Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2012, Vol. 48, No. 4, 989–1009
DOI: 10.1214/11-AIHP453

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

The critical barrier for the survival of branching random walk with absorption

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Received 9 November 2009; revised 6 July 2011; accepted 18 August 2011

Abstract. We study a branching random walk on $\mathbb{R}$ with an absorbing barrier. The position of the barrier depends on the generation. In each generation, only the individuals born below the barrier survive and reproduce. Given a reproduction law, Biggins et al. [*Ann. Appl. Probab.* **1** (1991) 573–581] determined whether a linear barrier allows the process to survive. In this paper, we refine their result: in the boundary case in which the speed of the barrier matches the speed of the minimal position of a particle in a given generation, we add a second order term $an^{1/3}$ to the position of the barrier for the nth generation and find an explicit critical value a_c such that the process dies when $a < a_c$ and survives when $a > a_c$. We also obtain the rate of extinction when $a < a_c$ and a lower bound for the population when it survives.

Résumé. Nous étudions une marche aléatoire branchante sur $\mathbb{R}$ avec une barrière absorbante. La position de la barrière dépend de la génération. À chaque génération, seuls les individus nés sous la barrière survivent et se reproduisent. Étant donnée une loi de reproduction, Biggins et al. [*Ann. Appl. Probab.* **1** (1991) 573–581] ont déterminé, pour une barrière linéaire, si le processus survit ou s'éteint. Dans cet article, nous affinons ce résultat : dans le cas frontière où la vitesse de la barrière correspond à la vitesse de la particule la plus à gauche d'une génération donnée, nous allons à l'ordre suivant en ajoutant un terme $an^{1/3}$ à la position de la barrière pour la $n$ième génération et obtenons une valeur critique explicite a_c telle que le processus s'éteint quand $a < a_c$ et survit quand $a > a_c$. Nous obtenons aussi le taux d'extinction lorsque $a < a_c$ et une borne inférieure sur la taille de la population lorsqu'il survit.

MSC: 60J80

Keywords: branching random walk; survival probability

1. Introduction

We study a discrete-time branching random walk on $\mathbb{R}$. The population forms a well-known Galton–Watson tree $\mathcal{T}$, and some extra information is added: to each individual $u \in \mathcal{T}$ we attach a displacement $\xi_u \in \mathbb{R}$ from the position of her parent. We set the initial ancestor ϱ at the origin, hence the individual u has position

$$V(u) = \sum_{\varrho < v \leq u} \xi_v = \sum_{i=1}^{|u|} \xi_{u_i},$$

where $|u|$ is the generation of u and u_i the ancestor of u in generation i. We denote by $\mathcal{T}_n := \{u \in \mathcal{T}\colon\ |u| = n\}$ the population at time n. We define an infinite path u through $\mathcal{T}$ as a sequence of individuals $u = (u_i)_{i \in \mathbb{N}}$ such that

$$\forall i \in \mathbb{N}, \quad |u_i| = i \quad \text{and} \quad u_i < u_{i+1}.$$

We denote their collection by $\mathcal{T}_\infty$.

Now we explain how the displacements $\xi_u, u \in \mathcal{T}$, are distributed. A simple choice, with very nice properties would be to take them i.i.d. but actually everything still works in a more general setting. All individuals still reproduce independently and the same way, but we allow correlations in the number and displacements of the children of every single individual. If we write $\Gamma(u)$ for the set of children of u, our requirement is that the point processes $\{\xi_v, v \in \Gamma(u)\}$ (with u running over all the potential individuals of the random tree $\mathcal{T}$) are i.i.d.

We define a barrier as a function $\varphi : \mathbb{N} \to \mathbb{R}$. In the branching random walk with absorption, the individuals u such that $V(u) > \varphi(|u|)$, i.e. born above the barrier are removed: they are immediately killed and do not reproduce.

Kesten [12], Derrida and Simon [8,9], Harris and Harris [11] have studied the continuous analog of this process, the branching Brownian motion with absorption. The understanding of what happens in the continuous setting, more convenient to handle from technical point of view, greatly helps us in the discrete one. In particular, we borrow here some ideas from Kesten [12].

Biggins et al. [7] introduced the branching random walk with an absorbing barrier in order to answer questions about parallel simulations. Pemantle [14] and Gantert et al. [10] also studied this model.

A natural question that arises is whether the process survives. This obviously depends on the walk as well as on the barrier. The case of the linear barriers has been solved by Biggins et al. [7].

Before stating their result, we need to introduce some notation:

We denote the intensity measure of the point process by μ, and its Laplace–Stieljes transform by Φ:

$$\Phi(t) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-t\xi_u}\right] = \int_{\mathbb{R}} \mathrm{e}^{-tz}\mu(\mathrm{d}z).$$

We assume that the expected number of children $\Phi(0)$ is finite and that negative displacements occur, i.e. that $\mu((-\infty, 0)) > 0$.

We also define $\Psi = \log \Phi$, this is a strictly convex function that takes values in $(-\infty, +\infty]$.

We call critical the case where

$$\Phi(1) = \mathbb{E}\left[\sum_{|u|=1} \mathrm{e}^{-\xi_u}\right] = 1 \quad \text{and} \quad \Phi'(1) := \mathbb{E}\left[\sum_{|u|=1} \xi_u \mathrm{e}^{-\xi_u}\right] = 0.$$

This can also be written $\Psi(1) = 0$ and $\Psi'(1) = 0$.

Theorem 1.1 (Biggins et al. [7]). *In the critical case, we have*:

$$\mathbb{P}\big(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq i\varepsilon\big) \begin{cases} = 0 & \text{if } \varepsilon \leq 0, \\ > 0 & \text{if } \varepsilon > 0. \end{cases}$$

The aim of this article is to refine this result by replacing the linear barrier $i \mapsto i\varepsilon$ with a more general barrier $i \mapsto \varphi(i)$.

Given a barrier φ we do not know in general whether $\mathbb{P}(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq \varphi(i)) = 0$ or not. We assume from now on that we are in the critical case. It is well known that many noncritical random walks can be transformed into critical ones by a linear modification of the displacements, so we do not loose much in generality. Theorem 1.1 leads us to focus on barriers such that $\frac{\varphi(i)}{i} \to 0$.

We introduce the parameter

$$\sigma^2 := \Phi''(1) = \mathbb{E}\left[\sum_{|u|=1} \xi_u^2 \mathrm{e}^{\xi_u}\right]$$

and assume through the following that it is finite.

Some specific technical difficulties arise in the computation of the second moment (that we use in order to give a lower bound for the survival probability or to prove survival) when dealing with Galton–Watson trees of unbounded degree. Actually individuals with many children may cause trouble. In order to have a sufficient control, we assume from now on that the number of children of a single individual is uniformly bounded or that the following condition holds:

$$\exists\delta_1 > 0, \quad \Phi(1+\delta_1) < +\infty \quad \text{and} \quad \exists\delta_2 > 0, \quad \mathbb{E}\big[\#\mathcal{T}_1^{1+\delta_2}\big] < +\infty. \tag{1.1}$$

Under these assumptions, we obtain the following result:

Theorem 1.2. *Let* $a_c = \frac{3}{2}(3\pi^2\sigma^2)^{1/3}$. *We have*:

$$\mathbb{P}\big(\exists u \in \mathcal{T}_\infty, \forall i \geq 1, V(u_i) \leq a i^{1/3}\big) \begin{cases} = 0 & \text{if } a < a_c, \\ > 0 & \text{if } a > a_c. \end{cases}$$

Unfortunately, we are not able to conclude in the case $a = a_c$, nor to give a necessary and sufficient condition on a general barrier for a line of descent to survive below it.

Theorem 1.2 has the following corollary:

Corollary 1.3. *Under the hypothesis of Theorem* 1.2, *we have*, *almost surely*, *on the set of ultimate survival of the underlying Galton–Watson process*,

$$\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} = a_c.$$

While proving Theorem 1.2, we actually obtain stronger results. The two following propositions together imply the theorem.

Proposition 1.4. *If* $a > a_c$, *then the equation* $a = b + \frac{3\pi^2\sigma^2}{2b^2}$ *has two solutions in* b, *let* b_a *be the one such that* $b_a > \frac{2a_c}{3}$. *For any* $\varepsilon > 0$, *for any* $N \in \mathbb{N}$ *large enough*, *we have with positive probability*:

$$\forall k \geq 1, \quad \#\big\{u \in \mathcal{T}_{N^k}\colon \forall i \leq N^k, (a - b_a) i^{1/3} \leq V(u_i) \leq a i^{1/3}\big\} \geq \exp\big(N^{k/3}(b_a - \varepsilon)\big).$$

Proposition 1.5. *If* $a < a_c$, *then there exists some constant* $c > 0$ *such that*

$$\frac{1}{n^{1/3}} \log \mathbb{P}\big(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq a i^{1/3}\big) \to -c.$$

The constant c, which depends on a, is determined in Section 5.

