Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2010, Vol. 46, No. 4, 1001–1024
DOI: 10.1214/09-AIHP337

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Heat kernel for random walk trace on $\mathbb{Z}^3$ and $\mathbb{Z}^4$

Daisuke Shiraishi

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-85 02, Japan. E-mail: daisuke@math.kyoto-u.ac.jp

Received 24 February 2009; revised 8 August 2009; accepted 11 August 2009

Abstract. We study the simple random walk X on the range of simple random walk on $\mathbb{Z}^3$ and $\mathbb{Z}^4$. In dimension four, we establish quenched bounds for the heat kernel of X and $\max_{0\le k\le n}|X_k|$ which require extra logarithmic correction terms to the higher-dimensional case. In dimension three, we demonstrate anomalous behavior of X at the quenched level. In order to establish these estimates, we obtain several asymptotic estimates for cut times of simple random walk and asymptotic estimates for loop-erased random walk, which are of independent interest.

Résumé. Nous étudions la marche aléatoire simple sur l'ensemble des points visités par une marche aléatoire simple sur $\mathbb{Z}^3$ et $\mathbb{Z}^4$. En dimension quatre, nous établissons des bornes presque sûres pour le noyau de la chaleur de X et pour $\max_{0\le k\le n}|X_k|$ qui nécessitent des termes correctifs logarithmiques. En dimension trois, nous montrons que X à un comportement non diffusif presque sûrement. Pour démontrer ces résultats, nous obtenons des estimées asymptotiques pour les temps de coupure de la marche aléatoire simple et pour la marche à boucles effacées qui sont intéressantes en elles-mêmes.

MSC: 82C41

Keywords: Random walk in random environment; Random walk trace; Heat kernel estimates; Cut time; Loop erased random walk

1. Introduction and main results

1.1. *Introduction*

The problem of the simple random walk X on the range (trace) of the simple random walk on $\mathbb{Z}^d$ has received attention both in the literature of physics and mathematics (see Section 1.2, for a precise definition of this process). Recently, several papers have studied the behavior of X [2,3]. For example, for an electrical network on a locally finite graph, it is shown in [2] that the trace, the set of edges traversed by the associated random walk, considered as an electrical network with unit conductances placed along each edge, is recurrent a.s.

Given the general recurrence result of [2], it is natural to study further properties of X when the original graph is $\mathbb{Z}^d$. For $d\ge 5$, X behaves in a diffusive fashion similar to the simple random walk on $\mathbb{Z}$. Roughly speaking, the intersections of the original simple random walk path are sparse and give no effect on the asymptotic behavior of X. See [3] for details. On the other hand, since the original simple random walk path intersects itself more complicatedly for $d\le 4$, it is interesting to consider this problem when $d=3,4$. (Note that when $d=1,2$ the recurrence of the original random walk easily implies that the range of the random walk is equal to the whole lattice a.s., so the law of X is the same as the original random walk.) For $d=4$, it is shown in [3] that logarithmic corrections are required in describing the asymptotic behavior of X compared to higher dimensions. Indeed, in [3], annealed bounds for the heat kernel of X and $\max_{0\le k\le n}|X_k|$ are obtained and they require extra logarithmic correction terms to the higher-dimensional case. (Here $|\cdot|$ denotes the Euclidean distance.)

In this paper, we establish further properties of X for $d=4$ and obtain an upper heat kernel estimate of X for $d=3$. For $d=4$, we prove quenched bounds for the heat kernel of X (Theorem 1.2.1) and $\max_{0\le k\le n}|X_k|$ (Theorem 1.2.2) with logarithmic corrections. For $d=3$, our heat kernel estimate (Theorem 1.2.3) shows anomalous behavior of X at the quenched level. Indeed, it allows us to conclude that $d_s>1$, where d_s denotes the quenched spectral dimension of the random walk X (see (1.1) for definition) which contrasts with $d\ge 4$ where $d_s=1$. Thus, in the language of statistical mechanics, our results provide further justification for the claim of [3] that the critical dimension of the random walk on the range of random walk is 4.

