Spectral gaps and exponential integrability of hitting times for linear diffusions

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Abstract. Let X be a regular continuous positively recurrent Markov process with state space $\mathbb{R}$, scale function S and speed measure m. For $a \in \mathbb{R}$ denote

$$B_a^+ = \sup_{x \geq a} m\big(]x, +\infty[\big)\big(S(x) - S(a)\big),$$

$$B_a^- = \sup_{x \leq a} m\big(]-\infty; x[\big)\big(S(a) - S(x)\big).$$

It is well known that the finiteness of $B_a^{\pm}$ is equivalent to the existence of spectral gaps of generators associated with X. We show how these quantities appear independently in the study of the exponential moments of hitting times of X. Then we establish a very direct relation between exponential moments and spectral gaps, all by improving their classical bounds.

Résumé. Soit X un processus de Markov récurrent positif à trajectoires continues et à valeurs dans $\mathbb{R}$. Soient S sa fonction d'échelle et m sa mesure de vitesse. Pour $a \in \mathbb{R}$ notons

$$B_a^+ = \sup_{x \geq a} m\big(]x, +\infty[\big)\big(S(x) - S(a)\big),$$

$$B_a^- = \sup_{x \leq a} m\big(]-\infty; x[\big)\big(S(a) - S(x)\big).$$

Il est bien connu que la finitude de $B_a^{\pm}$ est équivalente à l'existence d'un trou spectral du générateur associé à X. Nous montrons comment ces quantités apparaissent d'une manière indépendante dans l'étude des temps d'atteinte de X. Ensuite nous établissons une relation directe entre les moments exponentiels et le trou spectral, en améliorant en plus leurs encadrements classiques.

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Introduction

Let $(X_t; \ t \geq 0)$ be a regular linear continuous Markov process with the state space $\mathbb{R}$. We assume throughout the paper that X is positively recurrent and conservative (the killing time is identically $+\infty$). Denote by $S(x)$ a scale function of X and $m(\mathrm{d}x)$ the speed measure associated with S (cf. [23], Chapter VII). Recall that S is a continuous strictly increasing function and $m(\mathrm{d}x)$ is a symmetric measure for X, charging every not empty open set. Moreover, the positive recurrence of X implies $\lim_{x \to \pm\infty} S(x) = \pm\infty$ and $m(\mathbb{R}) < \infty$.

In this paper we study some relations between the exponential moments of hitting times of X, the finiteness of the quantities B_a^+, B_a^- given by (0.1), the Hardy and Poincaré inequalities for Dirichlet forms associated with X. As a consequence we give a chain of equalities linking all these objects.

Let $a \in \mathbb{R}$ and $T_a = \inf\{t \geq 0 \colon X_t = a\}$ be the hitting time of a by X. The first question we are interested in is the estimation of exponential moments $\mathbb{E}_x[e^{\lambda T_a}]$, $x \in \mathbb{R}$, $\lambda > 0$.

In some particular cases such moments have been well studied. We mention, for example, Ditlevsen [7] for Ornstein–Uhlenbeck process, Giorno et al. [12] for Bessel and Ornstein–Uhlenbeck processes, Deaconu and Wantz [6] for diffusion with strong drift, and the book of Borodin and Salminen [3] for an overview of known formulas. But we were not able to find in the literature a simple general estimate of exponential moments in terms of the scale function $S(x)$ and the speed measure $m(\mathrm{d}x)$.

In the present paper this question receives a very satisfactory response with the quantities:

$$\Phi_a^+(x) = m\big(]x, +\infty[\big)\big(S(x) - S(a)\big),$$
$$\Phi_a^-(x) = m\big(]-\infty; x[\big)\big(S(a) - S(x)\big).$$

Namely, let λ_a^+ be the supremum of $\lambda > 0$ such that $\mathbb{E}_x e^{\lambda T_a} < \infty$ for some $x > a$ (hence for all $x > a$, see the "all-or-none" property, Proposition 1.2). Respectively, let λ_a^- be the supremum of λ such that $\mathbb{E}_x e^{\lambda T_a} < \infty$ for some (all) $x < a$. Put

$$B_a^+ = \sup_{x \geq a} \Phi_a^+(x), \qquad B_a^- = \sup_{x \leq a} \Phi_a^-(x) \tag{0.1}$$

and

$$C_a^\pm = \liminf_{x \to \pm\infty} \Phi_a^\pm(x).$$

Our first result (see Section 1, Theorem 1.1) asserts that

$$\frac{1}{4B_a^\pm} \leq \lambda_a^\pm \leq \frac{1}{4C_a^\pm} \wedge \frac{1}{B_a^\pm}, \tag{0.2}$$

where B_a^+ or B_a^- can eventually be infinite. The positivity of $\lambda_a^\pm$ is thereby equivalent to the finiteness of $B_a^\pm$. Moreover, if $\lambda_a^\pm > 0$ for some $a \in \mathbb{R}$, it is so for all $a \in \mathbb{R}$.

As it turns out, the importance of the quantities $B_a^\pm$ is well-known in some different context. Using Krein's method, Kac and Krein [15] and Kotani and Watanabe [17] have shown that the spectral gaps γ_a^+ (respectively γ_a^-) of the generator of X killed when it exits $]a, +\infty[$ (resp. $]-\infty, a[$) satisfy

$$\frac{1}{4B_a^\pm} \leq \gamma_a^\pm \leq \frac{1}{B_a^\pm}. \tag{0.3}$$

We see therefore that the positivity of $\lambda_a^\pm$ is equivalent to this of $\gamma_a^\pm$, a fact that can actually be derived e.g. from the works of Down et al. [8] and Bakry [1] or Rökner and Wang [24]. But comparing (0.2) and (0.3) leads to the more explicit conjecture $\lambda_a^\pm = \gamma_a^\pm$. This identity would not be surprising, since for exit times from a bounded domain D it is well known from the works of Khasminskii [16] and Friedman [10]. Namely, if τ_D is the exit time from D and X^D is a process killed at τ_D, then the equality holds between the width γ_D of the spectral gap of the generator of X^D and the supremum λ_D of λ such that $\mathbb{E}_x e^{\lambda \tau_D} < \infty$ for all $x \in D$. However, the PDE methods of [16] require $\mathbb{E}_x \tau$ to be bounded, which is not the case in general, and in particular for $D =]a, \infty[$.

What was surprising, is that we have not found in the literature an analogue of Khasminskii identity for unbounded domains. So in the second section of this article we firstly show (Theorem 2.1) in the setting of m-symmetric Hunt processes that

$$\lambda_D = \gamma_D \quad \text{if } \lambda_D = \sup\big\{\lambda \colon \mathbb{E}_x e^{\lambda \tau_D} \in \mathbb{L}^1(m\mathbb{I}_D)\big\} > 0.$$

The proof is based on the spectral calculus, available thanks to the symmetry of the generator of X^D. Since this symmetry is automatically fulfilled in dimension one, we get for any $a \in \mathbb{R}$,

$$\gamma_a^+ = \lambda_a^+ \quad \text{and} \quad \gamma_a^- = \lambda_a^-. \tag{0.4}$$

Notice that we cannot establish such kind of equality directly on $\mathbb{R}$, because our process is conservative and the exit time from $\mathbb{R}$ is identically infinite. But it is well known that the spectral gap γ of a m-symmetric operator can be characterized in terms of Poincaré inequality for its Dirichlet form. We recall in the sequel of Section 2 that

$$\gamma_a^\pm = \frac{1}{A_a^\pm} \quad \text{and} \quad \gamma = \frac{1}{c_P},$$

where $A_a^\pm$ and c_P are the best possible constants satisfying

$$\int_a^\infty \left(F(x) - F(a) \right)^2 m(\mathrm{d}x) \le A_a^+ \int_a^\infty \left(\frac{\mathrm{d}F}{\mathrm{d}S} \right)^2 (t) \, \mathrm{d}S(t),$$

$$\int_{-\infty}^a \left(F(x) - F(a) \right)^2 m(\mathrm{d}x) \le A_a^- \int_{-\infty}^a \left(\frac{\mathrm{d}F}{\mathrm{d}S} \right)^2 (t) \, \mathrm{d}S(t)$$

and

$$\int_{-\infty}^{+\infty} \left(F(x) - \frac{m(F)}{m(\mathbb{R})} \right)^2 m(\mathrm{d}x) \le c_P \int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}F}{\mathrm{d}S} \right)^2 (x) \, \mathrm{d}S(x)$$

for all F in an appropriate functional space $\mathcal{F}$.