When $a < a_c$, extinction means that the total progeny Z is almost surely finite. This random variable has infinite mean, since the expected number of surviving individuals in generation n is $\exp(a n^{1/3}(1 + o(1)))$. We can estimate the tail of the distribution of Z:

Proposition 1.6. *If* $a < a_c$, *then let* g *be the optimal function determined in Section* 5, $c := g(0)$ *and* $d := \max_{[0,1]} g$.

$$\mathbb{P}(Z > k) = k^{-(c/d)(1+o(1))}.$$

Remark 1.7. *In the case* $a \leq 0$, g *is decreasing*, *hence* $d = c$ *and the claim of Proposition* 1.6 *is weaker than a known result*, *conjectured by Aldous and proved by Addario-Berry and Broutin* [1], *and improved by Aidékon* [2] *that for* $a = 0$, $\mathbb{E}[Z] < +\infty$ *and* $\mathbb{E}[Z \log Z] = +\infty$. *Exponents less than* -1 *are obtained* (*see Aidékon*, *Hu and Zindy* [3]) *for linear barriers* $i \mapsto -\varepsilon i$, *which corresponds to what is often referred as the subcritical case.*

Consider a general barrier $\varphi : \mathbb{N} \to \mathbb{R}$. We define

$$a^+ := \limsup_{n\to\infty} \frac{\varphi(n)}{n^{1/3}} \quad \text{and} \quad a^- := \liminf_{n\to\infty} \frac{\varphi(n)}{n^{1/3}}.$$

We deduce from Theorem 1.2 that there is extinction when $a^+ < a_c$ and survival when $a^- > a_c$. Making some modifications to the computations of Section 3, we can prove the following result:

Theorem 1.8. *Assume $a^+ \geq a_c$. The equation $a^+ = b + \frac{3\pi^2\sigma^2}{2b^2}$ admits a unique solution $b_{a_c} = \frac{2a_c}{3}$ if $a^+ = a_c$, and two solutions if $a > a_c$. Let $b_{a^+} \geq \frac{2a_c}{3}$ be the larger solution.*

If $a^- < \frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, then there is extinction. We have a partial converse: if $a^+ \geq a_c$ and a^- are reals such that $a^- < \frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, then we can find a corresponding barrier φ such that the absorbed branching random walk survives with positive probability.

The rest of the paper is organized as follows:

Section 2 introduces the tools we will use in the proof of our main results.

Section 3 is devoted to the proof of the upper bound in Proposition 1.5, which contains the first part of Theorem 1.2.

In Section 4, we prove Proposition 1.4 which implies the second part of Theorem 1.2.

In Section 5, we complete the proof of Proposition 1.5. We skip many details of technical arguments already exposed in Section 4 to obtain the lower bound and go back over some results of Section 3 in order to prove that the two bounds agree.

In Section 6, we prove Theorem 1.8, Proposition 1.6 and Corollary 1.3.

2. Some preliminaries

2.1. *Many-to-one lemma*

Since $\mathbb{E}[\sum_{|u|=1} \mathrm{e}^{-\xi_u}] = 1$, we can define the law of a random variable X such that for any measurable nonnegative function f,

$$\mathbb{E}\big[f(X)\big] = \mathbb{E}\Bigg[\sum_{|u|=1} \mathrm{e}^{-\xi_u} f(\xi_u)\Bigg].$$

Then $\mathbb{E}[X] = \mathbb{E}[\sum_{|u|=1} \xi_u \mathrm{e}^{-\xi_u}]$ so that X is centered by hypothesis.

We write $\mathbb{N}^*$ for the set of positive integers. Let $(X_i)_{i\in\mathbb{N}^*}$ be a i.i.d. sequence of copies of X. Write for any $n \in \mathbb{N}$, $S_n := \sum_{0<i\leq n} X_i$. S is then a mean-zero random walk starting from the origin.

We can now state the many-to-one lemma (this is exactly Lemma 4.1(iii) of Biggins and Kyprianou [6]):

Lemma 2.1 (Biggins and Kyprianou [6]). *For any $n \geq 1$ and any measurable function $F : \mathbb{R}^n \to [0, +\infty)$,*

$$\mathbb{E}\Bigg[\sum_{|u|=n} \mathrm{e}^{-V(u)} F\big(V(u_i), 1 \leq i \leq n\big)\Bigg] = \mathbb{E}\big[F(S_i, 1 \leq i \leq n)\big].$$

The proof of the lower bound for the survival probability also requires the following bivariate version of the many-to-one lemma.

Lemma 2.2 (Gantert, Hu and Shi [10]). *Let (X, ν) be a random variable taking values in $\mathbb{R} \times \mathbb{N}^*$ such that for any measurable nonnegative function f,*

$$\mathbb{E}\big[f(X, \nu)\big] = \mathbb{E}\Bigg[\sum_{|u|=1} \mathrm{e}^{-\xi_u} f(\xi_u, \#\mathcal{T}_1)\Bigg].$$

Let $n \geq 1$ and $(X_i, \nu_i)_{1 \leq i \leq n}$ be i.i.d. copies of (X, ν). Write for any $0 \leq k \leq n$, $S_k := \sum_{0<i\leq k} X_i$. Then for any measurable function $F : (\mathbb{R} \times \mathbb{N}^*)^n \to [0, +\infty)$,

$$\mathbb{E}\Big[\sum_{|u|=n} \mathrm{e}^{-V(u)} F\big(V(u_i), \#\Gamma(u_{i-1}), 1 \leq i \leq n\big)\Big] = \mathbb{E}\big[F(S_i, \nu_i, 1 \leq i \leq n)\big].$$

The proof, very similar to the one of Lemma 2.1, is omitted.

2.2. *Mogul'skii's estimate*

Let $\mathcal{F}[0,1]$ (respectively $\mathcal{C}[0,1]$) be the set of functions (respectively continuous functions) $[0,1] \mapsto \mathbb{R}$.

For any $L, \widetilde{L} \in \mathcal{F}[0,1]$, we write $L < \widetilde{L}$ when $\forall t \in [0,1]$, $L(t) < \widetilde{L}(t)$ and $L \leq \widetilde{L}$ when $\forall t \in [0,1]$, $L(t) \leq \widetilde{L}(t)$. If $n \geq 1$, we write $L <_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L(\frac{k}{n}) < \widetilde{L}(\frac{k}{n})$ and $L \leq_n \widetilde{L}$ when $\forall 1 \leq k \leq n$, $L(\frac{k}{n}) \leq \widetilde{L}(\frac{k}{n})$.

Theorem 2.3 (Mogul'skii). *Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables such that $E[\xi_1] = 0$ and $\sigma^2 := E[\xi_1^2] < \infty$. Let $(x_n, n \geq 0)$ be a sequence of positive numbers such that*

$$\lim_{n\to\infty} x_n = +\infty,$$

$$\lim_{n\to\infty} \frac{x_n}{\sqrt{n}} = 0.$$

Define for any $n \geq 0$

$$S_n := S_0 + \xi_1 + \xi_2 + \cdots + \xi_n,$$

where $S_0 = z$ almost surely under the probability $\mathbb{P}^z$ ($z \in \mathbb{R}$).

When $z = 0$, write $\mathbb{P} := \mathbb{P}^0$ and define, for any $t \in [0,1]$,

$$s_n(t) := \frac{S_{\lfloor tn \rfloor}}{x_n} = \frac{\xi_1 + \xi_2 + \cdots + \xi_k}{x_n} \quad \text{for } k/n \leq t < (k+1)/n.$$

Then, for any $L_1, L_2 \in \mathcal{C}[0,1]$, with

$$L_1 < L_2 \quad \text{and} \quad L_1(0) < 0 < L_2(0), \tag{2.1}$$

we have, as $n \to \infty$,

$$\log\big(\mathbb{P}(L_1 < s_n < L_2)\big) \sim -C_{L_1,L_2} n x_n^{-2},$$

where

$$C_{L_1,L_2} := \frac{\pi^2\sigma^2}{2} \int_0^1 \frac{\mathrm{d}t}{(L_2(t) - L_1(t))^2}.$$

We keep the notations and assumptions of Theorem 2.3 throughout this section. For the proof, we refer to [13].

We actually need more sophisticated versions of this estimate. For the proofs of the following results, we refer to [4]. In all this section, changing strict inequalities into weak ones in the definition of the events we are interested (but not in (2.1)) does not change the estimate of the probability.

Lemma 2.4 (Lemma 4.4 of [4]). *Set L_1 and L_2 like in Theorem 2.3. For any sequences $(L_1^n)_n$ and $(L_2^n)_n$ of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$, we have*

$$\log\big(\mathbb{P}\big(L_1^n <_n s_n <_n L_2^n\big)\big) \sim \log\big(\mathbb{P}\big(L_1^n < s_n < L_2^n\big)\big) \sim -n x_n^{-2} C_{L_1,L_2}.$$

From now on, we set:

$$\forall n \geq 1, \quad x_n := n^{1/3}. \tag{2.2}$$

Proposition 2.5 (Proposition 4.7 of [4]). *Set $L_1, L_2 \in \mathcal{C}[0,1]$, with*

$$L_1(0) \leq 0 \leq L_2(0) \quad \text{and} \quad \forall t \in [0,1], \quad L_1(t) \leq L_2(t). \tag{2.3}$$

Let $(L_1^n)_n$ and $(L_2^n)_n$ be sequences of $\mathcal{F}[0,1]$ such that $\|L_1^n - L_1\|_\infty \to 0$ and $\|L_2^n - L_2\|_\infty \to 0$ as $n \to \infty$. We assume B and C are mappings $[0,1] \times \mathbb{N}^* \mapsto \mathbb{N}^*$, nondecreasing in the first component and such that, for any $\alpha \in [0,1]$, the sequences $(B(\alpha,n) - \alpha n)_n$ and $(C(\alpha,n) - \alpha n)_n$ are bounded.

Uniformly in $0 \leq \beta < \gamma \leq 1$, we have

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\sup_z \mathbb{P}^z\left(L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(\beta,n)}}{x_n} < L_2^n\left(\frac{k}{n}\right), \forall B(\beta,n) < k \leq C(\gamma,n)\right)\right)$$
$$\leq -C_{L_1,L_2}^{\beta,\gamma} := -\frac{\pi^2\sigma^2}{2}\int_\beta^\gamma \frac{\mathrm{d}t}{(L_2(t) - L_1(t))^2},$$

where the $\sup_z$ is over the $z \in \mathbb{R}$ such that $x_n L_1^n(\frac{B(\beta,n)}{n}) \leq z \leq x_n L_2^n(\frac{B(\beta,n)}{n})$.

Remark 2.6. *The upper bound in Theorem 2.3 and Lemma 2.4 is still valid with condition (2.1) replaced by (2.3).*

In order to deal with Galton–Watson trees of infinite degree, we borrow Lemma 2.1 from [10]. Combined with the arguments leading to Proposition 2.5, it gives us the following estimate.

