

Lower large deviations and laws of large numbers for maximal flows through a box in first passage percolation

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Abstract. We consider the standard first passage percolation model in \mathbb{Z}^d for $d \geq 2$. We are interested in two quantities, the maximal flow τ between the lower half and the upper half of the box, and the maximal flow ϕ between the top and the bottom of the box. A standard subadditive argument yields the law of large numbers for τ in rational directions. Kesten and Zhang have proved the law of large numbers for τ and ϕ when the sides of the box are parallel to the coordinate hyperplanes: the two variables grow linearly with the surface s of the basis of the box, with the same deterministic speed. We study the probabilities that the rescaled variables τ/s and ϕ/s are abnormally small. For τ , the box can have any orientation, whereas for ϕ , we require either that the box is sufficiently flat, or that its sides are parallel to the coordinate hyperplanes. We show that these probabilities decay exponentially fast with s , when s grows to infinity. Moreover, we prove an associated large deviation principle of speed s for τ/s and ϕ/s , and we improve the conditions required to obtain the law of large numbers for these variables.

Résumé. Nous considérons le modèle standard de percolation de premier passage dans \mathbb{Z}^d pour $d \geq 2$. Nous nous intéressons à deux quantités, le flux maximal τ entre la moitié inférieure et la moitié supérieure d'une boîte, et le flux maximal ϕ entre le sommet et la base de la boîte. Un argument sous-additif standard implique une loi des grands nombres pour τ dans les directions rationnelles. Kesten et Zhang ont prouvé que τ et ϕ suivent une loi des grands nombres quand les faces de la boîte sont parallèles aux hyperplans des coordonnées: les deux variables grandissent linéairement en la surface s de la base de la boîte, avec la même vitesse déterministe. Nous étudions les probabilités que les variables renormalisées τ/s et ϕ/s soient anormalement petites. Pour τ , la boîte peut avoir n'importe quelle orientation, tandis que pour ϕ , nous imposons soit que la boîte soit suffisamment plate, soit que ses faces soient parallèles aux hyperplans des coordonnées. Nous montrons que ces probabilités décroissent exponentiellement vite avec s , quand s tend vers l'infini. De plus, nous prouvons les principes de grandes déviations de vitesse s associés pour τ/s et ϕ/s , et nous améliorons les conditions requises pour obtenir la loi des grands nombres pour ces variables.

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1. Introduction

The model of maximal flow in a randomly porous medium with independent and identically distributed capacities has been initially studied by Kesten (see [14]), who introduced it as a “higher dimensional version of First Passage Percolation.” The purpose of this model is to understand the behaviour of the maximum amount of flow that can cross the medium from one part to another.

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All the precise definitions will be given in Section 2, but let us be a little more accurate. The random medium is represented by the lattice \mathbb{Z}^d . We see each edge as a microscopic pipe which the fluid can flow through. To each edge e , we attach a nonnegative capacity $t(e)$ which represents the amount of fluid (or the amount of fluid per unit of time) that can effectively go through the edge e . Capacities are then supposed to be random, identically and independently distributed with common distribution function F . Let A be some hyperrectangle in \mathbb{R}^d (i.e., a box of dimension $d - 1$) and n be an integer. The portion of media that we will look at is a box B_n of basis nA and of height $2h(n)$, which nA splits into two equal parts. The boundary of B_n is thus split into two parts, A_n^1 and A_n^2 . We define two flows through B_n : the maximal flow τ_n for which the fluid can enter the box through A_n^1 and leave it through A_n^2 , and the maximal flow ϕ_n for which the fluid enters B_n only through its bottom side and leaves it through its top side. Existing results for ϕ_n and τ_n are essentially of two types: laws of large numbers and large deviation results. Subadditivity implies a law of large numbers for τ_n , when B_n is oriented to a rational direction (as defined in [3]). It is important to note that all the results concerning ϕ_n we present now were obtained for “straight” hyperrectangles A , i.e. hyperrectangles of the form $\prod_{i=1}^{d-1} [0, a_i] \times \{0\}$. Due to the symmetries of the lattice \mathbb{Z}^d , this simplifies considerably the task. Kesten proved a law of large numbers for ϕ_n in straight cylinders in \mathbb{Z}^3 (see [14]), under various conditions on the height $h(n)$, the value of $F(0)$ and an exponential moment condition on F . In a remarkable paper, Zhang (see [22]) recently optimized Kesten’s condition on $F(0)$ and extended the result to \mathbb{Z}^d , $d \geq 2$ (see Theorem 3.4 below). Thérét proved a large deviation principle for ϕ_n at volume order for upper deviations (see [17]). Lower large deviations for ϕ_n far from its asymptotic behaviour were investigated for Bernoulli capacities in [7], and for general functions in [18], and are shown to be of surface order, although a full large deviation principle was not proved.

The main results of this paper are the lower large deviation principles for τ_n and ϕ_n under various conditions, and the improvement of the moment conditions required to state the law of large numbers for these variables. More precisely, we shall show lower large deviation principles at the surface order for τ_n for general A and height $h(n)$, and for ϕ_n when $h(n)$ is small compared to n (see Theorem 3.10 and Corollary 3.14). We also show a lower large deviation principle at the surface order for ϕ_n when $\log h(n)$ is small compared to n^{d-1} and when A is straight (see Theorem 3.17). Unfortunately, when $d \geq 3$, we are not able to prove the lower large deviation principle for ϕ_n through general hyperrectangles and heights (see Remark 6.3). Incidentally, we prove deviation results which are interesting on their own for ϕ_n and τ_n , for general hyperrectangles A (see Theorems 3.9, 3.13 and 3.18). A consequence of these deviation results is a law of large numbers for τ_n in any fixed direction, even irrational, under an optimal moment condition (see Theorem 3.8). We also obtain a law of large numbers for ϕ_n in straight boxes under an optimal condition on the height of the box, and a weak moment condition. We stress the fact that we do not use any subadditive ergodic theorem for the law of large numbers for τ_n , since in our general setting, subadditivity of τ_n is lost in irrational directions. Instead, we use the “almost subadditivity” of τ_n combined with a lower deviation inequality.

The paper is organized as follows. In Section 2, we give the precise definitions and notations. In Section 3, we state the important background we shall rely on and the main results of the paper. In Section 4, we prove the deviation results for τ_n , and for ϕ_n in flat cylinders, the proof of the corresponding result for ϕ_n in straight boxes being completed at the end of the paper. We also obtain also the law of large numbers for τ_n in this section. Section 5 is devoted to the large deviation principle for τ_n , and its corollary, the large deviation principle for ϕ_n in flat boxes. Finally, we prove the law of large numbers, the order of the lower large deviations and the large deviation principle for ϕ_n in straight boxes in Section 6.

2. Definitions and notations

The most important notations are gathered in this section.

2.1. Maximal flow on a graph

First, let us define the notion of a flow on a finite unoriented graph $G = (V, \mathcal{E})$ with set of vertices V and set of edges \mathcal{E} . We write $x \sim y$ when x and y are two neighbouring vertices in G . Let $t = (t(e))_{e \in \mathcal{E}}$ be a collection of non-negative real numbers, which are called *capacities*. It means that $t(e)$ is the maximal amount of fluid that can go through the edge e per unit of time. To each edge e , one may associate two oriented edges, and we shall denote by $\vec{\mathcal{E}}$ the set of all these oriented edges. Let Y and Z be two finite, disjoint, non-empty sets of vertices of G : Y denotes

the source of the network, and Z the sink. A function θ on $\vec{\mathcal{E}}$ is called a flow from Y to Z with strength $\|\theta\|$ and capacities t if it is antisymmetric, i.e. $\theta_{xy} = -\theta_{yx}$, if it satisfies the node law at each vertex x of $V \setminus (Y \cup Z)$:

$$\sum_{y \sim x} \theta_{xy} = 0,$$

if it satisfies the capacity constraints:

$$\forall e \in \mathcal{E} \quad |\theta(e)| \leq t(e),$$

and if the “flow in” at Y and the “flow out” at Z equal $\|\theta\|$:

$$\|\theta\| = \sum_{\substack{y \in Y \\ x \notin Y}} \sum_{x \sim y} \theta(yx) = \sum_{\substack{z \in Z \\ x \notin Z}} \sum_{x \sim z} \theta(xz).$$

The maximal flow from Y to Z , denoted by $\phi_t(G, Y, Z)$, is defined as the maximum strength of all flows from Y to Z with capacities t . We stress the fact that $\phi_t(G, Y, Z)$ is non-negative for any choice of G, Y and Z . We shall in general omit the subscript t when it is understood from the context. The max-flow min-cut theorem (see [4], for instance) asserts that the maximal flow from Y to Z equals the minimal capacity of a cut between Y and Z . Precisely, let us say that $E \subset \mathcal{E}$ is a cut between Y and Z in G if every path from Y to Z borrows at least one edge of E . Define $V(E) = \sum_{e \in E} t(e)$ to be the capacity of a cut E . Then,

$$\phi_t(G, Y, Z) = \min\{V(E) \text{ s.t. } E \text{ is a cut between } Y \text{ and } Z \text{ in } G\}. \tag{1}$$

2.2. On the cubic lattice

We use many notations introduced in [13] and [14]. Let $d \geq 2$. We consider the graph $(\mathbb{Z}^d, \mathbb{E}^d)$ having for vertices \mathbb{Z}^d and for edges \mathbb{E}^d , the set of pairs of nearest neighbours for the standard L^1 norm: $\|z\|_1 = \sum_{i=1}^d |z_i|$ for $z = (z_1, \dots, z_d) \in \mathbb{R}^d$. To each edge e in \mathbb{E}^d we assign a random capacity $t(e)$ with values in \mathbb{R}^+ . We suppose that the family $(t(e), e \in \mathbb{E}^d)$ is independent and identically distributed, with a common distribution function F : this is the standard model of first passage percolation on the graph $(\mathbb{Z}^d, \mathbb{E}^d)$. More formally, we take the product measure \mathbb{P} on $\Omega = \prod_{e \in \mathbb{E}^d} [0, \infty[$, and we write its expectation \mathbb{E} .

For a subset X of \mathbb{R}^d , we denote by $\mathcal{H}^s(X)$ the s -dimensional Hausdorff measure of X (we will use $s = d - 1$ and $s = d - 2$). Let $A \subset \mathbb{R}^d$ be a non-degenerate hyperrectangle (for the usual scalar product), i.e., a box of dimension $d - 1$ in \mathbb{R}^d . All hyperrectangles will be supposed to be closed and non-degenerate in \mathbb{R}^d . Thus, every hyperrectangle A we will consider is the image by an isometry of \mathbb{R}^d of a set of the form $\prod_{i=1}^{d-1} [0, k_i] \times \{0\}$ for strictly positive real numbers k_i . With this notation, we define the smallest length of A , denoted by $l_{\min}(A)$ as

$$l_{\min}(A) = \min_{i=1, \dots, d-1} k_i,$$

i.e. the smallest length of a side of A . We denote by \vec{v} one of the two vectors of unit Euclidean norm, orthogonal to $\text{hyp}(A)$, the hyperplane spanned by A . For h a positive real number, we denote by $\text{cyl}(A, h)$ the cylinder of basis A and height $2h$, i.e., the set

$$\text{cyl}(A, h) = \{x + t\vec{v} \mid x \in A, t \in [-h, h]\}.$$

The set $\text{cyl}(A, h) \setminus \text{hyp}(A)$ has two connected components, which we denote by $C_1(A, h)$ and $C_2(A, h)$. For $i = 1, 2$, let A_i^h be the set of the points in $C_i(A, h) \cap \mathbb{Z}^d$ which have a nearest neighbour in $\mathbb{Z}^d \setminus \text{cyl}(A, h)$:

$$A_i^h = \{x \in C_i(A, h) \cap \mathbb{Z}^d \mid \exists y \in \mathbb{Z}^d \setminus \text{cyl}(A, h), \|x - y\|_1 = 1\}.$$

Let $T(A, h)$ (respectively, $B(A, h)$) be the top (respectively, the bottom) of $\text{cyl}(A, h)$, i.e.,

$$T(A, h) = \{x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}^d \text{ and } \langle x, y \rangle \text{ intersects } A + h\vec{v}\}$$

and

$$B(A, h) = \{x \in \text{cyl}(A, h) \mid \exists y \notin \text{cyl}(A, h), \langle x, y \rangle \in \mathbb{E}^d \text{ and } \langle x, y \rangle \text{ intersects } A - h\vec{v}\}.$$

The notation $\langle x, y \rangle$ corresponds to the edge of endpoints x and y . We define also the r -neighbourhood $\mathcal{V}(H, r)$ of a subset H of \mathbb{R}^d as

$$\mathcal{V}(H, r) = \{x \in \mathbb{R}^d \mid d(x, H) < r\},$$

where the distance is the Euclidean one, i.e. $d(x, H) = \inf\{\|x - y\|_2 \mid y \in H\}$ and $\|z\|_2 = \sqrt{\sum_{i=1}^d z_i^2}$ for $z = (z_1, \dots, z_d) \in \mathbb{R}^d$.

The main characters. For a given realization $(t(e), e \in \mathbb{E}^d)$ we define $\tau(A, h)$ by

$$\tau(A, h) = \phi_t(\text{cyl}(A, h) \cap \mathbb{Z}^d, A_1^h, A_2^h),$$

where ϕ_t is defined in Section 2.1 and $\text{cyl}(A, h) \cap \mathbb{Z}^d$ denotes the induced subgraph of \mathbb{Z}^d with set of vertices $\text{cyl}(A, h) \cap \mathbb{Z}^d$, equipped with capacities t . This definition makes sense if A_1^h and A_2^h are non-empty, otherwise we put $\tau(A, h) = 0$. Notice that if $h > 2\sqrt{d}$ and $l_{\min}(A) > \sqrt{d}$, then A_1^h and A_2^h are non-empty. Similarly, we define the variable $\phi(A, h)$ by

$$\phi(A, h) = \phi_t(\text{cyl}(A, h) \cap \mathbb{Z}^d, B(A, h), T(A, h)).$$

Finally, $p_c(d)$ denotes the critical parameter for the Bernoulli bond percolation on \mathbb{Z}^d .

3. Background and main results

3.1. Background

The following result allows to define the flow constant $\nu(\vec{v}_0)$ when $\vec{v}_0 \in \mathbb{R}^d$ is the vector $(0, \dots, 0, 1)$. It follows from the subadditive ergodic theorems of [1,15,16]. Let $\mathbf{k} = (k_1, \dots, k_{d-1}) \in (\mathbb{N}^*)^n$, and define $A_{\mathbf{k}} = \prod_{i=1}^{d-1} [0, k_i] \times \{0\}$.

Theorem 3.1 ([14]). *Suppose that $h(n)$ goes to infinity when n goes to infinity, and that*

$$\int_0^\infty x \, dF(x) < \infty.$$

Then, $\tau(nA_{\mathbf{k}}, h(n)) / (n^{d-1} \prod_{i=1}^{d-1} k_i)$ converges almost surely and in L^1 , when n goes to infinity, to a non-negative, finite constant $\nu(\vec{v}_0)$ which does not depend on \mathbf{k} .

An important problem is to know when $\nu(\vec{v}_0)$ equals zero. It is proved in [18] (see also [7] for capacities equal to zero or one) that $F(0) < 1 - p_c(d)$ implies $\nu(\vec{v}_0) > 0$. Conversely, Zhang proved in [21], Theorem 1.10, that $\nu(\vec{v}_0) = 0$ if $F(0) = 1 - p_c(d)$, and so by a simple coupling of probability if $F(0) \geq 1 - p_c(d)$. Actually, he wrote the proof for $d = 3$ but said himself that the argument works for $d \geq 3$ (see Remark 1 of [21]). This property is also satisfied in dimension $d = 2$ where we can use duality arguments (see [13], Theorem 6.1 and Remark 6.2). We gather these results in the following theorem.

Theorem 3.2. *Suppose that $\int_0^\infty x \, dF(x)$ is finite. Then, $\nu(\vec{v}_0) = 0$ if and only if $F(0) \geq 1 - p_c(d)$.*

Finally, a crucial result is the following theorem of Zhang, which allows to control the number of edges in a cut of minimal capacity. Let $\mathbf{k} = (k_1, \dots, k_{d-1}) \in (\mathbb{N}^*)^n$, $m \in \mathbb{N}^*$ and define

$$\mathbf{B}(\mathbf{k}, m) = \prod_{i=1}^{d-1} [0, k_i] \times [0, m].$$

Table 1
Hypotheses on the distribution and the height

Hypotheses on the distribution		Hypotheses on the height	
(F1)	$F(0) < 1 - p_c(d)$	(H1)	$\lim_{n \rightarrow \infty} h(n) = +\infty$
(F2)	$\int_0^\infty x \, dF(x) < \infty$	(F4)	$\exists \gamma > 0, \int_0^\infty e^{\gamma x} \, dF(x) < \infty$
(F3)	$\int_0^\infty x^{1+1/(d-1)} \, dF(x) < \infty$	(F5)	$\forall \gamma > 0, \int_0^\infty e^{\gamma x} \, dF(x) < \infty$
		(H2)	$\lim_{n \rightarrow \infty} \frac{\log h(n)}{n^{d-1}} = 0$
		(H3)	$\lim_{n \rightarrow \infty} \frac{h(n)}{n} = 0$

Let $N(\mathbf{k}, m)$ be the number of edges of a cut E between $\mathbf{B}(\mathbf{k}, m)$ and ∞ which achieves the minimal capacity $V(E) = \sum_{e \in E} t(e)$ among all these cuts. If there are more than one cut achieving the minimum, we use a deterministic method to select a unique one with the minimum number of edges among these.

Theorem 3.3 ([22], Theorem 1). *If $F(0) < 1 - p_c(d)$, then there exists constants $\beta = \beta(F, d)$, $m_0(F, d)$ and $C_i = C_i(F, d)$, for $i = 1, 2$ such that for all $n \geq \beta \prod_{i=1}^{d-1} k_i$ and $m_0 \leq m \leq \min_{i=1, \dots, d-1} k_i$,*

$$\mathbb{P}(N(\mathbf{k}, m) \geq n) \leq C_1 e^{-C_2 n}.$$

An analogue result is obtained in [22], Theorem 2, for the minimal cut between the top and the bottom of $\mathbf{B}(\mathbf{k}, m)$ inside $\mathbf{B}(\mathbf{k}, m)$. We shall make use of Theorem 3.3 through a slight modification, Proposition 4.2 below.

Finally, Kesten proved in 1987 the law of large numbers for ϕ in vertical boxes in dimension 3 under the additional assumption that $F(0)$ is sufficiently small and $h(n)$ not too large, plus an assumption of finite exponential moment (see Theorem 2.12 in [14]). In a remarkable paper, Zhang recently improved Kesten’s result by relaxing the assumption on $F(0)$ to the relevant one $F(0) < 1 - p_c(d)$, and extended it to any dimension $d \geq 3$ (see [22]). Zhang proved the following result.

Theorem 3.4 ([22]). *Suppose $F(0) < 1 - p_c(d)$, and there exists $\gamma > 0$ such that*

$$\int e^{\gamma x} \, dF(x) < \infty.$$

If k_1, \dots, k_{d-1}, m go to infinity in such a way that for some $0 < \eta \leq 1$, we have

$$\log m \leq \max_{1 \leq i \leq d-1} k_i^{1-\eta},$$

then

$$\lim_{k_1, \dots, k_{d-1}, m \rightarrow \infty} \frac{\phi(A_{\mathbf{k}}, m)}{k_1 \cdots k_{d-1}} = \nu(\vec{v}_0) \quad \text{a.s. and in } L^1.$$

3.2. Hypotheses on the distribution F and the height h

We gather and present in Table 1 the main hypotheses that we shall do on F and on the height h . Notice that (F5) \Rightarrow (F4) \Rightarrow (F3) \Rightarrow (F2) and (H3) \Rightarrow (H2).

Table 2 summarizes the needed hypotheses for the main results presented in the next sections. SLLN stands for Strong Law of Large Numbers, LDP for Large Deviation Principle and R.F. for Rate Function (of the large deviation principles).