A classical Mukenhoupt result [22] yields

$$B_a^+ \le A_a^+ \le 4B_a^+ \quad \text{and} \quad B_a^- \le A_a^- \le 4B_a^-.$$

Looking at (0.2) and (0.4), we see that the lower bounds above can actually be replaced by $4C_a^\pm \vee B_a^\pm$, which are sometimes more precise. They also allow to recover (in our settings) the case of equality $A_a^\pm = 4B_a^\pm$ studied in Miclo [21].

Further, c_P can be easily related to $A_a^\pm$ by

$$\sup_a \left(A_a^+ \wedge A_a^- \right) \le c_P \le \inf_a \left(A_a^+ \vee A_a^- \right)$$

which yields

$$\sup_a \left(B_a^+ \wedge B_a^- \right) \le c_P \le 4 \inf_a \left(B_a^+ \vee B_a^- \right).$$

In the works of Bobkov and Götze [2] and Malrieu and Roberto [20] it was shown that these inequalities can be replaced by

$$\frac{1}{2} B_{me}^+ \vee B_{me}^- \le c_P \le 4 \left(B_{me}^+ \vee B_{me}^- \right),$$

where me is a median of m (see [20], Theorem 6.6.2). At the end of Section 2 we give another refinement of the bounds on c_P. After proving that $A_a^\pm$ are S-Hölder in a, we show that there exists a point $c \in \mathbb{R}$ such that $A_c^+ = A_c^- = c_P$, which yields

$$B_c^+ \vee B_c^- \vee 4C_c^+ \vee 4C_c^- \le c_P \le 4 \left(B_c^+ \wedge B_c^- \right).$$

The point c is, however, unknown (except for obvious symmetric cases) and worths further investigations.

Finally, the last section contains some examples to illustrate the above-mentioned results.

1. Exponential integrability of hitting times

In this section we study the exponential moments of hitting times T_a. For $a \in \mathbb{R}$, denote

$$\lambda_a^+ = \sup\{\lambda \geq 0 \colon \forall x > a, \mathbb{E}_x e^{\lambda T_a} < \infty\};$$

and

$$\lambda_a^- = \sup\{\lambda \geq 0 \colon \forall x < a, \mathbb{E}_x e^{\lambda T_a} < \infty\}.$$

As we will see (Proposition 1.2), an important "all-or-none" property holds for any $\lambda > 0$:

$$\exists x > a, \quad \mathbb{E}_x e^{\lambda T_a} < \infty \quad \Longleftrightarrow \quad \forall x > a, \quad \mathbb{E}_x e^{\lambda T_a} < \infty,$$

the same being true for $x < a$.

Recall the definitions

$$B_a^+ = \sup_{x \geq a} m(]x, +\infty[)(S(x) - S(a)),$$

$$B_a^- = \sup_{x \leq a} m(]-\infty; x[)(S(a) - S(x))$$

and

$$C_a^+ = \liminf_{x \to \infty} m(]x, +\infty[)(S(x) - S(a)),$$

$$C_a^- = \liminf_{x \to -\infty} m(]-\infty; x[)(S(a) - S(x)).$$

The main result of this section is

Theorem 1.1. *For all $a \in \mathbb{R}$,*

$$\frac{1}{4B_a^+} \leq \lambda_a^+ \leq \frac{1}{4C_a^+} \wedge \frac{1}{B_a^+} \quad and \quad \frac{1}{4B_a^-} \leq \lambda_a^- \leq \frac{1}{4C_a^-} \wedge \frac{1}{B_a^-}$$

with the convention $1/\infty = 0$.

In the sequel we often prove only assertions concerning B_a^+ and λ_a^+, since the proofs of their "left" counterparts are completely similar.

1.1. *Kac formula*

The Kac formula, first derived in [13,14] for linear Brownian motion, then generalized in [5] and in [9], permits to calculate the moments of $A_v = \int_0^T v(X_t) \, dt$ for a function v of a Markov process (X) and a suitable random time T. In our proof we need a particular case of this formula, where $v = 1$ and T is an exit time from an interval or a hitting time.

For $a < x < b$ consider

$$T_{a,b} = \inf\{t \geq 0; X_t \notin \,]a, b[\}.$$

The Green potential kernel on $[a, b]$ is given by (see e.g. [23], Chapter VII)

$$G(a, b, x, y) = \frac{(S(b) - S(x \vee y))(S(x \wedge y) - S(a))}{S(b) - S(a)}.$$

This kernel defines the Green operator

$$Gf(x) = \int_a^b G(a, b, x, y) f(y) \, dm(y).$$

Notice that since $G(a, b, x, a) = G(a, b, x, b) = 0$, the integration interval may or may not include a and b.

With the help of this operator we can calculate the moments of $T_{a,b}$ using Kac formula

$$\mathbb{E}_x T_{a,b}^n = n \int_a^b G(a, b, x, y) \mathbb{E}_y T_{a,b}^{n-1} \, dm(y) = n! G^n 1(x). \tag{1.1}$$

To obtain an analogous formula for the moments of hitting times, recall that $\lim_{t \to \pm\infty} S(t) = \pm\infty$, and consider the limits of $G(a, b, x, y)$ when $a \to -\infty$ (resp. $b \to \infty$):

$$G(-\infty, b, x, \xi) = \begin{cases} (S(b) - S(\xi)), & x \leq \xi \leq b, \\ (S(b) - S(x)), & -\infty < \xi \leq x, \end{cases}$$

$$G(a, +\infty, x, \xi) = \begin{cases} (S(\xi) - S(a)), & a \leq \xi \leq x, \\ (S(x) - S(a)), & x \leq \xi < \infty. \end{cases}$$

Taking monotone limits in (1.1), we get a formula for the nth moment of hitting times (see also [18]):

$$\begin{aligned} \mathbb{E}_x T_b^n &= n \int_{-\infty}^b G(-\infty, b, x, \xi) \mathbb{E}_\xi T_b^{n-1} \, dm(\xi) \quad \text{if } x < b, \\ \mathbb{E}_x T_a^n &= n \int_a^{+\infty} G(a, +\infty, x, \xi) \mathbb{E}_\xi T_a^{n-1} \, dm(\xi) \quad \text{if } x > a. \end{aligned} \tag{1.2}$$

The summation over n yields a formula for exponential moments:

$$\begin{aligned} \mathbb{E}_x \exp(\lambda T_b) &= 1 + \lambda \int_{-\infty}^b G(-\infty, b, x, \xi) \mathbb{E}_\xi \exp(\lambda T_b) \, dm(\xi), \quad x < b, \\ \mathbb{E}_x \exp(\lambda T_a) &= 1 + \lambda \int_a^{+\infty} G(a, +\infty, x, \xi) \mathbb{E}_\xi \exp(\lambda T_a) \, dm(\xi), \quad x > a. \end{aligned} \tag{1.3}$$

Remark. *The expressions* (1.2)–(1.3) *are always defined, since all functions therein are positive.*

The following proposition will be referred to as "all-or-none" property in the sequel:

Proposition 1.2 (All-or-none). *Let $a \in \mathbb{R}$ and $\lambda > 0$. The following properties are equivalent*:

- *for some $x > a$, $\mathbb{E}_x \exp(\lambda T_a) < \infty$,*
- *for all $x > a$, $\mathbb{E}_x \exp(\lambda T_a) < \infty$,*
- *$\int_a^{+\infty} \mathbb{E}_\xi \exp(\lambda T_a) \, dm(\xi) < \infty$.*

The same holds for $x < a$.