Proposition 2.7. *For each $n \geq 1$, let $X_i^{(n)}$, $1 \leq i \leq n$ be i.i.d. real-valued random variables. We define $S_i^{(n)} = S_0^{(n)} + X_1^{(n)} + \cdots + X_i^{(n)}$ for $1 \leq i \leq n$.*

Assume that there exist constants $\delta > 0$ and $\sigma^2 > 0$ such that

$$\sup_{n\geq 1} \mathbb{E}\left[\left|X_1^{(n)}\right|^{2+\delta}\right] < +\infty, \quad \mathbb{E}\left[X_1^{(n)}\right] = \mathrm{o}\left(n^{-2/3}\right) \quad \text{and} \quad \mathrm{Var}\left(X_1^{(n)}\right) \to \sigma^2. \tag{2.4}$$

Let $L_1, L_2, (L_1^n)_n$ and $(L_2^n)_n$ be like in Lemma 2.4. Let β and γ be real numbers such that $0 \leq \beta < \gamma \leq 1$. Let $(B(n))_n$ and $(C(n))_n$ be sequences of reals such that the sequences $(B(n) - \beta n)_n$ and $(C(n) - \gamma n)_n$ are bounded and $\forall n \geq 1$, $1 \leq B(n) < C(n) \leq n$.

Let u^* and v^* be real numbers such that $L_1(\beta) < u^* < v^* < L_2(\beta)$. Let u_n and v_n be sequences of real numbers such that

$$\frac{u_n}{x_n} \to u^*, \qquad \frac{v_n}{x_n} \to v^*, \qquad L_1^n\left(\frac{B(n)}{n}\right)x_n \leq u_n \leq v_n \leq L_2^n\left(\frac{B(n)}{n}\right)x_n \quad \forall n \geq 1.$$

We have, for any $\varepsilon > 0$,

$$\liminf_{n\to\infty} \frac{1}{n^{1/3}} \log\left(\inf_z \mathbb{P}^z\left(\forall B(n) < k \leq C(n), L_1^n\left(\frac{k}{n}\right) < \frac{S_{k-B(n)}^{(n)}}{n^{1/3}} < L_2^n\left(\frac{k}{n}\right);\right.\right.$$
$$\left.\left.L_2^n\left(\frac{C(n)}{n}\right) - \varepsilon < \frac{S_{C(n)-B(n)}^{(n)}}{n^{1/3}} < L_2^n\left(\frac{C(n)}{n}\right)\right)\right) \geq -C_{L_1,L_2}^{\beta,\gamma},$$

where the $\inf_z$ is over the $z \in \mathbb{R}$ such that $u_n \leq z \leq v_n$.

2.3. *A rough estimate*

In Section 4, we need a lower bound in a particular case with $L_1(0) < 0 = L_2(0)$ and $L_2 \geq 0$ on $[0, 1]$. In order to apply the results above, the following lemma will be useful:

Lemma 2.8. *There are constants $M \geq 1$, and $\varepsilon_1 > 0$ such that, with $k := \lfloor \varepsilon_2 n^{1/3} \rfloor$, such that the probability $P_n(M, \varepsilon_1, \varepsilon_2)$ defined as*

$$\mathbb{P}\left(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M, L_1\left(\frac{i}{n}\right) \leq \frac{V(u_i)}{n^{1/3}} \leq L_2\left(\frac{i}{n}\right); -M\varepsilon_2 \leq \frac{V(u_k)}{n^{1/3}} \leq -\varepsilon_1\varepsilon_2\right)$$

satisfies

$$\lim_{\varepsilon_2 \to 0} \liminf_{n \to \infty} \frac{1}{n^{1/3}} \log P_n(M, \varepsilon_1, \varepsilon_2) = 0.$$

Proof. Let $\varepsilon_1 > 0$ and such that for some $M \geq 1$,

$$p := \mathbb{P}\big(\#\mathcal{T}_1 \leq M; \exists u \in \mathcal{T}_1, -M \leq V(u) \leq -\varepsilon_1\big) > 0.$$

By independence,

$$\mathbb{P}\big(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M, \forall i \leq k, -iM \leq V(u_i) \leq -i\varepsilon_1\big) \geq p^k.$$

Let $\varepsilon_2 > 0$ such that $M\varepsilon_2 < -L_1(0)$. For any integer n large enough, we take $k := \lfloor \varepsilon_2 n^{1/3} \rfloor$. Hence, for $\varepsilon_2 > 0$ small enough, we have

$$\mathbb{P}\left(\exists u \in \mathcal{T}_k, \forall i < k, \#\Gamma(u_i) \leq M; L_1\left(\frac{i}{n}\right) \leq \frac{V(u_i)}{n^{1/3}} \leq L_2\left(\frac{i}{n}\right); -M\varepsilon_2 \leq \frac{V(u_k)}{n^{1/3}} \leq -\varepsilon_1\varepsilon_2\right) \geq p^k. \qquad \square$$

3. Upper bound for the survival probability

3.1. *Splitting the survival probability*

Fix $a > 0$. Obviously,

$$\mathbb{P}\big(\exists u \in \mathcal{T}_\infty, \forall i, V(u_i) \leq ai^{1/3}\big) = \lim_{n \to \infty} \mathbb{P}\big(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq ai^{1/3}\big).$$

From now on, $n \geq 1$ is fixed.

We set a second barrier $i \mapsto ai^{1/3} - b_{i,n}$ (with $b_{i,n} > 0$ for $1 \leq i \leq n$ yet to be determined) below the first one $i \mapsto ai^{1/3}$: if a particle crosses it, then its descendants will be likely to stay below the first one until generation n.

Let $H(u) := \inf\{k \leq n \colon V(u_k) < ak^{1/3} - b_{k,n}\}$ be the first time the line of descent of a particle $u \in \mathcal{T}_n$ crosses this second barrier ($H(u) = \infty$ if the particle stays between the barriers until time n). We split the sum accordingly:

$$\mathbb{P}\big(\exists u \in \mathcal{T}_n, \forall i \leq n, V(u_i) \leq ai^{1/3}\big) \leq R_\infty + \sum_{j=1}^{n} R_j, \tag{3.1}$$

where

$$R_j = \mathbb{P}\big(\exists u \in \mathcal{T}_n, H(u) = j, \forall i \leq n, V(u_i) \leq ai^{1/3}\big) \quad \text{for } j = 1, \dots, n, \infty.$$

By Chebyshev's inequality and then Lemma 2.1, we get

$$\begin{aligned} R_\infty &\le \mathbb{E}\biggl[\sum_{u\in\mathcal{T}_n}\mathbb{1}_{\{\forall i\le n, ai^{1/3}-b_{i,n}\le V(u_i)\le ai^{1/3}\}}\biggr] \\ &\le \mathbb{E}\bigl[\mathrm{e}^{S_n}\mathbb{1}_{\{\forall i\le n, ai^{1/3}-b_{i,n}\le S_i\le ai^{1/3}\}}\bigr] \\ &\le \mathrm{e}^{an^{1/3}}\mathbb{P}\bigl(\forall i\le n, ai^{1/3}-b_{i,n}\le S_i\le ai^{1/3}\bigr). \end{aligned} \tag{3.2}$$

For $1\le j\le n$,

$$\begin{aligned} R_j &\le \mathbb{E}\biggl[\sum_{v\in\mathcal{T}_j}\mathbb{1}_{\{\forall i<j, ai^{1/3}-b_{i,n}\le V(v_i)\le ai^{1/3}, V(v)<aj^{1/3}-b_{j,n}\}}\biggr] \\ &\le \mathbb{E}\bigl[\mathrm{e}^{S_j}\mathbb{1}_{\{\forall i<j, ai^{1/3}-b_{i,n}\le S_i\le ai^{1/3}, V(S_j)<aj^{1/3}-b_{j,n}\}}\bigr] \\ &\le \mathrm{e}^{aj^{1/3}-b_{j,n}}\mathbb{P}\bigl(\forall i<j, ai^{1/3}-b_{i,n}\le S_i\le ai^{1/3}\bigr). \end{aligned} \tag{3.3}$$

3.2. *Asymptotics for R_∞*

In order to apply Lemma 2.4 (combined with Remark 2.6), we set $b_{i,n}:=n^{1/3}g(\frac{i}{n})$ for some continuous function $g:[0,1]\mapsto[0,+\infty)$. We take for any $t\in[0,1]$, $g_2(t):=at^{1/3}$ and $g_1(t)=g_2(t)-g(t)$. Then we have

$$\limsup_{n\to\infty}\frac{\log\mathbb{P}(\forall i\le n, ai^{1/3}-b_{i,n}\le S_i\le ai^{1/3})}{n^{1/3}}\le -C_{g_1,g_2}. \tag{3.4}$$

Putting together equations (3.2) and (3.4), we get

$$\limsup_{n\to\infty}\frac{\log R_\infty}{n^{1/3}}\le -s_1, \tag{3.5}$$

where

$$s_1:=-a+C_{g_1,g_2}=-a+\frac{\pi^2\sigma^2}{2}\int_0^1\frac{\mathrm{d}t}{g(t)^2}. \tag{3.6}$$

3.3. *Asymptotics for R_j*

We define B and C by $B(\alpha,n):=0$ and $C(\alpha,n):=\lfloor\alpha n\rfloor+1$ and write, for any $\alpha\in(0,1)$, $j:=C(\alpha,n)$. Proposition 2.5 yields that, uniformly in $\alpha\in(0,1)$,

$$\limsup_{n\to\infty}\frac{1}{n^{1/3}}\log\mathbb{P}\left(\forall i<j, ai^{1/3}-n^{1/3}g\left(\frac{i}{n}\right)\le S_i\le ai^{1/3}\right)\le -C^{0,\alpha}_{g_1,g_2}. \tag{3.7}$$

Putting together equations (3.3) and (3.7), we get that, uniformly in $\alpha\in(0,1)$,

$$\limsup_{n\to\infty}\frac{1}{n^{1/3}}\log R_j\le a\alpha^{1/3}-g(\alpha)-C^{0,\alpha}_{g_1,g_2}. \tag{3.8}$$

Obviously, for any $n\ge 1$,

$$\sum_{j=1}^n R_j(n)\le n\sup_{1\le j\le n}R_j(n)=n\sup_{0<\alpha<1}R_{C(\alpha,n)}(n). \tag{3.9}$$

As a consequence,

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \sum_{j=1}^{n} R_j(n) \le -s_2, \tag{3.10}$$

where

$$s_2 := \min_{0\le\alpha\le 1}\left\{-a\alpha^{1/3} + g(\alpha) + C^{0,\alpha}_{g_1,g_2}\right\} = \min_{0\le\alpha\le 1}\left\{-a\alpha^{1/3} + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2}\right\}. \tag{3.11}$$

Combining (3.10) with (3.5) and (3.1), we obtain

$$\limsup_{n\to\infty} \frac{1}{n^{1/3}} \log \mathbb{P}\big(\exists u \in \mathcal{T}_n, \forall i \le n, V(u_i) \le a i^{1/3}\big) \le -s,$$

where $s := \min(s_1, s_2)$.