Let us comment this table a bit. First, assumption (H1) is not necessary to study the flows τ and ϕ , but it is necessary to obtain a flow constant $\nu(\vec{v})$ which does not depend on the height h , and moreover it is natural when we interpret our system as a model for porous media. All the other assumptions are optimal concerning τ (cf. Remarks 3.6, 3.11 and 5.7) except perhaps (F3) (see Remark 4.10). In addition, our assumptions are also essentially optimal concerning ϕ (cf. Remarks 3.11 and 3.21) except perhaps assumption (F4) (see Remark 6.5) and (F3), for the same reason as for τ . Finally, assumption (H3), used to obtain results for cylinders which are not straight, is certainly far from optimality (see Remark 6.3). This assumption gives results only for “flat cylinders.”

Table 2
Hypotheses need for the main results to hold

	Existence of the R.F.	Positivity of the R.F.	SLLN for τ (and flat ϕ)		SLLN for straight ϕ		LDP for τ (and flat ϕ)	LDP for straight ϕ
			$0 \in A$	$0 \notin A$	$0 \in A$	$0 \notin A$		
(F1)		×					×	×
(F2)		×	×	×	×	×	×	×
(F3)				×		×	×	×
(F4)							×	×
(F5)							×	
(H1)	×	×	×	×	×	×	×	×
(H2)					×	×		×
(H3)			(×)	(×)			(×)	

3.3. Results concerning τ

First, we will extend the definition of $\nu(\cdot)$ in all directions.

Proposition 3.5 (Definition of ν). *We suppose that **(F2)** and **(H1)** hold. For every non-degenerate hyperrectangle A , the limit*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)}$$

exists and depends on the direction of \vec{v} , one of the two unit vectors orthogonal to $\text{hyp}(A)$, and not on A itself nor h . We denote it by $\nu(\vec{v})$ (the dependence in F and d is implicit).

Remark 3.6. *We chose to define simply the flow constant ν from the convergence of the rescaled expectations. Having made this choice, condition **(F2)** is necessary for the limit to be finite. Indeed, for most orientations, there exists two vertices $x \in A_1^h$ and $y \in A_2^h$ which are neighbours in \mathbb{Z}^d . Thus, the corresponding edge must belong to any cutsets, and this implies that the mean of $\tau(nA, h(n))$ is finite only if **(F2)** holds. Notice however that with some extra work, one could probably define a flow constant without any moment condition as in [13], Section 2.*

The following proposition states some basic properties of ν , and notably settles the question of its positivity.

Proposition 3.7 (Properties of ν). *Suppose that **(F2)** and **(H1)** hold. Let $\delta = \inf\{\lambda \mid \mathbb{P}(t(e) \leq \lambda) > 0\}$. Then:*

- (i) *For every unit vector \vec{v} , $\nu(\vec{v}) \geq \delta \|\vec{v}\|_1$.*
- (ii) *If $F(\delta) < 1 - p_c(d)$, then $\nu(\vec{v}) > \delta \|\vec{v}\|_1$ for all unit vector \vec{v} . In the case $\delta = 0$, the previous implication is in fact an equivalence.*
- (iii) *For every unit vector \vec{v} , and every non-degenerate hyperrectangle A orthogonal to \vec{v} ,*

$$\nu(\vec{v}) \leq \inf_{n \in \mathbb{N}} \left\{ \frac{\mathbb{E}(t(e))K(d, A)}{n} + \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \right\},$$

where $K(d, A) = c(d)\mathcal{H}^{d-2}(\partial A)/\mathcal{H}^{d-1}(A)$, and $c(d)$ is a constant depending only on the dimension d .

We will derive the law of large numbers for $\tau(A, h)$ in big cylinders $\text{cyl}(A, h)$ as a consequence of an almost subadditive argument.

Theorem 3.8 (LLN for τ). *We suppose that **(F2)** and **(H1)** hold. Then*

$$\lim_{n \rightarrow \infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = \nu(\vec{v}) \quad \text{in } L^1.$$

Moreover, if $0 \in A$, where 0 denotes the origin of \mathbb{Z}^d , or if **(F3)** holds,

$$\lim_{n \rightarrow \infty} \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = v(\vec{v}) \quad \text{a.s.}$$

Propositions 3.5, 3.7 and Theorem 3.8 will be proven in Section 4.3. Concerning large deviations, we will show two results: the first gives the speed of decay of the probability that the rescaled flow τ is abnormally small, and the second one states a large deviation principle for the rescaled variable τ .

The estimate of lower large deviations is the following. Notice that Theorem 3.3 is the key to obtain the relevant condition $F(0) < 1 - p_c(d)$.

Theorem 3.9 (Lower deviations for τ). *Suppose that **(F1)**, **(F2)** and **(H1)** hold. Then for every $\varepsilon > 0$ there exists a positive constant $C(d, F, \varepsilon)$ such that for every unit vector \vec{v} and every non-degenerate hyperrectangle A orthogonal to \vec{v} , there exists a constant $\tilde{C}(d, F, A, \varepsilon)$ (possibly depending on all the parameters d, F, A, ε) such that*

$$\mathbb{P}\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq v(\vec{v}) - \varepsilon\right) \leq \tilde{C}(d, F, A, \varepsilon) \exp(-C(d, F, \varepsilon) \mathcal{H}^{d-1}(A) n^{d-1}).$$

Now we can state a large deviation principle.

Theorem 3.10 (LDP for τ). *Suppose that **(F1)**, **(F5)** and **(H1)** hold. Then for every unit vector \vec{v} and every non-degenerate hyperrectangle A orthogonal to \vec{v} , the sequence*

$$\left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N}\right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$. Moreover, we know that $\mathcal{J}_{\vec{v}}$ is convex on \mathbb{R}^+ , infinite on $[0, \delta \|\vec{v}\|_1[\cup]v(\vec{v}), +\infty[$, where $\delta = \inf\{\lambda \mid \mathbb{P}(t(e) \leq \lambda) > 0\}$, equal to 0 at $v(\vec{v})$, and if $\delta \|\vec{v}\|_1 < v(\vec{v})$ we also know that $\mathcal{J}_{\vec{v}}$ is finite on $]\delta \|\vec{v}\|_1, v(\vec{v})]$, continuous and strictly decreasing on $[\delta \|\vec{v}\|_1, v(\vec{v})]$ and strictly positive on $[\delta \|\vec{v}\|_1, v(\vec{v})[$.

Remark 3.11. *Notice that, from Proposition 3.7, assumption **(F1)** is necessary to have positive asymptotic rescaled maximal flow, and thus to give a sense to the study of lower large deviations. Moreover, Theorem 3.10 is interesting only if $v(\vec{v}) > \delta \|\vec{v}\|_1$. Proposition 3.7 states that it is the case at least if $F(\delta) < 1 - p_c(d)$, and in the case $\delta = 0$, this condition is optimal. We do not know the optimal condition on $F(\delta)$ when $\delta \neq 0$.*

Remark 3.12. *In [20], Wouts shows a similar lower large deviations result in the context of the dilute Ising model. More precisely, for every temperature T , a Gibbs measure $\Phi_{n,T}$ with i.i.d. non-negative, bounded random interactions $(J_e)_{e \in \mathbb{E}^d}$ is constructed on the set of configurations $\{0, 1\}^{E_n}$, where E_n is the set of edges of a cube B_n of length n , and 0 (resp. 1) means the edge is closed (resp. open). Wouts defines the quenched surface tension in this box as the normalized logarithm of the $\Phi_{n,T}$ -probability of the event that there is a disconnection between the upper and lower parts of the boundary of B_n . Then, Wouts shows that for Lebesgue-almost every temperature T , the quenched surface tension satisfies a large deviation principle at surface order. A remarkable feature of this work is that the proof, quite simple, relies on a concentration property that avoids the use of any estimate like that of Theorem 3.3. A similar treatment could be done in our setting, with the value of $F(0)$ playing the role of the inverse temperature. Of course, this is quite artificial and unsatisfactory for our purpose, since one would not obtain any information for a precise distribution function F , but rather for almost all distributions of the form $p\delta_0 + (1-p)dF$, $p \in [0, 1]$. Still, it seems to us that Wouts' method deserves further investigation.*

3.4. Results concerning ϕ in flat cylinders

Under the additional assumption that the cylinder we study is sufficiently flat, in the sense that we suppose $\lim_{n \rightarrow \infty} h(n)/n = 0$, we can transport results from τ to ϕ even in non-straight boxes, because the behaviour of these two variables are very similar in that case. We obtain the following two results:

Theorem 3.13 (Lower deviations for flat ϕ). *Suppose (F1), (F2), (H1) and (H3) hold. Then for every $\varepsilon > 0$ there exists a positive constant $C'(d, F, \varepsilon)$ such that for every unit vector \vec{v} , every non-degenerate hyperrectangle A orthogonal to \vec{v} , there exists a constant $\tilde{C}'(d, F, A, h, \varepsilon)$ (possibly depending on all the parameters d, F, A, h, ε) such that*

$$\mathbb{P}\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq v(\vec{v}) - \varepsilon\right) \leq \tilde{C}'(d, F, A, h, \varepsilon) \exp(-C'(d, F, \varepsilon)\mathcal{H}^{d-1}(A)n^{d-1}).$$

Corollary 3.14 (of Theorem 3.10, LDP for flat ϕ). *Suppose (F1), (F5), (H1) and (H3) hold. Then, for every unit vector \vec{v} and every non-degenerate hyperrectangle A orthogonal to \vec{v} , the sequence*

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N}\right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$ (the same as in Theorem 3.10).

Remark 3.15. *Theorem 3.13 will be proven exactly as Theorem 3.9, using the fact that the convergence of $\mathbb{E}[\tau(nA, h(n))]/\mathcal{H}^{d-1}(nA)$ implies the convergence of $\mathbb{E}[\phi(nA, h(n))]/\mathcal{H}^{d-1}(nA)$ under the hypotheses (F2) and (H3). Corollary 3.14 will be proven using the exponential equivalence of the rescaled variables $\tau(nA, h(n))$ and $\phi(nA, h(n))$ under hypotheses (F5) and (H3).*

3.5. Results concerning ϕ in straight but high cylinders

We shall say that a hyperrectangle A is *straight* if it is of the form $\prod_{i=1}^{d-1} [0, a_i] \times \{0\}$ ($a_i \in \mathbb{R}_*^+$ for all i , so a straight hyperrectangle is non-degenerate). In particular, Theorem 3.4 implies that for a straight hyperrectangle A , for every function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying $\lim_{n \rightarrow \infty} h(n) = +\infty$ and $\log h(n) \leq n^{1-\eta}$ for some $0 < \eta \leq 1$, we have

$$\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = v((0, \dots, 0, 1)) \quad \text{a.s. and in } L^1.$$

We obtain three results for the rescaled variable ϕ in straight cylinders. Using subadditivity and symmetry arguments, we can prove the law of large numbers for ϕ in straight boxes under a minimal moment condition, and the hypothesis (H2) on h .

Theorem 3.16 (LLN for straight ϕ). *Suppose that (F2), (H1) and (H2) hold, and that A is a straight hyperrectangle. Then*

$$\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} = v(\vec{v}_0) \quad \text{a.s. and in } L^1,$$

where $\vec{v}_0 = (0, \dots, 0, 1)$.

Under the additional condition of an exponential moment for F , we can prove a large deviation principle for ϕ in straight boxes.

Theorem 3.17 (LDP for straight ϕ). *Suppose (F1), (F4), (H1) and (H2) hold. Then for every straight hyperrectangle A , the sequence*

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N}\right)$$

satisfies a large deviation principle of speed $\mathcal{H}^{d-1}(nA)$ with the good rate function $\mathcal{J}_{\vec{v}}$ with $\vec{v} = (0, \dots, 0, 1)$ (the same as in Theorem 3.10).

We also obtain a result similar to Theorem 3.9 for ϕ .

Theorem 3.18 (Lower deviations for straight ϕ). *Suppose (F1), (F2), (H1) and (H2) hold. Then, for every $\varepsilon > 0$ there exists a positive constant $C''(d, F, \varepsilon)$ such that for every straight hyperrectangle A , there exists a strictly positive constant $\tilde{C}''(d, F, A, h, \varepsilon)$ (possibly depending on all the parameters d, F, A, h and ε) such that*

$$\mathbb{P}\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \nu((0, \dots, 0, 1)) - \varepsilon\right) \leq \tilde{C}''(d, F, A, h, \varepsilon) \exp(-C''(d, F, \varepsilon) \mathcal{H}^{d-1}(A) n^{d-1}).$$

This result answers question (2.25) in [14]. We have to comment these three theorems by some remarks.

Remark 3.19. *We decided to state the law of large numbers (Theorem 3.16) in the case where the origin of the graph belongs to the straight hyperrectangle A since it is the case in the literature (see [14,22]). We also could state the same result for a hyperrectangle A of the form $\prod_{i=1}^{d-1} [a_i, b_i] \times \{c\}$ for real numbers $a_i < b_i$ and c . In this case, exactly as in Theorem 3.8, the same hypotheses (F2), (H1) and (H2) are required to obtain the convergence of $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ in L^1 , but we need moreover the stronger hypothesis (F3) to obtain the a.s. convergence of the variable if the origin of the graph does not belong to A .*

Remark 3.20. *The proofs of these three theorems are a little bit tangled. It comes from our willingness to obtain the best hypotheses on F each time. Indeed, we stress the fact that Theorem 3.18 is not a simple consequence of Theorem 3.17 when (F4) does not hold. In fact, we will prove first a proposition, Proposition 6.1, that will lead to Theorem 3.16 and Theorem 3.17 independently. Theorem 3.18 will be proven exactly as Theorems 3.9 and 3.13, using Theorem 3.16.*

Remark 3.21. *Actually the condition (H2), i.e. $\lim_{n \rightarrow \infty} \log h(n)/n^{d-1} = 0$, is essentially the good one. For instance, if $A = [0, 1]^{d-1} \times \{0\}$, $h(n) \geq \exp(kn^{d-1})$ for a constant k sufficiently large and $F(0) > 0$, then the maximal flow $\phi(nA, h(n))$ eventually equals 0, almost surely. Indeed if the n^{d-1} vertical edges of the cylinder that intersect one fixed horizontal plane have all 0 for capacity then $\phi(nA, h(n)) = 0$. By independence and translation invariance of the model, we obtain*

$$\mathbb{P}[\phi(nA, h(n)) \neq 0] \leq [1 - F(0)n^{d-1}]^{2\exp(kn^{d-1})},$$

which is summable for k large enough, and so we conclude by the Borel–Cantelli lemma.

Remark 3.22. *Notice that our setting in Theorem 3.16 is not entirely similar to the one of [22] since each side of nA grows at the same speed, whereas Zhang considers $A = \prod_{i=1}^{d-1} [0, k_i] \times \{0\}$ and lets all the k_i go to infinity, possibly at different speeds. In the case we consider, we improve the height and moment conditions in Theorem 3 of [22] to the relevant one, and so partially answer the question contained in Remark (2.17) and question (2.24) in [14]. See also Remark 6.4.*

4. Lower large deviations for τ and ϕ and law of large numbers for τ

In Section 4.2, we derive the crucial deviation inequalities from their means of the flows τ and ϕ . This will lead to the law of large numbers for τ rescaled in Section 4.5, and the deviations from ν of τ and flat ϕ rescaled in Section 4.6. Of course, we need to define properly ν in any direction, and this is done in Section 4.3, whereas properties of ν are proven in Section 4.4, using a combinatorial result stated in Section 4.1.

4.1. Minimal size of a cutset

For every hyperrectangle A , we denote by $\mathcal{N}(A, h)$ the minimal number of edges in A that can disconnect A_1^h from A_2^h in $\text{cyl}(A, h)$, if A_1^h and A_2^h are non-empty. The following lemma gives the asymptotic order of $\mathcal{N}(nA, h(n))$ when n goes to infinity.

Lemma 4.1. *Let \vec{v} be a unitary vector. Then for all hyperrectangle A orthogonal to \vec{v} , for all function $h : \mathbb{N} \rightarrow]2\sqrt{d}, +\infty[$, and for every $n \in \mathbb{N}$ such that $l_{\min}(nA) > \sqrt{d}$,*

$$\left| \frac{\mathcal{N}(nA, h(n))}{\mathcal{H}^{d-1}(nA)} - \|\vec{v}\|_1 \right| \leq \frac{d\mathcal{H}^{d-2}(\partial A)}{n\mathcal{H}^{d-1}(A)}.$$

Proof. We introduce some definitions. For A a hyperrectangle orthogonal to \vec{v} , we denote by $P_i(A)$ the orthogonal projection of A on the i th hyperplane of coordinates, i.e., the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_i = 0\}$. We have the property

$$\frac{\sum_{i=1}^d \mathcal{H}^{d-1}(P_i(A))}{\mathcal{H}^{d-1}(A)} = \|\vec{v}\|_1.$$

Indeed, $\mathcal{H}^{d-1}(P_i(A)) = |v_i| \mathcal{H}^{d-1}(A)$, where $\vec{v} = (v_1, \dots, v_d)$. We define now $E_i(nA)$ the set of edges orthogonal to the i th hyperplane of coordinates that ‘intersect’ the hyperrectangle nA in the following sense:

$$E_i(nA) = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid y_i = x_i + 1 \text{ and } [x, y[\cap nA \neq \emptyset \text{ and } [x, y[\not\subset nA\}.$$

We exclude here the extremity y in the segment $[x, y[$ to avoid problems of non uniqueness of such an edge intersecting nA at a given point. On the one hand, we have a straight path that goes from $(nA)_1^{h(n)}$ to $(nA)_2^{h(n)}$ through each edge of $E_i(nA)$, $i = 1, \dots, d$, except maybe the edges that intersect nA along $\partial(nA)$, and these paths are disjoint, so a set of edges that disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$ must cut each one of these paths, thus

$$\mathcal{N}(nA, h(n)) \geq \sum_{i=1}^d (\mathcal{H}^{d-1}(P_i(nA)) - \mathcal{H}^{d-2}(\partial P_i(nA))) \geq \left(\|\vec{v}\|_1 - d \frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)} \right) \mathcal{H}^{d-1}(nA).$$

On the other hand, each path from $(nA)_1^{h(n)}$ to $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$ must go through nA and so contains an edge of one of the $E_i(nA)$, $i = 1, \dots, d$. It suffices then to remove all the edges in the union of the sets $E_i(nA)$, $i = 1, \dots, d$, to disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$, and so

$$\mathcal{N}(nA, h(n)) \leq \sum_{i=1}^d (\mathcal{H}^{d-1}(P_i(nA)) + \mathcal{H}^{d-2}(\partial P_i(nA))) \leq \left(\|\vec{v}\|_1 + d \frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)} \right) \mathcal{H}^{d-1}(nA).$$

We conclude that

$$\left| \frac{\mathcal{N}(nA, h(n))}{\mathcal{H}^{d-1}(nA)} - \|\vec{v}\|_1 \right| \leq d \frac{\mathcal{H}^{d-2}(\partial(nA))}{\mathcal{H}^{d-1}(nA)} = \frac{d\mathcal{H}^{d-2}(\partial A)}{n\mathcal{H}^{d-1}(A)}. \quad \square$$

4.2. Lower deviations of the maximal flows from their means

Let A be a non-degenerate hyperrectangle. In this section, we obtain deviation inequalities for $\phi(A, h)$ and $\tau(A, h)$ from their means. These inequalities, stated below in Proposition 4.3, give the right speed for the lower large deviation probabilities as soon as the convergence of the rescaled expectation of the variables is known. This will be used in Section 4.3 to prove the law of large numbers for τ , but above all this will be essential to show the positivity of the rate function for lower large deviations in Section 5.4. This positivity will be used to prove Theorem 3.16 in Section 6.2.

To get this result, we state below in Proposition 4.2 a slight modification of Zhang’s Theorem 3.3, which allows to control the number of edges in a cut of minimal capacity. Notice that in this precise form, Proposition 4.2 is almost a strict analogue for flow problems of Proposition 5.8 in [13], the latter being of utmost importance in the study of First Passage Percolation.