Proof. Observe that $G(a, +\infty, x, \xi) \equiv \text{const} > 0$ for $\xi > x$, and that $\mathbb{E}_\bullet \exp(\lambda T_a)$ is increasing on $]a, \infty[$. Using the exponential Kac formula we then see that for $x > a$, $\mathbb{E}_x \exp(\lambda T_a) < \infty$ if and only if $\mathbb{E}_\xi \exp(\lambda T_a)$ is m-integrable on $]a, \infty[$. In this case, since m charges every interval of $\mathbb{R}$, $\mathbb{E}_\xi \exp(\lambda T_a) < \infty$ for all $\xi > a$ by monotonicity of $\mathbb{E}_\xi \exp(\lambda T_a)$. □

1.2. *Exit time from an interval*

It is known for a while (Khasminskii condition, see [9]) that the exit time $T_{a,b}$ from a bounded interval $[a, b]$ admits an exponential moment for some $\lambda > 0$. Let

$$\lambda_{a,b} = \sup\{\lambda \colon \forall x \in \,]a, b[, \mathbb{E}_x \exp(\lambda T_{a,b}) < \infty\}.$$

In this subsection we establish some upper and lower bounds for $\lambda_{a,b}$. The upper bound will be particularly important in the proof of the Theorem 1.1.

Lemma 1.3. $\lambda_{a,b} \geq 1/C_{a,b}$, *where*

$$C_{a,b} = \frac{1}{S(b) - S(a)} \int_a^b \big(S(b) - S(y)\big)\big(S(y) - S(a)\big)m(\mathrm{d}y).$$

Proof. Observe that

$$G(a, b, x, y) \leq \frac{(S(b) - S(y))(S(y) - S(a))}{S(b) - S(a)}.$$

If f is positive and bounded, then

$$Gf(x) \leq \frac{1}{S(b) - S(a)} \int_a^b \big(S(b) - S(y)\big)\big(S(y) - S(a)\big)f(y)m(\mathrm{d}y) \leq C_{a,b}\|f\|_\infty.$$

It follows that $\mathbb{E}_x T_{a,b}^n/n! = G^n 1(x) \leq C_{a,b}^n$, whence $\mathbb{E}_x \mathrm{e}^{\lambda T_{a,b}} < \infty$ for $\lambda < 1/C_{a,b}$. $\qquad\square$

Making $b \to \infty$, we get $\mathbb{E}_x \mathrm{e}^{\lambda T_a} < \infty$ for all $x > a$ if

$$\frac{1}{\lambda} > C_{a,\infty} = \int_a^\infty \big(S(y) - S(a)\big)m(\mathrm{d}y),$$

but the last integral is generally infinite, so this inequality does not provide a satisfactory lower bound for λ_a^+. Observe, however, that when $C_{a,\infty} < \infty$, we get $\mathbb{E}_x \mathrm{e}^{\lambda T_a} \leq (1 - \lambda C_{a,\infty})^{-1}$ for all $x > a$ and $\lambda < 1/C_{a,\infty}$ (we postpone an example for the last section).

To find an upper bound for $\lambda_{a,b}$, fix some interval $[a', b'] \subset \,]a, b[$. Define $\kappa_1 > 0$ and $\kappa_2 > 0$ by

$$S(a') - S(a) = \kappa_1\big(S(b') - S(a')\big), \qquad S(b) - S(b') = \kappa_2\big(S(b') - S(a')\big) \tag{1.4}$$

and denote

$$c = \frac{\kappa_1\kappa_2}{1 + \kappa_1 + \kappa_2}\big(S(b') - S(a')\big)m\big([a', b']\big).$$

Lemma 1.4. *If $\lambda \geq 1/c$ then for all $x \in [a', b']$, $\mathbb{E}_x \mathrm{e}^{\lambda T_{a,b}} = \infty$. In particular, $\lambda_{a,b} \leq 1/c$ for any choice of $[a', b']$.*

Proof. Observe that for all x, y in $[a', b']$,

$$G(a, b, x, y) \geq \frac{(S(b) - S(b'))(S(a') - S(a))}{S(b) - S(a)} = \frac{\kappa_1\kappa_2}{1 + \kappa_1 + \kappa_2}\big(S(b') - S(a')\big).$$

It follows that for all $x \in [a', b']$

$$\mathbb{E}_x T_{a,b} \geq \int_{a'}^{b'} G(a, b, x, y)\,\mathrm{d}m(y) \geq \frac{\kappa_1\kappa_2}{1 + \kappa_1 + \kappa_2}\big(S(b') - S(a')\big)m\big([a', b']\big) = c.$$

By induction $\mathbb{E}_x T_{a,b}^n \geq n! c^n$, as seen from

$$\mathbb{E}_x T_{a,b}^n \geq n \int_{a'}^{b'} G(a,b,x,y) \mathbb{E}_y T_{a,b}^{n-1} \, dm(y) \geq n(n-1)! c^{n-1} \int_{a'}^{b'} G(a,b,x,y) \, dm(y) = n! c^n.$$

Hence $\mathbb{E}_x e^{\lambda T_{a,b}} = \infty$ pour $\lambda \geq 1/c$ and $x \in [a',b']$. □

At this stage let us mention a theorem of Carmona–Klein [4]

"spectral gap" $\Rightarrow \mathbb{E}_x e^{\lambda T_U} < \infty,$

where U is a set of positive invariant measure. The formulation of this theorem is somewhat confusing, since it does not precise that λ depends on U. In fact, the following corollary shows that the property $\mathbb{E}_x e^{\lambda T_a} < \infty$ can not hold simultaneously for all (x,a) with a common $\lambda > 0$.

Corollary 1.5. $\forall \lambda > 0, \forall x \in \mathbb{R}$, *there exist $a < x$ and $b > x$ such that $\mathbb{E}_x e^{\lambda T_{a,b}} = \mathbb{E}_x e^{\lambda T_a} = \mathbb{E}_x e^{\lambda T_b} = \infty$.*

Proof. Fix $\lambda > 0$ and $x \in \mathbb{R}$. Put, for example, $\kappa_1 = \kappa_2 = 1$ and chose $[a',b']$ and $[a,b]$ in such a way that $x \in [a',b'] \subset]a,b[$ and the equalities (1.4) hold. Then

$$\frac{1}{c} = \frac{3}{(S(b') - S(a'))m([a',b'])} < \lambda$$

as soon as $(S(b') - S(a'))m([a',b']) > 3\lambda$, which can always be achieved taking a' or b' large enough. Hence, according to Lemma 1.4, $\mathbb{E}_x e^{\lambda T_{a,b}} = \infty$ and thereby $\mathbb{E}_x e^{\lambda T_a} = \mathbb{E}_x e^{\lambda T_b} = \infty$ for such a and b. □

1.3. *Hitting time moments*

In this subsection we will prove the Theorem 1.1:

$$\frac{1}{4B_a^+} \leq \lambda_a^+ \leq \frac{1}{4C_a^+} \wedge \frac{1}{B_a^+} \quad \text{and} \quad \frac{1}{4B_a^-} \leq \lambda_a^- \leq \frac{1}{4C_a^-} \wedge \frac{1}{B_a^-}.$$

The proofs of two parts being completely similar, we only give one for λ_a^+. It will be split in a number of propositions.