3.4. *Choice of g for the upper bound*

Set $a \in (0, a_c)$. We are looking for a function g such that $s > 0$. The existence of such a function implies extinction and ends the proof the first part of Theorem 1.2.

We add the constraint $g(1) = 0$ (but assume $\int_0^1 \frac{\mathrm{d}u}{g(u)^2} < \infty$). Taking $\alpha = 1$, we see from (3.11) and (3.6) that this implies $s_2 \le s_1$ and, as a result, $s = s_2$.

We choose g in such a way that the quantity $-a\alpha^{1/3} + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2}$ which appears in (3.11) does not depend on α. Hence g is defined as the solution of the equation:

$$\forall t \in [0, 1], \quad -a t^{1/3} + f(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{f(u)^2} = s, \tag{3.12}$$

where s is some positive constant, the value of which is to be set later in such a way that $f(1-) = 0$. According to the computations above, this value of s will give a bound for the rate of decay of the survival probability.

Equivalently, equation (3.12) may be written $f(0) = s$ and $\forall t \in (0, 1)$,

$$f'(t) = \frac{a}{3} t^{-2/3} - \frac{\pi^2\sigma^2}{2 f(t)^2}. \tag{3.13}$$

By the Picard–Lindelöf theorem (see for example [5]), such an ordinary differential equation admits a unique maximal solution f defined on an interval $[0, t_{\max})$ with $t_{\max} \in (0, +\infty]$. And if $t_{\max} < +\infty$, then f has limit 0 or $+\infty$ when t goes to $t_{\max}$.

Remark 3.1. *The fact that $f'(0)$ does not exist here is not troublesome at all since the proof of the theorem, using Picard iterates, actually relies on equation (3.12).*

In order to prove that there exists an initial value s such that $t_{\max} = 1$ and $\lim_{t\to 1} f(t) = 0$, we get a closer look at the differential equation.

First we state three simple results specific to this differential equation.

Proposition 3.2. *Let $\lambda > 0$ and f a continuous function $[0, t_0) \mapsto (0, +\infty)$. Define $f_\lambda : (0, \lambda^{-1} t_0) \mapsto (0, +\infty)$ by*

$$f_\lambda(t) = \lambda^{-1/3} f(\lambda t).$$

Then f satisfies equation (3.13) on $(0, t_0)$ if and only if f_λ does on $(0, \lambda^{-1} t_0)$.

Proof. Assume that f satisfies equation (3.13) for any $0 < t < t_0$. Then for any $0 < t < \lambda^{-1} t_0$,

$$\begin{aligned} f'_\lambda(t) &= \lambda^{2/3} f'(\lambda t) \\ &= \lambda^{2/3}\left(\frac{a}{3}(\lambda t)^{-2/3} - \frac{\pi^2\sigma^2}{2f(\lambda t)^2}\right) \\ &= \frac{a}{3} t^{-2/3} - \frac{\pi^2\sigma^2}{2f_\lambda(t)^2}. \end{aligned}$$

This means that f_λ also satisfies equation (3.13) for any $0 < t < \lambda^{-1} t_0$.

Conversely, assume that f_λ satisfies equation (3.13) on $(0, \lambda^{-1} t_0)$. We notice that if $\lambda' > 0$, then $(f_\lambda)_{\lambda'} = f_{\lambda\lambda'}$. We take $\lambda' = \lambda^{-1}$. Hence $(f_\lambda)_{\lambda'} = f$ also satisfies equation (3.13) for any $0 < t < (\lambda\lambda')^{-1} t_0 = t_0$. □

Proposition 3.3. *Set* $0 < a_1 < a_2$ *and* $s > 0$. *Let* f_1 *and* f_2 *be functions* $[0, t_{\max}) \mapsto (0, +\infty)$ *such that*

$$\forall 0 \le t < t_{\max}, \forall i \in \{1, 2\}, \quad -a_i t^{1/3} + f_i(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{\mathrm{d}u}{f_i(u)^2} = s.$$

Then, for all $0 \le t < t_{\max}$, $f_1(t) \le f_2(t)$.

Proof. It suffices to prove that, if $0 \le t_{\text{start}}$, $0 < a_1 < a_2$ and $0 < x_1 \le x_2$, then there exist $t_{\text{next}} > t_{\text{start}}$ such that there are functions f_1 and $f_2 : [t_{\text{start}}, t_{\text{next}}) \mapsto (0, +\infty)$ such that

$$\forall t_{\text{start}} \le t < t_{\text{next}}, \forall i \in \{1, 2\}, \quad -a_i\left(t^{1/3} - t_{\text{start}}^{1/3}\right) + f_i(t) + \frac{\pi^2\sigma^2}{2}\int_{t_{\text{start}}}^t \frac{\mathrm{d}u}{f_i(u)^2} = x_i;$$

then, for any $t_{\text{start}} \le t < t_{\text{next}}$, $f_1(t) \le f_2(t)$.

We choose t_{next} such that the Picard interates f_i^n defined, for $i \in \{1, 2\}$, by:

$$\forall t_{\text{start}} \le t < t_{\text{next}}, \quad f_i^0(t) = x_i;$$

$$\forall n \in \mathbb{N}, \forall t_{\text{start}} \le t < t_{\text{next}}, \quad f_i^{n+1}(t) = f_i^n(t_{\text{start}}) + a_i\left(t^{1/3} - t_{\text{start}}^{1/3}\right) - \frac{\pi^2\sigma^2}{2}\int_{t_{\text{start}}}^t \frac{\mathrm{d}u}{f_i^n(u)^2},$$

exist and converge on $[t_{\text{start}}, t_{\text{next}})$. The limits f_i are solutions of the integral equations for $i \in \{1, 2\}$.

It is easy to prove by induction on n that

$$\forall n \in \mathbb{N}, \forall t_{\text{start}} \le t < t_{\text{next}}, \quad f_1^n(t) \le f_2^n(t).$$

Letting n tend to infinity gives us the desired conclusion. □

Proposition 3.4. *Let* f *be as above. Then we are in one of the following cases*:

(A) $t_{\max} = +\infty$ *and* $f(t) \to +\infty$ *as* $t \to +\infty$;
(B) $t_{\max} < +\infty$ *and* $f(t) \to 0$ *as* $t \to t_{\max}$.

Proof. First notice that for any $0 < t < t_{\max}$, $f(t) \le s + at^{1/3}$. A consequence of this inequality is that if $t_{\max} < +\infty$, then the limit of f when t goes to $t_{\max}$ can only be 0.

Now, suppose that $t_{\max} = +\infty$ but that f does not go to infinity. Then there are $M > 0$ and a sequence $(t_n)_n \ge 1$ with $\lim_n t_n = +\infty$ such that for any $n \ge 1$, $f(t_n) \le M$. We can choose n such that $\frac{a}{3} t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2} < 0$.

Then it is easy to see that f decreases after t_n. Indeed, consider

$$t_* := \inf\left\{t \ge t_n, f'(t) > \frac{a}{3} t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2}\right\}.$$

We have, for $t_n \le t \le t_*$,

$$f'(t) = \frac{a}{3}t^{-2/3} - \frac{\pi^2\sigma^2}{2f(t)^2} \le \frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2} < 0.$$

If we assume $t_* < +\infty$, then $f'(t_*) < 0$, then f decreases in a neighborhood of t_* and the inequality $f'(t) \le \frac{a}{3}t_n^{-2/3} - \frac{\pi^2\sigma^2}{2M^2}$ still holds on this neighborhood, which contradicts the definition of t_*.

We have proved that $f'(t)$ is less than a negative constant for $t \ge t_n$, which implies that f reaches zero in finite time. □

Assume we are in the second case of Proposition 3.4. We set $\lambda := t_{\max}^{-1}$ and define the function f_λ like in Proposition 3.2 (with $t_0 = t_{\max}$). We choose $g = f_\lambda$ and set $g(1) = 0$ so that g is continuous over $[0, 1]$ and satisfies (3.13) for all $t \in (0, 1)$.

Remark 3.5. *A consequence of Proposition 3.2 is that the choice of the value s of $f(0)$ does not matter at all. If we replace $s > 0$ with another $\widetilde{s} > 0$, we then replace λ with $\widetilde{\lambda} = \lambda(\frac{\widetilde{s}}{s})^3$ and finally get the same g.*

So we only have to prove that, when $a < a_c$, we are in case (B) of Proposition 3.4, and we will deduce the upper bound in Theorem 1.2. This is contained in the following:

Proposition 3.6. *Let f be the solution of equation (3.13) with initial condition $f(0) = 1$.*

(i) *If $a > a_c$, then $t_{\max} = +\infty$ and $f(t) \sim bt^{1/3}$ as $t \to +\infty$ with b defined by $b > \frac{2a_c}{3}$ and $a = b + \frac{3\pi^2\sigma^2}{2b^2}$.*
(ii) *If $a = a_c$, then $t_{\max} = +\infty$ and $f(t) \sim \frac{2a_c}{3}t^{1/3}$ as $t \to +\infty$.*
(iii) *If $a < a_c$, then $t_{\max} < +\infty$ and $f(t) \to 0$ as $t \to t_{\max}$.*

In the proof of the proposition, we will need the following lemma.