It has recently been established [1,8] that in order to obtain heat kernel bounds, it is enough to obtain estimates on volume and effective resistance. Thus, applying these results, to obtain our conclusions, we are only required to consider volume and resistance. To deduce upper and lower bounds for the volume, it is useful to estimate the cut-times and loop-erasure of the original simple random walk, respectively (cf. [3]). There are many deep results regarding cut-times and the loop-erased random walk (see, for example [9–12]), but unfortunately some of the estimates we need do not appear in the literature. We thus establish a number of further estimates for cut-times and the loop-erased random walk (for example Corollary 2.2.4, Proposition 4.2.4), which are of independent interest.

Throughout this paper we use $c, c_1, c_2, \dots$ to denote arbitrary positive constants, depending only on dimensions, which may change from line to line. If a constant is to depend on some other quantity, this will be made explicit. For example, if c depends on δ, we write c_δ.

If $g(x), h(x)$ are functions we write $g\sim h$ if they are asymptotic, i.e.,

$$\lim_{x\to\infty}\frac{h(x)}{g(x)}=1.$$

We write $g\asymp h$ if there exist $c_1, c_2>0$ such that

$$c_1 g(x)\le h(x)\le c_2 g(x)\quad \text{for all } x.$$

1.2. *Framework and main results*

Let $S=(S_n)_{n\ge 0}$ be the simple random walk on $\mathbb{Z}^d$ starting from 0, built on underlying probability space $(\Omega, \mathcal{F}, P)$. Define the range of the random walk $S(\omega)$ to be the graph $\mathcal{G}(\omega)=(V(\mathcal{G}(\omega)), E(\mathcal{G}(\omega)))$ with vertex set

$$V\big(\mathcal{G}(\omega)\big):=\big\{S_n(\omega)\colon n\ge 0\big\},$$

and edge set

$$E\big(\mathcal{G}(\omega)\big):=\big\{\big\{S_n(\omega), S_{n+1}(\omega)\big\}\colon n\ge 0\big\},$$

where ω is an element of Ω. (For simplicity, we often omit ω.) Let $\mu_{\mathcal{G}}(x)$ be the number of bonds that contain x, i.e.,

$$\mu_{\mathcal{G}}(x)=\sharp\big\{\{x,y\}\in E\big(\mathcal{G}\big)\big\}.$$

We extend $\mu_{\mathcal{G}}$ to a measure on $\mathcal{G}$ by setting $\mu_{\mathcal{G}}(A)=\sum_{x\in A}\mu_{\mathcal{G}}(x)$ for $A\subset\mathcal{G}$.

We denote the simple random walk on $\mathcal{G}(\omega)$ by

$$X=\big((X_n)_{n\ge 0}, P_x^{\mathcal{G}(\omega)}, x\in V\big(\mathcal{G}(\omega)\big)\big),$$

and its heat kernel (transition density) with respect to $\mu_{\mathcal{G}(\omega)}$ by $h_n^{\mathcal{G}(\omega)}(x,y)$, i.e.,

$$h_n^{\mathcal{G}(\omega)}(x,y)=P_x^{\mathcal{G}(\omega)}(X_n=y)\frac{1}{\mu_{\mathcal{G}(\omega)}(y)}.$$

To define X we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega\times\overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$.

The following theorems are our main results in this paper.

Theorem 1.2.1. *Let $d = 4$. For each $\delta \in (0,1)$, there exist a $c > 0$ and $\Omega_1 \subset \Omega$ with $P(\Omega_1) = 1$ satisfying: for each $\omega \in \Omega_1$, there exists $N_1(\omega) < \infty$ such that*

$$n^{-1/2}(\log n)^{-3/2-\delta} \le h_{2n}^{\mathcal{G}(\omega)}(0,0) \le cn^{-1/2}(\log n)^{-1/6} \quad \forall n \ge N_1(\omega).$$

Theorem 1.2.2. *Let $d = 4$. For each $\delta \in (0,1)$, there exists $\Omega_2 \subset \Omega$ with $P(\Omega_2) = 1$ satisfying: for each $\omega \in \Omega_2$, there exists $N_2(\omega, \overline{\omega})$ with $P_0^{\mathcal{G}(\omega)}(N_2(\omega,\overline{\omega}) < \infty) = 1$ such that*

$$n^{1/4}(\log n)^{1/24-\delta} \le \max_{1\le k\le n} |X_k| \le n^{1/4}(\log n)^{13/12+\delta} \quad \forall n \ge N_2(\omega, \overline{\omega}),$$

where $|\cdot|$ denotes the Euclidean distance.

