and it gives an upper bound that is sharp up to, but not including, the exponent of the polynomial factor. This is the content of Lemma 6. As $a \to \infty$ we conclude

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_- < \infty) \le \exp\biggl(-\frac{2}{3}\beta a^{3/2} + \mathrm{O}(\ln a)\biggr), \quad \varepsilon = \frac{4}{\sqrt{\beta}}\sqrt{\ln a}/\sqrt[4]{a}. \tag{15}$$

In addition, Lemma 6 also shows that with $\xi = c_3 \ln a/\sqrt{a}$ we have

$$\mathbb{P}_{\sqrt{a}-\delta}(\xi \le T_{-\sqrt{a}+\delta} < \infty) \le \exp\biggl(-\frac{2}{3}\beta a^{3/2} - \beta \ln a\biggr), \quad \delta = \sqrt[4]{\ln a/a} \tag{16}$$

i.e. long times have polynomially smaller probability than what we expect for normal times.

2.3. *Final upper bound inside the parabola*

Decomposition according to the time the process spends near $\sqrt{a}$
Let us introduce the last passage time to the level $\sqrt{a} - \delta$:

$$L := \sup\bigl\{t \ge 0\colon\ X_t = \sqrt{a} - \delta\bigr\}, \quad \delta = \sqrt[4]{\ln a/a}$$

and use the temporary notation $\tau = c \ln a/\sqrt{a}$. We can use the less precise result and, similarly to the part above the parabola, make a change of variable $\hat{a} := a + \tau$. The strong Markov property and the monotonicity gives

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta} < \infty, L > \tau) \le \mathbb{P}_{(\tau,\sqrt{a}-\delta)}(T_{-\sqrt{a}+\delta} < \infty). \tag{17}$$

Since $\sqrt{a} - \delta \ge \sqrt{\hat{a}} - \hat{\varepsilon}$ we get the upper bound

$$\mathbb{P}_{(0,\sqrt{\hat{a}}-\hat{\varepsilon})}(\tilde{T}_{-\sqrt{\hat{a}}+\hat{\varepsilon}} < \infty) \le \exp\bigl(-2/3\beta a^{3/2} - \beta \ln a\bigr)$$

as long as c (depending on β) is large enough. The last inequality for some constant $c > 0$ follows from the preliminary bound (15). This again is polynomially smaller than the probability we expect for the main event.

For finer information about the last passage time, one can divide the paths according to the value of this last passage time to formalize the idea the process does not earn a lot when it stays near $\sqrt{a} - \delta$.

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta} < \infty, L < \tau) \le \sum_{k=0}^{\lfloor c \ln a\rfloor} \mathbb{P}\biggl(L \in \biggl[\frac{k}{\sqrt{a}}, \frac{k+1}{\sqrt{a}}\biggr), T_{-\sqrt{a}+\delta} < +\infty\biggr).$$

The event in the sum implies that X visits $\sqrt{a}-\delta$ in the time interval but not later. By the strong Markov property for the first visit after time $k/\sqrt{a}$ in that time interval and monotonicity, the sum can be bounded above by

$$\lfloor 1+c\ln a\rfloor \mathbb{P}_{\sqrt{a}-\delta}\left(L<\frac{1}{\sqrt{a}}, T_{-\sqrt{a}+\delta}<\infty\right).$$

To complete the picture, we find an upper bound of the process between the times 0 and $1/\sqrt{a}$. This will be possible thanks to a comparison with reflected Brownian motion. Indeed, the drift is non-positive above the critical parabola, and so up to time $1/\sqrt{a}$ the process X_t started at $\sqrt{a+1/\sqrt{a}}$ is stochastically dominated by $\sqrt{a+1/\sqrt{a}}$ plus reflected Brownian motion. This leads to the very rough estimate:

$$\begin{aligned}\mathbb{P}_{\sqrt{a+1/\sqrt{a}}}\Big(\sup_{t\in[0,1/\sqrt{a}]} X_t>c_2\sqrt{a}\Big) &\le P\Big(\sup_{s\in[0,1/\sqrt{a}]}|B_s|>(c_2-2)\sqrt{a}\Big)\\ &\le \exp\left(-\frac{1}{2}(c_2-2)^2a^{3/2}\right).\end{aligned}$$

If c_2 is large enough (precisely if $c_2>2/\sqrt{3}\sqrt{\beta}+2$), this event becomes negligible compared to the probability to cross the whole parabola. Therefore, we can examine the studied probability under the event that X_t is bounded from above by $c_2\sqrt{a}$ for $t\le 1/\sqrt{a}$.

We denote by $\mathcal{C}$ the event under which the above conditions are satisfied

$$\mathcal{C}:=\Big\{L<1/\sqrt{a}, \sup_{t\in[0,1/\sqrt{a}]}X_t<c_2\sqrt{a}\Big\}.$$

We have just seen that

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta}<\infty)\le (2c\ln a)\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta}<\infty,\mathcal{C})+\mathrm{O}\big(\exp\big(-2/3\beta a^{3/2}-\beta\ln a\big)\big). \tag{18}$$

Application of the Girsanov formula

We will apply the Girsanov formula with a function $\varphi=\varphi_2$ such that it compensates exactly the integral

$$\int_0^{T'_{-\sqrt{a}+\delta}}\left(\frac{8}{\beta}-2\right)Y_t-2\sqrt{a}\,\mathrm{d}t.$$

The suitable function φ_2 blows up at $-\sqrt{a}$ and $\sqrt{a}$. Therefore we only use it in the interval $[-\sqrt{a}+\delta,\sqrt{a}-\delta]$ and set $\varphi_2:=0$ outside $[-\sqrt{a},\sqrt{a}]$. Partially because of those blowups, this function creates error terms involving the first passage time to the level $-\sqrt{a}+\delta$, which, if finite, by (16) can be assumed to satisfy $T_{-\sqrt{a}+\delta}<\xi$. Girsanov's formula applied to the event $\{T_{\sqrt{a}+\delta}<\xi,\mathcal{C}\}$ with (18) leads to the fundamental upper bound of Proposition 8:

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta}<\xi)\le\exp\left(-\frac{2}{3}\beta a^{3/2}-\frac{3}{8}\beta\ln a+\mathrm{O}\big(\sqrt{\ln a}\big)\right). \tag{19}$$

Conclusion for the upper bound

Using (19) with $\tilde{a}$ as in the inequality (10) of the part above the parabola, we deduce the upper bound part of (6).