Lemma 3.7. *Assume that f is a solution on $[0, +\infty)$ of the differential equation and that*:

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \le b.$$

Then we have

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \le b' := a - \frac{3\pi^2\sigma^2}{2b^2}.$$

Proof. Let $\varepsilon > 0$. By hypothesis, for any t greater than some t_0, we have $f(t) \le (b+\varepsilon)t^{1/3}$. For some real constants c_0 and c_0' and any $t \ge t_0$, we have, by equation (3.12):

$$f(t) \le c_0 + at^{1/3} - \frac{\pi^2\sigma^2}{2(b+\varepsilon)^2}\int_{t_0}^{t}\frac{\mathrm{d}u}{u^{2/3}} = c_0' + \left(a - \frac{3\pi^2\sigma^2}{2(b+\varepsilon)^2}\right)t^{1/3}.$$

Hence

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \le \left(a - \frac{3\pi^2\sigma^2}{2(b+\varepsilon)^2}\right).$$

Letting ε tend to 0 ends the proof of the lemma. □

Iterating Lemma 3.7, we obtain:

Lemma 3.8. *Assume that f is a solution on $[0,+\infty)$ of the differential equation and let b_0 be a real such that $b_0 \geq \limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}}$. We define the sequence $(b_n)_{n\in\mathbb{N}}$ recursively by $b_{n+1} := a - \frac{3\pi^2\sigma^2}{2b_n^2}$. Then*

$$\forall n \geq 1, \quad \limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b_n.$$

Proof of Proposition 3.6.

(i) Assume $a \geq a_c$ and let b such that $a = b + \frac{3\pi^2\sigma^2}{2b^2}$. Define, for $0 \leq t \leq t_{\max}$, $f_0(t) := bt^{1/3}$. Then f_0 satisfies equation (3.13) as f does, with initial condition $f_0(0) = 0 < f(0) = s$. Hence

$$\forall 0 \leq t \leq t_{\max}, \quad f(t) \geq f_0(t).$$

This implies $t_{\max} = +\infty$. Now let $h = f - f_0$. Then, by equation (3.12), we have, for $t \geq 0$,

$$\begin{aligned} h(t) &= s + (a-b)t^{1/3} - \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2f(u)^2} \\ &= s + \left(a - b - \frac{3\pi^2\sigma^2}{2b^2}\right)t^{1/3} + \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2}\left(\frac{1}{f_0(u)^2} - \frac{1}{f(u)^2}\right). \end{aligned}$$

Since $a = b + \frac{3\pi^2\sigma^2}{2b^2}$,

$$h(t) = s + \int_0^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{2}\left(\frac{1}{f_0(u)^2} - \frac{1}{f(u)^2}\right) \leq s + \int_0^t \frac{\pi^2\sigma^2}{2}\frac{2h(u)\,\mathrm{d}u}{f_0(u)^3}.$$

We apply Gronwall's lemma and obtain, for any $0 < t_0 < t$,

$$h(t) \leq h(t_0)\exp\left(\int_{t_0}^t \frac{\pi^2\sigma^2\,\mathrm{d}u}{b^3 u}\right) = h(t_0)\left(\frac{t}{t_0}\right)^{\pi^2\sigma^2/b^3}. \tag{3.14}$$

Notice that $\frac{\pi^2\sigma^2}{b^3} = \frac{1}{3}(\frac{2a_c}{3b})^3$. Then if $a > a_c$ and $b > \frac{2a_c}{3}$, the exponent in the right-hand side of (3.14) will be less than $\frac{1}{3}$. Hence inequality (3.14) implies (i).

(ii) Assume $a = a_c$ and $b = \frac{2a_c}{3}$. This is the same as when $a > a_c$, except that the exponent in the right-hand side of (3.14) is exactly $\frac{1}{3}$, which means that for some constant $b_0 > \frac{2a_c}{3}$,

$$\forall t \geq t_0, \quad f_0(t) \leq f(t) \leq b_0 t^{1/3}.$$

Apply Lemma 3.8. The result follows from that $\lim_n b_n = \frac{2a_c}{3}$.

(iii) Assume $a < a_c$ and $t_{\max} = +\infty$. Then, by (ii) and Proposition 3.3, we have that any $b_0 > \frac{2a_c}{3}$, for t large enough,

$$\limsup_{t\to+\infty} \frac{f(t)}{t^{1/3}} \leq b_0.$$

We apply Lemma 3.8. If b_0 is close enough to $\frac{2a_c}{3}$, we will have $b_1 < \frac{2a_c}{3}$ and $b_n \to -\infty$ as n goes to infinity, which is absurd. We conclude that the hypothesis $t_{\max} = +\infty$ is false, which proves the proposition. □

4. Lower bound for the survival probability

4.1. *Strategy of the estimate*

The basic idea is to consider only the population between two barriers (below $i \mapsto ai^{1/3}$ but above $i \mapsto (a-b)i^{1/3}$), estimate the first two moments of the number of individuals in generation n and then to use the Paley–Zygmund inequality to get the lower bound.

Unfortunately, Mogul'skii's estimate causes the appearance of a factor $e^{o(n^{1/3})}$ in the estimates of the moments of the surviving population at generation n, so we will not be able to prove directly that the population survives with positive probability.

Here is how to overcome this difficulty:

Set $\lambda > 0$ such that $e^\lambda \in \mathbb{N}$ and $(v_k)_{k\geq 1}$ a sequence of positive integers. We consider the population surviving below the barrier $i \mapsto ai^{1/3}$: any individual that would be born above this barrier is removed and consequently does not reproduce. For any $k \in \mathbb{N}$, we pick a single individual z at position $V(z)$ in generation $e^{\lambda k}$ and consider the number $Y_k(z)$ of descendants she eventually has in generation $e^{\lambda(k+1)}$.

We get a lower bound for $Y_k(z)$ by considering, instead of z, a virtual individual $\widetilde{z}$ in the same generation $e^{\lambda k}$ but positioned on the barrier at $V(\widetilde{z}) := ae^{\lambda k/3} \geq V(z)$. The number and displacements of the descendants of $\widetilde{z}$ are exactly the same as those of z. Then the descendants of $\widetilde{z}$ are more likely to cross the barrier and be killed, which means that $Y_k(\widetilde{z}) \leq Y_k(z)$.

In order to apply Mogul'skii's estimate, we add a second absorbing barrier $i \mapsto (a-b)i^{1/3}$ for some $b > 0$ and kill any descendant of $\widetilde{z}$ that is born below it. This way, we obtain that, almost surely, $Z_k \leq Y_k(\widetilde{z}) \leq Y_k(z)$, where

$$Z_k := \#\bigl\{\widetilde{u} \in \mathcal{T}_{e^{\lambda(k+1)}} \colon \widetilde{u} > \widetilde{z}, \forall e^{\lambda k} < i \leq e^{\lambda(k+1)}, (a-b)i^{1/3} \leq V(\widetilde{u}_i) \leq ai^{1/3}\bigr\}.$$

Clearly, Z_k depends on z but its law and in particular $A_k := \mathbb{P}(Z_k \geq v_k) \leq \mathbb{P}(Y_k(z) \geq v_k)$ do not.

We define, for any $n \geq 1$:

$$\mathcal{P}_n := \mathbb{P}\bigl(\forall 1 \leq k \leq n, \#\bigl\{u \in \mathcal{T}_{e^{\lambda k}} \colon \forall i \leq e^{\lambda k}, V(u_i) \leq ai^{1/3}\bigr\} \geq v_{k-1}\bigr).$$

If $1 \leq n_0 \leq n$, then we have:

$$\mathcal{P}_{n+1} \geq \mathcal{P}_n\bigl(1 - (1-A_n)^{v_{n-1}}\bigr).$$

By induction, we obtain:

$$\mathcal{P}_n \geq \mathcal{P}_{n_0} \prod_{k=n_0}^{n-1} \bigl(1 - (1-A_k)^{v_{k-1}}\bigr) \geq \mathcal{P}_{n_0} \prod_{k=n_0}^{n} \bigl(1 - e^{-v_{k-1}A_k}\bigr).$$

$$\log \mathcal{P}_n \geq \log \mathcal{P}_{n_0} + \sum_{k=n_0}^{n} \log\bigl(1 - e^{-v_{k-1}A_k}\bigr).$$

With the equivalent $\log(1+x) \sim x$ for small values of x, the previous inequality makes Proposition 1.4 a consequence of the following lemma:

Lemma 4.1. *If $a > a_c$, we can choose $(v_k)_{k\in\mathbb{N}}$ such that, when λ is large enough and such that $e^\lambda \in \mathbb{N}$, we have*

$$\sum_{k=0}^{\infty} e^{-v_k A_{k+1}} < +\infty. \tag{4.1}$$

Fix $\theta \in (0,1)$, for example $\theta = \frac{1}{2}$. The Paley–Zygmund inequality, with $v_k := \theta\mathbb{E}[Z_k]$ will provide us with the lower bound on A_k needed to prove Lemma 4.1:

$$A_k \geq (1-\theta)^2 \frac{(\mathbb{E}[Z_k])^2}{\mathbb{E}[Z_k^2]}. \tag{4.2}$$

We set $k \geq 0$ and consider, as stated above, the descendants of an individual $\widetilde{z}$ starting at time $e^{\lambda k}$ at position $ae^{\lambda k/3}$ over $\ell_k := e^{\lambda(k+1)} - e^{\lambda k}$ generations. The individuals of generation i are killed and have no descendant if they are out of the interval:

$$I_i := \bigl[(a-b)i^{1/3}, ai^{1/3}\bigr].$$

We set, for $k=0$ for example (then the equations also hold for all $k\in\mathbb{N}$ with the same functions):

$$g_2(t):=a\left(\left(t+\frac{e^{\lambda k}}{\ell_k}\right)^{1/3}-\left(\frac{e^{\lambda k}}{\ell_k}\right)^{1/3}\right),\ g(t):=b\left(t+\frac{e^{\lambda k}}{\ell_k}\right)^{1/3},\quad g_1(t):=g_2(t)-g(t). \tag{4.3}$$