We introduce the following notation: $E_{\tau(A, h)}$ (resp. $E_{\phi(A, h)}$) is a cut whose capacity achieves the minimum in the dual definition (1) of $\tau(A, h)$ (resp. $\phi(A, h)$). If there are more than one cut achieving the minimum, we use a deterministic method to select a unique one with the minimum number of edges among these. Recall also that for a

hyperrectangle A , we defined $l_{\min}(A)$ as the “smallest length of A ,” i.e. the number t such that A is the isometric image of $\prod_{i=1}^{d-1} [0, t_i] \times \{0\}$, with $t = t_1 \leq \dots \leq t_{d-1}$.

Proposition 4.2. *Suppose that (F1) holds, i.e. $F(0) < 1 - p_c(d)$. Then, there are constants $\varepsilon(F, d)$, $C_1(F(0), d)$, $C_2(F(0), d)$ and $t_0(F(0), d)$, such that, for every $s \in \mathbb{R}$, every non-degenerate hyperrectangle A such that $l_{\min}(A) \geq t_0$, and every $h > 2\sqrt{d}$, we have*

$$\mathbb{P}(\text{card}(E_{\tau(A,h)}) \geq s \text{ and } \tau(A, h) \leq \varepsilon(F, d)s) \leq C_1(F(0), d)e^{-C_2(F(0), d)s}$$

and

$$\mathbb{P}(\text{card}(E_{\phi(A,h)}) \geq s \text{ and } \phi(A, h) \leq \varepsilon(F, d)s) \leq C_1(F(0), d)he^{-C_2(F(0), d)s}.$$

Furthermore, the constant ε depends on F only on the neighbourhood of 0 in the sense that if $(F_n)_{n \in \mathbb{N}}$ is a sequence of possible distribution functions for $t(e)$, which coincide on $[0, \eta]$ for some $\eta > 0$, then one can take the same constants ε , C_1 and C_2 for the whole sequence in the above inequalities.

Proof. First notice that when $d = 2$, this is a consequence of Proposition 5.8 in [13], through duality. In fact, when $d = 3$, all the hard work has been done by Zhang, giving Theorem 3.3, so we only stress the minor differences for the reader who would like to check how one goes from the proof of Theorem 3.3 (i.e., Theorem 1 in [22]) to Proposition 4.2, and we rely heavily on the proof and notations of [22].

The first thing is to see that one can perform the renormalization argument of Section 2 of [22]. To do this for $\tau(A, h)$, replace ∞ by A_2^h and the box $\mathbf{B}(\mathbf{k}, m)$ by A_1^h . For $\phi(A, h)$, replace ∞ by $T(A, h)$ and $\mathbf{B}(\mathbf{k}, m)$ by $B(A, h)$. For both $\tau(A, h)$ and $\phi(A, h)$ also, one requires that all the connectedness properties happen “in $\text{cyl}(A, h)$.” Then, the construction of the linear cutset is identical, except for one thing: when $B_t(u)$ is a block of the “block cutset” such that $\overline{B}_t(u)$ intersects $\partial \text{cyl}(A, h)$, it has a property slightly different than the “blocked property” of Zhang. Define $\overline{B}'_t(u)$ to be the set of t -cubes which are \mathbf{L}^d -neighbours of the cubes in $\overline{B}_t(u)$. Let us say that a set of vertices V_0 of \mathbb{Z}^d is of *smallest length* t if there is a hyperrectangle H in \mathbb{R}^d , isometric image of $[0, t]^{d-1} \times \{0\}$, such that for each edge e of \mathbb{Z}^d intersecting H , there is an endpoint of e which belongs to V . Now, let us say that a block $B_t(u)$ has a “blocking surface property” if either one of the following holds:

- (i) there are two subsets of vertices V_1 and V_2 of smallest length $t/2$ in $\overline{B}'_t(u)$ which cannot be connected by an open path in $\overline{B}'_t(u)$,
- (ii) or there are a subset of vertices V_1 of smallest length $t/2$ and an open path γ connecting $B_t(u)$ to $\overline{B}_t(u)$ in $\overline{B}'_t(u)$ such that γ and V_1 cannot be connected by an open path in $\overline{B}'_t(u)$.

Then, if A is of *smallest length* larger than t , and if $B_t(u)$ is a block of the “block cutset” such that $\overline{B}_t(u)$ intersects $\partial \text{cyl}(A, h)$, it has a “blocking surface property.” Now, it is easy to see, using the same arguments as Zhang from [9], Section 7, that the probability that $\overline{B}_t(u)$ has a “blocking surface property” decays exponentially to zero as t goes to infinity, when $F(0) < 1 - p_c(d)$. This shows that the renormalization works if A is of smallest length larger than some $t_0(F(0), d)$, see the choice of t above (5.26) in [22]. Notice that to prove Lemma 8 in [22], Zhang appeals to Lemma 7.104 in [9] whereas it seems better to see this as a direct consequence of the fact that percolation in slabs occurs.

The rest of the proof is almost identical. Note however that when considering $\tau(A, h)$, there is no need to put a sum over the possible intersections of the cutset with \mathbf{L} (in (5.4), and before (5.26)), since we know there is a constant $R(d)$ such that there is a set of $R(d)$ edges that a cut needs to intersect (it is essentially “pinned” at the border of A). This is why we do not have any condition on the height in the first inequality of Proposition 4.2, and why on the contrary h appears in our second inequality: for $\phi(A, h)$, we only know a set of h edges that a cut needs to intersect.

Finally, notice that we do not have any condition of moment on F , since we are bounding the probability that $\{\text{card}(E_{\tau(A,h)}) \geq k\}$ and $\{\tau(A, h) \leq \varepsilon k\}$ occur, not only $\mathbb{P}(\text{card}(E_{\tau(A,h)}) \geq k)$, and Zhang uses the moment condition only to bound $\mathbb{P}(\tau(A, h) \leq \varepsilon k)$. Also, the last statement on the constants is easily seen by tracking the choice of ε (see (5.2) and below (5.10)). \square

Thanks to Proposition 4.2 and general deviation inequalities due to [5], we obtain the following deviation result for the maximal flows $\tau(nA, h(n))$ and $\phi(nA, h(n))$.

Proposition 4.3. *Suppose that hypotheses (F1) and (F2) occur. Then, for any $\eta \in]0, 1]$, there are positive constants $C(\eta, F, d)$, $C_3(F(0), d)$ and $t_0(F, d)$ such that, for every $n \in \mathbb{N}^*$, every non-degenerate hyperrectangle A such that nA has smallest length at least t_0 :*

$$\mathbb{P}(\tau(nA, h(n)) \leq \mathbb{E}(\tau(nA, h(n)))(1 - \eta)) \leq C_3(F(0), d)e^{-C(\eta, F, d)\mathbb{E}(\tau(nA, h(n)))} \tag{2}$$

and

$$\mathbb{P}(\phi(nA, h(n)) \leq \mathbb{E}(\phi(nA, h(n)))(1 - \eta)) \leq C_3(F(0), d)h(n)e^{-C(\eta, F, d)\mathbb{E}(\phi(nA, h(n)))}. \tag{3}$$

Proof. To shorten the notations, define $\tau_n = \tau(nA, h(n))$ and $\phi_n = \phi(nA, h(n))$. We prove the result for τ_n , the variant for ϕ_n being entirely similar. Since $\mathbb{P}(\tau_n \leq \mathbb{E}(\tau_n)(1 - \eta))$ is a decreasing function of η , it is enough to prove the result for all η less or equal to some absolute $\eta_0 \in]0, 1[$. We use this remark to exclude the case $\eta = 1$ in our study, thus, from now on, let η be a fixed real number in $]0, 1[$.

Fix A a non-degenerate hyperrectangle, and n such that nA has smallest length at least $t_0(F, d)$, with t_0 as in Proposition 4.2. We order the edges in $\text{cyl}(nA, h(n))$ as e_1, \dots, e_{m_n} . For every hyperrectangle A , we denote by $\mathcal{N}(A, h)$ the minimal number of edges in A that can disconnect A_1^h from A_2^h in $\text{cyl}(A, h)$, as in Section 4.1. For any real number $r \geq \mathcal{N}(nA, h(n))$, we define

$$\tau_n^r = \min\{V(E) \text{ s.t. } \text{card}(E) \leq r \text{ and } E \text{ cuts } (nA)_1^{h(n)} \text{ from } (nA)_2^{h(n)} \text{ in } \text{cyl}(nA, h(n))\}.$$

Now, suppose that $F(0) < 1 - p_c(d)$, let ε , C_1 and C_2 be as in Proposition 4.2, and define $r = (1 - \eta)\mathbb{E}(\tau_n)/\varepsilon$. Suppose first that $r < \mathcal{N}(nA, h(n))$. Then,

$$\begin{aligned} \mathbb{P}(\tau_n \leq (1 - \eta)\mathbb{E}(\tau_n)) &= \mathbb{P}(\tau_n \leq (1 - \eta)\mathbb{E}(\tau_n) \text{ and } \text{card}(E_{\tau_n}) \geq (1 - \eta)\mathbb{E}(\tau_n)/\varepsilon) \\ &\leq C_1 e^{-C_2(1-\eta)\mathbb{E}(\tau_n)/\varepsilon} \end{aligned}$$

from Proposition 4.2, and the desired inequality is obtained. Suppose now that $r \geq \mathcal{N}(nA, h(n))$. Then,

$$\begin{aligned} \mathbb{P}(\tau_n \leq (1 - \eta)\mathbb{E}(\tau_n)) &\leq \mathbb{P}(\tau_n \leq (1 - \eta)\mathbb{E}(\tau_n) \text{ and } \tau_n^r \neq \tau_n) + \mathbb{P}(\tau_n^r \leq (1 - \eta)\mathbb{E}(\tau_n)) \\ &\leq C_1 e^{-C_2 r} + \mathbb{P}(\tau_n^r \leq (1 - \eta)\mathbb{E}(\tau_n^r)) \end{aligned} \tag{4}$$

from Proposition 4.2 and the fact that $\tau_n^r \geq \tau_n$. Now, we truncate our variables $t(e)$. Let a be a positive real number to be chosen later, and define $\tilde{t}(e) = t(e) \wedge a$. Let

$$\tilde{\tau}_n^r = \min\left\{\sum_{e \in E} \tilde{t}(e) \text{ s.t. } \text{card}(E) \leq r \text{ and } E \text{ cuts } (nA)_1^{h(n)} \text{ from } (nA)_2^{h(n)} \text{ in } \text{cyl}(nA, h(n))\right\}.$$

Notice that $\tilde{\tau}_n^r \leq \tau_n^r$. We shall denote by $E_{\tilde{\tau}_n^r}$ a cut whose capacity achieves the minimum in the definition of $\tilde{\tau}_n^r$. If there are more than one cut achieving the minimum, we use a deterministic method to select a unique one with the minimum number of edges among these. Then

$$\begin{aligned} 0 \leq \mathbb{E}(\tau_n^r) - \mathbb{E}(\tilde{\tau}_n^r) &\leq \mathbb{E}\left[\sum_{e \in E_{\tilde{\tau}_n^r}} t(e) - \sum_{e \in E_{\tilde{\tau}_n^r}} \tilde{t}(e)\right] \leq \mathbb{E}\left[\sum_{e \in E_{\tilde{\tau}_n^r}} t(e)\mathbb{1}_{t(e) \geq a}\right] \\ &= \sum_{i=1}^{m_n} \mathbb{E}(t(e_i)\mathbb{1}_{t(e_i) \geq a}\mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}}) = \sum_{i=1}^{m_n} \mathbb{E}\left[\mathbb{E}(t(e_i)\mathbb{1}_{t(e_i) \geq a}\mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}} \mid (t(e_j))_{j \neq i})\right]. \end{aligned}$$

Now, when $(t(e_j))_{j \neq i}$ is fixed, $t(e_i) \mapsto \mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}}$ is a non-increasing function and $t(e_i) \mapsto t(e_i)\mathbb{1}_{t(e_i) \geq a}$ is of course non-decreasing. Furthermore, since the variables $(t(e_i))$ are independent, the conditional expectation $\mathbb{E}(\cdot \mid (t(e_j))_{j \neq i})$

corresponds to expectation over $t(e_i)$, keeping $(t(e_j))_{j \neq i}$ fixed. Thus, Chebyshev's association inequality (see [12], p. 43) implies

$$\begin{aligned} \mathbb{E}(t(e_i) \mathbb{1}_{t(e_i) \geq a} \mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}} | (t(e_j))_{j \neq i}) &\leq \mathbb{E}(t(e_i) \mathbb{1}_{t(e_i) \geq a} | (t(e_j))_{j \neq i}) \mathbb{E}(\mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}} | (t(e_j))_{j \neq i}) \\ &= \mathbb{E}(t(e_1) \mathbb{1}_{t(e_1) \geq a}) \mathbb{E}(\mathbb{1}_{e_i \in E_{\tilde{\tau}_n^r}} | (t(e_j))_{j \neq i}). \end{aligned}$$

Thus,

$$0 \leq \mathbb{E}(\tau_n^r) - \mathbb{E}(\tilde{\tau}_n^r) \leq \mathbb{E}(t(e_1) \mathbb{1}_{t(e_1) \geq a}) \mathbb{E}(\text{card}(E_{\tilde{\tau}_n^r})) \leq r \mathbb{E}(t(e_1) \mathbb{1}_{t(e_1) \geq a}). \tag{5}$$

Now, since F has a finite moment of order 1, we can choose $a(\eta, F, d)$ such that

$$\frac{1 - \eta}{\varepsilon} \mathbb{E}(t(e_1) \mathbb{1}_{t(e_1) \geq a}) \leq \frac{\eta}{2},$$

to get

$$\begin{aligned} \mathbb{E}(\tau_n^r) - \mathbb{E}(\tilde{\tau}_n^r) &\leq \frac{\eta}{2} \mathbb{E}(\tau_n) \leq \frac{\eta}{2} \mathbb{E}(\tau_n^r), \\ \mathbb{P}(\tau_n^r \leq (1 - \eta) \mathbb{E}(\tau_n^r)) &\leq \mathbb{P}\left(\tilde{\tau}_n^r \leq \mathbb{E}(\tilde{\tau}_n^r) - \frac{\eta}{2} \mathbb{E}(\tau_n^r)\right). \end{aligned} \tag{6}$$

Now, we shall use Corollary 3 in [5]. To this end, we need some notation. We take \tilde{t}' an independent collection of capacities with the same law as $\tilde{t} = (\tilde{t}(e_i))_{i=1, \dots, m_n}$. For each edge $e_i \in \text{cyl}(A, h)$, we denote by $\tilde{t}^{(i)}$ the collection of capacities obtained from \tilde{t} by replacing $\tilde{t}(e_i)$ by $\tilde{t}'(e_i)$, and leaving all other coordinates unchanged. Define

$$V_- := \mathbb{E} \left[\sum_{i=1}^{m_n} (\tilde{\tau}_n^r(t) - \tilde{\tau}_n^r(t^{(i)}))_-^2 \middle| t \right],$$

where $\tilde{\tau}_n^r(t)$ is the maximal flow through $\text{cyl}(nA, h(n))$ when capacities are given by t . We shall denote by $R_{\tilde{\tau}_n^r}$ the intersection of all the cuts whose capacity achieves the minimum in the definition of $\tilde{\tau}_n^r$. Observe that

$$\tilde{\tau}_n^r(t^{(i)}) - \tilde{\tau}_n^r(t) \leq (\tilde{t}'(e_i) - \tilde{t}(e_i)) \mathbb{1}_{e_i \in R_{\tilde{\tau}_n^r}},$$

and thus,

$$V_- \leq a^2 \mathbb{E}(\text{card}(R_{\tilde{\tau}_n^r})) \leq a^2 r = a^2 (1 - \eta) \mathbb{E}(\tau_n) / \varepsilon.$$

Thus, Corollary 3 in [5] implies that, for every $\eta \in]0, 1[$,

$$\mathbb{P}\left(\tilde{\tau}_n^r \leq \mathbb{E}(\tilde{\tau}_n^r) - \frac{\eta}{2} \mathbb{E}(\tau_n^r)\right) \leq \exp\left(-\frac{\mathbb{E}(\tau_n^r)^2 \eta^2 \varepsilon}{16a^2(1 - \eta) \mathbb{E}(\tau_n)}\right) \leq \exp\left(-\frac{\mathbb{E}(\tau_n) \eta^2 \varepsilon}{16a^2(1 - \eta)}\right),$$

which, with inequalities (6) and (4) finishes the proof of inequality (2). □

Remark 4.4. *If we suppose the existence of an exponential moment for F , then one can get concentration inequalities: there are positive constants D_1 and D_2 , depending only on F and d and such that, for every hyperrectangle A , every $h > 0$ and every $u > 0$,*

$$\mathbb{P}(|\tau(A, h) - \mathbb{E}(\tau(A, h))| \geq u) \leq D_1 \exp\left(-\frac{u^2}{D_2 \mathcal{H}^{d-1}(A)}\right) + D_1 \exp\left(-\frac{1}{D_2} \mathcal{H}^{d-1}(A)\right).$$

Furthermore, for every $h \leq \exp(\mathcal{H}^{d-1}(A))$ and every $u > 0$,

$$\mathbb{P}(|\phi(A, h) - \mathbb{E}(\phi(A, h))| \geq u) \leq D_1 \exp\left(-\frac{u^2}{D_2 \mathcal{H}^{d-1}(A)}\right) + D_1 \exp\left(-\frac{1}{D_2} \mathcal{H}^{d-1}(A)\right).$$

This can be proved much as in [22], Section 9. It should be noted that these results certainly do not give the right order of the “typical fluctuations,” i.e., fluctuations that occur with a non-negligible probability. Indeed, let S_n be the square

$$S_n = \partial\left(\left[-\frac{1}{2}, n - \frac{1}{2}\right]^{d-1} \times \left\{\frac{1}{2}\right\}\right).$$

We say that a set of edges E “is a cut based on S_n ” if it is finite, and if every closed path in \mathbb{Z}^d which is not contractible to one point in $\mathbb{R}^d \setminus S_n$ has to contain one edge of E . Let \mathcal{E}_n be the set of all sets of edges which are a cut based on S_n and define

$$\tilde{\tau}_n = \inf\{V(E) \mid E \in \mathcal{E}_n\}.$$

Then, mimicking the work of [2], one can prove that the variance of $\tilde{\tau}_n$ is at most of order $C(n^{d-1}/\log n)$ where C is a constant (and there is no reason for this bound to be optimal). It is then very reasonable to think that $\tau(A, h)$ and $\phi(A, h)$ will inherit this property to have “submean” variance, i.e. their typical fluctuations should be small with respect to $(\mathcal{H}^{d-1}(A))^{1/2}$ when the side lengths of A tend to infinity.

Remark also that these concentration inequalities, while they reflect the right order of lower large deviations, do not give the right asymptotic of upper large deviations, which are of volume order. We do not know a simple route to reach that which would avoid the work of [17].

4.3. Asymptotic of $\mathbb{E}(\tau)$

Here, we prove Proposition 3.5, so we suppose that the capacity of the edges is in L^1 . Let us consider two hyperrectangles A, A' which have a common orthogonal unit vector \vec{v} , and two functions $h, h' : \mathbb{N} \rightarrow \mathbb{R}^+$ such that $\lim_{n \rightarrow \infty} h(n) = \lim_{n \rightarrow \infty} h'(n) = +\infty$. We take $n, N \in \mathbb{N}$ such that $N \geq N_0(n)$ with $N_0(n)$ large enough to have $h(N) \geq h'(n) + 1$ and $N \text{ diam}(A) > n \text{ diam}(A')$ for all $N \geq N_0(n)$ (here $\text{diam}(A) = \sup\{\|x - y\|_2 \mid x, y \in A\}$). We define

$$D(n, N) = \{x \in NA \mid d(x, \partial(NA)) > 2n \text{ diam } A'\}.$$

There exists a finite collection of sets $(T(i), i \in I)$ such that each $T(i)$ is a translate of nA' intersecting the set $D(n, N)$, the sets $(T(i), i \in I)$ have pairwise disjoint interiors, and their union $\bigcup_{i \in I} T(i)$ contains the set $D(n, N)$ (see Fig. 1). For all i , there exists a vector \vec{t}_i in \mathbb{R}^d such that $\|\vec{t}_i\|_\infty < 1$ and $T'(i) = T(i) + \vec{t}_i$ is the image of nA' by an integer translation (that leaves \mathbb{Z}^d globally invariant). The cylinders $\text{cyl}(T'(i), h'(n))$ are still included in $\text{cyl}(NA, h(N))$ for all $i \in I$, and the family $(\tau(T'(i), h'(n)), i \in I)$ is identically distributed (but not independent in general). For each i , by the max-flow min-cut theorem, we know that $\tau(T'(i), h'(n))$ is equal to the minimal capacity $V(E) = \sum_{e \in E} t(e)$ of a set of edges $E \subset \text{cyl}(T'(i), h'(n))$ that cuts $T'(i)_1^{h'(n)}$ from $T'(i)_2^{h'(n)}$. For each $i \in I$, let E_i be such a set of edges of minimal capacity, i.e., $\tau(T'(i), h'(n)) = V(E_i)$.