2.4. *Outline of the lower bound*

The lower bound, as often in the literature, is easier. It suffices to consider the most probable paths. For the part above the parabola, we can write the inequality

$$\mathbb{P}_\infty(T_{-\infty}<\infty)\ge\mathbb{P}_\infty\left(T_{\sqrt{a}}\le\frac{3}{8}\frac{\ln a}{\sqrt{a}}\right)\mathbb{P}_{((3/8)\ln a/\sqrt{a},\sqrt{a})}(T_{-\infty}<\infty)$$

and use Proposition 2 to bound the first factor. The second factor can be bounded below by the following:

$$\mathbb{P}_{((3/8)\ln a/\sqrt{a},\sqrt{a})}\left(T_{\sqrt{\tilde a}-\tilde\delta} < \frac{1}{\sqrt{a}}\right) \times \mathbb{P}_{\sqrt{\tilde a}-\tilde\delta}(T_{-\sqrt{\tilde a}+\tilde\delta} < \tilde\xi) \times \mathbb{P}_{(\tilde\xi,-\sqrt{\tilde a}+\tilde\delta)}(T_{-\infty} < \infty), \tag{20}$$

where $\tilde a := 3/8\ln a/\sqrt{a} + 1/\sqrt{a}$.

A domination by a Brownian motion with drift permits to deal with the first term of (20) which is of the order $\exp(-\mathrm{O}(\sqrt{\ln a}))$. The middle term can be controlled with the same event $\tilde{\mathcal{C}}$ introduced for the upper bound with a replaced by $\tilde a$. We apply Girsanov formula directly with the SDE used for the precise result of the upper bound to obtain Proposition 8:

$$\mathbb{P}_{\sqrt{\tilde a}-\tilde\delta}(T_{-\sqrt{\tilde a}+\tilde\delta} < \tilde\xi, \tilde{\mathcal{C}}) \ge \exp\left(-\frac{2}{3}\beta\tilde a^{3/2} - \frac{3}{8}\beta\ln\tilde a + \mathrm{o}(\ln\tilde a)\right)\mathbb{P}_{\sqrt{\tilde a}-\tilde\delta}\left(T'_{-\sqrt{\tilde a}+\tilde\delta} < \tilde\xi, \tilde{\mathcal{C}}'\right)$$

(recall the “prime” notation deals with the “new” diffusion Y).

A comparison with the solution of a simple differential equation will show that the solution of the new SDE indeed has a “large” probability to go down to $-\sqrt{\tilde a} + \tilde\delta$ before the time $\tilde\xi$. This is the content of Lemma 9.

We conclude the proof of the lower bound by checking that the last term of (20) is also negligible: the proof is similar to the study above the parabola and can be found in Proposition 10.

3. Above the parabola

We show at first that we need a certain amount of time to reach the level $\sqrt{a}$, typically a time $\tau := 3/8\ln a/\sqrt{a}$. The following proposition proves indeed that the probability the process hits $\sqrt{a}$ significantly before τ is small, but becomes quite large if this hitting happens around the time τ.

Proposition 2. *The following upper bound holds for all sufficiently large* a:

$$\mathbb{P}_\infty\left(T_{\sqrt{a}} \le \left(\frac{3}{8} - \frac{1}{\ln a}\right)\frac{\ln a}{\sqrt{a}}\right) \le \exp\left(-\frac{4}{3}\beta \mathrm{e}^{2\sqrt{\ln a}}\right). \tag{21}$$

There exists $c_0 > 0$ *depending only on* β *such that we also have the lower bound*:

$$\mathbb{P}_\infty\left(T_{\sqrt{a}} \le \frac{3}{8}\frac{\ln a}{\sqrt{a}}\right) \ge c_0\frac{1}{\sqrt{\ln a}}. \tag{22}$$

Proof. *First part.* Fix $\sigma \in (0, 1/8)$ and let

$$\tau' := \left(\frac{3}{8} - \sigma\right)\frac{\ln a}{\sqrt{a}}.$$

It helps to remove the time dependence from the drift coefficient of (4). In this direction, consider the SDE:

$$\begin{cases} \mathrm{d}Y_t = \left(a - Y_t^2\right)\mathrm{d}t + \frac{2}{\sqrt{\beta}}\,\mathrm{d}B_t, \\ Y_0 = +\infty. \end{cases} \tag{23}$$

The process X stochastically dominates Y. Therefore, if $T^Y_{\sqrt{a}}$ is the first passage time to $\sqrt{a}$ for the diffusion Y, we have:

$$\mathbb{P}_\infty\left(T_{\sqrt{a}} \le \tau'\right) \le \mathbb{P}\left(T^Y_{\sqrt{a}} \le \tau'\right).$$

Now, we study the difference $Z_t := Y_t - \frac{2}{\sqrt{\beta}}B_t$ where B is the Brownian motion driving Y in (23). It satisfies the (random) ODE:

$$\mathrm{d}Z_t = \left[a - Z_t^2\left(1 + \frac{2}{\sqrt{\beta}}\frac{B_t}{Z_t}\right)^2\right]\mathrm{d}t. \tag{24}$$

Define $M := \sup\{|B_t|, t \in [0, \tau']\}$ and notice that thanks to the Brownian tail (7),

$$\mathbb{P}(M \geq 1) \leq 4\exp\left(-\frac{1}{2(3/8-\sigma)}\frac{\sqrt{a}}{\ln a}\right).$$

By definition, for all $t \in [0, \tau' \wedge T^Y_{\sqrt{a}}]$, the process Z_t is above $\sqrt{a} - 2/\sqrt{\beta}M$ and consequently above the (random) solution of the differential equation:

$$\begin{cases} F'(t) = a - Cf^2(t), \\ F(0) = +\infty, \end{cases}$$

where C has the following expression:

$$C := \left(1 + \frac{4}{\sqrt{\beta}}\frac{M}{\sqrt{a} - (2/\sqrt{\beta})M}\right)^2.$$

This differential equation admits almost surely the unique solution

$$\forall t \geq 0, \quad F(t) = \sqrt{\frac{a}{C}}\coth\bigl(\sqrt{aC}t\bigr).$$