From the above results, we see that for $d = 4$ the process X and its heat kernel do not satisfy the same scaling results as in the higher dimensional case, but exhibit logarithmic corrections to the leading polynomial order.

When $d = 3$, we have the following result that shows the anomalous behavior of X.

Theorem 1.2.3. *Let $d = 3$. There exist a $r > 0$ and $\Omega_3 \subset \Omega$ with $P(\Omega_3) = 1$ satisfying: for each $\omega \in \Omega_3$, there exists $N_3(\omega) < \infty$ such that*

$$h_{2n}^{\mathcal{G}(\omega)}(0,0) \le n^{-10/19}(\log n)^r \quad \forall n \ge N_3(\omega).$$

Define the quenched spectral dimension of the random walk X by the limits

$$d_s := \lim_{n\to\infty} \frac{2\log h_{2n}^{\mathcal{G}(\omega)}(0,0)}{-\log n}, \tag{1.1}$$

when it exists. From Theorem 1.2.1 and Theorem 1.2.3, we conclude that $d_s = 1$, P-a.s., for $d = 4$ and $d_s \ge \frac{20}{19} > 1$, P-a.s., for $d = 3$. Thus, we see that the critical dimension of this model is 4. We cannot determine the exact value of d_s for $d = 3$. Numerical simulations suggest that $d_s \approx \frac{8}{7}$ in this dimension [6].

We now begin to prove main theorems. We will give the full proofs of Theorems 1.2.1, 1.2.2 and 1.2.3 in Sections 2–4 respectively.

2. Proof of Theorem 1.2.1

2.1. *Upper bound*

In this subsection we will prove the upper bound of Theorem 1.2.1. By [8], Proposition 3.1, we need to estimate the lower bound of volume. We first give some notions that are used in the proof. Recall the setting described in Section 1.2. Let $d = 4$. For a finite simple random walk path $\lambda = [\lambda(0), \dots, \lambda(m)]$ of length m, assign a self-avoiding walk path $L\lambda$ in the following way. Let

$$\sigma_0 = \sup\{j\colon \lambda(j) = \lambda(0)\},$$

and for $i > 0$,

$$\sigma_i = \sup\{j\colon \lambda(j) = \lambda(\sigma_{i-1} + 1)\}.$$

Let

$$l = \inf\{i\colon \sigma_i = m\}.$$

Now define

$$\hat{\lambda}(i) = \lambda(\sigma_i),$$

and

$$L\lambda = \big[\hat{\lambda}(0), \hat{\lambda}(1), \dots, \hat{\lambda}(l)\big].$$

This self-avoiding path clearly satisfies $(L\lambda)(0) = \lambda(0)$ and $(L\lambda)(l) = \lambda(m)$.

We let $d_{\mathcal{G}}(\cdot,\cdot)$ be the shortest path graph distance on $\mathcal{G}$.

Proposition 2.1.1. *If $d = 4$, there exists $c > 0$ such that*

$$P\Big(\max_{1 \le m \le n} d_{\mathcal{G}}(0, S_m) \ge cn(\log n)^{-1/3}\Big) = \mathrm{O}\big((\log n)^{-5/3}\big)$$

Proof. For each n, choose

$$0 = j_0 < j_1 < \cdots < j_m = n$$

such that $(j_i - j_{i-1}) \sim n(\log n)^{-2}$, uniformly in i. (Note that this choice of $\{j_i\}$ is same as that used in the proof of [9], Theorem 7.7.5.) Then $m \sim (\log n)^2$. Erase loops on each interval $[j_{i-1}, j_i]$ separately (i.e., take $L(S[j_{i-1}, j_i])$). Let Y_k be the indicator function of the event "S_k is not erased in this procedure" i.e.,

$$Y_k = \mathbf{1}\big\{L\big(S[j_{i-1}, k]\big) \cap S[k+1, j_i] = \varnothing\big\} \quad \text{for } j_{i-1} \le k < j_i. \tag{2.1}$$