Proposition 1.6. $\forall a \in \mathbb{R}, \lambda_a^+ \leq \frac{1}{B_a^+}$, *where $1/\infty = 0$.*

Proof. Fix some $a \in \mathbb{R}$. From Lemma 1.4 we deduce that for $a < a' < x < b' < b$, for all $x \in [a',b']$, $\mathbb{E}_x e^{\lambda T_a} = \infty$ if $\lambda \geq 1/c$, where

$$\frac{1}{c} = \frac{1 + \kappa_1 + \kappa_2}{\kappa_1 \kappa_2 (S(b') - S(a'))m([a',b'])}.$$

Now fix a' and b' and make $k_2 \to \infty$ (so $b \to \infty$), then

$$\frac{1}{c} \to \frac{1}{\kappa_1 (S(b') - S(a'))m([a',b'])} = \frac{1}{(S(a') - S(a))m([a',b'])}.$$

We conclude that $\mathbb{E}_x e^{\lambda T_a} = \infty$ for $x \in [a',b']$ and

$$\lambda > \frac{1}{(S(a') - S(a))m([a',b'])}.$$

It follows by the "all-or-none" Proposition 1.2 that $\mathbb{E}_x e^{\lambda T_a} = \infty$ for any $x > a$, all a', b' and λ as above. Observing that

$$\sup_{a',b'} (S(a') - S(a)) m([a', b']) = \sup_{a'>a} (S(a') - S(a)) m([a', \infty[) = B_a^+$$

by the continuity of S, we get $\mathbb{E}_x e^{\lambda T_a} = \infty$ for any

$$\lambda > \inf_{a',b'} \frac{1}{(S(a') - S(a)) m([a', b'])} = \frac{1}{B_a^+}.$$

The inequality $\lambda_a^+ \leq 1/B_a^+$ is thereby proved. □

The bounds $(4C_a^+)^{-1} \geq \lambda_a^+ \geq (4B_a^+)^{-1}$ require more work. To simplify the notations we put $B_a^+ = B$. Let $b \geq a$ and define for $x > a$ and $f \geq 0$ two positive linear operators, J_b and K_b:

$$K_b f(x) = \int_a^x (S(y) - S(a)) f(y) 1_{b \leq y} m(\mathrm{d}y),$$

$$J_b f(x) = (S(x) - S(a)) 1_{b \leq x} \int_x^\infty f(y) m(\mathrm{d}y),$$

where $\int_x^y$ is understood as $\int_{]x,y]}$. Notice that

$$Gf = J_a f + K_a f \geq J_b f + K_b f = G(f 1_{[b,\infty[}).$$

Put

$$C(b) = \inf_{y>b} J_b 1(y), \qquad B(b) = \sup_{y>b} J_b 1(y),$$

so $B = B(a)$.

Proposition 1.7. *We have*

$$\sum_{l=0}^n a_{n,l} C^l(b) K_b^{n-l} 1(x) \leq (J_b + K_b)^n 1(x) \leq \sum_{l=0}^n a_{n,l} B^l(b) K_b^{n-l} 1(x)$$

where $a_{n,l} \geq 0$ satisfy

$$a_{n,l} = 0 \quad \text{if } l < 0 \text{ or } l > n,$$

$$a_{0,0} = 1, \qquad a_{n+1,l} = \sum_{i \leq l} a_{n,i}.$$

Proof. For any f we have

$$J_b K_b f(x) = (S(x) - S(a)) 1_{b \leq x} \int_x^\infty \mathrm{d}m(y) \int_a^y (S(u) - S(a)) f(u) 1_{b \leq u} \, \mathrm{d}m(u)$$

$$= (S(x) - S(a)) 1_{b \leq x} \int_x^\infty \mathrm{d}m(y) \int_a^x (S(u) - S(a)) f(u) 1_{b \leq u} \, \mathrm{d}m(u)$$

$$+ (S(x) - S(a)) 1_{b \leq x} \int_x^\infty \mathrm{d}m(y) \int_x^y (S(u) - S(a)) f(u) 1_{b \leq u} \, \mathrm{d}m(u)$$

$$= J_b 1(x) K_b f(x) + (S(x) - S(a)) 1_{b \leq x} \int_x^\infty f(u) \, \mathrm{d}m(u) (S(u) - S(a)) 1_{b \leq u} \int_u^\infty \mathrm{d}m(y)$$

$$= J_b 1(x) K_b f(x) + J_b(f J_b 1)(x)$$

whence

$$C(b)(K_b f + J_b f)(x) \le J_b K_b f(x) \le B(b)(K_b f + J_b f)(x),$$

the inequalities being trivially true for $x \le b$. By induction, we easily see that the following inequalities hold for all $n \in \mathbb{N}$:

$$C(b)\big(C^n(b) + C^{n-1}(b)K_b 1(x) + \cdots + K_b^n 1(x)\big)$$
$$\le J_b K_b^n 1(x) \le B(b)\big(B^n(b) + B^{n-1}(b)K_b 1(x) + \cdots + K_b^n 1(x)\big).$$

Now notice that for $n = 0$, $(J_b + K_b)^n 1(x) = 1 = a_{0,0}$. By induction again, for any $x > a$,

$$(J_b + K_b)^{n+1} 1(x) \le (J_b + K_b) \sum_{l \ge 0} a_{n,l} B^l(b) K_b^{n-l} 1(x)$$

$$= J_b \sum_{l \ge 0} a_{n,l} B^l(b) K_b^{n-l} 1(x) + \sum_{l \ge 0} a_{n,l} B^l(b) K_b^{n-l+1} 1(x)$$

$$\le B(b) \sum_{l \ge 0} a_{n,l} B^l(b) \sum_{i=0}^{n-l} B^{n-l-i}(b) K_b^i 1(x) + \sum_{l \ge 0} a_{n,l} B^l(b) K_b^{n-l+1} 1(x)$$

$$= \sum_{i \ge 0} B^{n+1-i}(b) K_b^i 1(x) \sum_{l \le n-i} a_{n,l} + \sum_{i \ge 0} a_{n,n+1-i} B^{n+1-i}(b) K_b^i 1(x)$$

$$= \sum_{i \ge 0} B^{n+1-i}(b) K_b^i 1(x) \sum_{l \le n+1-i} a_{n,l} = \sum_{j \ge 0} B^j(b) K_b^{n+1-j} 1(x) \sum_{l \le j} a_{n,l}$$

$$= \sum_{j \ge 0} a_{n+1,j} B^j(b) K_b^{n+1-j} 1(x).$$

The lower bound is proved in the same way. $\qquad\square$

An explicit formula for $a_{n,l}$ will now be derived.

Lemma 1.8. *For $0 \le l \le n$, $a_{n,l} = C_{n+l}^l - C_{n+l}^{l-1}$, with $C_n^{-1} = 0$. This implies*

$$\frac{4^n}{n+2} \frac{8}{\sqrt{\pi(n+2)}} \lesssim \sum_{k=0}^n a_{n,k} = a_{n+1,n} \le 4^n,$$

whence

$$\lim_{n \to \infty} \sqrt[n]{a_{n+1,n}} = 4.$$

Proof. The expression of $a_{n,l}$ follows by induction from the recursive definition of $a_{n,l}$ and the equality

$$C_n^0 + C_{n+1}^1 + \cdots + C_{n+l}^l = C_{n+l+1}^l.$$

The second assertion follows then easily by Stirling's formula. $\qquad\square$

Theorem 1.9. *Recall that*

$$B = B_a^+ = \sup_{x > a}(S(x) - S(a)) m([x, \infty])$$

and

$$C_a^+ = \liminf_{x \to \infty} \big(S(x) - S(a)\big) m\big(]x, \infty]\big).$$

- *If $0 \le \lambda < (4B_a^+)^{-1}$ then for all $x > a$, $\mathbb{E}_x e^{\lambda T_a} \, dm(x) < \infty$.*
- *If $\lambda > (4C_a^+)^{-1}$ then for all $x > a$, $\mathbb{E}_x e^{\lambda T_a} \, dm(x) = \infty$.*

Proof. Put $\|f\|_b = \int_b^\infty f(z) m(dz)$ for $b \ge a$. We have for $f \ge 0$

$$\|K_b f\|_b = \int_b^\infty m(dz) \int_b^z \big(S(u) - S(a)\big) f(u) m(du) = \int_b^\infty f(u) m(du) \big(S(u) - S(a)\big) \int_u^\infty m(dz),$$

whence

$$C(b)\|f\|_b \le \|K_b f\|_b \le B(b)\|f\|_b. \tag{1.5}$$

The first point of the theorem being obviously true for $B = \infty$, we can suppose that $B = B(a) < \infty$. Combining the inequality (1.5) for $b = a$ with the Proposition 1.7 and Lemma 1.8, we can write

$$\frac{1}{n!} \int_a^{+\infty} \mathbb{E}_x T_a^n \, dm(x) = \big\| G^n 1 \big\|_a \le \sum_{l=0}^n a_{n,l} B^l \big\| K_a^{n-l} 1 \big\|_a$$

$$\le \sum_{l=0}^n a_{n,l} B^l B^{n-l} \|1\|_a = a_{n+1,n} B^n m\big(]a, \infty[\big) \le 4^n B^n m\big(]a, \infty[\big),$$

which implies the first assertion.