Hence,

$$\begin{aligned} \mathbb{P}\bigl(T^Y_{\sqrt{a}} \leq \tau'\bigr) &\leq \mathbb{P}\left(\inf_{t\in[0,\tau']}\left(F(t) + \frac{2}{\sqrt{\beta}}B_t\right) \leq \sqrt{a}\right) \\ &\leq \mathbb{P}\left(F\bigl(\tau'\bigr) - \sqrt{a} \leq -\frac{2}{\sqrt{\beta}}\inf_{t\in[0,\tau']} B_t\right). \end{aligned} \tag{25}$$

Let us compute $F(\tau')$:

$$F\bigl(\tau'\bigr) = \sqrt{\frac{a}{C}}\frac{1+\mathrm{e}^{-2\tau'\sqrt{aC}}}{1-\mathrm{e}^{-2\tau'\sqrt{aC}}} = \sqrt{\frac{a}{C}}\bigl(1 + 2\mathrm{e}^{-2\tau'\sqrt{aC}} + \mathrm{O}\bigl(\mathrm{e}^{-4\tau'\sqrt{aC}}\bigr)\bigr).$$

Under the event $\{M \leq 1\}$,

$$\sqrt{C} = 1 + \frac{2}{\sqrt{\beta}}\frac{M}{\sqrt{a}} + \mathrm{O}\left(\frac{1}{a}\right).$$

This implies:

$$\sqrt{\frac{a}{C}} = \sqrt{a} - \frac{2}{\sqrt{\beta}}M + \mathrm{O}\left(\frac{1}{\sqrt{a}}\right) \tag{26}$$

and

$$\begin{aligned} \exp\bigl(-2\tau'\sqrt{aC}\bigr) &= \exp\left(-2(3/8-\sigma)\ln a\left(1 + \mathrm{O}\left(\frac{1}{\sqrt{a}}\right)\right)\right) \\ &= \frac{1}{a^{3/4-2\sigma}} + \mathrm{O}\left(\frac{\ln a}{a^{5/4-2\sigma}}\right). \end{aligned} \tag{27}$$

Taylor expansions (26) and (27) give:

$$\begin{aligned} F\bigl(\tau'\bigr) &= \left(\sqrt{a} - \frac{2}{\sqrt{\beta}}M + \mathrm{O}\left(\frac{1}{\sqrt{a}}\right)\right)\left(1 + \frac{2}{a^{3/4-2\sigma}} + \mathrm{O}\left(\frac{\ln a}{a^{5/4-2\sigma}}\right)\right) \\ &= \sqrt{a} - \frac{2}{\sqrt{\beta}}M + \frac{2}{a^{1/4-2\sigma}} + \mathrm{O}\left(\frac{1}{\sqrt{a}}\right). \end{aligned} \tag{28}$$

Inequality (25) becomes:

$$\begin{aligned}\mathbb{P}_\infty\big(T_{\sqrt{a}} \le \tau'\big) &\le \mathbb{P}\biggl(F\big(\tau'\big) - \sqrt{a} \le \frac{2}{\sqrt{\beta}} M, M \le 1\biggr) + \mathbb{P}(M > 1)\\ &\le \mathbb{P}\biggl(\sqrt{\beta}\frac{1}{a^{1/4-2\sigma}} + \mathrm{O}\biggl(\frac{1}{\sqrt{a}}\biggr) \le M, M \le 1\biggr) + \mathbb{P}(M > 1)\\ &\le 4\exp\biggl(-\frac{4}{3}\beta\frac{a^{4\sigma}}{\ln a} + \mathrm{o}\biggl(\frac{a^{4\sigma}}{\ln a}\biggr)\biggr).\end{aligned}$$

If we take $\sigma = \frac{1}{\sqrt{\ln a}}$, we have:

$$\mathbb{P}_\infty\biggl(T_{\sqrt{a}} \le \biggl(\frac{3}{8} - \frac{1}{\sqrt{\ln a}}\biggr)\frac{\ln a}{\sqrt{a}}\biggr) \le 4\exp\biggl(-\frac{4}{3}\beta \mathrm{e}^{4\sqrt{\ln a} - \ln\ln a}\biggr),$$

and so inequality (21) holds.

Second part. The proof is similar. Let us check the main lines. At first, we have:

$$\mathbb{P}_\infty(T_{\sqrt{a}} \le \tau) \ge \mathbb{P}_\infty\biggl(T_{\sqrt{a}+15/\sqrt[4]{a}} \le \tau - \frac{1}{\sqrt{a}}\biggr) \times \mathbb{P}_{\sqrt{a}+15/\sqrt[4]{a}}\biggl(T_{\sqrt{a}} \le \frac{1}{\sqrt{a}}\biggr).$$

Now the process $(X_t, t \in [0, T_{\sqrt{a}}])$ starting at a value above $\sqrt{a}$ has a non-positive drift and is therefore stochastically dominated by Brownian motion. Thus the second factor can be bounded below by

$$P\biggl(B_{1/\sqrt{a}} \ge \frac{15}{\sqrt[4]{a}}\biggr) = P(B_1 \ge 15).$$

For the first factor, instead of the SDE (23) we choose:

$$\begin{cases} \mathrm{d}Y_t = \big(a + \tau - Y_t^2\big)\,\mathrm{d}t + \frac{2}{\sqrt{\beta}}\,\mathrm{d}B_t, \\ Y_0 = +\infty \end{cases} \tag{29}$$

and study the difference

$$Z_t := Y_t - \frac{2}{\sqrt{\beta}} B_t.$$

Set $M' := \sup\{B_t, t \in [0, \tau - 1/\sqrt{a}]\}$. For every $t \in [0, (\tau - 1/\sqrt{a}) \wedge T^Y_{\sqrt{a}}]$, the process Z_t is below

$$F(t) := \sqrt{\frac{a+\tau}{C}}\coth\bigl(\sqrt{(a+\tau)C}t\bigr),$$

where

$$C := 1 - \frac{4}{\sqrt{\beta}}\frac{M'}{\sqrt{a} - (2/\sqrt{\beta})M'}.$$

The Taylor expansion of $F(\tau - 1/\sqrt{a})$ under $\{M' \le 1\}$ gives:

$$F\biggl(\tau - \frac{1}{\sqrt{a}}\biggr) = \sqrt{a} + \frac{2\mathrm{e}^2}{a^{1/4}} + \frac{2}{\sqrt{\beta}}M' + \mathrm{O}\biggl(\frac{1}{\sqrt{a}}\biggr).$$