Let $0 \le l \le n$ and let i be such that $j_{i-1} \le l < j_i$. Since there is a path from 0 to S_l which does not pass the loops arising on each interval, we see that

$$d_{\mathcal{G}}(0, S_l) \le \sum_{k=0}^{j_{i-1}} Y_k + (j_i - j_{i-1}). \tag{2.2}$$

Therefore,

$$\max_{1 \le l \le n} d_{\mathcal{G}}(0, S_l) \le \sum_{k=0}^{n} Y_k + 2n(\log n)^{-2}. \tag{2.3}$$

It follows from the proof of [9], Theorem 7.7.5 and [10], (1.2) that

$$E\left(\sum_{k=0}^{n} Y_k\right) \asymp n(\log n)^{-1/3},$$

and hence by (2.3), for $c > 0$ sufficiently large,

$$\begin{aligned}
P\Big(\max_{1 \le l \le n} d_{\mathcal{G}}(0, S_l) \ge cn(\log n)^{-1/3}\Big) &\le P\left(\sum_{k=0}^{n} Y_k + 2n(\log n)^{-2} \ge cn(\log n)^{-1/3}\right) \\
&\le P\left(\sum_{k=0}^{n} Y_k \ge \frac{c}{2} n(\log n)^{-1/3}\right) \\
&\le P\left(\left|\sum_{k=0}^{n} Y_k - E\left(\sum_{k=0}^{n} Y_k\right)\right| \ge E\left(\sum_{k=0}^{n} Y_k\right)\right) \\
&\le \frac{\mathrm{Var}(\sum_{k=0}^{n} Y_k)}{E(\sum_{k=0}^{n} Y_k)^2}.
\end{aligned}$$

Using the estimate

$$\mathrm{Var}\left(\sum_{k=0}^{n} Y_k\right) = E\left(\sum_{k=0}^{n} Y_k\right) \mathrm{O}\big(n(\log n)^{-2}\big),$$

(see proof of [9], Theorem 7.7.5), the proposition is proved. □

Let

$$B_{\mathcal{G}}(x, R) = \big\{y\colon\, d_{\mathcal{G}}(x, y) < R\big\}, \quad x \in \mathcal{G}, R \in (0, \infty).$$

We have the following result for the bounds on volume.

Proposition 2.1.2. *Let* $d = 4$. *Then there exists* $c > 0$ *such that*

$$P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}\big(0, cn(\log n)^{-1/3}\big)\big) \le n\big) = \mathrm{O}\big((\log n)^{-5/3}\big). \tag{2.4}$$

Proof. It follows by Proposition 2.1.1 that

$$\begin{aligned} P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}\big(0, cn(\log n)^{-1/3}\big)\big) \le rn\big) &\le P\big(\mu_{\mathcal{G}}\big(\{S_l\colon\, 0 \le l \le n\}\big) \le rn\big) \\ &\quad + P\Big(\max_{1 \le l \le n} d_{\mathcal{G}}(0, S_l) \ge cn(\log n)^{-1/3}\Big) \\ &\le P\big(\sharp\{S_l\colon\, 0 \le l \le n\} \le rn\big) + \mathrm{O}\big((\log n)^{-5/3}\big). \end{aligned} \tag{2.5}$$

But it follows from [4], (2.22) and [5], (4.1) that there exist $p \in (0, 1)$ and $\tilde{c} > 0$ such that

$$E\big(\sharp\{S_l\colon\, 0 \le l \le n\}\big) \sim pn,$$
$$\mathrm{Var}\big(\sharp\{S_l\colon\, 0 \le l \le n\}\big) = \tilde{c}n + \mathrm{O}\big(n^{1/2}(\log n)\big).$$

So, if we choose r sufficiently small so that $r < \frac{p}{2}$, then the right-hand side of (2.5) is bounded above by

$$\frac{4\,\mathrm{Var}(\sharp\{S_l\colon\, 0 \le l \le n\})}{E(\sharp\{S_l\colon\, 0 \le l \le n\})^2} + \mathrm{O}\big((\log n)^{-5/3}\big) = \mathrm{O}\big((\log n)^{-5/3}\big).$$

By a simple reparameterisation, we have (2.4). □

Using these propositions, it is now relatively straightforward to prove the second inequality of Theorem 1.2.1.