In the same way, for any $b \ge a$,

$$\big\| G^n 1 \big\|_b \ge \sum_{l=0}^n a_{n,l} C(b)^l C(b)^{n-l} \|1\|_b = a_{n+1,n} C(b)^n m\big(]b, \infty[\big).$$

Lemma 1.8 now implies that $\int_b^\infty \mathbb{E}_x e^{\lambda T_a} m(dx) = \infty$ for any $\lambda > (4C(b))^{-1}$. The second point follows by the "all-or-none" property, since $\lim_{b \to \infty} C(b) = C_a^+$. $\qquad\square$

Finally, the Propositions 1.6 and 1.9 jointly imply the assertion of Theorem 1.1, namely

$$\frac{1}{4B_a^+} \le \lambda_a^+ \le \frac{1}{4C_a^+} \wedge \frac{1}{B_a^+} \quad \text{and} \quad \frac{1}{4B_a^-} \le \lambda_a^- \le \frac{1}{4C_a^-} \wedge \frac{1}{B_a^-},$$

the inequalities concerning λ_a^- being proved in the same way.

Remark. *It is easy to see that $B_a^\pm < \infty$ implies $\forall a \in \mathbb{R}$, $B_a^\pm < \infty$. So the Theorem 1.1 yields yet another "all-or-none" property*:

$$\exists a \in \mathbb{R}, \quad \lambda_a^\pm > 0 \quad \Longleftrightarrow \quad \forall a \in \mathbb{R}, \quad \lambda_a^\pm > 0.$$

The Corollary 1.5 implies, however, that

$$\lim_{a \to \infty} \lambda_a^- = \lim_{a \to -\infty} \lambda_a^+ = 0.$$

2. Spectral gap

It turns out that the quantities $B_a^{\pm}$ have already appeared in the context of the operators theory (see, e.g., [15,17]). In this section we discuss some relations between exponential moments of a diffusion and spectral gaps of associated operators.

We begin with a general remark. Consider a Hunt process X on a Polish space $\mathbb{E}$ in the sense of Fukushima et al. [11]. Let m be a Radon measure on $\mathbb{E}$. Suppose that m is bounded and X is a m-symmetric process. Denote by $(P_t)_{t \geq 0}$ the transition semigroup of X. Denote by $\mathbb{P}_x$, $x \in \mathbb{E}$, the law of the process X issued from $x \in \mathbb{E}$.

For an open set $\mathbb{G} \subseteq \mathbb{E}$, set

$$\tau_{\mathbb{G}} = \inf\{t > 0 \colon\ X_t \notin \mathbb{G}\}$$

the exit time of X from $\mathbb{G}$. Introduce

$$P_t^{\mathbb{G}}[\mathbb{A}](x) = \mathbb{P}_x[X_t \in \mathbb{A};\, t < \tau_{\mathbb{G}}]$$

for measurable subset $\mathbb{A}$ of $\mathbb{E}$, and set

$$X_t^{\mathbb{G}} = \begin{cases} X_t, & 0 \leq t < \tau_{\mathbb{G}}, \\ \Delta, & t \geq \tau_{\mathbb{G}}. \end{cases}$$

Then, according to [11], $X^{\mathbb{G}}$ is a Hunt process on the state space $\mathbb{G}$, symmetric with respect to the measure $\mathbb{I}_{\mathbb{G}} \cdot m(\mathrm{d}x)$ with the transition semigroup $(P_t^{\mathbb{G}})$. If $A^{\mathbb{G}}$ denotes the infinitesimal generator of $(P_t^{\mathbb{G}})$ in $\mathbb{L}^2(\mathbb{I}_{\mathbb{G}} \cdot m(\mathrm{d}x))$, $A^{\mathbb{G}}$ is a self-adjoint negative operator. Let us denote by $(\cdot, \cdot)$ the scalar product in $\mathbb{L}^2(\mathbb{I}_{\mathbb{G}} \cdot m(\mathrm{d}x))$ and by $(E_\xi, \xi \geq 0)$ the spectral family of $-A^{\mathbb{G}}$.

Recall now the usual properties of the spectral decomposition. For any bounded and continuous f on $[0, \infty[$ one can define $f(-A^{\mathbb{G}})$ by

$$f\left(-A^{\mathbb{G}}\right)u = \int_{[0;\infty[} f(\xi)\,\mathrm{d}E_\xi u, \quad u \in \mathbb{L}^2\left(\mathbb{I}_{\mathbb{G}} \cdot m(\mathrm{d}x)\right),$$

with

$$\left(f\left(-A^{\mathbb{G}}\right)u, g\left(-A^{\mathbb{G}}\right)v\right) = \int_{[0;\infty[} f(\xi)g(\xi)\,\mathrm{d}(E_\xi u, v).$$

In particular,

$$P_t^{\mathbb{G}} = \exp\left(t A^{\mathbb{G}}\right) = \int_{[0;\infty[} \mathrm{e}^{-\xi t}\,\mathrm{d}E_\xi.$$

Denote by $\mathcal{E}^{\mathbb{G}}$ the Dirichlet form associated with $-A^{\mathbb{G}}$ on $\mathbb{L}^2(m)$. Let H_ξ be the image space of E_ξ (which is a projection operator). The elements of H_0 are those who satisfy $P_t^{\mathbb{G}}u = u$ for all $t > 0$. We know that $-A^{\mathbb{G}}$ has a spectral gap at 0 of width at least $\gamma > 0$ if and only if the following inequality

$$\gamma \|u - E_0 u\|^2 = \gamma \int_{]0,\infty[} \mathrm{d}(E_\xi u, u) \leq \int_{]0,\infty[} \xi\,\mathrm{d}(E_\xi u, u) = \mathcal{E}^{\mathbb{G}}(u, u) \tag{2.1}$$

holds for all u in the domain of $\mathcal{E}^{\mathbb{G}}$.

2.1. *Khasminskii identity*

The main theorem of this section concerns the case $\tau_{\mathbb{G}} < \infty$.

For any bounded non negative function $u(x) \in \mathbb{L}^2(\mathbb{I}_G \cdot m(dx))$, for all $\lambda > 0$, $0 < N < \infty$, we have the formulas

$$\int_0^N e^{\lambda t} P_t^G 1(x) \, dt = \frac{1}{\lambda} \mathbb{E}_x \big[e^{\lambda \tau_G \wedge N} - 1 \big],$$

$$\int_0^N e^{\lambda t} P_t^G u(x) \, dt \leq \|u\|_\infty \frac{1}{\lambda} \mathbb{E}_x \big[e^{\lambda \tau_G \wedge N} - 1 \big].$$

The spectral calculus yields

$$\frac{1}{\lambda} \mathbb{E}_x \big[e^{\lambda \tau_G \wedge N} - 1 \big] = \int_{[0,\infty[} \frac{e^{(\lambda - \xi)N} - 1}{\lambda - \xi} \, dE_\xi 1.$$

Hypothesis (λ_0). *$\lambda_0 > 0$ and for any $\lambda < \lambda_0$, $\mathbb{E}_x[e^{\lambda \tau_G}]$ is an element of $\mathbb{L}^1(\mathbb{I}_G \cdot m(dx))$.*

Theorem 2.1. *Hypothesis* (λ_0) *is equivalent to $E_{(\lambda_0-)} = 0$, i.e. $-A^G$ has a spectral gap of width at least equal to λ_0.*

Remark. *This equivalence for bounded domains $G \subset \mathbb{R}^n$ is well-known since the works of Khasminskii* [16] *and Friedman* [10]. *However, the proof of* [16], *Theorem* 2, *makes use of the boundedness of $\mathbb{E}_x \tau_G$ in G, which may not be the case in our general setting.*

The proof is divided in two parts.