Therefore

$$\mathbb{P}_\infty\biggl(T_{\sqrt{a}+15/\sqrt[4]{a}} \le \tau - \frac{1}{\sqrt{a}}\biggr) \ge P\biggl(F\biggl(\tau - \frac{1}{\sqrt{a}}\biggr) - \frac{2}{\sqrt{\beta}}B\biggl(\tau - \frac{1}{\sqrt{a}}\biggr) \le \sqrt{a} + \frac{15}{\sqrt[4]{a}}\biggr).$$

Since $M' - B(\tau - 1/\sqrt{a})$ has the same law as the reflected Brownian motion at time $\tau - 1/\sqrt{a}$ and $15 - 2e^2 > 0$, we get the lower bound

$$P\left(\frac{2}{\sqrt{\beta}}\left|B\left(\tau - \frac{1}{\sqrt{a}}\right)\right| \le \frac{15 - 2e^2}{\sqrt[4]{a}}\right) \ge c_0 \frac{1}{\sqrt{\ln a}}. \tag{30}$$

Here c_0 in represents an adequate constant depending only on β. □

4. Inside the parabola

The exponential cost comes from this stretch. This section will be devoted to the proof of the following proposition:

Proposition 3. *Recall* $\delta := \sqrt[4]{\ln a}/\sqrt[4]{a}$, *we have*:

$$\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta} < \infty) = \exp\left(-\frac{2}{3}\beta a^{3/2} - \frac{3}{8}\beta \ln a + \mathrm{O}\left(\sqrt{\ln a}\right)\right).$$

4.1. *Control of the path behavior*

Here we show a lemma about the return to $-\sqrt{a} + \varepsilon$.

Lemma 4. *For* $\varepsilon := \frac{4}{\sqrt{\beta}}\sqrt{\ln a}/\sqrt[4]{a}$ *as* $a \to \infty$ *we have* $\mathbb{P}_{\sqrt{a}-\varepsilon/2}(T_{\sqrt{a}-\varepsilon} = \infty) \to 1$.

Proof. Certainly, the probability that X begun at $\sqrt{a} - \varepsilon/2$ never reaches $\sqrt{a} - \varepsilon$ is bounded below by the same probability where X is replaced by its reflected (downward) at $\sqrt{a} - \varepsilon/2$ version. Further, when restricted to the space interval $[\sqrt{a} - \varepsilon, \sqrt{a} - \varepsilon/2]$, the X-diffusion has drift everywhere bounded below by $t + 1/2\sqrt{a}\varepsilon$. Thus, we may consider instead the same probability for the appropriate reflected Brownian motion with quadratic drift.

To formalize this, it is convenient to shift orientation. Let now

$$\bar{X} := \frac{2}{\sqrt{\beta}} B(t) - \frac{1}{2}t^2 - qt, \quad q := \frac{1}{2}\sqrt{a}\varepsilon.$$

Let X^* denote reflected (upward) at the origin. Namely,

$$X^*(t) = \bar{X}(t) - \inf_{s \le t} \bar{X}(s). \tag{31}$$

If we can show that $P(X^* \text{ never reaches } \varepsilon/2)$ tends to 1 when $a \to \infty$, then it will also be the case for the X-probability in question.

We need to introduce the first hitting time of level y for the new process $\bar{X}$: $\tau_y := \inf\{t \ge 0\colon \bar{X}(t) = y\}$. For each $n \in \mathbb{N}$, define the event:

$$D_n = \left\{\bar{X}(t) \text{ hits } -(n-1)\varepsilon/4 \text{ for some } t \text{ between } \tau_{-n\varepsilon/4} \text{ and } \tau_{-(n+1)\varepsilon/4}\right\}.$$

From the representation (31), one can see that $\{\sup_{t \ge 0} X^*(t) > \varepsilon/2\}$ implies that some D_n must occur. Indeed, for this $\bar{X}$ must go above its past minimum by at least $\varepsilon/2$, so in this case it would have to retreat at least one level before establishing reaching a new minimum level among multiples of $\varepsilon/4$. Define the event

$$\mathcal{E} = \left\{\bar{X}(t) \ge -\frac{1}{2}t^2 - 2qt - 1\right\}.$$

The event $\mathcal{E}^c$ is equivalent to the Brownian motion $\frac{2}{\sqrt{\beta}}B(t)$ hitting a line of slope $-q$ starting at -1. Since $|B(t)|$ is sublinear, this will not happen for large enough q. So

$$P\left(\mathcal{E}^c\right) \to 0 \quad \text{when } a \to +\infty. \tag{32}$$

On $\mathcal{E}$, when the following term under the square root is positive, we have

$$\tau_{-x} \geq \sqrt{2}\sqrt{2q^2 + x - 1} - 2q.$$

Assume that $q \geq 1$. A calculation shows that if $q \leq \sqrt{x}/2$ then the above inequality implies $\tau_{-x} \geq \sqrt{x}/2$. So on $\mathcal{E}$ for all $x \geq 0$ we have

$$\tau_{-x} + q \geq q \vee \sqrt{x}/2,$$

and so for all $t \geq 0$

$$\bar{X}(\tau_{-x} + t) - \bar{X}(\tau_{-x}) \leq \frac{2}{\sqrt{\beta}} B(\tau_{-x} + t) - \big(q \vee \sqrt{x}/2\big)t. \tag{33}$$

Setting $x = \varepsilon n/4$ we see that for all $n \geq 0$

$$P(D_n \cap \mathcal{E} | \mathcal{F}_{\tau_{-n\varepsilon/4}}) \leq P\big(\text{the process on the right of (33) hits } \varepsilon/4 \text{ before } -\varepsilon/4\big).$$

The distribution of that process is just Brownian motion with drift. By a stopping time argument for the exponential martingale $\exp(\gamma \hat{B}_t - t\gamma^2/2)$ with $\gamma = (q \vee \sqrt{n\varepsilon}/4)\sqrt{\beta}$ the above probability equals $1/(1 + \mathrm{e}^{\gamma\varepsilon\sqrt{\beta}/8}) \leq \mathrm{e}^{-\gamma\varepsilon/\sqrt{\beta}/8}$. We get

$$P(D_n \cap \mathcal{E}) \leq \exp\big(-\big(q \vee \sqrt{n\varepsilon}/4\big)\beta\varepsilon/8\big).$$