We fix $\zeta = 4d$. Let E_0^1 (resp. E_0^2, E_0) be the set of the edges included in \mathcal{E}_0^1 (resp. $\mathcal{E}_0^2, \mathcal{E}_0$), where we define

$$\mathcal{E}_0^1 = \bigcup_{i \in I} (\mathcal{V}(\text{cyl}(\partial T'(i), +\infty), \zeta) \cap \mathcal{V}(\text{hyp}(NA), \zeta)),$$

$$\mathcal{E}_0^2 = \text{cyl}(NA \setminus D(n, N), \zeta)$$

and

$$\mathcal{E}_0 = \mathcal{E}_0^1 \cup \mathcal{E}_0^2.$$

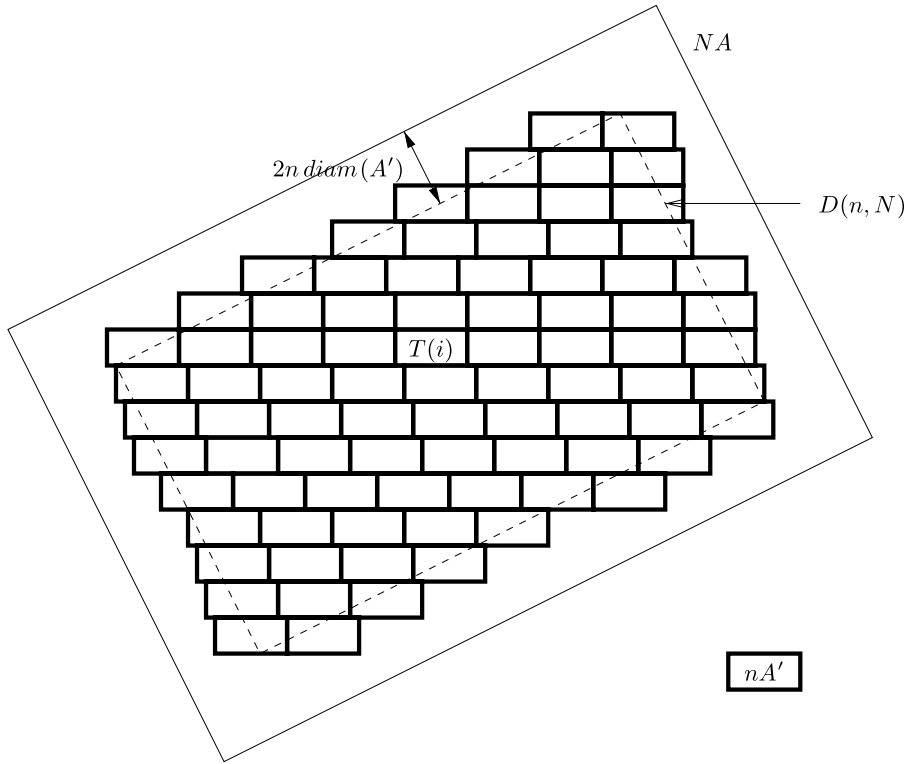


Fig. 1. The hyperplane $\text{hyp}(A)$.

The set of edges $E_0 \cup \bigcup_{i \in I} E_i$ cuts $(NA)_1^{h(N)}$ from $(NA)_2^{h(N)}$ in $\text{cyl}(NA, h(N))$, so

$$\tau(NA, h(N)) \leq V(E_0) + \sum_{i \in I} V(E_i) \leq V(E_0) + \sum_{i \in I} \tau(T'(i), h'(n)). \tag{7}$$

Taking the expectation of (7), we obtain

$$\begin{aligned} \frac{\mathbb{E}(\tau(NA, h(N)))}{\mathcal{H}^{d-1}(NA)} &\leq \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \mathbb{E}(t) + \frac{\text{card}(I) \mathbb{E}(\tau(nA', h'(n)))}{\mathcal{H}^{d-1}(NA)} \\ &\leq \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \mathbb{E}(t) + \frac{\mathbb{E}(\tau(nA', h'(n)))}{\mathcal{H}^{d-1}(nA')}. \end{aligned} \tag{8}$$

There exists a constant $c(d)$ such that

$$\text{card}(E_0^1) \leq c(d) \frac{\mathcal{H}^{d-1}(NA)}{\mathcal{H}^{d-1}(nA')} \mathcal{H}^{d-2}(\partial(nA')) \quad \text{and} \quad \text{card}(E_0^2) \leq c(d) \mathcal{H}^{d-2}(\partial(NA)) \text{diam}(nA'), \tag{9}$$

thus

$$\text{card}(E_0) \leq c(d) \left[\frac{\mathcal{H}^{d-1}(NA)}{\mathcal{H}^{d-1}(nA')} \mathcal{H}^{d-2}(\partial(nA')) + \mathcal{H}^{d-2}(\partial(NA)) \text{diam}(nA') \right], \tag{10}$$

and so

$$\lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} = 0.$$

By sending N to infinity, and then n to infinity, we obtain that

$$\limsup_{N \rightarrow \infty} \frac{\mathbb{E}(\tau(NA, h(N)))}{\mathcal{H}^{d-1}(NA)} \leq \liminf_{n \rightarrow \infty} \frac{\mathbb{E}(\tau(nA', h'(n)))}{\mathcal{H}^{d-1}(nA')}.$$

For $A = A'$ and $h = h'$, we deduce from this inequality that $\lim_{n \rightarrow \infty} \mathbb{E}(\tau(nA, h(n)))/\mathcal{H}^{d-1}(nA)$ exists. For different A, A' , and h, h' , we conclude that this limit does not depend on A and h , but only on the direction of \vec{v} (and on F and d of course). We denote this limit by $\nu(\vec{v})$.

4.4. Properties of ν

Here we prove Proposition 3.7. Lemma 4.1 implies that $\nu(\vec{v}) \geq \delta \|\vec{v}\|_1$ for every unit vector \vec{v} , so we only need to prove assertions (ii) and (iii) in Proposition 3.7. First, let us show that $\nu(\vec{v}) > 0$ is equivalent to $F(0) < 1 - p_c(d)$. We begin by stating the weak triangle inequality for $\nu(\vec{v})$.

Proposition 4.5. *We suppose that (F2) holds. Let (ABC) be a non-degenerate triangle in \mathbb{R}^d and let \vec{v}_A, \vec{v}_B and \vec{v}_C be the exterior normal unit vectors to the sides $[BC], [AC], [AB]$ in the plane spanned by A, B, C . Then*

$$\mathcal{H}^1([AB])\nu(\vec{v}_C) \leq \mathcal{H}^1([AC])\nu(\vec{v}_B) + \mathcal{H}^1([BC])\nu(\vec{v}_A).$$

We do not prove Proposition 4.5 as it is the strict analogue of Proposition 11.2 in [6]. We stress the fact that it uses only the definition of $\nu(\vec{v})$ as the limit of the expectation of the rescaled variable τ , i.e. Proposition 3.5. As in [13], one can extend ν as a function on \mathbb{R}^d as follows

$$\nu(\vec{0}) = 0 \quad \text{and} \quad \forall \vec{u} \neq \vec{0} \quad \nu(\vec{u}) := \|\vec{u}\| \cdot \nu\left(\frac{\vec{u}}{\|\vec{u}\|}\right).$$

Then, Proposition 4.5 shows that ν is convex (and even subadditive). Using this convexity, it is standard to obtain that

$$\exists \vec{v} \neq \vec{0} \quad \text{s.t.} \quad \nu(\vec{v}) = 0 \quad \iff \quad \forall \vec{v} \quad \nu(\vec{v}) = 0,$$

see, for example, (3.15) in [13]. We deduce that

$$F(0) \geq 1 - p_c(d) \quad \iff \quad \exists \vec{v} \neq 0 \quad \text{s.t.} \quad \nu(\vec{v}) = 0 \quad \iff \quad \forall \vec{v} \quad \nu(\vec{v}) = 0. \tag{11}$$

Now we study the case $\delta > 0$. For a given realization of $(t(e), e \in \mathbb{E}^d)$, we define the family of variables $(t'(e), e \in \mathbb{E}^d)$ by $t'(e) = t(e) - \delta$ for all e . Then the variables $(t'(e), e \in \mathbb{E}^d)$ are independent and identically distributed, and if we denote by F' their distribution function, we have $F'(\lambda) = F(\lambda + \delta)$ for all $\lambda \in \mathbb{R}$. We compare the variable $\tau(nA, h(n))$ and the corresponding variable $\tau'(nA, h(n))$ for the capacities $(t'(e))$, for a given hyperrectangle A of normal unit vector \vec{v} , and a given height function h such that $\lim_{n \rightarrow \infty} h(n) = +\infty$. We still denote by $\mathcal{N}(nA, h(n))$ the minimal number of edges that can disconnect $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$. By the max-flow min-cut theorem, we easily obtain that

$$\tau(nA, h(n)) \geq \tau'(nA, h(n)) + \delta \mathcal{N}(nA, h(n))$$

and so

$$\frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \geq \frac{\mathbb{E}(\tau'(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} + \delta \frac{\mathcal{N}(nA, h(n))}{\mathcal{H}^{d-1}(nA)}.$$

Proposition 3.5 and Lemma 4.1 give us that

$$\nu_F(\vec{v}) \geq \nu_{F'}(\vec{v}) + \delta \|\vec{v}\|_1$$

with trivial notations. Now $F(\delta) = F'(0) < 1 - p_c(d)$ implies that $\nu_{F'}(\vec{v}) > 0$, so (ii) is proved.

Finally, from inequalities (8) and (10), with $A = A'$ and letting N go to infinity, we get, for every non-degenerate hyperrectangle A orthogonal to some unit vector \vec{v} :

$$\nu(\vec{v}) \leq \inf_{n \in \mathbb{N}} \left\{ \frac{\mathbb{E}(t(e))c(d)\mathcal{H}^{d-2}(\partial A)}{n\mathcal{H}^{d-1}(A)} + \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \right\}.$$

Thus, Proposition 3.7 is proved.

4.5. Law of large numbers for τ

Here, we prove Theorem 3.8. We begin with the almost sure convergence of $\tau(nA, h(n))/\mathcal{H}^{d-1}(nA)$. To deduce it from the convergence of its expectation, we will use the following result.

Lemma 4.6. *Suppose that hypotheses (F1) and (F2) occur. Then*

$$\liminf_{n \rightarrow \infty} \frac{\tau(nA, h(n)) - \mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \geq 0 \quad \text{a.s.}$$

Proof. It is a simple consequence of Proposition 4.3 and the fact that $\mathbb{E}(\tau(nA, h(n)))$ is equivalent to $\mathcal{H}^{d-1}(nA)\nu(\vec{v})$, using Borel–Cantelli’s lemma. \square

We shall use (7) with $h = h'$ and $A = A'$, i.e. the sets $T'(i)$ are integer translates of nA . We emphasize the dependence on N and n by writing $E_0^i = E_0^i(N, n)$ for $i \in \{1, 2\}$, $I = I(N, n)$ and $T'(i) = T'_{N,n}(i)$. Suppose first that $0 \in A$. Then, we can construct the sets $T'_{N,n}(i)$ in order to have

$$\forall n \geq 1, \forall N' \geq N \geq N_0(n) \quad (T'_{N,n}(i))_{i \in I(N,n)} \subset (T'_{N',n}(i))_{i \in I(N',n)}.$$

We obtain that

$$\forall n \geq 1, \forall N' \geq N \geq N_0(n) \quad E_0^1(N, n) \subset E_0^1(N', n).$$

Thus, the strong law of large numbers for i.i.d. random variables implies, using inequality (9):

$$\limsup_{N \rightarrow \infty} \frac{V(E_0^1)}{\mathcal{H}^{d-1}(NA)} \leq \mathbb{E}(t(e)) \limsup_{N \rightarrow \infty} \frac{\text{card}(E_0^1)}{\mathcal{H}^{d-1}(NA)} \leq \frac{\mathbb{E}(t(e))K(d, A)}{n} \quad \text{a.s.}, \quad (12)$$

where $K(d, A) = c(d)\mathcal{H}^{d-2}(\partial A)/\mathcal{H}^{d-1}(A)$. Moreover, we know (see (9)) that

$$\text{card } E_0^2 \leq c(d)\mathcal{H}^{d-2}(\partial A) \text{diam}(A)N^{d-2}n.$$

Under the assumption (F2), Theorem 4.1 in [11] states that $V(E_0^2(N, n))/\mathcal{H}^{d-1}(NA)$ converges completely to 0, with the definition of the complete convergence given by Gut (Definition (1.1) in [11]). Complete convergence implies almost sure convergence through Borel–Cantelli’s lemma, thus

$$\lim_{N \rightarrow \infty} \frac{V(E_0^2(N, n))}{\mathcal{H}^{d-1}(NA)} = 0 \quad \text{a.s.}$$

Also, we claim that

$$\limsup_{N \rightarrow \infty} \frac{\sum_{i \in I(N,n)} \tau(T'(i), h'(n))}{\mathcal{H}^{d-1}(NA)} = \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \quad \text{a.s.} \quad (13)$$

Indeed, notice that (for n large enough) $\tau(T'(i), h'(n))$ is independent of all the other $\tau(T'(j), h'(n))$ except for at most $3^d - 1$ values of j corresponding to the $T'(j)$ that can intersect $T'(i)$. Thus, (13) follows by partitioning the

sets $T'(j)$ into $3^d - 1$ classes of i.i.d. variables, and then applying the strong law of large numbers for i.i.d. random variables. Thus, for n large enough,

$$\limsup_{N \rightarrow \infty} \frac{\tau(NA, h(N))}{\mathcal{H}^{d-1}(NA)} \leq \frac{\mathbb{E}(t(e))K(d, A)}{n} + \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \quad \text{a.s.}$$

and using Proposition 3.5:

$$\limsup_{N \rightarrow \infty} \frac{\tau(NA, h(N))}{\mathcal{H}^{d-1}(NA)} \leq \nu(\vec{v}) \quad \text{a.s.} \tag{14}$$

If $\nu(\vec{v}) = 0$, since τ is non-negative, we get the desired result. We suppose that $\nu(\vec{v}) > 0$. From Proposition 3.7, we know that $\nu(\vec{v}) > 0$ is equivalent to $F(0) < 1 - p_c(d)$. Then it follows from Lemma 4.6 and the convergence of $\mathbb{E}(\tau(nA, h(n)))/\mathcal{H}^{d-1}(nA)$ to $\nu(\vec{v})$ that

$$\nu(\vec{v}) \leq \liminf_{N \rightarrow \infty} \frac{\tau(NA, h(N))}{\mathcal{H}^{d-1}(NA)} \quad \text{a.s.}$$

which, together with (14) gives the law of large numbers for τ .

Now, what happens if $0 \notin A$? Then, we suppose that **(F3)** holds, and we can combine Borel–Cantelli’s lemma with the complete convergence in the law of large numbers for subsequences (Theorem 4.1 in [10], or more generally Theorem 4.1 in [11]) to replace the classical law of large numbers to prove (12) and (13).

This ends the proof of the almost sure convergence. Now, let us prove the convergence in L^1 . Suppose first that $0 \in A$. Then, one can find a sequence of sets of edges $(E(n))_{n \in \mathbb{N}}$ such that for each n , $E(n)$ is a cut between $(nA)_1^{h(n)}$ and $(nA)_2^{h(n)}$, $E(n) \subset E(n + 1)$ and

$$\frac{\text{card}(E(n))}{\mathcal{H}^{d-1}(nA)} \xrightarrow{n \rightarrow \infty} \|\vec{v}\|_1.$$

Now, define

$$f_n = \frac{\tau_n}{\mathcal{H}^{d-1}(nA)} \quad \text{and} \quad g_n = \frac{1}{\mathcal{H}^{d-1}(nA)} \sum_{e \in E(n)} t(e).$$

Then, we know the following:

- (i) $0 \leq f_n \leq g_n$ for every n ,
- (ii) $(g_n)_{n \in \mathbb{N}}$ converges almost surely and in L^1 , thanks to the usual law of large numbers,
- (iii) $(f_n)_{n \in \mathbb{N}}$ converges almost surely to $\nu(\vec{v})$, thanks to the almost sure convergence for $0 \in A$ that we have just proven,
- (iv) $(\mathbb{E}(f_n))_{n \in \mathbb{N}}$ converges to $\nu(\vec{v})$, thanks to Proposition 3.5.

It is then standard to show that f_n converges in L^1 to $\nu(\vec{v})$: apply the monotone convergence theorem to $b_n = \inf_{m \geq n} (g_m - f_m)$, and then show that $(g - f - b_n)_{n \in \mathbb{N}}$ and $(g_n - f_n - b_n)_{n \in \mathbb{N}}$ are positive sequences converging to zero in L^1 .

It remains to show the convergence in L^1 when we do not know whether $0 \in A$. Let A'' be the translate of A such that $0 \in A''$, and 0 is the center of A'' . For any fixed n , there exists a hyperrectangle A'_n which is a translate of nA by an integer vector and such that $d_\infty(0, nA'_n) < 1$ and $d_\infty(nA'', A'_n) < 1$, where d_∞ denotes the distance induced by $\|\cdot\|_\infty$. We want to compare the maximal flow through $\text{cyl}(nA'', h(n))$ to the maximal flow through $\text{cyl}(A'_n, h(n))$. The difficulty is that one of these cylinders is not included in the other. This is the reason why we will construct bigger and smaller version of $\text{cyl}(nA'', h(n))$. We recall that $l_{\min}(A)$ is the smallest length of A , i.e.,

$$l_{\min}(A) = \min_{i=1, \dots, d-1} k_i,$$

where A is the image by an isometry of the set $\prod_{i=1}^{d-1} [0, k_i] \times \{0\}$. We define the biggest length of A as

$$l_{\max}(A) = \max_{i=1, \dots, d-1} k_i$$

with the same notation. We only consider n large enough such that $h(n) > 1$. Thus the following inclusions holds

$$\text{cyl}\left(\left(n - \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) - 1\right) \subset \text{cyl}(A'_n, h(n)) \subset \text{cyl}\left(\left(n + \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) + 1\right),$$

where $\lceil x \rceil$ is the smallest integer bigger than or equal to x . For all n , we have

$$\partial\left[\left(n - \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A''\right] \subset \mathcal{V}\left(\partial A'_n, l_{\max}(A) \left\lceil \frac{2}{l_{\min}(A)} \right\rceil + 1\right)$$

and

$$\partial\left[\left(n + \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A''\right] \subset \mathcal{V}\left(\partial A'_n, l_{\max}(A) \left\lceil \frac{2}{l_{\min}(A)} \right\rceil + 1\right).$$

Arguing as in Section 4.3, let F_n be the edges included in \mathcal{F}_n defined as

$$\mathcal{F}_n = \mathcal{V}\left(\partial A'_n, l_{\max}(A) \left\lceil \frac{2}{l_{\min}(A)} \right\rceil + 1 + 4d\right).$$

We get, for n large enough,

$$\begin{aligned} & \tau\left(\left(n + \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) + 1\right) - V(F_n) \\ & \leq \tau(A'_n, h(n)) \leq \tau\left(\left(n - \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) - 1\right) + V(F_n). \end{aligned}$$

Using the convergence in L^1 for A'' which contains 0, we see that

$$\tau\left(\left(n + \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) + 1\right) / \mathcal{H}^{d-1}(nA) \quad \text{and} \quad \tau\left(\left(n - \left\lceil \frac{2}{l_{\min}(A)} \right\rceil\right)A'', h(n) - 1\right) / \mathcal{H}^{d-1}(nA)$$

converge to $\nu(\vec{v})$ in L^1 as n goes to infinity. Furthermore, since $\text{card}(F_n)$ is negligible compared to n^{d-1} , $V(F_n) / \mathcal{H}^{d-1}(nA)$ go to zero in L^1 , and we get the convergence of $\tau(A'_n, h(n)) / \mathcal{H}^{d-1}(nA)$ to $\nu(\vec{v})$ in L^1 . But since A'_n is an integer translate of nA , it implies the convergence of $\tau(nA, h(n)) / \mathcal{H}^{d-1}(nA)$ to $\nu(\vec{v})$ in L^1 .