Lemma 2.2. *Hypothesis* (λ_0) *implies $\int_{[0,\lambda_0[} dE_\xi = 0$, i.e. $E_{(\lambda_0-)} = 0$.*

Proof. Let $0 < \lambda < \lambda_0$. For any bounded non negative function $f(x) \in \mathbb{L}^2(\mathbb{I}_G \cdot m(dx))$, for all $\lambda > 0$, $0 < N < \infty$, we can write

$$\frac{\|f\|_\infty^2}{\lambda} \big(\mathbb{E}_\bullet \big[e^{\lambda \tau_G \wedge N} - 1 \big], 1 \big)$$

$$\geq \left(\int_0^N e^{\lambda t} P_t^G f \, dt, f \right)$$

$$= \int_{[0,\infty[} d(E_\xi f, f) \int_0^N e^{(\lambda - \xi)t} \, dt$$

$$\geq \int_{[0,\lambda[} d(E_\xi f, f) \int_0^N e^{(\lambda - \xi)t} \, dt$$

$$= \int_{[0,\lambda[} \frac{e^{(\lambda - \xi)N} - 1}{\lambda - \xi} \, d(E_\xi f, f).$$

Taking the limit when $N \uparrow \infty$, the preceding computation gives

$$(E_{(\lambda-)} f, f) = \int_{[0,\lambda[} d(E_\xi f, f) = 0.$$

The bounded non negative functions being dense in $\mathbb{L}^2(\mathbb{I}_G \cdot m(dx))$, we conclude that $E_{(\lambda-)} = 0$. Since this holds for any $0 < \lambda < \lambda_0$, the lemma is proved. $\qquad \square$

Lemma 2.3. *Let $0 < \lambda < \lambda_0$. Suppose that $E_{(\lambda_0-)} = 0$, i.e. $\int_{[0,\lambda_0[} dE_\xi = 0$. Then, $\mathbb{E}_\bullet[e^\lambda]$ is an element of $\mathbb{L}^1(\mathbb{I}_G \cdot m(dx))$ and therefore Hypothesis* (λ_0) *is true.*

Proof. For $0 < \lambda < \lambda_0$ we look at the formula

$$\left(\frac{1}{\lambda}\mathbb{E}_\bullet\big[e^{\lambda \tau_G \wedge N} - 1\big], 1\right) = \int_{[\lambda_0, \infty[} \left(\frac{1 - e^{(\lambda - \xi)N}}{\xi - \lambda}\right) d(E_\xi 1, 1).$$

Let $N \uparrow \infty$. The dominated convergence theorem yields

$$\frac{1}{\lambda}\big(\mathbb{E}_\bullet\big[e^{\lambda \tau_G} - 1\big], 1\big)$$

$$= \int_{[\lambda_0, \infty[} \frac{1}{(\xi - \lambda)} \, d(E_\xi 1, 1)$$

$$\le \frac{1}{(\lambda_0 - \lambda)}(1, 1)$$

$$= \frac{1}{(\lambda_0 - \lambda)}m(G) < \infty$$

whence

$$\mathbb{E}_m e^{\lambda \tau_G} \le \frac{m(G)}{1 - \lambda/\lambda_0}. \tag{2.2}$$

$\square$

Now, for $G = {]}a, \infty[$ and $G = {]}-\infty, a[$, denote by γ_a^+ and γ_a^- the spectral gaps of the corresponding operators. In virtue of the "all-or-none" Proposition 1.2, the Hypotheses $(\lambda_a^\pm)$ are verified, and we obtain

Theorem 2.4. *For any* $a \in \mathbb{R}$, $\gamma_a^\pm = \lambda_a^\pm$, *whence*

$$\frac{1}{4B_a^\pm} \le \gamma_a^\pm \le \frac{1}{4C_a^\pm} \wedge \frac{1}{B_a^\pm}.$$

2.2. *Hardy and Poincaré inequalities*

The Theorem 2.1 and the above (in)equalities only make sense under the condition $\tau_G < \infty$. In this subsection we would like to estimate the spectral gap of a (not-killed) diffusion X on $G = E = \mathbb{R}$. To do so, we use the well-known relations between spectral gaps and Poincaré inequalities.

Recall that S and m are a scale function and the corresponding speed measure of X. Denote by dS the measure induced by $S(x)$. Let $F(x)$ be a real function on $\mathbb{R}$. We shall write $dF \ll dS$, if there exists a function $f(x)$ in $\mathbb{L}^1(dS)$ such that

$$\int_a^b f(x) \, dS(x) = F(b) - F(a) \quad \forall a < b.$$

The function $f(x)$ will be denoted $\frac{dF}{dS}(x)$. Introduce then the function spaces

$$\mathcal{F} = \left\{F \in L^2(m) \colon dF \ll dS, \frac{dF}{dS} \in L^2(dS)\right\}, \tag{2.3}$$

$$\mathcal{F}_{]a, \infty[} = \big\{F \in \mathcal{F} \colon F(x) = 0, x \le a\big\},$$

$$\mathcal{F}_{]-\infty, a[} = \big\{F \in \mathcal{F} \colon F(x) = 0, x \ge a\big\}.$$

Theorem 2.5. *The diffusion* X *is* m*-symmetric. The Dirichlet space associated with* X *is the function space* $\mathcal{F}$ *given by* (2.3), *and the Dirichlet form has the expression*

$$\mathcal{E}(F, F) = \int_{-\infty}^{\infty} \left(\frac{dF}{dS}\right)^2 (x) \, dS(x), \quad F \in \mathcal{F}.$$

The restriction of the Dirichlet form $\mathcal{E}$ on $\mathcal{F}_{]a,\infty[}$ is the Dirichlet form $\mathcal{E}^{]a,\infty[}$ associated with the semigroup $(P_t^{]a,\infty[})_{t\geq 0}$ of the process X killed when it exits $]a,\infty[$. The killed process $X^{]a,\infty[}$ is symmetric with respect to $\mathbb{I}_{]a,\infty[} \cdot m(\mathrm{d}x)$.

The same is true (with obvious modifications) for $\mathcal{E}^{]-\infty,a[}$.

The proof of this theorem is given in [19].

For $a \in \mathbb{R}$ and $0 \leq A \leq \infty$, we shall call what follows Hardy inequalities associated with the function space $\mathcal{F}$, over the upper half space $]a,\infty[$ and the lower half space $]-\infty,a[$, with the constant A:

$$\int_a^\infty \big(F(x) - F(a)\big)^2 \, \mathrm{d}m(x) \leq A \int_a^\infty \left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2 (t) \, \mathrm{d}S(t) \quad \forall F \in \mathcal{F},$$

$$\int_{-\infty}^a \big(F(x) - F(a)\big)^2 \, \mathrm{d}m(x) \leq A \int_{-\infty}^a \left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2 (t) \, \mathrm{d}S(t) \quad \forall F \in \mathcal{F}. \tag{2.4}$$

Denote by A_a^+ (resp. A_a^-) the infimum of the constants A in Hardy inequality over the upper (resp. the lower) half state space.

Let $c \leq \infty$ be a constant. We call the following inequality Poincaré inequality associated with the function space $\mathcal{F}$, with the constant c:

$$\int_{-\infty}^{+\infty} \big(F(x) - \overline{m}(F)\big)^2 \, \mathrm{d}m(x) \leq c \int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2 (x) \, \mathrm{d}S(x) \quad \forall F \in \mathcal{F}, \tag{2.5}$$

where $\overline{m}(F) = \frac{1}{m(\mathbb{R})} \int F(x) \, \mathrm{d}m(x)$. Denote by c_P the lower bound of the constants c in Poincaré inequality.

It is easy to see now that the Poincaré inequality (2.5) with constant c_P can be written as (2.1).

Proposition 2.6. *The generator of the Dirichlet form associated with X has a spectral gap $\gamma > 0$ if and only if $c_P < \infty$. In this case, $\gamma = 1/c_P$.*

Proof. By the L^1-ergodicity of the process X (see, e.g., [3]), the space H_0 can contain only constants. As X is a conservative process, $P_t c = c$ for all $t > 0$, whence $H_0 = \mathbb{R}$. Notice that $\overline{m}$ is the orthogonal projection operator upon H_0, i.e. $E_0 = \overline{m}$. The equivalence

$$\gamma \|F - E_0 F\|^2 \leq \mathcal{E}(F,F) \quad \Longleftrightarrow \quad \gamma \|F - \overline{m}(f)\|^2 \leq \int_{-\infty}^\infty \left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2 \mathrm{d}S$$

proves the proposition. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Now we address the Hardy inequalities.