Proof of the upper bound of Theorem 1.2.1. Fix $n \in \mathbb{N}$ and we define R_0 so that $n = R_0^2(\log R_0)^{1/3}$. Then $R_0 \sim n^{1/2}(\log n)^{-1/6}$. On the set

$$\big\{\mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0, R_0)\big) \ge cR_0(\log R_0)^{1/3}\big\},$$

we have

$$\begin{aligned} h_{2n}^{\mathcal{G}}(0, 0) &\le c\big(R_0(\log R_0)^{1/3}\big)^{-1} \\ &\le 2c\big(n^{1/2}(\log n)^{1/6}\big)^{-1}. \end{aligned}$$

(See [8], Proposition 3.1. We apply this as $v(R) = cR(\log R)^{1/3}$ and $r(R) = R$.)
Therefore, by Proposition 2.1.2, for c sufficiently small,

$$\begin{aligned} P\big(h_{2n}^{\mathcal{G}}(0, 0) \le 2c\big(n^{1/2}(\log n)^{1/6}\big)^{-1}\big) &\ge P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0, R_0)\big) \ge cR_0(\log R_0)^{1/3}\big) \\ &= 1 - \mathrm{O}\big((\log n)^{-5/3}\big). \end{aligned}$$

Let $n_k = \lfloor \mathrm{e}^k \rfloor$. Then, since $\sum (\log n_k)^{-5/3} < \infty$, by the Borel–Cantelli lemma there exists $K_1(\omega)$ with $P(K_1 < \infty) = 1$ such that

$$h_{2n_k}^{\mathcal{G}(\omega)}(0,0) \le 2c\bigl(n_k^{1/2}(\log n_k)^{1/6}\bigr)^{-1} \quad \forall k \ge K_1(\omega).$$

This implies that the upper bound of Theorem 1.2.1 holds for the subsequence n_k. The spectral decomposition gives that $h_{2n}^{\mathcal{G}(\omega)}(0,0)$ is monotone decreasing in n. So, if $n > N_1(\omega) := \mathrm{e}^{K_1(\omega)} + 1$, let $k \ge K_1(\omega)$ be such that $n_k \le n < n_{k+1}$. Then

$$\begin{aligned} h_{2n}^{\mathcal{G}(\omega)}(0,0) &\le h_{2n_k}^{\mathcal{G}(\omega)}(0,0) \\ &\le 2c\bigl(n_k^{1/2}(\log n_k)^{1/6}\bigr)^{-1} \\ &\le \tilde{c}\bigl(n^{1/2}(\log n)^{1/6}\bigr)^{-1}. \end{aligned}$$

□

2.2. *Lower bound*

In this subsection we will prove the lower bound of Theorem 1.2.1 by analysing the cut-times of S, where we call a time k a cut-time for S if $S[0,k] \cap S(k,\infty) = \varnothing$. In four dimensions the set of cut-times

$$\mathcal{T} := \bigl\{k\colon\ S[0,k] \cap S(k,\infty) = \varnothing\bigr\},$$

is an infinite set, P-a.s., so we can write $T_1 < T_2 < \cdots$ to represent the elements of the set of cut-times $\mathcal{T}$. We let J_j be the indicator function of the event "j is a cut-time" that is, $J_j = 1$ if $S[0,j] \cap S(j,\infty) = \varnothing$, and write

$$R_n = \sum_{j=0}^{n} J_j.$$

We write C_n to represent the nth cut-point S_{T_n}. In order to prove the lower bound of Theorem 1.2.1, we need the upper bound of volume (see, for example, [8], Proposition 3.2). However, it is clear that $\mu_{\mathcal{G}}(B_{\mathcal{G}}(0,n)) \le 8T_n$. Thus, we start by a considering the upper bound of T_n (or lower bound of R_n).