Remark 4.7. Most likely, the almost sure convergence of $(\tau(nA, h(n)) / \mathcal{H}^{d-1}(nA))$, $n \in \mathbb{N}$ could also be obtained by adapting the proof of [1], and thus relaxing the independence hypothesis on $(t(e))_e$ to stationarity. In any case, general subadditive results existing in the literature are not well adapted to treat the case of irrational directions, i.e. directions \vec{v} such that $\tau(nA, h(n))$ is not exactly subadditive and stationary. Some authors circumvent this problem by proving that the almost sure convergence is uniform with respect to rational directions, which allows to extend the convergence to irrational directions, see [13] and [3], for instance. But for flows like τ , the uniform convergence requires a moment of order strictly larger than 1, see, for instance, Theorems 1.3, 1.9 and Section 4 in [3]. Notice also that Theorem 6.1 in [3] shows directly the convergence in any direction for First Passage Percolation in dimension 2, using techniques some of which are similar to ours and others belong to the realm of ergodic theory. In this paper, the strategy we adopt is to use the fact that our space \mathbb{R}^d has one dimension more than the hyperrectangles which are the indices of the almost subadditive family: we can move the hyperrectangles $T(i)$ out of the hypersurface spanned by nA to obtain the hyperrectangles $T'(i)$ that have good properties. Moreover, the non-negativity of our variables τ implies that it is simpler to use a concentration inequality than a maximal inequality as in the classical subadditive ergodic theorems.

Remark 4.8. We have obtained readily the independence of the limit with regard to the precise form of the hyperrectangle we consider and this is not surprising since it appears already in subadditive ergodic theorems like in [15].

Remark 4.9. The almost sure convergence of $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$ is not necessary to prove Theorem 3.9, but we need the convergence in probability to prove Theorem 3.10.

Remark 4.10. If $0 \notin A$, it is not clear to us whether condition **(F3)** is necessary or not: it is necessary for complete convergence to hold, but complete convergence is stronger than the a.s. convergence.

4.6. Lower deviations for τ and flat ϕ : Proofs of Theorems 3.9 and 3.13

Now, we can prove Theorem 3.9, and so we consider F, h, \vec{v} and A as in the statement of this theorem. If $\nu(\vec{v}) = 0$, there is nothing to prove. Suppose now that $\nu(\vec{v}) > 0$ and let $\varepsilon \leq \nu(\vec{v})$ be a positive real number. Let $u = \varepsilon/(2\nu_{\max})$, where $\nu_{\max} = \max\{\nu(\vec{v}) \mid \vec{v} \text{ unit vector}\}$. Then $u > 0$ and we have

$$\frac{\nu(\vec{v}) - \varepsilon}{\nu(\vec{v}) - \varepsilon/2} \leq 1 - u.$$

Using assertion (iii) in Proposition 3.7, we know that there exists a $n_0 = n_0(A)$ (not depending on h) large enough to have

$$\forall n \geq n_0 \quad \frac{\mathbb{E}(\tau(nA, h(n)))}{\mathcal{H}^{d-1}(nA)} \geq \nu(\vec{v}) - \frac{\varepsilon}{2}.$$

Then, for all $n \geq n_0$,

$$\mathbb{P}[\tau(nA, h(n)) \leq (\nu(\vec{v}) - \varepsilon)\mathcal{H}^{d-1}(nA)] \leq \mathbb{P}\left[\frac{\tau(nA, h(n))}{\mathbb{E}(\tau(nA, h(n)))} \leq 1 - u\right].$$

Now, the result follows easily from Proposition 4.3, for n larger than some $n_1 = n_1(A)$. Adapting the constant for $n \leq n_1$ leads to $\tilde{C}(d, F, A, \varepsilon)$.

Remark 4.11. Notice that for every hyperrectangle A :

$$\frac{2}{l_{\min}(A)} \leq \frac{\mathcal{H}^{d-2}(\partial A)}{\mathcal{H}^{d-1}(A)} \leq \frac{2(d-1)}{l_{\min}(A)}.$$

Thus, from the proof above, Proposition 3.7(iii) and Proposition 4.3, it can be seen that $n_1(A)$ and thus the constant $\tilde{C}(d, F, A, \varepsilon)$ depends on A only through $K(d, A)$, or equivalently, only through $l_{\min}(A)$.

We can do the same calculus for $\phi(nA, h(n))$ as soon as we know that $\mathbb{E}(\phi(nA, h(n)))/\mathcal{H}^{d-1}(nA)$ converges to $\nu(\vec{v})$. To prove Theorem 3.13, it is sufficient to prove that it is the case under hypotheses **(F2)**, **(H1)** and **(H3)**. We have to compare ϕ and τ . We suppose that $\lim_{n \rightarrow \infty} h(n)/n = 0$, and fix $\zeta \geq 2d$. We consider n large enough such that the sides of nA have length bigger than ζ , i.e., $l_{\min}(A) \geq \zeta$. Let E_1^+ be the set of the edges that belong to \mathcal{E}_1^+ , defined as

$$E_1^+ = \mathcal{V}(\text{cyl}(\partial(nA), h(n)), \zeta) \cap \text{cyl}(nA, h(n)).$$

We have, for all n large enough,

$$\tau(nA, h(n)) \geq \phi(nA, h(n)) \geq \tau(nA, h(n)) - V(E_1^+).$$

There exists a constant C^+ such that

$$\text{card}(E_1^+) \leq C^+ n^{d-2} h(n),$$

so we have

$$\frac{|\mathbb{E}[\phi(nA, h(n))] - \mathbb{E}[\tau(nA, h(n))]|}{\mathcal{H}^{d-1}(nA)} \leq \frac{C^+ n^{d-2} h(n)}{n^{d-1} \mathcal{H}^{d-1}(nA)} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and this proves the convergence of $\mathbb{E}[\phi(nA, h(n))]/\mathcal{H}^{d-1}(nA)$ to $\nu(\vec{v})$. Notice that the speed of convergence depends on h . Using Proposition 4.3, we can find $n_1(d, F, A, h, \varepsilon)$ such that for all $n \geq n_1(d, F, A, h, \varepsilon)$ we have

$$\begin{aligned} \mathbb{P}(\phi(nA, h(n)) \leq (\nu(\vec{v}) - \varepsilon)\mathcal{H}^{d-1}(nA)) \\ \leq C_3(F(0), d)h(n) \exp(-C(\varepsilon, F, d)(\nu(\vec{v}) - \varepsilon/2)\mathcal{H}^{d-1}(nA)) \\ \leq C_3(F(0), d) \exp\left(\frac{\log h(n)}{n^{d-1}} n^{d-1} - C(\varepsilon, F, d)(\nu(\vec{v}) - \varepsilon/2)\mathcal{H}^{d-1}(nA)\right). \end{aligned}$$

Using hypothesis **(H2)**, which is implied by **(H3)**, Theorem 3.13 is proved for $n \geq n_2(d, F, A, h, \varepsilon)$, for $n_2(d, F, A, h, \varepsilon)$ large enough. Adapting the constant $C_3(F(0), d)$ for the n_2 first terms, Theorem 3.13 is proved for all n with a constant C' depending on d, F, A, h, ε .

To prove Theorem 3.18, it remains to prove the convergence of $\mathbb{E}[\phi(nA, h(n))]/\mathcal{H}^{d-1}(nA)$ to $\nu(\vec{v})$ under the hypotheses **(F1)**, **(F2)**, **(H1)** and **(H2)**. This will be done during the proof of Theorem 3.16 in Section 6.2, so we postpone the end of the proof of Theorem 3.18 until Section 6.3.

Remark 4.12. Using Theorems 3.13, 3.8 and the fact that $\phi(nA, h(n)) \leq \tau(nA, h(n))$, we obtain the law of large numbers for $\phi(nA, h(n))$ in flat cylinders (i.e., under hypothesis **(H3)**) under the same hypothesis as the one for $\tau(nA, h(n))$.

5. Large deviation principle for τ and ϕ in flat cylinders

In this section, we show the large deviation principle for τ . We construct a precursor of the rate function in Section 5.1, and then study its properties. Precisely, we show it is convex in Section 5.2, finite (and thus continuous) on $]\delta\|\vec{v}\|_1, +\infty[$ in Section 5.3, and strictly positive on $[0, \nu(\vec{v})[$ in Section 5.4. After having shown in Section 5.5 that upper large deviations occur at an order bigger than the surface order, we can complete the proof of the full large deviation principle for τ in Section 5.6 and deduce the one for ϕ in flat cylinders in Section 5.7.

5.1. Construction of the rate function

We will prove the following lemma, for which no condition on F is required.

Lemma 5.1. For every function $h : \mathbb{N} \rightarrow \mathbb{R}^+$ satisfying **(H1)**, for every non-degenerate hyperrectangle A , for all λ in \mathbb{R}^+ , the limit

$$\lim_{n \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)\mathcal{H}^{d-1}(nA)\right]$$

exists in $[0, +\infty[$ and depends only on the direction of \vec{v} , one of the two unit vectors orthogonal to $\text{hyp}(A)$. We denote it by $\mathcal{I}_{\vec{v}}(\lambda)$.

We introduce a factor $1/\sqrt{n}$ in the definition of $\mathcal{I}_{\vec{v}}(\lambda)$ because we want to work with subadditive objects, but $\tau(A, h)$ is not subadditive in A , except for straight cylinders. Indeed, if A and B are two hyperrectangles with a common orthogonal vector and with a common side, to glue together a set of edges in $\text{cyl}(A, h)$ that cuts A_1^h from A_2^h and a set of edges in $\text{cyl}(B, h)$ that cuts B_1^h from B_2^h , we have to add edges at the common side of A and B (see the set of edges E_0 defined in Section 4.3). These edges may not have a capacity 0, so they perturb the subadditivity of τ . We add the factor $1/\sqrt{n}$ to compensate.

Remark 5.2. It is natural to have no condition on F in Lemma 5.1 since it comes essentially from an almost subadditive property for a non-random quantity.

Proof of Lemma 5.1. For the proof of Lemma 5.1, we consider the same construction as in Section 4.3 (see Fig. 1). From (7) we deduce that for all $\lambda \in \mathbb{R}_*^+$, we have

$$\begin{aligned} & \mathbb{P}\left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA)\right] \\ & \geq \mathbb{P}\left[V(E_0) + \sum_{i \in I} \tau(T'(i), h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA)\right]. \end{aligned}$$

Let $\mathcal{D} = \{\lambda \mid \mathbb{P}(t(e) \leq \lambda) > 0\}$, and $\delta = \inf \mathcal{D}$. We take $u = \delta + \zeta$, so $p = \mathbb{P}(t(e) \leq u) > 0$. We use first the FKG inequality and then the fact that the family $(\tau(T'(i), h'(n)), i \in I)$ is identically distributed to obtain that

$$\begin{aligned} & \mathbb{P}\left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA)\right] \\ & \geq \mathbb{P}[V(E_0) \leq u \text{card}(E_0)] \prod_{i \in I} \mathbb{P}\left[\tau(T'(i), h'(n)) \leq \frac{(\lambda - 1/\sqrt{N})\mathcal{H}^{d-1}(NA) - u \text{card}(E_0)}{\text{card}(I)}\right] \\ & \geq \mathbb{P}[t(e) \leq u]^{\text{card}(E_0)} \mathbb{P}\left[\tau(nA', h'(n)) \leq \frac{(\lambda - 1/\sqrt{N})\mathcal{H}^{d-1}(NA) - u \text{card}(E_0)}{\text{card}(I)}\right]^{\text{card}(I)}. \end{aligned}$$

We have immediately that $\text{card}(I) \leq \mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA')$, so

$$\begin{aligned} & \frac{-1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P}\left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA)\right] \\ & \leq \frac{-1}{\mathcal{H}^{d-1}(nA')} \log \mathbb{P}[\tau(nA', h'(n)) \leq \beta] - \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \log p, \end{aligned}$$

where

$$\beta = \frac{(\lambda - 1/\sqrt{N})\mathcal{H}^{d-1}(NA) - u \text{card}(E_0)}{\text{card}(I)}.$$

As we saw in Section 4.3, there exists a constant $c(d, \zeta, A, A')$ such that

$$\text{card}(E_0) \leq c(d, \zeta, A, A')(N^{d-2}n + N^{d-1}/n + 1).$$

On the one hand, we obtain that

$$\lim_{n \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \log p = 0.$$

On the other hand, we want to compare β with $(\lambda - 1/\sqrt{n})\mathcal{H}^{d-1}(nA')$. Obviously, we have

$$\frac{\lambda \mathcal{H}^{d-1}(NA)}{\text{card}(I)} \geq \lambda \mathcal{H}^{d-1}(nA').$$

We also know that

$$\text{card}(I) \geq \frac{\mathcal{H}^{d-1}(D(n, N))}{\mathcal{H}^{d-1}(nA')}$$

so there exist a constant $c'(d, A, A')$ and an integer $N_1(n)$ large enough to have, for all $N \geq N_1(n)$,

$$\text{card}(I) \geq c'(d, A, A') \left(\frac{N}{n}\right)^{d-1}.$$

Thus, there exist constants $c_i(d, \zeta, A, A')$ such that for all $N \geq N_1(n)$, we have

$$\frac{\mathcal{H}^{d-1}(NA)}{\text{card}(I)\sqrt{N}} \leq \frac{c_1(d, \zeta, A, A')}{\sqrt{N}} \mathcal{H}^{d-1}(nA')$$

and

$$\frac{u \text{card}(E_0)}{\text{card}(I)} \leq c_2(d, \zeta, A, A') \left(\frac{n}{N} + \frac{1}{n}\right) \mathcal{H}^{d-1}(nA').$$

There exists n_0 such that for all $n \geq n_0$, $c_2/n \leq 1/(4\sqrt{n})$. Then there exists $N_2(n) \geq N_0(n) \vee N_1(n)$ such that for all $N \geq N_2(n)$, $c_2n/N \leq 1/(4\sqrt{n})$ and $c_1/\sqrt{N} \leq 1/(2\sqrt{n})$. Thus for a fixed $n \geq n_0$, for all $N \geq N_2(n)$, we have

$$\beta \geq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA').$$

Now in the following inequality, obtained for $n \geq n_0$ and $N \geq N_2(n)$,

$$\begin{aligned} & \frac{-1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P} \left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA) \right] \\ & \leq \frac{-1}{\mathcal{H}^{d-1}(nA')} \log \mathbb{P} \left[\tau(nA', h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA') \right] - \frac{\text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \log p, \end{aligned}$$

we send N to infinity for a fixed $n \geq n_0$, and then we send n to infinity. We thus obtain

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(NA)} \log \mathbb{P} \left[\tau(NA, h(N)) \leq \left(\lambda - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA) \right] \\ & \leq \liminf_{n \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(nA')} \log \mathbb{P} \left[\tau(nA', h'(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA') \right]. \end{aligned}$$

For $A = A'$ and $h = h'$, this gives us the existence of

$$\lim_{n \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA) \right]$$

for all $\lambda \in \mathbb{R}_*^+$, and for different A, A', h, h' this shows that the limit is independent of A and h . We denote this limit by $\mathcal{I}_{\vec{v}}(\lambda)$.

For $\lambda = 0$,

$$\mathbb{P} \left[\tau(nA, h(n)) \leq -\frac{\mathcal{H}^{d-1}(nA)}{\sqrt{n}} \right] = 0$$

for all $n \in \mathbb{N}$, so the previous limit equals $+\infty$, independently of A and \vec{v} . This ends the proof of Lemma 5.1. \square

Remark 5.3. The function $\mathcal{I}_{\vec{v}}$ is not exactly the rate function we will consider later: we will change its value from 0 to $+\infty$ on $]\nu(\vec{v}), +\infty[$ and we will regularize it at $\|\vec{v}\|_1 \delta$.

5.2. Convexity of $\mathcal{I}_{\bar{v}}$

We will prove that $\mathcal{I}_{\bar{v}}$ is convex, i.e., for all $\lambda_1 \geq \lambda_2 \in \mathbb{R}^+$ and $\alpha \in]0, 1[$, we have

$$\mathcal{I}_{\bar{v}}(\alpha\lambda_1 + (1 - \alpha)\lambda_2) \leq \alpha\mathcal{I}_{\bar{v}}(\lambda_1) + (1 - \alpha)\mathcal{I}_{\bar{v}}(\lambda_2).$$

For $\lambda_2 = 0$, the result is obvious, so we suppose $\lambda_2 > 0$. We keep the same notations as in the previous section, for $D(n, N)$, $T(i)$, E_i , etc., except that we take $A = A'$. We define

$$\gamma = \lfloor \alpha \text{card}(I) \rfloor.$$

If we have

$$\tau(T'(i), h(n)) \leq (\lambda_1 - 1/\sqrt{n})\mathcal{H}^{d-1}(nA) \quad \text{for } i = 1, \dots, \gamma, \quad (15)$$

$$\tau(T'(i), h(n)) \leq (\lambda_2 - 1/\sqrt{n})\mathcal{H}^{d-1}(nA) \quad \text{for } i = \gamma + 1, \dots, \text{card}(I), \quad (16)$$

and

$$V(E_0) \leq u \text{card}(E_0),$$

then we obtain that

$$\begin{aligned} \tau(nA, h(N)) &\leq \left(\gamma \left(\lambda_1 - \frac{1}{\sqrt{n}} \right) + (\text{card}(I) - \gamma) \left(\lambda_2 - \frac{1}{\sqrt{n}} \right) \right) \mathcal{H}^{d-1}(nA) + u \text{card}(E_0) \\ &\leq (\alpha\lambda_1 + (1 - \alpha)\lambda_2) \text{card}(I) \mathcal{H}^{d-1}(nA) - \frac{\text{card}(I) \mathcal{H}^{d-1}(nA)}{\sqrt{n}} + u \text{card}(E_0) \\ &\leq (\alpha\lambda_1 + (1 - \alpha)\lambda_2) \mathcal{H}^{d-1}(NA) - \rho, \end{aligned}$$

where

$$\rho = \frac{\text{card}(I) \mathcal{H}^{d-1}(nA)}{\sqrt{n}} - u \text{card}(E_0).$$

We want to prove that $\rho \geq \mathcal{H}^{d-1}(NA)/\sqrt{N}$ for N large enough. We have seen in the previous section that there exists a constant $c(d, \zeta, A)$ such that

$$\text{card}(E_0) \leq c(d, \zeta, A) N^{d-1} \left(\frac{n}{N} + \frac{1}{n} \right),$$

and that there exists a constant $c'(d, A)$ and a $N_1(n)$ large enough to have, for all $N \geq N_1(n)$,

$$\text{card}(I) \geq c'(d, A) \left(\frac{N}{n} \right)^{d-1}.$$

There exists n_1 such that for all $n \geq n_1$, $2c/n \leq c'/(2\sqrt{n})$. For a fixed $n \geq n_1$, there exists constants $c_i(d, \zeta, A)$ and $N_3(n)$ such that for all $N \geq N_3(n)$ we have

$$\frac{u \text{card}(E_0)}{\mathcal{H}^{d-1}(NA)} \leq \frac{2c}{n} \leq \frac{c'}{2\sqrt{n}}, \quad \frac{\text{card}(I) \mathcal{H}^{d-1}(nA)}{\mathcal{H}^{d-1}(NA) \sqrt{n}} \geq \frac{c'}{\sqrt{n}} \quad \text{and} \quad \frac{c'}{2\sqrt{n}} \geq \frac{1}{\sqrt{N}}.$$

We conclude that for $n \geq n_1$ and $N \geq N_3(n)$, $\gamma \geq \mathcal{H}^{d-1}(NA)/\sqrt{N}$ and then

$$\tau(nA, h(N)) \leq \left(\alpha\lambda_1 + (1 - \alpha)\lambda_2 - \frac{1}{\sqrt{N}} \right) \mathcal{H}^{d-1}(NA),$$

as long as (15) and (16) hold. Then, for all $n \geq n_1$ and $N \geq N_3(n)$, we have, by the FKG inequality:

$$\begin{aligned} \mathbb{P}\left(\tau(NA, h(N)) \leq \left(\alpha\lambda_1 + (1-\alpha)\lambda_2 - \frac{1}{\sqrt{N}}\right) \mathcal{H}^{d-1}(NA)\right) \\ \geq \mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda_1 - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right)^\gamma \\ \times \mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda_2 - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right)^{\text{card}(I)-\gamma} p^{\text{card}(E_0)}. \end{aligned}$$

We take the logarithm of this expression, we divide it by $\mathcal{H}^{d-1}(NA)$, we send N to infinity and then n to infinity to obtain

$$\mathcal{I}_{\vec{v}}(\alpha\lambda_1 + (1-\alpha)\lambda_2) \leq \alpha\mathcal{I}_{\vec{v}}(\lambda_1) + (1-\alpha)\mathcal{I}_{\vec{v}}(\lambda_2).$$

The convexity of $\mathcal{I}_{\vec{v}}$ is so proved.