4.2. *Upper bound for the second moment*

We split the double sum over $u,v\in\mathcal{T}$ according to the generation j of $u_j=u\wedge v\in\mathcal{T}$ the lowest common ancestor of u and v:

$$\mathbb{E}\big[Z_k^2\big]=\mathbb{E}\Bigg[\sum_{\substack{u>\widetilde{z},v>\widetilde{z}\\ |u|=|v|=e^{\lambda(k+1)}}}\mathbb{1}_{\{\forall e^{\lambda k}<i\le e^{\lambda(k+1)},V(u_i)\in I_i,V(v_i)\in I_i\}}\Bigg]=\sum_{j=0}^{\ell_k}B_{k,j}, \tag{4.4}$$

where $B_{k,k}=Z_k$ (for each time $v=u=u_j$) and for $j<k$,

$$B_{k,j}:=\mathbb{E}\Bigg[\sum_{u>\widetilde{z},|u|=e^{\lambda(k+1)}}\mathbb{1}_{\{\forall e^{\lambda k}<i\le e^{\lambda(k+1)},V(u_i)\in I_i\}}\sum_{\substack{v>u_j,|v|=e^{\lambda(k+1)}\\ v_{j+1}\ne u_{j+1}}}\mathbb{1}_{\{\forall e^{\lambda k}+j<i\le e^{\lambda(k+1)},V(v_i)\in I_i\}}\Bigg]. \tag{4.5}$$

Thanks to Lemma 2.1, we have:

$$\begin{aligned}h_{k,j}(x)&:=\mathbb{E}\Bigg[\sum_{v\ge u_j,|v|=e^{\lambda(k+1)}}\mathbb{1}_{\{\forall e^{\lambda k}+j<i\le e^{\lambda(k+1)},V(v_i)\in I_i\}}\Bigg|V(u_j)=x\Bigg]\\&=\mathbb{E}\big[e^{S_{\ell_k-j}}\mathbb{1}_{\{\forall 0<i\le\ell_k-j,x+S_i\in I_{e^{\lambda k}+j+i}\}}\big]\\&\le\exp\big(a\big(e^{\lambda(k+1)/3}-\big(e^{\lambda k}+j\big)^{1/3}\big)+b_{e^{\lambda k}+j}\big)\mathbb{P}(\forall 0<i\le\ell_k-j,x+S_i\in I_{e^{\lambda k}+j+i}).\end{aligned} \tag{4.6}$$

Actually we need u in order to define $h_{k,j}(x)$ but the real number obtained actually does not depend on the choice of u.

By conditioning on the σ-algebra generated by the ξ_v, $v\in\Gamma(u_i)$, $e^{\lambda k}\le i\le e^{\lambda k}+j-1$ for each u, equation (4.5) gives, in the case of deterministic branching:

$$B_{k,j}\le\sup_{x\in I_{e^{\lambda k}+j}}h_{k,j}(x)\mathbb{E}[Z_k].$$

In the general case, this argument fails because the number (and the displacements) of the sisters of u_{j+1} are correlated with $\xi_{u_{j+1}}$ (and with the fact that this individual exists). We have independence of the σ-algebra mentioned above for the descendants of the sisters of u_{j+1}. If we assume that each individual has almost surely at most r children, we obtain

$$\sum_{\substack{v>u_j,|v|=e^{\lambda(k+1)}\\ v_{j+1}\ne u_{j+1}}}\mathbb{1}_{\{\forall e^{\lambda k}+j<i\le e^{\lambda(k+1)},V(v_i)\in I_i\}}\le(r-1)\sup_{x\in I_{e^{\lambda k}+j+1}}h_{k,j+1}(x).$$

Hence

$$B_{k,j}\le(r-1)\sup_{x\in I_{e^{\lambda k}+j+1}}h_{k,j+1}(x)\mathbb{E}[Z_k].$$

In the case of an unbounded number of children, we remove all the descendants of the individuals having a number of children greater than some number r_k to be set later. This obviously gives a lower bound, and that is what we want.

Formally, we keep the same notations and add a superscript $^{(k)}$ when dealing with this new process. Equation (4.4) becomes

$$\mathbb{E}\big[Z_k^{(k)2}\big] = \sum_{j=0}^{\ell_k} B_{k,j}^{(k)} \tag{4.7}$$

and we have the upper bound

$$B_{k,j}^{(k)} \le (r_k - 1) \sup_{x \in I_{\mathrm{e}^{\lambda k}+j+1}} h_{k,j+1}(x)^{(k)} \mathbb{E}\big[Z_k^{(k)}\big]$$

with, obviously from the definition, $h_{k,j}(x)^{(k)} \le h_{k,j}(x)$.

We define B and C by $B(\alpha,\ell) := \lfloor \alpha\ell \rfloor + 1$ and $C(\alpha,\ell) := \ell$ and write, for any $\alpha \in (0,1)$ $j := B(\alpha,\ell_k) - 1$. Proposition 2.5 (combined with Remark 2.6) yields that, uniformly in $\alpha \in (0,1)$ and $x \in I_{\mathrm{e}^{\lambda k}+B(\alpha,\ell_k)}$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \mathbb{P}\big(\forall 0 < i \le \ell_k - (j+1), x + S_i \in I_{\mathrm{e}^{\lambda k}+j+1+i}\big) \le -C_{g_1,g_2}^{\alpha,1}.$$

Combining with the bound (4.6) yields that, uniformly in $\alpha \in (0,1)$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \frac{B_{k,B(\alpha,\ell_k)-1}^{(k)}}{\mathbb{E}[Z_k^{(k)}]} \le g_2(1) - g_2(\alpha) + g(\alpha) - C_{g_1,g_2}^{\alpha,1}. \tag{4.8}$$

4.3. *Lower bound for the first moment*

For any $k \ge 1$, we consider i.i.d. random variables $X_i^{(k)}$, $1 \le i \le \ell_k$ with the same distribution as X conditioned on $\nu \le r_k$ (with (X,ν) defined in Lemma 2.2) and write $S_j^{(k)} := \sum_{i=1}^{j} X_i^{(k)}$ for any $0 \le j \le \ell_k$. Let $\varepsilon > 0$.

By Lemma 2.2,

$$\begin{aligned}
\mathbb{E}\big[Z_k^{(k)}\big] &= \mathbb{E}\Bigg[\sum_{u > \widetilde{z}, |u| = \mathrm{e}^{\lambda(k+1)}} \mathbb{1}_{\{\forall \mathrm{e}^{\lambda k} < i \le \mathrm{e}^{\lambda(k+1)}, V(u_i) \in I_i, \#\Gamma(u_{i-1}) \le r_k\}}\Bigg] \\
&= \mathbb{E}\big[\mathrm{e}^{S_{\ell_k}} \mathbb{1}_{\{\forall i \le \ell_k, a\mathrm{e}^{\lambda k/3} + S_i \in I_{\mathrm{e}^{\lambda k}+i}, \nu_i \le r_k\}}\big] \\
&= \mathbb{P}(\nu \le r_k)^{\ell_k} \mathbb{E}\big[\mathrm{e}^{S_{\ell_k}^{(k)}} \mathbb{1}_{\{\forall i \le \ell_k, a\mathrm{e}^{\lambda k/3} + S_i^{(k)} \in I_{\mathrm{e}^{\lambda k}+i}\}}\big] \\
&\ge \mathbb{P}(\nu \le r_k)^{\ell_k} \exp\big(l_k^{1/3}\big(g_2(1) - \varepsilon\big)\big) \mathbb{P}\big(g_1 \le_{\ell_k} s_{\ell_k}^{(k)} \le_{\ell_k} g_2; S_{\ell_k}^{(k)} \ge l_k^{1/3}\big(g_2(1) - \varepsilon\big)\big),
\end{aligned} \tag{4.9}$$

where, for any $t \in [0,1]$,

$$s_{\ell_k}^{(k)}(t) := \frac{S_{\lfloor t l_k \rfloor}^{(k)}}{l_k^{1/3}}.$$

Let δ_1 and δ_2 be like in condition (1.1), and let $\delta_3 := \frac{\delta_1}{1+\delta_1}$. Hölder's inequality yields

$$\begin{aligned}
\mathbb{P}(\nu > r_k) &= \mathbb{E}\Bigg[\mathbb{1}_{\{\#\mathcal{T}_1 > r_k\}} \sum_{|u|=1} \mathrm{e}^{-\xi_u}\Bigg] \\
&= \mathbb{E}\Bigg[\big(\#\mathcal{T}_1^{\delta_3} \mathbb{1}_{\{\#\mathcal{T}_1 > r_k\}}\big)\Bigg(\#\mathcal{T}_1^{-\delta_3} \sum_{|u|=1} \mathrm{e}^{-\xi_u}\Bigg)\Bigg]
\end{aligned}$$

$$\leq \mathbb{E}[\#\mathcal{T}_1 \mathbb{1}_{\{\#\mathcal{T}_1 > r_k\}}]^{\delta_3} \mathbb{E}\left[\left(\#\mathcal{T}_1^{-\delta_3} \sum_{|u|=1} \mathrm{e}^{-\xi_u}\right)^{1+\delta_1}\right]^{1/(1+\delta_1)}. \tag{4.10}$$

We begin with the second factor in the right-hand side of (4.10). The convexity of $t \mapsto t^{1+\delta_1}$ gives

$$\#\mathcal{T}_1^{-\delta_1}\left(\sum_{|u|=1} \mathrm{e}^{-\xi_u}\right)^{1+\delta_1} \leq \sum_{|u|=1} \mathrm{e}^{-\xi_u(1+\delta_1)}.$$

Hence

$$\mathbb{E}\left[\left(\#\mathcal{T}_1^{-\delta_3} \sum_{|u|=1} \mathrm{e}^{-\xi_u}\right)^{1+\delta_1}\right] \leq \Phi(1+\delta_1) < +\infty.$$

For the first factor in the right-hand side of (4.10), Markov's inequality yields

$$\mathbb{E}[\#\mathcal{T}_1 \mathbb{1}_{\{\#\mathcal{T}_1 > r_k\}}] \leq \frac{\mathbb{E}[\#\mathcal{T}_1^{1+\delta_2}]}{r_k^{\delta_2}}.$$

Finally, the bound (4.10) becomes

$$\mathbb{P}(\nu > r_k) \leq \frac{\mathbb{E}[\#\mathcal{T}_1^{1+\delta_2}]^{\delta_3}}{r_k^{\delta_2\delta_3}} \Phi(1+\delta_1)^{1/(1+\delta_1)}.$$

We choose $r_k := \lfloor \mathrm{e}^{l_k^{1/4}} \rfloor$. Therefore

$$\lim_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \mathbb{P}(\nu \leq r_k)^{l_k} = 0.$$

In order to apply Proposition 2.7 to the third factor of (4.9), we have to check the conditions (2.4). It is not hard to see that these conditions are consequences of the hypothesis (1.1) (see [10] for the details).