Our main estimates for the lower bound of R_n is given in Proposition 2.2.3. This can be proved by using the fact that "short-range" and "long-range" intersections of random walks are asymptotically independent in four dimensions. In order to establish this proposition, we need Lemmas 2.2.1 and 2.2.2. So we will show these lemmas first, and then Proposition 2.2.3.

Fix $\delta \in (0,1)$. We write

$$a_{i,n} = \bigl\lfloor n(\log n)^{i\delta}\bigr\rfloor \quad \text{for } i = 1,2,3. \tag{2.6}$$

Lemma 2.2.1. *Let $d = 4$. Then,*

$$P\bigl(S[0,n] \cap S[a_{1,n},\infty) \ne \varnothing\bigr) = \mathrm{O}\bigl((\log n)^{-1-\delta}\bigr).$$

Proof. Let S^1, S^2 denote independent simple random walks starting at the origin in $\mathbb{Z}^4$. It follows from the proof of [9], Corollary 4.2.5 that

$$\begin{aligned} P\bigl(S[0,n] \cap S[a_{1,n},\infty) \ne \varnothing\bigr) &= P\bigl(S^1[0,n] \cap S^2[a_{1,n}-n,\infty) \ne \varnothing\bigr) \\ &\le c(\log n)^{-1} \sum_{i=0}^{n} \sum_{j=a_{1,n}-n}^{\infty} P\bigl(S_i^1 = S_j^2\bigr). \end{aligned} \tag{2.7}$$

By the local central limit theorem of [9], Theorem 1.2.1, the right-hand side of (2.7) can be bounded above by

$$c(\log n)^{-1}\sum_{i=0}^{n}\sum_{j=a_{1,n}-n}^{\infty}\frac{1}{(i+j)^2}=\mathrm{O}\big((\log n)^{-1-\delta}\big).$$

□

Let

$$A_1:=\big\{S[0,n]\cap S[a_{1,n},\infty)=\varnothing, S[0,a_{1,n}]\cap S[a_{2,n},\infty)=\varnothing\big\}. \tag{2.8}$$

By Lemma 2.2.1, we have

$$P\big(A_1^c\big)=\mathrm{O}\big((\log n)^{-1-\delta}\big). \tag{2.9}$$

If we also let

$$\tilde{J}_j=\begin{cases}\mathbf{1}\big\{S[0,j]\cap S(j,a_{1,n}]=\varnothing\big\}, & 0\le j\le n,\\ \mathbf{1}\big\{S[a_{1,n},j]\cap S(j,\infty)=\varnothing\big\}, & a_{2,n}\le j\le a_{3,n}.\end{cases} \tag{2.10}$$

Then it is easy to see that $J_j=\tilde{J}_j$ on the event A_1.

Lemma 2.2.2. *Let $d=4$. There exists $c>0$ such that*

$$P\Bigg(\sum_{j=0}^{n}\tilde{J}_j\le cn(\log n)^{-1/2}\Bigg)=\mathrm{O}\bigg(\frac{\log\log n}{\log n}\bigg), \tag{2.11}$$

$$P\Bigg(\sum_{j=a_{2,n}}^{a_{3,n}}\tilde{J}_j\le cn(\log n)^{-1/2}\Bigg)=\mathrm{O}\bigg(\frac{\log\log n}{\log n}\bigg). \tag{2.12}$$

Proof. The proof of (2.11) and (2.12) are similar, we will only prove (2.11). Let $0\le j\le n$. It is well known that

$$E(J_j)=P\big(S[0,j]\cap S(j,\infty)=\varnothing\big)\sim\tilde{c}(\log j)^{-1/2}$$

for some $\tilde{c}>0$; see Introduction in [11], for example.
Therefore, by Lemma 2.2.1,

$$E(\tilde{J}_j)\le E(J_j)+P\big(S[0,j]\cap S[a_{1,n},\infty)\ne\varnothing\big)\sim\tilde{c}(\log j)^{-1/2},$$