Recall that

$$P\Big(\sup_{t\geq 0} X^*(t) > \varepsilon/2\Big) \leq P\big(\mathcal{E}^c\big) + \sum_{n\geq 0} P(\mathcal{E} \cap D_n).$$

The sum of terms where $n\varepsilon \leq (4q)^2$ is bounded above by $(1 + (4q)^2\varepsilon^{-1})\mathrm{e}^{-\varepsilon q\beta/8}$. The sum of the rest is not more than

$$\sum_{n \geq (4q)^2/\varepsilon} \exp\big(-\varepsilon^{3/2}\beta\sqrt{n}/32\big) \leq c_1 \frac{q}{\varepsilon^2} \exp(-\varepsilon q\beta/8),$$

where c_1 depends on β only. Replacing ε and q by their expressions in terms of a, and using (32) we obtain:

$$P\Big(\sup_{t\geq 0} X^*(t) > \varepsilon/2\Big) \leq P\big(\mathcal{E}^c\big) + a^{3/4+\mathrm{o}(1)}\mathrm{e}^{-(\beta/16)(16/\beta)\ln a} = \mathrm{o}(1).$$

□

4.2. *Application of the Girsanov formula*

For every $\varphi \in C^2(\mathbb{R}, \mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |\varphi(x)| \leq \sqrt{a}$ (the function φ is in fact small compared to the other terms, and it will be chosen after), we would like to consider the following SDE (defined on the same probability spaces as (4)):

$$\mathrm{d}Y_t = \big(-a + Y_t^2 - t + \varphi(Y_t)\big)\,\mathrm{d}t + \frac{2}{\sqrt{\beta}}\,\mathrm{d}B_t. \tag{34}$$

Remark 5. *The drift of this SDE is the reversal of the drift in the initial SDE* (4). *The solution of the new SDE starting around $\sqrt{a}$ is a good candidate for the process X conditioned to blow up to $-\infty$ when a goes to $+\infty$. Its expression comes from minimizing* (*approximately*) *the potential*:

$$\int_0^s \big(g'(u) - \big(u + a - g^2(u)\big)^2\big)\,\mathrm{d}u$$

over the set of functions g such that $g(0) = \sqrt{a}$ and $g(s) = -\sqrt{a}$.

If we look at events under which the diffusion is bounded, it is easy to modify the drift of the new diffusion outside the studied domain and prove that the Novikov condition is satisfied as long as the examined events are in the space $\mathcal{F}_t$ for a fixed $t > 0$. Let us fix a time $t > 0$, a level $l \in (0, 1)$ and denote by $T_\pm := T_{\pm(\sqrt{a}-l)}$ the first passage times to $\pm(\sqrt{a} - l)$ for the diffusion X (respectively $T'_\pm := T'_{\pm(\sqrt{a}-\delta)}$ for Y). We take an event $E \in \mathcal{F}_t \cap \mathcal{F}_{T'_-}$ under which the paths of the diffusion are bounded by a deterministic value, which can depend on a and β. Since E is $\mathcal{F}_t$-measurable, Girsanov's theorem gives

$$\mathbb{P}_{\sqrt{a}-l}(E) = \mathbb{E}_{\sqrt{a}-l}\big(1_E \exp\big(G_t(Y)\big)\big).$$

The Radon–Nikodym density $\exp(G_{\cdot\wedge t\wedge T'_-}(Y))$ is a bounded martingale, and E is $\mathcal{F}_{T'_-}$-measurable, so by the optional stopping theorem the above quantity equals

$$\mathbb{E}_{\sqrt{a}-l}\big(\mathbb{E}_{\sqrt{a}-l}\big(1_E \exp\big(G_t(Y)\big) \mid \mathcal{F}_{T'_-}\big)\big) = \mathbb{E}_{\sqrt{a}-l}\big(1_E \exp\big(G_{T'_-\wedge t}(Y)\big)\big). \tag{35}$$

In the following, we consider E of the form $E = \{T_- < \infty\} \cap E_1$. The assumption on E requires E_1 being an event under which the diffusion is bounded from above. Taking the limit $t \to +\infty$ leads to the fundamental formula:

$$\mathbb{P}_{\sqrt{a}-l}(T_- < \infty, E_1) = \mathbb{E}_{\sqrt{a}-l}\big(1_{T'_-<\infty, E'_1} \exp\big(G_{T'_-}(Y)\big)\big). \tag{36}$$

Thanks to Itô's formula we can write the exponential martingale $\frac{4}{\beta} G_{T'_-}(Y)$ as

$$2\int_0^{T'_-} \big(t + a - Y_t^2\big)\,\mathrm{d}Y_t \tag{37}$$

$$+\,\phi(Y_0) - \phi(Y_{T'_-})$$

$$+\int_0^{T'_-} \frac{2}{\beta}\varphi'(Y_t) + \frac{1}{2}\varphi(Y_t)^2 + \varphi(Y_t)\big(Y_t^2 - a - t\big)\,\mathrm{d}t, \tag{38}$$

where φ' denotes the derivative of φ and ϕ the indefinite integral. Again by Itô's formula, we can compute the first term (37) above.

$$(37) = 2a(Y_{T'_-} - Y_0) - \frac{2}{3}\big(Y^3_{T'_-} - Y_0^3\big) + \left(\frac{8}{\beta} - 2\right)\int_0^{T'_-} Y_t\,\mathrm{d}t + 2T'_- Y_{T'_-}.$$

Replacing $Y_{T'_-}$ by its value, we obtain the expression (valid under E):

$$(37) = -\frac{8}{3}a^{3/2} - \frac{4}{3}l^3 + 4\sqrt{a}l^2 + 2lT'_- - 2\sqrt{a}T'_- + \left(\frac{8}{\beta} - 2\right)\int_0^{T'_-} Y_t\,\mathrm{d}t. \tag{39}$$

4.3. *The preliminary upper bound*

Let c_1 be a constant such that $c_1 > (|8/\beta - 2| - 2) \vee 0$. Recall that

$$\delta := \sqrt[4]{\ln a/a}, \qquad \varepsilon = \frac{4}{\sqrt{\beta}}\sqrt{\ln a}\big/\sqrt[4]{a}$$

and take the function φ_1 defined by

$$\varphi_1 : x \mapsto \begin{cases} \frac{c_1\sqrt{a}}{a-x^2} & \text{if } x \in \big(-\sqrt{a}+\delta, \sqrt{a}-\delta\big), \\ 0 & \text{if } x \notin \big(-\sqrt{a}, \sqrt{a}\big) \end{cases} \tag{40}$$

and extend this function on the entire real line such that φ_1 remains a smooth function supported on $[-\sqrt{a}, \sqrt{a}]$ (this is possible for all large enough "a"). Of course, there are many functions satisfying the above conditions but we just need to fix one. Let ϕ_1 be an antiderivative of φ_1.