5.3. Continuity of $\mathcal{I}_{\vec{v}}$

Now we come back to the problem of the continuity of $\mathcal{I}_{\vec{v}}$. Since $\mathcal{I}_{\vec{v}}$ is convex, we first try to determine its domain. Recall that $\delta = \delta(F) = \inf\{\lambda \mid \mathbb{P}(t(e) \leq \lambda) > 0\}$.

- $\lambda > \|\vec{v}\|_1\delta$: there exists $\varepsilon > 0$ such that $\lambda > (\|\vec{v}\|_1 + \varepsilon)(\delta + 2\varepsilon)$. Then there exists n_0 such that, for all $n \geq n_0$, there exists a set of edges $E_0(n)$ that disconnects $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$ and such that $\text{card}(E_0(n)) \leq (\|\vec{v}\|_1 + \varepsilon)\mathcal{H}^{d-1}(nA)$. We obtain for $n \geq n_0$,

$$\begin{aligned} \mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right) &\geq \mathbb{P}\left(V(E_0(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right) \\ &\geq \mathbb{P}\left(t(e) \leq \frac{\lambda - 1/\sqrt{n}}{\|\vec{v}\|_1 + \varepsilon}\right)^{\lfloor (\|\vec{v}\|_1 + \varepsilon)\mathcal{H}^{d-1}(nA) \rfloor}. \end{aligned}$$

But there exists n_1 large enough to have for all $n \geq n_1$, $\lambda - 1/\sqrt{n} \geq (\|\vec{v}\|_1 + \varepsilon)(\delta + \varepsilon)$, so for all $n \geq n_0 \vee n_1$, we have

$$\mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right) \geq \mathbb{P}(t(e) \leq \delta + \varepsilon)^{\lfloor (\|\vec{v}\|_1 + \varepsilon)\mathcal{H}^{d-1}(nA) \rfloor},$$

and finally

$$\mathcal{I}_{\vec{v}}(\lambda) \leq -(\|\vec{v}\|_1 + \varepsilon) \log \mathbb{P}(t(e) \leq \delta + \varepsilon) < \infty.$$

- $\lambda \leq \|\vec{v}\|_1\delta$: for $\lambda > 0$, there exists n_0 such that for all $n \geq n_0$,

$$\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \delta \frac{\mathcal{N}(nA, h(n))}{\mathcal{H}^{d-1}(nA, h(n))} \geq \delta\|\vec{v}\|_1 - \frac{1}{2\sqrt{n}} > \lambda - \frac{1}{\sqrt{n}},$$

and so for all $n \geq n_0$,

$$\mathbb{P}\left(\tau(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right) = 0.$$

The same result is true for $\lambda = 0$. We obtain that $\mathcal{I}_{\vec{v}}(\lambda) = +\infty$.

Now, we know that $\mathcal{I}_{\vec{v}}$ is convex and finite on $]\delta\|\vec{v}\|_1, +\infty[$ so it is continuous on $]\delta\|\vec{v}\|_1, +\infty[$, and it is infinite on $[0, \delta\|\vec{v}\|_1]$.

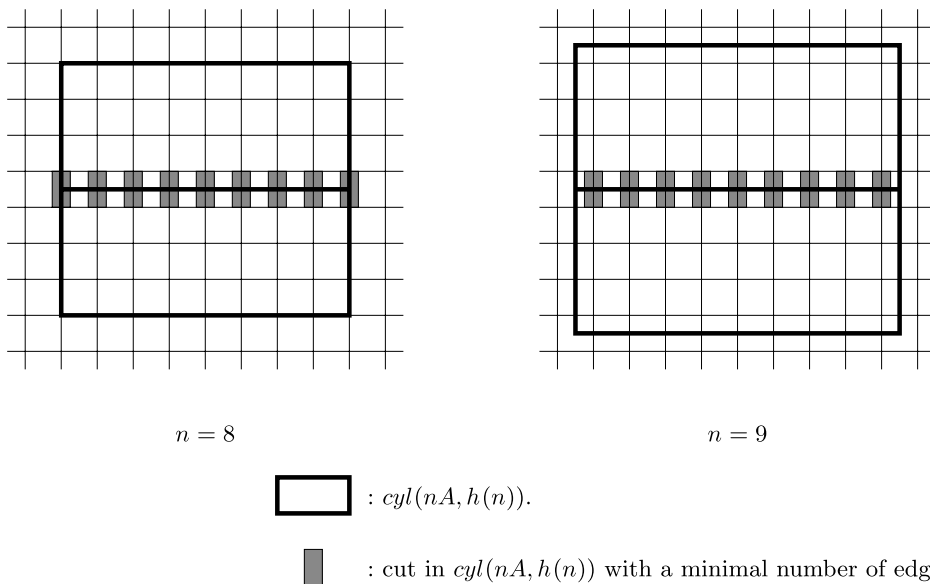


Fig. 2. Examples of cuts.

Remark 5.4. The only point we did not study is the behaviour of the function near $\delta \|\vec{v}\|_1$. In fact, we will eventually change the value of $\mathcal{I}_{\vec{v}}(\delta \|\vec{v}\|_1)$ to obtain a lower semicontinuous function. Moreover, the fact that $\mathcal{I}_{\vec{v}}(\delta \|\vec{v}\|_1) = +\infty$ even if there exists an atom of the law of $t(e)$ at δ is linked with the fact that we added a term $1/\sqrt{n}$ and not with the behaviour of $\mathbb{P}(\tau(nA, h(n)) \leq \delta \|\vec{v}\|_1 \mathcal{H}^{d-1}(nA))$. This remark can be illustrated by an example in dimension 2: let $A = [-1/2, 1/2] \times \{1/2\}$. Here $\vec{v} = (0, 1)$ so $\|\vec{v}\|_1 = 1$. We consider a law of capacities with an atom at δ . We remark (see Fig. 2) that $\mathcal{N}((2n + 1)A, 2n + 1) = 2n + 1$. Moreover, there exists a unique cut $E_0(2n + 1)$ in $cyl((2n + 1)A, 2n + 1)$ composed by $2n + 1$ edges (see it on the figure). So we have

$$\mathbb{P}(\tau((2n + 1)A, 2n + 1) \leq (2n + 1)\delta) = \mathbb{P}(V(E_0(2n + 1)) = (2n + 1)\delta) = \mathbb{P}(t(e) = \delta)^{2n+1}$$

and

$$\lim_{n \rightarrow \infty} \frac{-1}{2n + 1} \log \mathbb{P}(\tau((2n + 1)A, 2n + 1) \leq (2n + 1)\delta) = -\log \mathbb{P}(t(e) = \delta) < \infty.$$

We also remark that $\mathcal{N}(2nA, 2n) = 2n + 1$ because a cut in $cyl(2nA, 2n)$ must contain a vertical edge of first coordinate i for $i = 0, \dots, 2n$. Then we have

$$\mathbb{P}(\tau(2nA, 2n) \leq 2n\delta) = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{-1}{2n} \log \mathbb{P}(\tau(2nA, 2n) \leq 2n\delta) = +\infty.$$

This example shows that the behaviour of $\mathbb{P}(\tau(nA, h(n)) \leq \delta \|\vec{v}\|_1 \mathcal{H}^{d-1}(nA))$ is not clear, and we will avoid the problem by taking later at $\|\vec{v}\|_1 \delta$ the value of the limit

$$\lim_{\lambda > \|\vec{v}\|_1 \delta, \lambda \rightarrow \|\vec{v}\|_1 \delta} \mathcal{I}_{\vec{v}}(\lambda)$$

instead of $\mathcal{I}_{\vec{v}}(\|\vec{v}\|_1 \delta)$.

5.4. Positivity of $\mathcal{I}_{\vec{v}}$

From now on we need the assumptions **(F1)** and **(F2)**, i.e. $F(0) < 1 - p_c(d)$, and F admits a moment of order 1. It is an immediate consequence of Theorem 3.8 that $\mathcal{I}_{\vec{v}}$ is equal to zero on $]v(\vec{v}), +\infty[$, and Theorem 3.9 implies immediately too that $\mathcal{I}_{\vec{v}}$ is strictly positive on $[0, v(\vec{v})[$ if $v(\vec{v}) > 0$.

Remark 5.5. We did not study the function $\mathcal{I}_{\vec{v}}$ at $v(\vec{v})$, i.e., if $\mathcal{I}_{\vec{v}}(v(\vec{v})) = 0$ or not. If $v(\vec{v}) > \delta \|\vec{v}\|_1$, then $\mathcal{I}_{\vec{v}}$ is continuous at $v(\vec{v})$ and so $\mathcal{I}_{\vec{v}}(v(\vec{v})) = 0$. If $v(\vec{v}) = \delta \|\vec{v}\|_1$, the value of $\mathcal{I}_{\vec{v}}(v(\vec{v}))$ is not relevant for the understanding of the system as explained in Remark 5.4. Finally, Proposition 3.7 gives a sufficient condition to have $v(\vec{v}) > \delta \|\vec{v}\|_1$, and this condition is also necessary when $\delta = 0$.

5.5. Upper large deviations for τ

We will need the following result to prove the large deviation principle for τ in the next section.

Lemma 5.6. Suppose that **(H1)** and **(F5)** hold. Then we have, for all $\lambda > v(\vec{v})$,

$$\lim_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda \right] = -\infty. \tag{17}$$

We do not prove Lemma 5.6 here. The proof is an adaptation of Section 3.7 in [17], that proves that the upper large deviations for $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ in straight boxes are of volume order. It is written completely in [19], where other assumptions on F are also considered. We describe here only the two adaptations required to get Lemma 5.6 from the proof in [17]. The proof for ϕ is based on a comparison between the variable $\phi(NA, h(N))$ in a big cylinder, and the minimum over $h(N)/h(n)$ possible choices of sums of $\mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA)$ independent variables equal in law with $\tau(nA, h(n))$, where n is small compared to N . This comparison is obtained by dividing the big cylinder $\text{cyl}(NA, h(N))$ into $h(N)/h(n)$ slabs, and diving each slab in $\mathcal{H}^{d-1}(NA)/\mathcal{H}^{d-1}(nA)$ translates of $\text{cyl}(nA, h(n))$. Then, in any fixed slab, if we glue together cutsets in the small cylinder of size n , we can construct a cutset in $\text{cyl}(NA, h(N))$. There are two difficulties to replace $\phi(NA, h(N))$ by $\tau(NA, h(N))$ in this construction, and to consider potentially tilted cylinders. First, the fact that the cylinders we consider may be tilted implies a default of subadditivity of the variable τ , so we have to add edges between the small cylinders of size n to glue together the different cutsets, and we have to control the number of the edges we must consider. Then, when the small cutsets are glued together, they form a set of edges that cuts the top from the bottom of $\text{cyl}(NA, h(N))$. It remains to link this cutset to the boundary of NA to obtain a cutset corresponding to the variable $\tau(NA, h(N))$. To obtain a control on the number of edges we must add at this step, we have to consider only slabs whose distance to NA is negligible compared to N . Using Cramér theorem for each possible sum of independent variables in a slab, and optimizing over the possible choices of slab, we obtain the desired result.

Remark 5.7. For the variable ϕ , it suffices to have one exponential moment for the law F to obtain this speed of decay (see [17]). For τ , one exponential moment is not a sufficiently strong condition. Consider, for example, an exponential law of parameter 1 for the capacities of the edges. We know that $\mathbb{E}(\exp(\gamma\tau)) < \infty$ for all $\gamma < 1$. Let x_0 be a fixed point of the boundary $\partial(nA)$. There are, at distance at most $4d$ of x_0 , one vertex of $(nA)_1^{h(n)}$ and another of $(nA)_2^{h(n)}$. Let γ be some smallest path in $\text{cyl}(nA, h(n))$ joining those two vertices. Its length is at most some constant $R(d)$, and we know that every set of edges that cuts $(nA)_1^{h(n)}$ from $(nA)_2^{h(n)}$ in $\text{cyl}(nA, h(n))$ must contain one of the edges of γ . The probability that all of them have a capacity bigger than $\lambda \mathcal{H}^{d-1}(nA)$ for some $\lambda > v(\vec{v})$, and therefore that $\tau(nA, h(n))$ is bigger than $\lambda \mathcal{H}^{d-1}(nA)$, is greater than $\exp(-R(d)\lambda \mathcal{H}^{d-1}(nA))$. Then the property (17) cannot hold.

Remark 5.8. It is also proved in [19] that if the capacity of the edges is bounded, the upper large deviations are of order $n^{d-1} \min(n, h(n))$, and this is the right order of the upper large deviations in this case.

5.6. Proof of Theorem 3.10

We define the function $\mathcal{J}_{\vec{v}}$ on \mathbb{R}^+ by

$$\mathcal{J}_{\vec{v}}(\lambda) = \begin{cases} \mathcal{I}_{\vec{v}}(\lambda) & \text{if } \lambda \leq \nu(\vec{v}) \text{ and } \lambda \neq \|\vec{v}\|_1 \delta, \\ \lim_{\mu \rightarrow \|\vec{v}\|_1 \delta, \mu > \|\vec{v}\|_1 \delta} \mathcal{I}_{\vec{v}}(\mu) & \text{if } \lambda = \|\vec{v}\|_1 \delta, \\ +\infty & \text{if } \lambda > \nu(\vec{v}). \end{cases}$$

The study of the function $\mathcal{I}_{\vec{v}}$ made previously and the construction of $\mathcal{J}_{\vec{v}}$ gives us immediately that the function $\mathcal{J}_{\vec{v}}$ is a good rate function. As soon as we know that the upper large deviations are of order bigger than the lower large deviations, the techniques we will use to prove the large deviation principle are standard (see, for example, [6]).

• **Lower bound**

We have to prove that for all open subset \mathcal{O} of \mathbb{R}^+ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{O} \right] \geq -\inf_{\mathcal{O}} \mathcal{J}_{\vec{v}}.$$

Classically, it suffices to prove the local lower bound:

$$\forall \alpha \in \mathbb{R}^+, \forall \varepsilon > 0 \quad \liminf_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \geq -\mathcal{J}_{\vec{v}}(\alpha).$$

If $\mathcal{J}_{\vec{v}}(\alpha) = +\infty$, the result is trivial. Otherwise, suppose $\mathcal{J}_{\vec{v}}(\alpha) < +\infty$. The function $\mathcal{I}_{\vec{v}}$ is convex, equal to zero on $[\nu(\vec{v}), +\infty[$, positive on $[0, \nu(\vec{v})[$ and finite on $] \|\vec{v}\|_1 \delta, +\infty[$. Then $\mathcal{I}_{\vec{v}}$ is strictly decreasing on $] \|\vec{v}\|_1 \delta, \nu(\vec{v})]$, and so is $\mathcal{J}_{\vec{v}}$ (because $\mathcal{I}_{\vec{v}} = \mathcal{J}_{\vec{v}}$ on $] \|\vec{v}\|_1 \delta, \nu(\vec{v})]$). Yet $\mathcal{J}_{\vec{v}}(\alpha) < +\infty$ implies that $\alpha \in] \|\vec{v}\|_1 \delta, \nu(\vec{v})]$ or $\alpha = \|\vec{v}\|_1 \delta$ and $\mathcal{J}_{\vec{v}}(\|\vec{v}\|_1 \delta) < +\infty$. In both cases, we so obtain that $\mathcal{J}_{\vec{v}}(\alpha) < \mathcal{J}_{\vec{v}}(\alpha - \varepsilon/2)$. Then the following inequality, true for $n > 4/\varepsilon^2$,

$$\begin{aligned} & \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \\ & \geq \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \alpha - \frac{1}{\sqrt{n}} \right] - \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq \alpha - \frac{\varepsilon}{2} - \frac{1}{\sqrt{n}} \right] \end{aligned}$$

leads to

$$\liminf_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in]\alpha - \varepsilon, \alpha + \varepsilon[\right] \geq -\mathcal{J}_{\vec{v}}(\alpha).$$

• **Upper bound**

We have to prove that for all closed subset \mathcal{F} of \mathbb{R}^+ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F} \right] \leq -\inf_{\mathcal{F}} \mathcal{J}_{\vec{v}}.$$

Let \mathcal{F} be a closed subset of \mathbb{R}^+ . If $\nu(\vec{v}) \in \mathcal{F}$, the result is obvious. We suppose now that $\nu(\vec{v}) \notin \mathcal{F}$. We consider $\mathcal{F}_1 = \mathcal{F} \cap [0, \nu(\vec{v})]$ and $\mathcal{F}_2 = \mathcal{F} \cap]\nu(\vec{v}), +\infty[$. Let $f_1 = \sup \mathcal{F}_1$ ($f_1 < \nu(\vec{v})$ because \mathcal{F} is closed) and $f_2 = \inf \mathcal{F}_2$ ($f_2 > \nu(\vec{v})$ for the same reason). Then,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F} \right] \\ & \leq \limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \left(\mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1 \right] + \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq f_2 \right] \right). \end{aligned}$$

We know that:

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1 \right) \\ & \leq \lim_{\eta \rightarrow 0} \limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left(\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \leq f_1 + \eta - \frac{1}{\sqrt{n}} \right) \\ & = - \lim_{\eta \rightarrow 0} \mathcal{I}_{\bar{v}}(f_1 + \eta) = -\mathcal{J}_{\bar{v}}(f_1), \end{aligned}$$

and since $\mathcal{J}_{\bar{v}}$ is non-increasing on $[0, v_{\bar{v}}]$ and the upper large deviations of $\tau(nA, h(n))$ are of order bigger than n^{d-1} , we obtain

$$\limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \in \mathcal{F} \right] \leq -\mathcal{J}_{\bar{v}}(f_1) = -\inf_{\mathcal{F}} \mathcal{J}_{\bar{v}}.$$

5.7. Large deviation principle for ϕ in small boxes

In this section, we shall prove Corollary 3.14, i.e., under the assumption that $\lim_{n \rightarrow \infty} h(n)/n = 0$, the sequence

$$\left(\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)}, n \in \mathbb{N} \right)$$

satisfies the same large deviation principle as $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA), n \in \mathbb{N})$.