With the notations of Lemma 2.8

$$\mathbb{P}\left(g_1 \leq_{\ell_k} s_{\ell_k}^{(k)} \leq_{\ell_k} g_2; S_{\ell_k}^{(k)} \geq \ell_k^{1/3}\left(g_2(1) - \varepsilon\right)\right)$$
$$\geq P_{\ell_k}(M, \varepsilon_1, \varepsilon_2) \inf_{-M\varepsilon_2\ell_k^{1/3} \leq z \leq -\varepsilon_1\varepsilon_2\ell_k^{1/3}} Q_{\ell_k}(z, \varepsilon_1, \varepsilon_2, g_1, g_2),$$

where

$$Q_{\ell_k}(z, \varepsilon_1, \varepsilon_2, g_1, g_2) := \mathbb{P}^z\left(S^{(k)}_{\ell_k - \lfloor \varepsilon_2 \ell_k^{1/3} \rfloor} \geq \ell_k^{1/3}\left(g_2(1) - \varepsilon\right);\right.$$
$$\left.\forall i \leq \ell_k - \left\lfloor \varepsilon_2 \ell_k^{1/3} \right\rfloor, g_1\left(\frac{\lfloor \varepsilon_2 \ell_k^{1/3} \rfloor + i}{\ell_k}\right) \leq \frac{S_i^{(k)}}{\ell_k^{1/3}} \leq g_2\left(\frac{\lfloor \varepsilon_2 \ell_k^{1/3} \rfloor + i}{\ell_k}\right)\right).$$

Proposition 2.7, with $\beta = 0$, $\gamma = 1$, $B(\ell_k) = \lfloor \varepsilon_2 \ell_k^{1/3} \rfloor$, $C(\ell_k) = \ell_k$, $u_{\ell_k} = -M\varepsilon_2\ell_k^{1/3}$ and $v_{\ell_k} = -\varepsilon_1\varepsilon_2\ell_k^{1/3}$, yields

$$\liminf_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \inf_{u_{\ell_k} \leq z \leq v_{\ell_k}} Q_{\ell_k}(z, \varepsilon_1, \varepsilon_2, g_1, g_2) \geq -C_{g_1,g_2}.$$

Then letting $\varepsilon_2 \to 0$ in (4.11) thanks to Lemma 2.8, the bound (4.9) becomes

$$\liminf_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \mathbb{E}\left[Z_k^{(k)}\right] \geq g_2(1) - \varepsilon - C_{g_1,g_2}.$$

This inequality holds for any $\varepsilon > 0$ small enough, hence also for $\varepsilon = 0$.

4.4. *Proof of Proposition* 1.4

Combining with (4.8) yields that, uniformly in $\alpha \in (0, 1)$,

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log \frac{B^{(k)}_{k,B(\alpha,\ell_k)-1}}{(\mathbb{E}[Z_k^{(k)}])^2} \le -g_2(\alpha) + g(\alpha) + C^{0,\alpha}_{g_1,g_2}.$$

Consequently, in view of (4.7) and (4.2)

$$\limsup_{k\to\infty} \frac{1}{\ell_k^{1/3}} \log A_k^{(k)} \ge \min_{0\le\alpha\le 1} g_2(\alpha) - g(\alpha) - C^{0,\alpha}_{g_1,g_2}.$$

Lemma 4.1 yields

$$\max_{0\le\alpha\le 1} G_\lambda(\alpha) < 0 \quad \Rightarrow \quad \mathbb{P}\big(\exists u \in \mathcal{T}_\infty, \forall i \ge 1, V(u_i) \le a i^{1/3}\big) > 0, \tag{4.11}$$

where

$$G_\lambda(\alpha) := -g_2(\alpha) + g(\alpha) + \frac{\pi^2\sigma^2}{2}\int_0^\alpha \frac{\mathrm{d}t}{g(t)^2} + \mathrm{e}^{-\lambda/3}\left[-g_2(1) + \frac{\pi^2\sigma^2}{2}\int_0^1 \frac{\mathrm{d}t}{g(t)^2}\right].$$

We denote

$$\forall t \in [0, 1], \quad f(t) := \left(t + \frac{1}{\mathrm{e}^\lambda - 1}\right)^{1/3}.$$

We have $g_2 = af - af(0)$. We choosed for the width of the pipe the function $g := bf$. This gives

$$G_\lambda(\alpha) = af(0) + (b-a)f(\alpha) + \frac{\pi^2\sigma^2}{2b^2}\int_0^\alpha \frac{\mathrm{d}t}{f(t)^2} + \mathrm{e}^{-\lambda/3}\left[af(0) - af(1) + \frac{\pi^2\sigma^2}{2b^2}\int_0^1 \frac{\mathrm{d}t}{f(t)^2}\right].$$

Since $f(1) = \mathrm{e}^{\lambda/3} f(0)$ and $f' = \frac{1}{3} f^{-2}$, this becomes:

$$G_\lambda(\alpha) = \left(b + \frac{3\pi^2\sigma^2}{2b^2} - a\right) f(\alpha) + \mathrm{e}^{-\lambda/3}\left[af(0) - \frac{3\pi^2\sigma^2}{2b^2} f(0)\right].$$

Assuming $a > a_c$, we can choose b such that $b + \frac{3\pi^2\sigma^2}{2b^2} < a$. Since f is increasing on [0, 1],

$$\max_{0\le\alpha\le 1} G_\lambda(\alpha) = G_\lambda(0) = f(0)\left[\left(b + \frac{3\pi^2\sigma^2}{2b^2} - a\right) + \mathrm{e}^{-\lambda/3}\left(a - \frac{3\pi^2\sigma^2}{2b^2}\right)\right].$$

This value is negative for sufficiently large λ (that we can choose such that we also have $\mathrm{e}^\lambda \in \mathbb{N}$), which, in view of (4.11), completes the proof.

5. The extinction rate

Throughout this section, we assume $a < a_c$.

5.1. *Upper bound*

It follows from the computations of Section 3 that, for any continuous function $g:[0,1] \mapsto [0,+\infty)$ such that $g(0)=1$,

$$\limsup_n \frac{1}{n^{1/3}} \log \mathbb{P}\bigl(\exists u \in \mathcal{T}_n, \forall i \le n, V(u_i) \le a i^{1/3}\bigr) \le -c_g,$$

where

$$c_g := \min_{0\le t\le 1}\left(g(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{du}{g(u)^2} - a t^{1/3}\right).$$

The best choice for g is the one described in the end of Section 3: it is the solution of the integral equation (3.12) with $s = c_g$ such that $g(1) = 0$ (or equivalently, $t_{\max} = 1$). We can make this choice thanks to Proposition 3.2 and Proposition 3.6(iii).

5.2. *Lower bound*

For the sake of clarity, we treat only the regular case. The modifications required by the general case are the same as above. We directly apply the Paley–Zygmund inequality to the number W_n of individuals $u \in \mathcal{T}_n$ such that

$$\forall i \le n, \quad a i^{1/3} - n^{1/3} g\left(\frac{i}{n}\right) \ge V(u) \le a i^{1/3}.$$

Following the computations of Section 4, we obtain

$$\liminf_n \frac{1}{n^{1/3}} \log \mathbb{P}\bigl(\exists u \in \mathcal{T}_n, \forall i \le n, V(u_i) \le a i^{1/3}\bigr) \ge \liminf_n \frac{1}{n^{1/3}} \log\bigl(\mathbb{P}(W_n \ge 1) \ge\bigr) \ge -d_g,$$

where

$$d_g := \max_{0\le t\le 1}\left(g(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{du}{g(u)^2} - a t^{1/3}\right).$$

The optimal g would be exactly the same as in the upper bound, except that we are forced to take approximations because g must be strictly positive on $[0,1]$. Since this optimal g is such that $g(t) + \frac{\pi^2\sigma^2}{2}\int_0^t \frac{du}{g(u)^2} - a t^{1/3}$ does not depend on t, we have proved

$$c := \sup_g c_g = \inf_g d_g.$$

This completes the proof of Proposition 1.5.

6. Some refinements

6.1. *About more general barriers*

We are going to give a sketch of the proof of Theorem 1.8. We notice that $\frac{3\pi^2\sigma^2}{2b_{a^+}^2} \le \frac{3\pi^2\sigma^2}{2b_{a_c}^2} = \frac{a_c}{3} < a_c$.