A first step is to prove a less precise upper bound which does not give us the constant in front of the logarithm term of (6). In this subsection, the notations Y, T'_-, $\mathcal{A}'$ will always refer to the definitions using this particular φ_1. We have the events

$$\mathcal{A}=\{T_{\varepsilon-\sqrt{a}}<T_{\sqrt{a}-\varepsilon/2}\},\qquad \mathcal{C}=\left\{T_{\delta-\sqrt{a}}<T_{c_2\sqrt{a}}, L\le\frac{1}{\sqrt{a}}\right\}.$$

Lemma 6. (a) *The following inequality holds*:

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_{\varepsilon-\sqrt{a}}<\infty,\mathcal{A})\le\exp\left(-\frac{2}{3}\beta a^{3/2}+\mathrm{O}(\ln a)\right). \tag{41}$$

(b) *For some $c_3\ge 1$ we also have*

$$\mathbb{P}_{\sqrt{a}-\delta}\big(c_3\ln a/\sqrt{a}\le T_{\delta-\sqrt{a}}<\infty,\mathcal{C}\big)\le\exp\left(-\frac{2}{3}\beta a^{3/2}-\beta\ln a\right). \tag{42}$$

Part (a) with Lemma 4 immediately give the following.

Corollary 7. *The following upper bound holds*:

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_-<\infty)\le\exp\left(-\frac{2}{3}\beta a^{3/2}+\mathrm{O}(\ln a)\right).$$

Proof. *Part* (a). First let $T_-=T_{\sqrt{a}-\varepsilon}$. Consider the process Y defined with the function φ_1. The equality (35) gives:

$$\mathbb{P}_{\sqrt{a}-\varepsilon}(T_-<\infty,\mathcal{A})=E_{\sqrt{a}-\varepsilon}\big(1_{\{T'_-<\infty\mathcal{A}'\}}\exp\big(G_{T'_-}(Y)\big)\big)$$

with $G_{T'_-}(Y)$ given by (37)–(38).

To find an upper bound of the last term (38) in G, we remark that the chosen function φ_1 on $[-\sqrt{a}+\varepsilon,\sqrt{a}-\varepsilon/2]$ attains its maximum at time $\sqrt{a}-\varepsilon/2$. Under the event $\mathcal{A}'$, it gives:

$$(38)\le\left(\frac{2c_1^2+8/\beta c_1}{\varepsilon^2}-c_1\sqrt{a}\right)T'_-. \tag{43}$$

Moreover:

$$\phi_1(Y_0)-\phi_1(Y_{T'_-})=c_1\ln\left(2\frac{\sqrt{a}}{\varepsilon}+1\right). \tag{44}$$

Thanks to (39), (43) and (44), we deduce:

$$\begin{aligned}\frac{4}{\beta}G_{T'_-}(Y)+\frac{8}{3}a^{3/2}&\le 4\sqrt{a}\varepsilon^2+2\varepsilon T'_-+\left(\left|\frac{8}{\beta}-2\right|-2-c_1\right)\sqrt{a}T'_-\\&\quad+\left(\frac{8}{\beta}c_1+2c_1^2\right)\frac{1}{\varepsilon^2}T'_-+c_1\ln\left(2\frac{\sqrt{a}}{\varepsilon}+1\right).\end{aligned}$$

We now take c_1 such that $|8/\beta-2|-2-c_1<0$ and the coefficient in front of $\sqrt{a}T'_-$ becomes negative and dominates the terms involving T'_-. The last one creates the logarithmic error, and (41) follows from

$$G_{T'_-}(Y)\le-\frac{2}{3}\beta a^{3/2}+\left(\frac{3}{16}c_1\beta+16\right)\ln a+\mathrm{o}(\ln a).$$

Part (b). Now let $T_- = T_{\sqrt{a}-\delta}$. Just as in part (a), we need to bound the Girsanov terms. To find an upper bound of the last term (38), namely

$$\int_0^{T'_-} \frac{2}{\beta}\varphi'(Y_t) + \frac{1}{2}\varphi(Y_t)^2 + \varphi(Y_t)\big(Y_t^2 - a - t\big)\,\mathrm{d}t$$

note that φ^2 and φ' are both uniformly $\mathrm{o}(\sqrt{a})$. On the other hand, we have $\varphi(Y_t)(a - Y^2)$ is non-negative. Moreover, it is greater than $c_1\sqrt{a} + \mathrm{o}(\sqrt{a})$ as long as $L' \le t \le T'_-$. So on $\mathcal{C}'$ we have the lower bound

$$(38) \le \mathrm{o}\big(\sqrt{a}\big)T'_- - c_1\sqrt{a}\big(T'_- - L'\big). \tag{45}$$

The other inequalities are similar to part (a) with δ replaced by ε, except the last term in (39) gives an extra term due to the fact that Y is only bounded by $c_2\sqrt{a}$ up to time L'.

Thanks to (39), (45) and (44), we deduce:

$$\frac{4}{\beta}G_{T'_-}(Y) + \frac{8}{3}a^{3/2} \le 4\sqrt{a}\delta^2 + 2\delta T'_- + \left(\left|\frac{8}{\beta} - 2\right| - 2 - c_1\right)\sqrt{a}T'_- \\ + (c_1 + c_2)\sqrt{a}L' + \mathrm{o}\big(\sqrt{a}\big)T'_- + c_1 \ln\left(2\frac{\sqrt{a}}{\delta} + 1\right).$$

From part (a), we have $c := |8/\beta - 2| - 2 - c_1 < 0$. We chose $c_3 \ge 1$ large enough so that the terms involving $T'_- \ge c_3 \ln a/a$ together with the $\ln a$ term coming from the antiderivative are less than $-\beta \ln a$, i.e. $\frac{\beta}{4}c_3 c + \frac{3}{16}c_1\beta < -\beta$. This completes the proof of (42) since

$$G_{T'_-}(Y) \le -\frac{2}{3}\beta a^{3/2} - \beta \ln a + \mathrm{o}(\ln a). \qquad \square$$

4.4. *Precise asymptotics for the exponent*

Recall $\delta := \sqrt[4]{\ln a/a}$, L the last passage time to $\sqrt{a} - \delta$, and the event introduced for technical reasons:

$$\mathcal{C} := \left\{L < 1/\sqrt{a},\ \sup_{t \in [0, 1/\sqrt{a}]} X_t < c_2\sqrt{a}\right\}$$

defined in the outline of the proof. We will study in this section $\mathbb{P}_{\sqrt{a}-\delta}(T_{-\sqrt{a}+\delta} < \infty, \mathcal{C})$.