We will use a result of exponential equivalence. For (X_n) and (Y_n) two sequences of random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and for a given speed function $v(n)$ which goes to infinity with n , we say that (X_n) and (Y_n) are exponentially equivalent with regard to $v(n)$ if and only if for all positive ε , we have

$$\limsup_{n \rightarrow \infty} \frac{1}{v(n)} \log \mathbb{P}(|X_n - Y_n| \geq \varepsilon) = -\infty.$$

The following result is classical in large deviations theory (see [8], Theorem 4.2.13).

Theorem 5.9. *Let (X_n) and (Y_n) be two sequences of random variables defined on the same probability space $(\Omega, \mathcal{A}, \mathbb{P})$. If (X_n) satisfies a large deviation principle of speed $v(n)$ with a good rate function, and if (X_n) and (Y_n) are exponentially equivalent with regard to $v(n)$, then (Y_n) satisfies the same large deviation principle as (X_n) .*

We will prove that the sequences $(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ and $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA))$ are exponentially equivalent with regard to $\mathcal{H}^{d-1}(nA)$ under the assumptions that there exist exponential moment of the law of capacity of all orders and for height functions h satisfying $\lim_{n \rightarrow \infty} h(n)/n = 0$.

We take a hyperrectangle A and use the same notations as in Section 4.6. Let $\zeta \geq 2d$, and n large enough such that the sides of nA have length bigger than ζ . Let E_1^+ be the set of the edges that belong to \mathcal{E}_1^+ defined as

$$E_1^+ = \mathcal{V}(\text{cyl}(\partial(nA), h(n)), \zeta) \cap \text{cyl}(nA, h(n)).$$

We have for all $n \geq p$

$$\phi(nA, h(n)) \leq \tau(nA, h(n)) \leq \phi(nA, h(n)) + V(E_1^+).$$

Thus for all $\varepsilon > 0$, for all $n \geq p$, we obtain

$$\mathbb{P} \left(\left| \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} - \frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right| \geq \varepsilon \right) \leq \mathbb{P}(V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)).$$

We know that there exists a constant C^+ such that

$$\text{card}(E_1^+) \leq C^+ n^{d-2} h(n),$$

so for all $\varepsilon > 0$, for all $\gamma > 0$, for a family (t_k) of independent variables with the same law as the capacities of the edges, we have

$$\begin{aligned} \mathbb{P}[V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)] &\leq \mathbb{P}\left[\sum_{k=1}^{C^+ n^{d-2} h(n)} t_k \geq \varepsilon \mathcal{H}^{d-1}(nA)\right] \\ &\leq \mathbb{E}(e^{\gamma t})^{C^+ n^{d-2} h(n)} \exp(-\gamma \varepsilon \mathcal{H}^{d-1}(nA)) \\ &\leq \exp\left(-\mathcal{H}^{d-1}(nA) \left(\gamma \varepsilon - C^+ \frac{n^{d-2} h(n)}{\mathcal{H}^{d-1}(nA)} \log \mathbb{E}(e^{\gamma t})\right)\right). \end{aligned}$$

For a fixed $R > 0$, we can choose γ large enough to have $\gamma \varepsilon \geq 2R$, and also there exists n_2 such that for all $n \geq n_2$, we have

$$C^+ \frac{n^{d-2} h(n)}{\mathcal{H}^{d-1}(nA)} \log \mathbb{E}(e^{\gamma t}) \leq R,$$

so for all $R > 0$

$$\limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)] \leq -R$$

and then

$$\limsup_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}[V(E_1^+) \geq \varepsilon \mathcal{H}^{d-1}(nA)] = -\infty.$$

We obtain immediately that $(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ and $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA))$ are exponentially equivalent with regard to $\mathcal{H}^{d-1}(nA)$, and so by Theorem 5.9, $(\phi(nA, h(n))/\mathcal{H}^{d-1}(nA))$ satisfies the same large deviation principle as $(\tau(nA, h(n))/\mathcal{H}^{d-1}(nA))$.

6. Law of large numbers, large deviation principle and lower large deviations for ϕ in straight boxes

The main work is done in Section 6.1, where one proves that ϕ and τ share the same rate function in straight boxes. Then, the law of large numbers is proven in Section 6.2. The large deviation principle is proven in Section 6.3, as well as the deviation inequality from ν (Theorem 3.18).

6.1. Comparison between ϕ and τ

We prove in this section that under hypotheses **(F1)**, **(F4)**, **(H1)** and **(H2)**, the lower large deviations of $\phi(nA, h(n))$ and $\tau(nA, h(n))$ are of the same exponential order. The following proposition is the key to prove both Theorems 3.16 and 3.17.

Proposition 6.1. *Suppose that **(F1)**, **(F4)**, **(H1)** and **(H2)** hold. Let A be a non-degenerate straight hyperrectangle. Then, for every λ in \mathbb{R}^+ ,*

$$\lim_{n \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P}\left[\phi(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}}\right) \mathcal{H}^{d-1}(nA)\right] = \mathcal{I}_{\vec{v}}(\lambda),$$

where $\vec{v} = (0, \dots, 0, 1)$.

Proof. Since $\phi(nA, h(n)) \leq \tau(nA, h(n))$, we only need to show that

$$\liminf_{n \rightarrow \infty} \frac{-1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\phi(nA, h(n)) \leq \left(\lambda - \frac{1}{\sqrt{n}} \right) \mathcal{H}^{d-1}(nA) \right] \geq \mathcal{I}_{\bar{v}}(\lambda).$$

To shorten the notations, we shall suppose that

$$A = [0, 1]^{d-1} \times \{0\},$$

the general case of a straight hyperrectangle being handled exactly along the same lines. Notice that $\mathcal{H}^{d-1}(nA) = n^{d-1}$. As in Section 4.2, we shall write ϕ_n instead of $\phi(nA, h(n))$, and denote by E_{ϕ_n} a cut whose capacity achieves the minimum in the dual definition (1) of ϕ_n .

The idea of the proof is the following. The minimal cut E_{ϕ_n} has a certain intersection with the sides of the cylinder $\text{cyl}(A, h)$. Thanks to Zhang's result, Theorem 3.10, and after having eventually reduced a little the cylinder, one can prove that the intersection of E_{ϕ_n} with the sides of this reduced cylinder has less than $Cn^{d-1}/n^{1/3}$ edges with very high probability (here C is a constant). This shows that (with very high probability) ϕ_n is larger than the minimum of a collection of random variables $(\tau_F)_{F \in I_n}$, where F designs a possible trace of E_{ϕ_n} , i.e., its intersection with the sides of the reduced cylinder, and where I_n is the set of all the possible choices for F . Since E_{ϕ_n} itself has less than Cn^{d-1} edges, and since it is connected (in the dual sense), a trivial bound for the cardinal of I_n is roughly:

$$\text{card}(I_n) \leq h(n) (C'n^{2d-3})^{Cn^{d-1}/n^{1/3}}.$$

The important point here is that $\log \text{card}(I_n)$ is small compared to n^{d-1} . Having done this, a subadditive argument using symmetries can be performed to show that in fact the smallest τ_F (in distribution) behaves essentially like $\tau(nA, h(n))$, which has $\mathcal{I}_{\bar{v}}$ as a rate function.

Now, we turn to a formal proof. In the sequel, we shall suppose that n is large enough to ensure that

$$\log h(n) \leq n^{d-1}, \quad nl_{\min}(A) \geq t_0 \quad \text{and} \quad h(n) > 2\sqrt{d},$$

where t_0 is defined in Proposition 4.2. We consider $\gamma > 0$ such that $\mathbb{E}(\exp(\gamma t(e))) < \infty$. Let E_n be the cutset defined by $E_n = \{e = \langle x, y \rangle \in \mathbb{E}^d \mid x \in nA \text{ and } y_d = 1\}$. Notice that for n large enough,

$$\text{card}(E_n) \leq 2n^{d-1}.$$

For a fixed L , and for ε, C_1 and C_2 as in Proposition 4.2, using this proposition we obtain

$$\begin{aligned} \mathbb{P}(\text{card}(E_{\phi_n}) \geq Ln^{d-1}) &\leq \mathbb{P}(\text{card}(E_{\phi_n}) \geq Ln^{d-1} \text{ and } \phi_n \leq \varepsilon Ln^{d-1}) + \mathbb{P}(\phi_n \geq \varepsilon Ln^{d-1}) \\ &\leq C_1 h(n) \exp[-C_2 Ln^{d-1}] + \mathbb{P}(V(E_n) \geq \varepsilon Ln^{d-1}) \\ &\leq C_1 \exp[-(C_2 L - 1)n^{d-1}] + \mathbb{P}\left(\sum_{j=1}^{2n^{d-1}} t(e_j) \geq \varepsilon Ln^{d-1}\right) \\ &\leq C_1 \exp[-(C_2 L - 1)n^{d-1}] + \exp[-(\gamma \varepsilon L - 2 \log \mathbb{E}(e^{\gamma t(e)}))n^{d-1}]. \end{aligned}$$

Thus, there exist constants $\beta(F, d)$ and $C'_i(F, d)$ for $i = 1, 2$ such that for all $L \geq \beta$ and every n , we have

$$\mathbb{P}(\text{card}(E_{\phi_n}) \geq Ln^{d-1}) \leq C'_1(F, d) e^{-C'_2(F, d) Ln^{d-1}}.$$

We fix a real number $L \geq \beta$ to be chosen later. Define

$$\phi_{L,n} = \min\{V(E) \mid E \text{ is a } (B(nA, h(n)), T(nA, h(n)))\text{-cut in } \text{cyl}(nA, h(n)) \text{ and } \text{card}(E) \leq Ln^{d-1}\}.$$

Thus,

$$\mathbb{P}(\phi_n \leq \lambda n^{d-1}) \leq \mathbb{P}(\phi_{L,n} \leq \lambda n^{d-1}) + C'_1 e^{-C'_2 L n^{d-1}}. \quad (18)$$

We shall now concentrate on the first summand in the right-hand side of the last inequality. Let $\psi(n) = \lceil n^{1/3} \rceil$. For any k in $\{1, \dots, \psi(n)\}$, define

$$\begin{aligned} A_{n,k} &= [k, n-k]^{d-1} \times \{0\}, \\ B_{n,k} &= [k, n-k]^{d-1} \times [-h(n), h(n)] \end{aligned}$$

and

$$S_{n,k} = \partial([k, n-k]^{d-1}) \times [-h(n), h(n)].$$

In order to perform the announced subadditive argument, we shall need to patch together two cuts of neighbouring boxes which share a same trace in the intersection of these boxes. It is not so trivial to show that one obtains a cut doing so. This is why we shall impose a kind of ‘‘connection trace’’ on the sides of the box, which remembers if a vertex of the side is connected to the top or the bottom of the cylinder once the cut E_{ϕ_n} has been removed. Let us precise the needed definitions. If X is a subset of vertices of a subgraph G of \mathbb{Z}^d , we denote by $C_G(X)$ the union of all the connected components of G intersecting X . If v is a vertex of G , we write $C_G(v)$ instead of $C_G(\{v\})$. We shall say that a function x from some $S_{n,k}$ to $\{0, 1, 2\}$ is a *weak connection function* for F in $S_{n,k}$ if for every u and v in $S_{n,k}$,

$$u \in C_{S_{n,k} \setminus F}(v) \quad \Rightarrow \quad x(u) = x(v).$$

If E cuts $B(A_{n,k}, h(n))$ from $T(A_{n,k}, h(n))$ in $B_{k,n}$, we define x_E , the *connection function of E (in $B_{k,n}$)* as follows

$$\forall u \in \text{cyl}(A_{n,k}, h(n)) \quad x_E(u) = \begin{cases} 1 & \text{if } u \in C_{B_{k,n} \setminus E}(T(A_{n,k}, h(n))), \\ 0 & \text{if } u \in C_{B_{k,n} \setminus E}(B(A_{n,k}, h(n))), \\ 2 & \text{else.} \end{cases}$$

Clearly, \tilde{x}_E , the restriction of x_E to $S_{n,k}$ is a weak connection function for $E \cap S_{n,k}$. Then, define the following set of ‘‘good’’ couples (F, x) of a trace F and a weak connection function x :

$$\begin{aligned} I_n &= \bigcup_{k=1}^{\psi(n)} \bigcup_{h=-h(n)}^{h(n)-Ln^{d-1}} \left\{ (F, x) \mid F \subset \mathbb{E}^d \cap \partial([k, n-k]^{d-1}) \times [h, h+Ln^{d-1}], \text{card}(F) \leq \frac{Ln^{d-1}}{\psi(n)}, \right. \\ &\quad \left. x \text{ is a weak connection function for } F \text{ in } S_{n,k} \right\}. \end{aligned}$$

If F satisfies the conditions in the above definition, then there are at most $2^d Ln^{d-1} / \psi(n)$ distinct connected components in $S_{n,k} \setminus F$. Thus, for a fixed F , there are at most $3^{2^d Ln^{d-1} / \psi(n)}$ distinct weak configuration functions x such that (F, x) belongs to I_n . Thus, there is a constant C_3 , which depends only on d , such that

$$(2h(n) + 1) \leq \text{card}(I_n) \leq 2h(n)\psi(n)(C_3 Ln^{2d-3})^{Ln^{d-1} / \psi(n)}. \quad (19)$$

On the other hand, define, for (F, x) in I_n and k such that $F \subset S_{n,k}$,

$$\begin{aligned} C_{F,x} &= \{E \subset \mathbb{E}^d \mid E \text{ is a } (B(A_{n,k}, h(n)), T(A_{n,k}, h(n)))\text{-cut in } B_{n,k}, \\ &\quad E \cap S_{n,k} = F, \text{card}(E) \leq Ln^{d-1} \text{ and } \tilde{x}_E = x\} \end{aligned}$$

and

$$\tau_{(F,x)} = \min\{V(E) \mid E \in C_{F,x}\}.$$

We claim that

$$\phi_{L,n} \geq \min_{(F,x) \in I_n} \tau_{(F,x)}. \tag{20}$$

To see why (20) is true, notice that for any k in $\{1, \dots, \psi(n)\}$, $E_{\phi_{L,n}} \cap B_{n,k}$ cuts $B(A_{n,k}, h(n))$ from $T(A_{n,k}, h(n))$ in $B_{n,k}$, and has less than Ln^{d-1} edges. $E_{\phi_{L,n}}$ is connected in the dual sense (see the proof of Lemma 12 in [22]), and has less than Ln^{d-1} edges. Then there is an h such that $E_{\phi_{L,n}}$ is included in $[0, n]^{d-1} \times [h, h + Ln^{d-1}]$. Thus, there is an h such that $E_{\phi_{L,n}} \cap B_{n,k}$ is included in $[k, n - k]^{d-1} \times [h, h + Ln^{d-1}]$. Furthermore, since $S_{n,1}, \dots, S_{n,\psi(n)}$ are pairwise disjoint, there is at least one k in $\{1, \dots, \psi(n)\}$ such that

$$\text{card}(E_{\phi_{L,n}} \cap S_{n,k}) \leq \frac{\text{card}(E_{\phi_{L,n}})}{\psi(n)} \leq \frac{Ln^{d-1}}{\psi(n)}.$$

Thus, denoting $F = E_{\phi_{L,n}} \cap S_{n,k}$ and $x = \tilde{x}_{E_{\phi_{L,n}} \cap S_{n,k}}$, this shows that $\phi_{L,n} \geq \tau_{(F,x)}$, and claim (20) is proved.

Now, we need to show that $\min_{(F,x) \in I_n} \tau_{(F,x)} / n^{d-1}$ has lower large deviations given by $\mathcal{I}_{\tilde{v}}$. First, notice that

$$\mathbb{P}(\phi_{L,n} \leq \lambda n^{d-1}) \leq \mathbb{P}\left(\min_{(F,x) \in I_n} \tau_{(F,x)} \leq \lambda n^{d-1}\right) \leq \sum_{(F,x) \in I_n} \mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1}). \tag{21}$$

Since, according to inequality (19), $\log \text{card}(I_n)$ is small compared to n^{d-1} , we shall be done if we can show that, uniformly in $(F, x) \in I_n$, the probability of deviation $\mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1})$ is asymptotically of order at most $\exp(-\mathcal{I}_{\tilde{v}}(\lambda)n^{d-1})$. We shall do this using a subadditivity argument. From now on, we fix (F, x) in I_n and k such that $F \subset S_{n,k}$. The notations and rigorous proofs are a little cumbersome, but everything can be guessed in two stages, looking at Figs 3 and 4.

Let N be an integer such that for every $N' \geq N$, $h(2(n - 2k)N') \geq h(n)$. Define, for $i = 1, \dots, d - 1$, the following hyperplanes:

$$H_i = \mathbb{R}^{i-1} \times \{n - k\} \times \mathbb{R}^{d-i}.$$

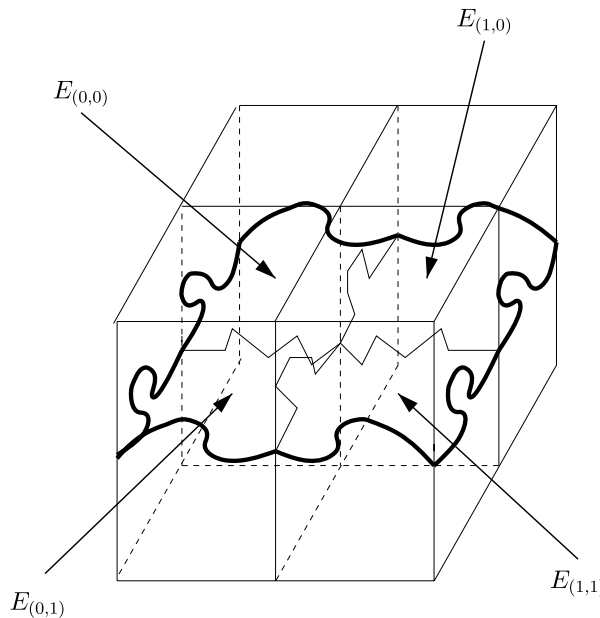


Fig. 3. Patching E_b for $b \in \{0, 1\}^{d-1}$ when $d = 3$.

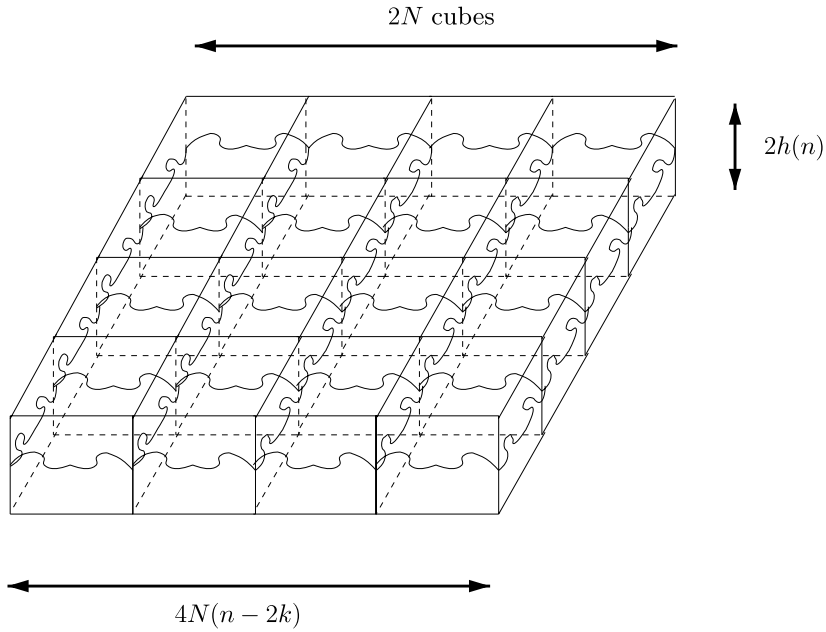


Fig. 4. Patching cuts with the same perimeter.