Proposition 2.7. *The generator of $\mathcal{E}^{]a,\infty[}$ has a spectral gap $\gamma_a^+ > 0$ if and only if the Hardy inequality (2.4) holds with $A_a^+ < \infty$. In this case, $\gamma_a^+ = 1/A_a^+$.*

Proof. Recall that if $u \in H_0$, $P_t^{]a,\infty[}u = u$. Take a bounded non negative fonction v. We have

$$(u,v) = \big(P_t^{]a,\infty[}u, v\big) = \big(u, P_t^{]a,\infty[}v\big).$$

But

$$\lim_{t \to \infty} P_t^{]a,\infty[}v(x) \leq \|v\|_\infty \mathbb{P}_x[t < T_a] = 0$$

due to the positive recurrence property of X. We get $(u,v) = 0$ for any such function v. This means that $u = 0$ and therefore $E_0 = 0$.

Now, for all $F(x) \in \mathcal{F}_{]a,\infty[}$,

$$\gamma \|F - E_0 F\|^2 \le \mathcal{E}_{]a,\infty[}(F, F) \iff \gamma \|F\|^2 \le \int_a^\infty \left(\frac{\mathrm{d}F}{\mathrm{d}S}\right)^2 \mathrm{d}S.$$

Clearly, for any $F(x) \in \mathcal{F}$, $F(x) - F(x \wedge a) \in \mathcal{F}_{]a,\infty[}$, which finishes the proof. $\qquad\square$

The same property evidently holds for A_a^-. From Theorems 1.1 and 2.4 we deduce that for all $a \in \mathbb{R}$,

$$B_a^\pm \vee 4C_a^\pm \le A_a^\pm \le 4B_a^\pm,$$

which slightly improves the celebrated Muckenhoupt bounds [22] in our case.

Now, a well-known technique (see, e.g., [19]) allows to tie c_P with A_a^+ and A_a^-:

$$\sup_a A_a^+ \wedge A_a^- \le c_P \le \inf_a A_a^+ \vee A_a^-,$$

whence

$$\sup_a B_a^+ \wedge B_a^- \le c_P \le 4 \inf_a B_a^+ \vee B_a^-.$$

These inequalities can be made somewhat more precise, if a is a median of m. Namely, Bobkov and Götze [2] have shown that for such a,

$$\frac{1}{2}\left(B_a^+ \vee B_a^-\right) \le c_P \le 4\left(B_a^+ \vee B_a^-\right).$$

We will give yet another estimation of c_P, firstly showing that $A_a^\pm$ are locally Hölder w.r.t. S (see Miclo [21] for some other regularity results).

Proposition 2.8. *Let $a \le b$. Then*

$$0 \le \sqrt{A_a^+} - \sqrt{A_b^+} \le \sqrt{\left(S(b) - S(a)\right)m\left(]a, \infty]\right)}$$

and

$$0 \le \sqrt{A_b^-} - \sqrt{A_a^-} \le \sqrt{\left(S(b) - S(a)\right)m\left(]-\infty, b]\right)}.$$

Proof. Let $a < b$ and $F \in \mathcal{F}$, $f = \mathrm{d}F/\mathrm{d}S$. Then

$$\int_a^\infty \left(F(x) - F(a)\right)^2 m(\mathrm{d}x) = \int_a^b \left(F(x) - F(a)\right)^2 m(\mathrm{d}x) + \int_b^\infty \left(F(x) - F(a)\right)^2 m(\mathrm{d}x).$$

Notice that

$$\left(F(x) - F(a)\right)^2 = \left(\int_a^x f(u)\,\mathrm{d}S(u)\right)^2 \le \left(S(x) - S(a)\right)\int_a^x f^2(u)\,\mathrm{d}S(u).$$

The first integral above is hence bounded by:

$$\int_a^b \left(F(x) - F(a)\right)^2 m(\mathrm{d}x) \le \int_a^b \left(S(x) - S(a)\right)\int_a^x f^2(u)\,\mathrm{d}S(u)m(\mathrm{d}x)$$

$$\le \left(S(b) - S(a)\right)m\left(]a, b]\right)\int_a^b f^2(u)\,\mathrm{d}S(u)$$

$$\le \left(S(b) - S(a)\right)m\left(]a, b]\right)\int_a^\infty f^2(u)\,\mathrm{d}S(u).$$

The second one:

$$\int_b^\infty \left(F(x) - F(a)\right)^2 m(\mathrm{d}x)$$

$$= \int_b^\infty \left(F(x) - F(b)\right)^2 m(\mathrm{d}x) + \left(F(b) - F(a)\right)^2 \int_b^\infty m(\mathrm{d}x) + 2\left(F(b) - F(a)\right) \int_b^\infty \left(F(x) - F(b)\right) m(\mathrm{d}x)$$

$$\le A_b^+ \int_b^\infty f^2(x)\,\mathrm{d}S(x) + \left(S(b) - S(a)\right) \int_a^b f^2(u)\,\mathrm{d}S(u) m\big(]b, \infty]\big)$$

$$+ 2\sqrt{\left(S(b) - S(a)\right) \int_a^b f^2(u)\,\mathrm{d}S(u) m\big(]b, \infty]\big) \int_b^\infty \left(F(x) - F(b)\right)^2 m(\mathrm{d}x)}$$

$$= \left(\sqrt{A_b^+ \int_b^\infty f^2(x)\,\mathrm{d}S(x)} + \sqrt{\left(S(b) - S(a)\right) \int_a^b f^2(u)\,\mathrm{d}S(u) m\big(]b, \infty]\big)}\right)^2$$

$$\le \left(\sqrt{A_b^+} + \sqrt{\left(S(b) - S(a)\right) m\big(]b, \infty]\big)}\right)^2 \int_a^\infty f^2(x)\,\mathrm{d}S(x).$$

Together with the first bound it yields

$$\sqrt{A_a^+} - \sqrt{A_b^+} \le \sqrt{\left(S(b) - S(a)\right) m\big(]a, \infty]\big)}.$$

The A^- counterpart is proved in the same way.   □

Recall that S is continuous, hence so are $A^\pm$. Since

$$\lim_{a \to \pm\infty} A_a^\pm = \lim_{a \to \pm\infty} \frac{1}{\lambda_a^\pm} = \infty,$$

we deduce that there exists a point $c \in \mathbb{R}$ such that $A_c^+ = A_c^- = c_P$. This finally gives us the following inequalities:

$$B_c^+ \vee B_c^- \vee 4C_c^+ \vee 4C_c^- \le c_P \le 4\left(B_c^+ \wedge B_c^-\right).$$

To resume the main results of this section, let us state a concluding theorem.

Theorem 2.9. *For any $a \in \mathbb{R}$,*

$$\frac{1}{A_a^+} = \gamma_a^+ = \lambda_a^+ \quad and \quad \frac{1}{A_a^-} = \gamma_a^- = \lambda_a^-,$$

whence

$$\sup_a \left(\frac{1}{\lambda_a^+} \wedge \frac{1}{\lambda_a^-}\right) \le c_P \le \inf_a \left(\frac{1}{\lambda_a^+} \vee \frac{1}{\lambda_a^-}\right)$$

or, equivalently,

$$\sup_a \left(\lambda_a^+ \wedge \lambda_a^-\right) \le \gamma \le \inf_a \left(\lambda_a^+ \vee \lambda_a^-\right),$$

where $\gamma = 1/c_P$ is the spectral gap of X on $\mathbb{R}$.
There exists a point $c \in \mathbb{R}$ such that $A_c^+ = A_c^- = c_P$, whence

$$B_c^+ \vee B_c^- \vee 4C_c^+ \vee 4C_c^- \le c_P \le 4\left(B_c^+ \wedge B_c^-\right).$$

3. Examples

In this last section we give some examples to illustrate the above results. We suppose in what follows that $a < x$ and study $\mathbb{E}_x e^{\lambda T_a}$, so it suffices to define X_t in some vicinity of $[a, \infty[$.