In order to obtain the coefficient in front of the logarithm term, we need to be more precise in our analysis and we will look more carefully at T_-, the first passage time to $-\sqrt{a} + \delta$ of X.

Our tool is again the Cameron–Martin–Girsanov formula with a drift containing a different function φ. Let us define φ_2 in the following way:

$$\varphi_2 : x \mapsto \begin{cases} \frac{(8/\beta - 2)x - 2\sqrt{a}}{a - x^2} & \text{if } x \in \big(-\sqrt{a} + \delta/2, \sqrt{a} - \delta\big), \\ 0 & \text{if } x \notin \big(-\sqrt{a}, \sqrt{a}\big) \end{cases} \tag{46}$$

and extend it such that it remains a smooth function on $\mathbb{R}$ satisfying: $\sup|\varphi| \le \sqrt[4]{a}$ and $\sup|\varphi'| \le \sqrt{a}$ (this is possible for a large enough "a"). Similarly to the previous subsection, the notations Y, T'_-, $\mathcal{C}'$ etc. refers to definitions with this chosen function.

Proposition 8. *We have*

$$\mathbb{P}_{\sqrt{a}-\delta}\big(T_- < c_3 \ln a/\sqrt{a}, \mathcal{C}\big) \\ = \exp\left(-\frac{2}{3}\beta a^{3/2} - \frac{3}{8}\beta \ln a + \mathrm{O}\big(\sqrt{\ln a}\big)\right)\mathbb{P}_{\sqrt{a}-\delta}\big(T'_- < c_3 \ln a/\sqrt{a}, \mathcal{C}'\big).$$

Proof. Let us compute the new Radon–Nikodym derivative according to the position of Y using the relations (37)–(38) and (13).

At first, the term $\phi(Y_0)-\phi(Y_{T'_-})$ is equal to $-3/2\ln a$. Moreover,

$$\forall y\in\left[-\sqrt{a}+\delta,\sqrt{a}-\delta\right],\quad -2\sqrt{a}+\left(\frac{8}{\beta}-2\right)y+\left(y^2-a\right)\varphi(y)=0.$$

Consequently, there exists a constant $c'>0$, depending only on β, such that for every $y\in[-\sqrt{a}+\delta,\sqrt{a}-\delta]$,

$$\left|-2\sqrt{a}+\left(\frac{8}{\beta}-2\right)y+\left(y^2-a\right)\varphi(y)+\frac{2}{\beta}\varphi'(y)+\frac{1}{2}\varphi(y)^2\right|\le\frac{c'a}{(a-y^2)^2}\le\frac{2c'}{\delta^2}.$$

There is another constant $c''>0$ such that

$$\left|\int_0^{T'_-}u\varphi(Y_u)\,\mathrm{d}u\right|\le\frac{c''}{\delta}T'^2_-.$$

For every $y\ge\sqrt{a}-\delta$,

$$\left|\frac{2}{\beta}\varphi'(y)+\frac{1}{2}\varphi(y)^2+\varphi(y)\left(y^2-a-t\right)\right|\le\left(\frac{2}{\beta}+1\right)\sqrt{a}.$$

Putting all together and using the upper bound on the last passage time to $\sqrt{a}-\delta$ contained in $\mathcal{C}$, we obtain:

$$\left|\frac{4}{\beta}G_{T'_-}(Y)+\frac{8}{3}a^{3/2}+\frac{3}{2}\ln a\right|\le 4\sqrt{a}\delta^2+\frac{2c'}{\delta^2}T'_-+\frac{c''}{\delta}T'^2_-+2\delta T'_-+\frac{4}{3}\delta^3+\left(\frac{2}{\beta}+1\right).$$

If $\{T'_-\le c_3\ln a/\sqrt{a}\}$ holds,

$$\frac{2c'}{\delta^2}T'_-+\frac{c''}{\delta}T'^2_-+2\delta T'_-+\frac{4}{3}\delta^3\le 2c'\sqrt{\ln a}+\mathrm{O}(1).$$

Under $\{T'_-<c_3\ln a/\sqrt{a}\}\cap\mathcal{C}'$, we conclude

$$G_{T'_-}(Y)=-\frac{2}{3}\beta a^{3/2}-\frac{3}{8}\beta\ln a+\mathrm{O}\left(\sqrt{\ln a}\right).$$

□

To complete the study inside the parabola for the lower bound, we prove:

Lemma 9. *There exists $c_4>0$ depending only on β such that with $\xi=c_3\ln a/\sqrt{a}$*

$$\mathbb{P}_{\sqrt{a}-\delta}\left(T'_-<\xi,\mathcal{C}'\right)\ge\exp\left(-c_4\sqrt{\ln a}\right).$$

Proof. For the lower bound we can replace the event C' by the event that $T'_+=T'_{\sqrt{a}-\delta/2}$ is infinite, i.e. the corresponding level is never hit. We will show that this events happens as long as

$$M:=\sup\left\{|B_t|,t\in[0,\xi]\right\}\le\delta\sqrt{\beta}/5,$$

which by the Brownian motion estimate (8) has the right probability.