and hence

$$E(\tilde{J}_j)\sim\tilde{c}(\log j)^{-1/2}. \tag{2.13}$$

Therefore, letting $c=\frac{\tilde{c}}{4}$, we have

$$\begin{aligned}P\Bigg(\sum_{j=0}^{n}\tilde{J}_j\le cn(\log n)^{-1/2}\Bigg)&\le P\Bigg(\sum_{j=0}^{n}\tilde{J}_j\le\frac{1}{2}E\Bigg(\sum_{j=0}^{n}\tilde{J}_j\Bigg)\Bigg)\\&\le P\Bigg(\Bigg|\sum_{j=0}^{n}\tilde{J}_j-E\Bigg(\sum_{j=0}^{n}\tilde{J}_j\Bigg)\Bigg|\ge\frac{1}{2}E\Bigg(\sum_{j=0}^{n}\tilde{J}_j\Bigg)\Bigg)\\&\le\frac{4\operatorname{Var}(\sum_{j=0}^{n}\tilde{J}_j)}{E(\sum_{j=0}^{n}\tilde{J}_j)^2}.\end{aligned}$$

To complete the proof of (2.11) it remains to establish that

$$\operatorname{Var}\left(\sum_{j=0}^{n} \tilde{J}_j\right) = E\left(\sum_{j=0}^{n} \tilde{J}_j\right)^2 \mathrm{O}\left(\frac{\log\log n}{\log n}\right). \tag{2.14}$$

We have

$$\begin{aligned}\operatorname{Var}\left(\sum_{j=0}^{n} \tilde{J}_j\right) &= E\left(\left(\sum_{j=0}^{n} \tilde{J}_j\right)^2\right) - E\left(\sum_{j=0}^{n} \tilde{J}_j\right)^2 \\ &= \sum_{j=0}^{n} E\big(\tilde{J}_j^{\,2}\big) + 2\sum_{0\le j<k\le n} E(\tilde{J}_j\tilde{J}_k) \\ &\quad - \sum_{j=0}^{n} E(\tilde{J}_j)^2 - 2\sum_{0\le j<k\le n} E(\tilde{J}_j)E(\tilde{J}_k),\end{aligned}$$

and

$$\sum_{j=0}^{n} E\big(\tilde{J}_j^{\,2}\big) - \sum_{j=0}^{n} E(\tilde{J}_j)^2 = \sum_{j=0}^{n} E(\tilde{J}_j) - \sum_{j=0}^{n} E(\tilde{J}_j)^2 \le n+1.$$

Therefore we only need to consider

$$\sum_{0\le j<k\le n} E(\tilde{J}_j\tilde{J}_k) - \sum_{0\le j<k\le n} E(\tilde{J}_j)E(\tilde{J}_k).$$

Let $A = \{(j,k)\colon 0\le j<k\le n\}$ and $a_n = \lfloor n(\log n)^{-9}\rfloor$. We write

$$\begin{aligned}A^1 &= \big\{(j,k)\in A\colon 0\le j\le a_n\big\},\\ A^2 &= \big\{(j,k)\in A\colon a_n<j<k<n-a_n\big\},\\ A^3 &= \big\{(j,k)\in A\colon n-a_n\le k\le n\big\},\end{aligned}$$

and partition A^2 into two sets:

$$\begin{aligned}A^2_+ &= \big\{(j,k)\in A^2\colon k-j>2a_n\big\},\\ A^2_- &= \big\{(j,k)\in A^2\colon k-j\le 2a_n\big\}.\end{aligned}$$

Since $\sharp A^1 + \sharp A^2_- + \sharp A^3 \le 4n^2(\log n)^{-9}$, the sum over $A^1\cup A^2_-\cup A^3$ can be bounded above by

$$4n^2(\log n)^{-9} = E\left(\sum_{j=0}^{n} \tilde{J}_j\right)^2 \mathrm{O}\big((\log n)^{-8}\big).$$

For the sum over A^2_+, we need to be a little careful. Let $(j,k)\in A^2_+$. By independence,

$$\begin{aligned}E(\tilde{J}_j\tilde{J}_k) &\le P\big(S[j-a_n,j]\cap S(j,j+a_n]=\varnothing,\\ &\qquad S[k-a_n,k]\cap S(k,k+a_n]=\varnothing\big)\\ &= P\big(S[j-a_n,j]\cap S(j,j+a_n]=\varnothing\big)\end{aligned}$$

$$
\begin{aligned}
&\times P\big(S[k-a_n,k]\cap S(k,k+a_n]=\varnothing\big)\\
&=b_n^2,
\end{aligned}
$$

where $b_n := P(S[0,a_n]\cap S(a_n,2a_n]=\varnothing)$.