Let X_t be given by

$$dX_t = \sigma(X_t)\, dW_t + \beta(X_t)\, dt.$$

Recall that in this case we can choose

$$S'(x) = s(x) = \exp\left(-2 \int_\bullet^x \frac{\beta(u)}{\sigma^2(u)}\, du\right), \qquad m(dx) = \frac{2\, dx}{\sigma^2(x)s(x)}.$$

We refer to Borodin and Salminen [3] for the known facts and formulas used below.

3.1. *Drifted Brownian motion*

Let $0 \leq a < x$ and

$$dX_t = dW_t - \beta\, dt, \quad \beta > 0,$$

on $]0, \infty[$, then $\mathbb{E}_x e^{\lambda T_a} < \infty$ if and only if $\lambda \leq \beta^2/2$. On the other hand,

$$s(x) = \exp\left(2\int_0^x \beta\, du\right) = \exp(2\beta x), \qquad S(x) = \frac{1}{2\beta}\big(\exp(2\beta x) - 1\big)$$

and

$$m\big(]x, +\infty[\big) = 2 \int_x^\infty \frac{dt}{s(t)} = \frac{1}{\beta} \exp(-2\beta x),$$

whence

$$B_a^+ = \sup_{x>a}\big(S(x) - S(a)\big)m\big(]x, +\infty[\big) = \frac{1}{2\beta^2},$$

so the product $\lambda_a^+ B_a^+$ equals $1/4$, hence the lower bound $\lambda_a^+ \geq (4B_a^+)^{-1}$ is sharp.

We actually see that

$$B_a^+ = \lim_{x\to\infty}\big(S(x) - S(a)\big)m\big([x, +\infty[\big) = C_a^+,$$

so $\lambda_a^+ = (4B_a^+)^{-1}$ was to be expected. As shown in [21], the above condition is also necessary for the equality $\lambda_a^+ = (4B_a^+)^{-1}$.

The exact value of $\mathbb{E}_m e^{\lambda T_a}$ is (here m means $m|_{]a,\infty[}$)

$$\mathbb{E}_m e^{\lambda T_a} = 2\frac{e^{-2a\beta}}{\beta + \sqrt{\beta^2 - 2\lambda}}, \quad \lambda < \frac{\beta^2}{2}.$$

The estimate (2.2) yields

$$\mathbb{E}_m e^{\lambda T_a} \leq \frac{e^{-2a\beta}}{\beta(1 - 2\lambda/\beta^2)}, \quad \lambda < \frac{\beta^2}{2},$$

which gives the exact exponential rate of $\mathbb{E}_m e^{\lambda T_a}$ in a, but is irrelevant in λ.

3.2. "Geometric" Brownian motion on the natural scale

Suppose that $0 < a < x$ and that X is given by

$$dX_t = \sigma(X_t)\,dW_t, \quad 0 < \sigma_1 x \leq \sigma(x) \leq \sigma_2 x,$$

so $S(x) - S(a) = x - a$ and

$$\frac{2\,dx}{\sigma_2^2 x^2} \leq m(dx) = \frac{2\,dx}{\sigma^2(x)} \leq \frac{2\,dx}{\sigma_1^2 x^2}.$$

We then get

$$B_a^+ = \sup_x \big(S(x) - S(a)\big)m\big(]x, \infty[\big) \leq \frac{2}{\sigma_1^2},$$

$$C_a^+ = \liminf_{x \to \infty} \big(S(x) - S(a)\big)m\big(]x, \infty[\big) \geq \frac{2}{\sigma_2^2},$$

so our bound $\lambda_a^+ \leq (4C_a^+)^{-1}$ is better than $\lambda_a^+ \leq (B_a^+)^{-1}$ as soon as $\sigma_2 < 2\sigma_1$. In particular, for $\sigma_1 = \sigma_2$ (geometric Brownian motion) it gives the well-known value $\lambda_a^+ = \sigma_1^2/8$.

3.3. Ornstein–Uhlenbeck process

Take

$$dX_t = -\frac{1}{2}X_t\,dt + dW_t,$$

then

$$m(dx) = 2\exp\big(-x^2/2\big)\,dx, \qquad S(x) = \int_0^x \exp\big(u^2/2\big)\,du.$$

On the other hand, according to Borodin and Salminen [3], for $x > a$

$$\mathbb{E}_x \exp(\lambda T_a) = \frac{\mathrm{He}_{2\lambda}(x)}{\mathrm{He}_{2\lambda}(a)},$$

where He_ν are the (modified) Hermite functions. We deduce that $\lambda_a^+ = \nu/2$, where ν is the least positive index such that $\mathrm{He}_\nu(a) = 0$. Equivalently, a is the largest zero of He_ν, whence $\lambda_a^+ \approx a^2/2$ for $a \gg 1$.

For $a = 0$, $\lambda_a^+ = 1/2$ and

$$\lambda_a^+ B_a^+ = 2 \sup_{x>0} \int_0^x \exp\big(u^2/2\big)\,du \times \int_x^\infty \exp\big(-u^2/2\big)\,du = 0.4788\dots.$$

For $a = -0.765$: $\lambda_a^+ = 1/4$ and

$$\lambda_a^+ B_a^+ = 2\lambda_a^+ \sup_{x>a} \int_a^x \exp\big(u^2/2\big)\,du \times \int_x^\infty \exp\big(-u^2/2\big)\,du = 0.553\dots$$

so $\lambda_a^+ B_a^+$ (varying between 1/4 and 1) is not constant here.

3.4. *Strongly entering diffusions*

Consider the case when $\mathbb{E}_x e^{\lambda T_a} \leq D$ for all $x > a$, as in [6]. Note that it implies $K_a 1(x) \leq G1(x) = \mathbb{E}_x T_a \leq D/\lambda$, so

$$\sup_{x > a} K_a 1(x) = \int_a^\infty \big(S(y) - S(a)\big) m(y)\, \mathrm{d}y = C < \infty, \tag{3.1}$$

which is the condition of entering boundary. On the other hand, (3.1) implies that $B \leq C$ and

$$\frac{1}{n!} \mathbb{E}_x T_a^n \leq \frac{C}{B} (4B)^n,$$

as can be seen from the proof of Proposition 1.7 by switching the roles of J and K. Hence, for all $x > a$ and $\lambda < (4B)^{-1}$,

$$\mathbb{E}_x e^{\lambda T_a} \leq \frac{C}{B} (1 - 4B\lambda)^{-1}.$$

We conclude that the entering boundary condition (3.1) is equivalent to the uniform boundedness of $\mathbb{E}_x e^{\lambda T_a}$ for $x > a$ (at least for $\lambda < (4B)^{-1}$).

To give an example, let $a = 0$ and $S(0) = 0$, and consider a diffusion

$$\mathrm{d}X_t = \mathrm{d}W_t - \frac{1}{2} b(X_t)\, \mathrm{d}t.$$

One can write $S(x) = e^{v(x)} - 1$ with $v(x) = \ln(S(x) + 1)$, hence $s(x) = v'(x) e^{v(x)}$ and

$$b(x) = \frac{v''(x)}{v'(x)} + v'(x).$$

Then we have

$$\int_0^\infty S(y) m(y)\, \mathrm{d}y = \int_0^\infty \frac{S(y)}{s(y)}\, \mathrm{d}y \leq \int_0^\infty \frac{\mathrm{d}y}{v'(y)}. \tag{3.2}$$

The entering boundary condition (3.1) holds if and only if the last integral is finite. This is the case e.g. if $v'(x) \gtrsim x^{1+\varepsilon}$ as $x \to \infty$, which leads to the main example $b(x) = x^{1+\varepsilon}$ of [6]. However, the condition (3.2) is more pertinent with respect to the uniform boundedness of $\mathbb{E}_x e^{\lambda T_a}$ than those given in [6].

Acknowledgments

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