The main idea is to consider the function g_2 defined by

$$\forall t \in [0,1), \quad g_2(t) := a^+ t^{1/3}; \qquad g_2(1) = a^-.$$

We compute the quantities R_j and follow the arguments of Section 3. Almost everything goes as before with $a = a^+$, except that R_∞ is less than before. We search for the optimal g (with still $g_1 = g_2 - g$). This is a solution of (3.12) (with a^+ instead of a) over $[0,1]$, but the boundary condition $g(1)=0$ is replaced by $g(1-)=a^+-a^-$. If a^- is the critical value $\frac{3\pi^2\sigma^2}{2b_{a^+}}$ of Theorem 1.8, then the function $g(t)=b_{a^+}t^{1/3}$ almost works, but the factor giving the exponential decay of the probability is $0=g(0)$. If a^- is smaller than the critical value, we obtain some solution (starting with the boundary condition at 1 and solving the differential equation) with $g(0)>0$, which implies that there is extinction and that, roughly speaking, the probability decays at least like $\exp(-g(0)n_k^{1/3}(1+o(1)))$ along a subsequence n_k such that $\varphi(n_k)/n_k^{1/3}$ is close to a^-.

Conversely, when we consider the same function g_2 with a^- greater than the critical value, we can find a solution of the differential equation with arbitrarily small $g(0)>0$ such that $g(1-)>a^+-a^-$, which means that the probability to have an exponential population $\exp((a^--a^++g(1-))n_k^{1/3}(1+o(1)))$ is of order $\exp(-g(0)n_k^{1/3}(1+o(1)))$. Then we can apply the arguments of Section 4. Let $a^+\ge a_c$ and $a^->\frac{3\pi^2\sigma^2}{2b_{a^+}^2}$. We construct a barrier φ satisfying $\limsup_{n\to\infty}\frac{\varphi(n)}{n^{1/3}}=a^+$ and $\liminf_{n\to\infty}\frac{\varphi(n)}{n^{1/3}}=a^-$ such that the process survives with positive probability. It suffices to take $\varphi(n)=a^-n^{1/3}$ if $n\in\{N^k\colon k\in\mathbb{N}\}$ and $\varphi(n)=a^+n^{1/3}$ otherwise, for some integer N big enough, depending on a^+ and a^-. The proof of this is essentially identical to the proof of the lower bound contained in Section 4. When $a^+\ge a_c$, $a^->\frac{3\pi^2\sigma^2}{2b_{a^+}^2}$, there is not always survival. For example, if $\varphi(n)$ equals $a^+n^{1/3}$ for n even and $a^-n^{1/3}$ for n odd, then a^+ does not matter, it is easy to see that there is extinction if $a^-<a_c$: staying below this barrier is almost as difficult as for the barrier $n\mapsto a^-n^{1/3}$. The trouble comes from the fact that $\frac{\varphi(n)}{n^{1/3}}$ is too often close to a^-.

6.2. *Sketch of the proof of Proposition* 1.6

First we give the upper bound. Let $a<a_c$ and let g be with the optimal function seen before. Let $n\ge 1$. We add a second absorbing barrier $i\mapsto ai^{1/3}-n^{1/3}g(\frac{i}{n})$. We write Z_i (resp. Z_i^*) for the number of individuals in generation i that survive below the barrier $i\mapsto ai^{1/3}$ (resp. between the barriers). We have

$$\mathbb{P}(Z>k)\le\mathbb{P}(Z_{n+1}>0)+\sum_{i=0}^{n}\mathbb{P}\left(Z_i^*>\frac{k}{n+1}\right)+\mathbb{P}\big(Z_i>Z_i^*\big).$$

The terms $\mathbb{P}(Z_i>Z_i^*)$ correspond to the R_j. We know that they are, like $\mathbb{P}(Z_{n+1}>0)$, $\exp(-g(0)n^{1/3}(1+o(1)))$. It is not hard to see from the integral equation satisfied by g that

$$\mathbb{E}\big[Z^*_{\lfloor\alpha n\rfloor}\big]=\exp\big(\big(g(\alpha)-g(0)\big)n^{1/3}\big(1+o(1)\big)\big).$$

Therefore, by Markov's inequality,

$$\sum_{i=0}^{n}\mathbb{P}\left(Z_i^*>\frac{k}{n+1}\right)\le\frac{(n+1)^2}{k}\exp\big((d-c)n^{1/3}\big(1+o(1)\big)\big).$$

Finally we choose a sequence $n=n(k)$ such that $k\sim\exp(dn^{1/3})$, and we obtain the upper bound by letting $k\to\infty$.

For the lower bound, we consider the same barriers as above. For any $\alpha\in[0,1]$, the probability that at least one individual survives between the barriers until generation $\lfloor\alpha n\rfloor$ and is close to the lower barrier at time $\lfloor\alpha n\rfloor$ is $\exp(cn^{1/3}(1+o(1)))$. We choose α maximizing g. An individual close to the lower barrier at time $\lfloor\alpha n\rfloor$ gives, with probability at least $\exp((c-\varepsilon_1)n^{1/3})$, in around $\varepsilon_2n^{1/3}$ generations a number of children at least $\exp((d-\varepsilon_3)n^{1/3})$. Taking $\varepsilon_1,\varepsilon_2,\varepsilon_3$ small and making the same choice of n as for the upper bound yield the result.

6.3. *Proof of Corollary* 1.3 *from Theorem* 1.2

Lemma 6.1. *We assume that the underlying Galton–Watson process is supercritical, and we denote by q the extinction probability. Then, conditional on survival of the Galton–Watson process, the random variable* $\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}$ *is almost surely constant.*

Proof. Let f be the generating function of the underlying supercritical Galton–Watson process. We mean that for any $s \in [0,1]$, $f(s) = \mathbb{E}[s^{\#\mathcal{T}_1}]$. It is well known that f has exactly two fixed points, 1 and $q \in [0,1)$ the extinction probability. Let $a \in \mathbb{R}$. In order to prove the lemma, it suffices to show that the number

$$\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} > a\right)$$

is a fixed point of the generating function f.

We write the boundary of the tree

$$\mathcal{T}_\infty = \bigcup_{|v|=1} \mathcal{T}_\infty^v,$$

where

$$\mathcal{T}_\infty^v = \{u \in \mathcal{T}_\infty, u > v\}$$

is the boundary of the tree $\mathcal{T}^v$ rooted at v. Hence

$$\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} = \inf_{|v|=1} \inf_{u\in\mathcal{T}_\infty^v} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} = \inf_{|v|=1} \inf_{u\in\mathcal{T}_\infty^v} \limsup_{n\to\infty} \frac{V(v)+V^v(u_n)}{n^{1/3}}. \tag{6.1}$$

For any $v \in \mathcal{T}_1$, $\frac{V(v)}{n^{1/3}} \to 0$, hence for any $u \in \mathcal{T}_\infty^v$,

$$\limsup_{n\to\infty} \frac{V(v)+V^v(u_n)}{n^{1/3}} = \limsup_{n\to\infty} \frac{V^v(u_n)}{(n-1)^{1/3}}.$$

From this last equality and the independence properties of the branching random walk, we deduce that the random variables

$$\inf_{u\in\mathcal{T}_\infty^v} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}}, \quad v \in \mathcal{T}_1,$$

form an i.i.d. family, and are distributed like $\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}}$.

With this in mind, equation (6.1) yields, for any $a \in \mathbb{R}$,

$$\begin{aligned}
\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} > a\right) &= \mathbb{P}\left(\forall v \in \mathcal{T}_1, \inf_{u\in\mathcal{T}_\infty^v} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} > a\right) \\
&= \mathbb{E}\left[\prod_{v\in\mathcal{T}_1} \mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty^v} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} > a\right)\right] \\
&= \mathbb{E}\left[\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}}\right)^{\#\mathcal{T}_1}\right] \\
&= f\left(\mathbb{P}\left(\inf_{u\in\mathcal{T}_\infty} \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}}\right)\right).
\end{aligned}$$

This completes the proof of the lemma. □

Proof of Corollary 1.3. Let $a > a_c$. By Theorem 1.2, the branching random walk absorbed by the barrier $i \mapsto ai^{1/3}$ survives with positive probability. Hence, with at least the same positive probability,

$$\exists u \in \mathcal{T}_\infty, \quad \limsup_{n\to\infty} \frac{V(u_n)}{n^{1/3}} \le a.$$

In light of the lemma, this implies that on the set of ultimate survival of the underlying Galton–Watson tree,

$$\inf_{u\in\mathcal{T}_\infty}\limsup_{n\to\infty}\frac{V(u_n)}{n^{1/3}}\le a$$

for any $a > a_c$, hence also for $a = a_c$.

It remains to prove that this random variable is at least a_c, almost surely (remind that it equals $+\infty$ when there is extinction). We reason ab absurdo. We assume that with positive probability, this random variable is less than some positive real $a < a_c$. We deduce the existence of an integer N such that, with positive probability,

$$\exists u \in \mathcal{T}_\infty, \forall n \ge N, \quad \frac{V(u_n)}{n^{1/3}} \le a.$$

Focusing on the value of $V(u_n)$ on this event yields that there exist some real x satisfying the following conditions:

(i) With positive probability, there exists $u \in \mathcal{T}_N$ such that $V(u) \le x$;
(ii) With positive probability, there exists $v \in \mathcal{T}_\infty$ such that $\forall i \ge 1$, $V(v_i) + x \le a(N+i)^{1/3}$.

If condition (ii) holds for some x, then it obviously also holds for any smaller value. Hence, since (i) holds for $x = 0$, we may assume $x \le 0$ (if $x > 0$ we take $x = 0$).

Actually condition (i) is equivalent to $\mathbb{P}(\exists u \in \mathcal{T}_1\colon\ V(u) \le \frac{x}{N}) > 0$ and implies that with positive probability

$$\exists u \in \mathcal{T}_N, \forall n \le N, \quad \xi_{u_n} \le \frac{x}{N}.$$

Hence, with at least the same probability,

$$\exists u \in \mathcal{T}_N, \forall n \le N, \quad V(u_n) \le \frac{xn}{N} \le 0 \le an^{1/3}.$$

This, combined with (ii) yields that the branching random walk absorbed by the barrier $i \mapsto ai^{1/3}$ survives with positive probability.

By Theorem 1.2, this implies $a \ge a_c$, which contradicts our assumption $a < a_c$. □

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