We define σ_i to be the affine orthogonal reflection relative to H_i , and tr_i to be the following translation along coordinate i :

$$\text{tr}_i(z) = z + 2(n - 2k)\mathbf{e}_i,$$

where $(\mathbf{e}_1, \dots, \mathbf{e}_d)$ is the canonical orthonormal basis of \mathbb{R}^d . For any $b \in \{-2N, \dots, 2N - 1\}^{d-1}$, we define the map σ_b as follows. For every i in $\{1, \dots, d - 1\}$, let $a_i = \lfloor b_i/2 \rfloor$ and $c_i = b_i - 2a_i$. Then, we denote by σ_b the (commutative) product of translations and reflections $\prod_{i=1}^{d-1} \text{tr}_i^{a_i} \circ \prod_{i=1}^{d-1} \sigma_i^{c_i}$, where $\sigma_i^{c_i}$ (resp., $\text{tr}_i^{a_i}$) is the c_i th iterate of σ_i (respectively the a_i th iterate of tr_i). Finally, we define also, for any set of vertices or set of edges X ,

$$\sigma_N(X) = \bigcup_{b \in \{-2N, \dots, 2N-1\}^{d-1}} \sigma_b(X)$$

and

$$\tilde{\sigma}_N(X) = \sigma_N(X) \cap S_N,$$

where

$$S_N = \partial([k - 2N(n - 2k), k + 2N(n - 2k)]^{d-1}) \times [-h(n), h(n)].$$

The following lemma should be intuitive looking at Figs 3 and 4. In words, the main message of this lemma (assertion (ii)) is the following. Let E (resp., E') be a cut between the top and the bottom in some box B (resp., B'). Suppose that B and B' share exactly a face, and that the connection functions of E and E' coincide on this face. Then, $E \cup E'$ is a cut between the top and the bottom in $B \cup B'$. Notice that assertion (i) is just an obvious property of symmetry: if you take a cut E between the top and the bottom in a box B , then $\sigma_b(E)$ is a cut between the top and the bottom in $\sigma_b(B)$, for any b .

Lemma 6.2. Let (F, x) be fixed in I_n . Suppose that for every $b \in \{-2N, \dots, 2N-1\}^{d-1}$, we are given a set E_b of edges that cuts $B(\sigma_b(A_{n,k}), h(n))$ from $T(\sigma_b(A_{n,k}), h(n))$ in $\sigma_b(B_{n,k})$. Let $\mathbf{0}$ denote $(0, \dots, 0)$ and define

$$E = \bigcup_{b \in \{-2N, \dots, 2N-1\}^{d-1}} E_b.$$

(i) If $E_{\mathbf{0}} \cap S_{n,k} = F$, and $\tilde{x}_{E_{\mathbf{0}}} = x$, then for every $b \in \{-2N, \dots, 2N-1\}^{d-1}$, the set of edges $\sigma_b(E_{(0, \dots, 0)})$ cuts $B(\sigma_b(A_{n,k}), h(n))$ from $T(\sigma_b(A_{n,k}), h(n))$ in $\sigma_b(B_{n,k})$, has configuration function $x \circ \sigma_b^{-1}$, and satisfies

$$\sigma_b(E_{(0, \dots, 0)}) \cap \sigma_b(S_{n,k}) = \sigma_b(F).$$

(ii) If, for every $b \in \{-2N, \dots, 2N-1\}^{d-1}$,

$$\tilde{x}_{E_b} \circ \sigma_b = x,$$

then E cuts $B(\sigma_N(A_{n,k}), h(n))$ from $T(\sigma_N(A_{n,k}), h(n))$ in $\sigma_N(B_{n,k})$.

Proof. Assertion (ii) is the only non-trivial point to show. Let b and b' be two members of $\{-2N, \dots, 2N-1\}^{d-1}$. The hypotheses on the cuts E_b and $E_{b'}$ ensure that x_{E_b} and $x_{E_{b'}}$ coincide on $\sigma_b(B_{n,k}) \cap \sigma_{b'}(B_{n,k})$. Thus, we can extend all the functions $(x_b)_{b \in \{-2N, \dots, 2N-1\}^{d-1}}$ in a single function x on $\sigma_N(B_{n,k})$. This implies that for every two neighbours u and v in $\sigma_N(B_{n,k})$, if $\langle u, v \rangle \notin E$, then $x(u) = x(v)$. Thus, x is constant on each connected component of $\sigma_N(B_{n,k}) \setminus E$. Since in each box $\sigma_b(B_{n,k})$, E_b cuts $B(\sigma_b(A_{n,k}), h(n))$ from $T(\sigma_b(A_{n,k}), h(n))$, we have that x takes the value 1 on $B(\sigma_N(A_{n,k}), h(n))$, and 0 on $T(\sigma_N(A_{n,k}), h(n))$. Thus, these two sets are disconnected in $\sigma_N(B_{n,k}) \setminus E$. \square

Now, for every $b \in \mathbb{Z}^{d-1}$, define

$$\begin{aligned} \mathcal{C}_{F,x,b} = \{ E \subset \mathbb{E}^d \mid E \text{ is a } (B(\sigma_b(A_{n,k}), h(n)), T(\sigma_b(A_{n,k}), h(n)))\text{-cut in } \sigma_b(B_{n,k}), \\ E \cap \sigma_b(S_{n,k}) = \sigma_b(F), \text{card}(E) \leq Ln^{d-1} \text{ and } \tilde{x}_E \circ \sigma_b = x \} \end{aligned}$$

and

$$\tau_{(F,x,b)} = \min\{V(E) \mid E \in \mathcal{C}_{F,x,b}\}.$$

For every N , let E_N denote the set of the edges e in $\sigma_N(B_{n,k})$ such that at least one endpoint of e belongs to S_N . Define $M(N) = N + \psi(N)$ and, for $N' \in \{N, M(N)\}$,

$$\tau_{N'} = \tau(\sigma_{N'}(A_{n,k}), h(N')).$$

If E is a $(B(\sigma_N(A_{n,k}), h(n)), T(\sigma_N(A_{n,k}), h(n)))$ -cut in $\sigma_N(A_{n,k})$, $E \cup E_{N'}$ clearly cuts $\sigma_N(A_{n,k})_1^{h(n)}$ from $\sigma_N(A_{n,k})_2^{h(n)}$. Thus, part (ii) of Lemma 6.2 gives us that

$$\sum_{b \in \{-2M(N), \dots, 2M(N)-1\}^{d-1}} \tau_{(F,x,b)} + \min_{N' \in \{N, \dots, M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \min_{N' \in \{N, \dots, M(N)\}} \tau_{N'}.$$

Notice that some edges are counted twice on the left-hand side of the preceding inequality. From part (i) of Lemma 6.2, we know that the random variables $(\tau_{(F,x,b)})_{b \in \{-2M(N), \dots, 2M(N)-1\}^{d-1}}$ are identically distributed, with the same distribution as $\tau_{(F,x)}$. Using the FKG inequality,

$$\begin{aligned} \mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1})^{(4M(N))^{d-1}} &= \prod_{b \in \{-2M(N), \dots, 2M(N)-1\}^{d-1}} \mathbb{P}(\tau_{(F,x,b)} \leq \lambda n^{d-1}) \\ &\leq \mathbb{P}(\forall b \in \{-2M(N), \dots, 2M(N)-1\}^{d-1}, \tau_{(F,x,b)} \leq \lambda n^{d-1}) \end{aligned}$$

$$\begin{aligned} &\leq \mathbb{P}\left(\sum_{b \in \{-2M(N), \dots, 2M(N)-1\}^{d-1}} \tau_{(F,x,b)} \leq \lambda n^{d-1} (4M(N))^{d-1}\right) \\ &\leq \mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \tau_{N'} - \min_{N' \in \{N, \dots, M(N)\}} \sum_{e \in E_{N'}} t(e) \leq \lambda n^{d-1} (4M(N))^{d-1}\right). \end{aligned}$$

Let $\varepsilon > 0$ be a fixed positive real number:

$$\begin{aligned} \mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1}) &\leq \left(\mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \tau_{N'} \leq (\lambda + \varepsilon)n^{d-1} (4M(N))^{d-1}\right)\right. \\ &\quad \left.+ \mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\right)\right)^{1/(4M(N))^{d-1}}. \end{aligned} \tag{22}$$

Now we shall let N go to infinity. Using Lemma 5.1, the fact that $\lim_{N \rightarrow \infty} \psi(N)/N = 0$ and a union bound,

$$\begin{aligned} &\liminf_{N \rightarrow \infty} -\frac{1}{(4N(n-2k))^{d-1}} \log \mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \tau_{N'} \leq (\lambda + \varepsilon)n^{d-1} (4M(N))^{d-1}\right) \\ &\geq \liminf_{N \rightarrow \infty} -\frac{1}{(4N(n-2k))^{d-1}} \max_{N' \in \{N, \dots, M(N)\}} \log \mathbb{P}\left(\tau_{N'} \leq \left(\lambda + 2\varepsilon - \frac{1}{\sqrt{4(n-2k)N'}}\right)n^{d-1} (4N')^{d-1}\right) \\ &\geq \mathcal{I}_{\bar{v}}\left((\lambda + 2\varepsilon)\left(\frac{n}{n-2k}\right)^{d-1}\right). \end{aligned} \tag{23}$$

Now, we use the fact that F possesses an exponential moment, and that the sets $E_{N'}$ are disjoint. Using Chebyshev inequality, there are positive constants C_4 and C_5 , depending only on F and d , such that

$$\begin{aligned} &\mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\right) \\ &= \prod_{N' \in \{N, \dots, M(N)\}} \mathbb{P}\left(\sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\right), \\ &\leq (e^{C_4 h(n)n^{d-2} M(N)^{d-2} - C_5 n^{d-1} (4M(N))^{d-1}})^{\psi(N)}. \end{aligned}$$

Thus,

$$\liminf_{N \rightarrow \infty} -\frac{1}{(4N(n-2k))^{d-1}} \log \mathbb{P}\left(\min_{N' \in \{N, \dots, M(N)\}} \sum_{e \in E_{N'}} t(e) \geq \varepsilon n^{d-1} (4M(N))^{d-1}\right) = +\infty.$$

Therefore, inequalities (22) and (23) imply

$$-\frac{1}{n^{d-1}} \log \mathbb{P}(\tau_{(F,x)} \leq \lambda n^{d-1}) \geq \frac{(n-2k)^{d-1}}{n^{d-1}} \mathcal{I}_{\bar{v}}\left((\lambda + 2\varepsilon)\left(\frac{n}{n-2k}\right)^{d-1}\right).$$

We choose $\varepsilon = \frac{1}{n}$, and replace λ by $\lambda - \frac{1}{\sqrt{n}}$ to get

$$-\frac{1}{n^{d-1}} \log \mathbb{P}\left(\tau_{(F,x)} \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)n^{d-1}\right) \geq \frac{(n-2k)^{d-1}}{n^{d-1}} \mathcal{I}_{\bar{v}}\left(\left(\lambda - \frac{1}{\sqrt{n}} + \frac{2}{n}\right)\left(\frac{n}{n-2k}\right)^{d-1}\right).$$

Since $k \leq \psi(n)$ and $\psi(n)$ is small compared to \sqrt{n} , and since $\mathcal{I}_{\vec{v}}$ is non-increasing, for n large enough,

$$-\frac{1}{n^{d-1}} \log \mathbb{P}\left(\tau_{(F,x)} \leq \left(\lambda - \frac{1}{\sqrt{n}}\right)n^{d-1}\right) \geq \frac{(n-2k)^{d-1}}{n^{d-1}} \mathcal{I}_{\vec{v}}(\lambda).$$

Using inequalities (19) and (21),

$$\liminf_{n \rightarrow \infty} -\frac{1}{n^{d-1}} \log \mathbb{P}(\phi_{L,n} \leq \lambda n^{d-1}) \geq \mathcal{I}_{\vec{v}}(\lambda).$$

And thus, from inequality (18),

$$\liminf_{n \rightarrow \infty} -\frac{1}{n^{d-1}} \log \mathbb{P}(\phi_n \leq \lambda n^{d-1}) \geq \min\{\mathcal{I}_{\vec{v}}(\lambda), C'_2 L\}.$$

Letting L tend to infinity finishes the proof of Proposition 6.1. □

Remark 6.3. This “symmetric-subadditive” argument does not work in the “non-straight” case. It is perhaps important to note that in this case, it is not obvious at all to know in advance for which F the random variable τ_F has the “minimal” distribution. It is natural to conjecture that this “minimal” F is a hyperrectangle, but we do not know how to prove this for all dimensions. When $d = 2$, though, we are able to solve this problem and to show that if $h(n)/n$ converges towards $\tan(\alpha)$ for some α in $[0, \pi/2]$, and if $\vec{v} = (\cos \theta, \sin \theta) = \vec{v}_\theta$ is orthogonal to $A = A_\theta$, then $\phi(nA_\theta, h(n))/n$ converges towards $\min\{v(\vec{v}_\theta)/\cos(\tilde{\theta} - \theta) \text{ s.t. } |\tilde{\theta} - \theta| \leq \alpha\}$. A similar method gives an analog result for the lower large deviations. This will be done rigorously in a forthcoming paper.

6.2. Law of large numbers

In this section, we prove Theorem 3.16. So we suppose that **(F2)**, **(H1)** and **(H2)** hold, and that A is a straight (so non-degenerate) hyperrectangle. Notice first that if **(F1)** does not hold, then v always equals zero (cf. Proposition 3.7) and the law of large numbers for ϕ is a consequence of the one for τ , Theorem 3.8. Thus, we may suppose that **(F1)** holds. We first prove the a.s. convergence of the rescaled variable. Since $\phi(nA, h(n)) \leq \tau(nA, h(n))$, we only need to show that

$$\liminf_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq v(\vec{v}_0) \quad \text{a.s.}, \tag{24}$$

where $\vec{v}_0 = (0, \dots, 0, 1)$.

Suppose first that F has bounded support. Then, **(F4)** is obviously satisfied, and we deduce from Proposition 6.1, the positivity of $\mathcal{I}_{\vec{v}}$ on $[0, v(\vec{v})[$ and Borel–Cantelli’s lemma that (24) is true.

Now, let F be general, i.e. satisfy **(F2)**. We rely on the ideas of Proposition 4.3. Let $a > 1/2$ be a real number to be chosen later, define $\tilde{t}(e) = t(e) \wedge a$ and let F_a be the distribution function of $\tilde{t}(e)$. We define

$$\tilde{\tau}_n = \min \left\{ \sum_{e \in E} \tilde{t}(e) \text{ s.t. } E \text{ cuts } (nA)_1^{h(n)} \text{ from } (nA)_2^{h(n)} \text{ in } \text{cyl}(nA, h(n)) \right\},$$

and define analogously $\tilde{\phi}_n$. We use the notations $v_F(\vec{v})$ (resp. $v_{F_a}(\vec{v})$) to denote the limit of the rescaled flow τ corresponding to capacities of distribution function F (resp., F_a). As we obtained (5), we get

$$\mathbb{E}(\tau_n) - \mathbb{E}(\tilde{\tau}_n) \leq \mathbb{E}(t(e_1) \mathbb{1}_{t(e_1) \geq a}) \mathbb{E}(\text{card } E_{\tilde{\tau}_n}).$$

Proposition 4.2 implies that there are constants ε , C_1 and C_2 such that

$$\mathbb{E}(\text{card } E_{\tilde{\tau}_n}) \leq \frac{C_1}{C_2} + \frac{1}{\varepsilon} \mathbb{E}(\tilde{\tau}_n) \leq \frac{C_1}{C_2} + \frac{1}{\varepsilon} \mathbb{E}(\tau_n),$$

where the constants ε , C_1 and C_2 depend only on d and F , and not on a , since F and F_a coincide on $[0, 1/2]$. Then, for any $\varepsilon' > 0$, one can choose a large enough so that

$$\mathbb{E}(\tau_n) - \mathbb{E}(\tilde{\tau}_n) \leq \varepsilon \mathcal{H}^{d-1}(nA),$$

leading to $\nu_F(\vec{v}_0) - \nu_{F_a}(\vec{v}_0) \leq \varepsilon$. Since $\phi_n \geq \tilde{\phi}_n$, and using the result for F_a which has bounded support, we get for every $\varepsilon' > 0$:

$$\liminf_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \nu_{F_a}(\vec{v}_0) \geq \nu_F(\vec{v}_0) - \varepsilon \quad \text{a.s.}$$

which gives the desired result.

It remains to prove the convergence in L^1 . It may be derived exactly as in the proof of the convergence in L^1 of Theorem 3.8 as soon as we have proved the convergence of the expectation of the rescaled maximal flow. But this is immediate thanks to Fatou's lemma:

$$\begin{aligned} \nu(\vec{v}_0) &= \mathbb{E} \left[\lim_{n \rightarrow \infty} \frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right] \leq \liminf_{n \rightarrow \infty} \mathbb{E} \left[\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right] \\ &\leq \limsup_{n \rightarrow \infty} \mathbb{E} \left[\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right] \leq \lim_{n \rightarrow \infty} \mathbb{E} \left[\frac{\tau(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \right] = \nu(\vec{v}_0). \end{aligned}$$

This ends the proof of Theorem 3.16.

Remark 6.4. Our proof of Proposition 6.1 can be carried out in Kesten and Zhang's setting [22], who consider $A_{\mathbf{k}} = \prod_{i=1}^{d-1} [0, k_i] \times \{0\}$ with $k_1 \leq \dots \leq k_{d-1}$ and let all the k_i go to infinity, possibly at different speeds. The only obstacle to do this is when one reduces the sides of the box: $\psi(n)$ has to be replaced by $\psi(k_1)$, and the set I_n by a set $I_{\mathbf{k}}$ satisfying:

$$\text{card}(I_{\mathbf{k}}) \leq C_3 h(\mathbf{k}) \psi(k_1) \left(C_4 L \prod_{i=1}^{d-1} k_i \right)^{LC_5 \prod_{i=1}^{d-1} k_i / \psi(k_1)},$$

where C_3 , C_4 and C_5 are constants depending on d and $h(\mathbf{k})$ is the height of the box. Then, the proof works as long as $\log \text{card}(I_{\mathbf{k}})$ is small with respect to $\prod_{i=1}^{d-1} k_i$, which is the case if $\log h(\mathbf{k})$ is small with respect to $\prod_{i=1}^{d-1} k_i$ and $\log k_{d-1}$ is small with respect to k_1 . Thus, we obtain the law of large numbers (and also Proposition 6.1) under the conditions:

$$\begin{cases} (\log h(\mathbf{k})) / \prod_{i=1}^{d-1} k_i \xrightarrow{\mathbf{k} \rightarrow \infty} 0, \\ (\log k_{d-1}) / k_1 \xrightarrow{\mathbf{k} \rightarrow \infty} 0 \end{cases}$$

and condition **(F1)**. So, in a sense, the height condition is better than in Theorem 3.4 (and essentially optimal), however we are not able to get rid of the second ugly condition – which imposes that the sides of A do not have too different asymptotic behaviours – without requiring a stronger condition on h , similar to the one of Kesten and Zhang.

6.3. Final steps of the proofs of Theorems 3.18 and 3.10

The proof of Theorem 3.18 is exactly the same as the one of Theorem 3.9, using Theorem 3.16 and Proposition 4.3. It remains to end the proof of Theorem 3.17. Proposition 6.1 states in a sense that ϕ and τ share the same rate function. Since this function has already been studied, and since the upper large deviations of ϕ have been studied in [17], the construction of the rate function of ϕ was the main work to do in order to show the large deviation principle for ϕ in straight boxes. Indeed, the only thing we have to prove is that for all $\lambda > \nu(\vec{v})$,

$$\lim_{n \rightarrow \infty} \frac{1}{\mathcal{H}^{d-1}(nA)} \log \mathbb{P} \left[\frac{\phi(nA, h(n))}{\mathcal{H}^{d-1}(nA)} \geq \lambda \right] = -\infty. \tag{25}$$

As soon as we have (25), we can write exactly the same proof for Theorem 3.17 as for Theorem 3.10 (see Section 5.6), since we have proved that $\phi(nA, h(n))/\mathcal{H}^{d-1}(nA)$ converges a.s. to $v(\vec{v}_0)$. To obtain (25), we can refer to Section 3.7 in [17] (here only the existence of one exponential moment is required).

Remark 6.5. *We leave the following questions open: is condition (F4) necessary to obtain the existence of a rate function for ϕ ? If this rate function exists under weaker hypothesis than (F4), is it necessarily the same as the one for τ ? When the rate function exists, do we necessarily obtain the corresponding large deviation principle?*

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