Let $\xi=c_3\ln a/\sqrt{a}$. Again, we compare our equation to an ODE. The quantity $Z=X-B$ on $[0,\xi]$ satisfies

$$Z'=-(Z-B)^2+t-a\le Z^2+a+\frac{4}{\sqrt{\beta}}MZ+\xi,$$

so let H be the solution of the (random) ODE:

$$\begin{cases} H'(t) = H^2(t) - C, \\ H(0) = \sqrt{a} - \delta, \end{cases} \tag{47}$$

where the random constant satisfies

$$C = a - \frac{4}{\sqrt{\beta}}\sqrt{a}M + \mathrm{O}(1).$$

By the same argument of comparison as in Section 3, when $M \le c\delta$ the diffusion Y is under $t \mapsto H(t) + 2/\sqrt{\beta}B_t$ up to the minimum of ξ and the exit time from $[-\sqrt{a}-1, \sqrt{a}]$. Therefore we will have $T'_- < \xi$, $T'_- < T'_+$ as long as

$$H(\xi) + \frac{2}{\sqrt{\beta}}B_\xi \le -\sqrt{a} + \delta \quad \text{and} \quad \sup_{s\in[0,\xi]}\left(H(s) + \frac{2}{\sqrt{\beta}}B_s\right) < \sqrt{a} - \delta/2.$$

Since $H(s)$ is decreasing in s, the second event is implied by our assumption on M.

The solution H takes the form:

$$H(t) = -\sqrt{C}\tanh\bigl(\sqrt{C}t - \operatorname{arctanh}(b)\bigr) = \frac{\sqrt{C}(\tanh(\sqrt{C}t) - b)}{b\tanh(\sqrt{C}t) - 1}, \quad b = \frac{\sqrt{a}-\delta}{\sqrt{C}}.$$

When $c_3 \ge 1$ we have $\tanh(\sqrt{C}t) = 1 + \mathrm{O}(a^{-2})$. So we get the asymptotics

$$H(\xi) = -\sqrt{a} + 2M/\sqrt{\beta} + \mathrm{o}(\delta),$$

and we indeed have

$$H(\xi) + \frac{2}{\sqrt{\beta}}B_\xi \le -\sqrt{a} + \frac{4M}{\sqrt{\beta}} \le \frac{4}{5}\delta + \mathrm{o}(\delta).$$

□

5. Under the parabola, lower bound

We will prove:

Proposition 10. *There exists $c_5 > 0$ depending only on β such that*,

$$\mathbb{P}_{-\sqrt{a}+\delta}(T_{-\infty} < \infty) \ge \exp\bigl(-c_5\sqrt{\ln a}\bigr).$$

Proof. Using the strong Markov property and the increasing property, we can lower bound the left-hand side by

$$\mathbb{P}_{-\sqrt{a}+\delta}\left(T_{-\sqrt{a}-\delta} < \frac{1}{\sqrt{a}} \wedge T_{-\sqrt{a}+2\delta}\right) \times \mathbb{P}_{(1/\sqrt{a}, -\sqrt{a}-\varepsilon)}\left(T_{-\sqrt{a}-\sqrt{\ln a}/\sqrt[4]{a}} < \frac{\ln a}{2\sqrt{a}} \wedge T_{-\sqrt{a}}\right)$$
$$\times \mathbb{P}_{(\ln a/\sqrt{a}, -\sqrt{a}-\sqrt{\ln a}/\sqrt[4]{a})}(T_{-\infty} < \infty).$$

- The first probability gives the main cost. Under this event, the process X is stochastically dominated by the drifted Brownian motion:

  $$t \mapsto -\sqrt{a} + \delta + 2\sqrt{a}\delta t + \frac{2}{\sqrt{\beta}}B_t.$$

  Thus,

  $$\mathbb{P}_{-\sqrt{a}+\varepsilon}\left(T_{-\sqrt{a}-\varepsilon} < \frac{1}{\sqrt{a}} \wedge T_{-\sqrt{a}+2\varepsilon}\right) \ge \mathbb{P}\left(B_1 < -\frac{3}{2}\sqrt{\beta}\sqrt[4]{a}\varepsilon, \sup_{s\in[0,1]} B_s \le \frac{1}{2}\sqrt{\beta}\sqrt[4]{a}\varepsilon\right).$$

By the reflection principle, this equals

$$\mathbb{P}\left(\frac{5}{2}\sqrt{\beta}\sqrt[4]{a}\varepsilon \geq B_1 \geq \frac{3}{2}\sqrt{\beta}\sqrt[4]{a}\varepsilon\right) \geq \sqrt{\beta}\sqrt[4]{\ln a}\exp\left(-\frac{25}{4}\beta\sqrt{\ln a}\right).$$

- For the second part, under the studied event, the diffusion $(X_t, t \geq 1/\sqrt{a})$ is stochastically dominated by

$$\frac{1}{4\sqrt{a}} + t + \frac{2}{\sqrt{\beta}}B_t - \sqrt{a} - \varepsilon.$$

Thus the studied probability is bounded from below by a constant depending only on β.

- For the last part, we need to compare the diffusion with the solution of a simple differential equation. Similarly to the previous comparisons, under the event $\{X(\ln a/\sqrt{a}) = -\sqrt{a} - \sqrt{\ln a}/\sqrt[4]{a}, T_{-\infty} < T_{-\sqrt{a}} \wedge (3/8 \ln a/\sqrt{a})\}$, the diffusion X is stochastically dominated by $G(t) + 2/\sqrt{\beta}B_t$ where G is the solution of the differential equation:

$$\begin{cases} G'(t) = a + \frac{11}{8}\frac{\ln a}{\sqrt{a}} - \left(1 - \frac{4}{\sqrt{\beta}}\frac{M}{\sqrt{a}}\right)G^2(t), \\ G(0) = -\sqrt{a} - \frac{\sqrt{\ln a}}{\sqrt[4]{a}} \end{cases}$$

and

$$M = \sup_{s \in [0, (3/8)\ln a/\sqrt{a}]} |B_s|.$$

Whenever we have

$$\left\{M \leq \frac{2}{2\sqrt{\beta}}\frac{\sqrt{\ln a}}{\sqrt[4]{a}}\right\},$$

the function G blows up to $-\infty$ at a time smaller than $3/8 \ln a/\sqrt{a}$ and the diffusion $(X_t, t \in [0, 3/8 \ln a/\sqrt{a}])$ stays under $-\sqrt{a}$. Therefore:

$$\mathbb{P}_{(\ln a/\sqrt{a}, -\sqrt{a} - \sqrt{\ln a}/\sqrt[4]{a})}(T_{-\infty} < \infty) \geq P\left(M \leq \frac{2}{2\sqrt{\beta}}\frac{\sqrt{\ln a}}{\sqrt[4]{a}}\right)$$

which is greater than a constant depending only on β. It leads to the result. □

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