On the other hand, the following holds for $a_n < j \le n$,

$$
E(\tilde J_j) = b_n\left(1-\mathrm{O}\left(\frac{\log\log n}{\log n}\right)\right). \tag{2.15}
$$

Indeed, let

$$
\begin{aligned}
U &= \big\{S[0,j]\cap S(j,\infty)=\varnothing\big\},\\
\overline{V} &= \big\{S[0,j]\cap S(j,a_{1,n}]=\varnothing\big\},\\
V &= \big\{S[j-a_n,j]\cap S(j,j+a_n]=\varnothing\big\},
\end{aligned}
$$

so that $U\subset\overline{V}\subset V$. Then,

$$
1-\mathrm{O}\left(\frac{\log\log n}{\log n}\right)=\frac{P(U)}{P(V)}\le\frac{P(\overline{V})}{P(V)},
$$

where the first equality is due to [9], Lemma 7.7.3. This implies (2.15).

Since $b_n\sim\tilde c(\log n)^{-1/2}$ (see, for example, [9], Appendix A), we can combine this with the above results to conclude

$$
\begin{aligned}
\sum_{(j,k)\in A_+^2}\big\{E(\tilde J_j\tilde J_k)-E(\tilde J_j)E(\tilde J_k)\big\} &\le \sum_{(j,k)\in A_+^2}\left\{b_n^2-b_n^2\left(1-\mathrm{O}\left(\frac{\log\log n}{\log n}\right)\right)\right\}\\
&=\sum_{(j,k)\in A_+^2} b_n^2\mathrm{O}\left(\frac{\log\log n}{\log n}\right)\\
&=n^2b_n^2\mathrm{O}\left(\frac{\log\log n}{\log n}\right)\\
&=E\left(\sum_{j=0}^n\tilde J_j\right)^2\mathrm{O}\left(\frac{\log\log n}{\log n}\right).
\end{aligned}
$$

□

The key result in four dimensions in this paper is the following proposition.

Proposition 2.2.3. *Let $d=4$. There exists $c>0$ such that*

$$
P\big(R_{a_{3,n}}\le cn(\log n)^{-1/2}\big)=\mathrm{O}\big((\log n)^{-1-\delta}\big). \tag{2.16}
$$

Proof. By (2.9), the independence of $\sum_{j=0}^n\tilde J_j$ and $\sum_{j=a_{2,n}}^{a_{3,n}}\tilde J_j$, and Lemma 2.2.2,

$$
\begin{aligned}
P\big(R_{a_{3,n}}\le cn(\log n)^{-1/2}\big) &\le P\left(\sum_{j=0}^n J_j+\sum_{j=a_{2,n}}^{a_{3,n}} J_j\le cn(\log n)^{-1/2}\right)\\
&\le P\left(\sum_{j=0}^n J_j+\sum_{j=a_{2,n}}^{a_{3,n}} J_j\le cn(\log n)^{-1/2}, A_1\text{ holds}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ P(A_1 \text{ does not hold}) \\
&\le P\left(\sum_{j=0}^{n} \tilde{J}_j + \sum_{j=a_{2,n}}^{a_{3,n}} \tilde{J}_j \le cn(\log n)^{-1/2}\right) \\
&+ \mathrm{O}\big((\log n)^{-1-\delta}\big) \\
&\le P\left(\sum_{j=0}^{n} \tilde{J}_j \le cn(\log n)^{-1/2}\right) P\left(\sum_{j=a_{2,n}}^{a_{3,n}} \tilde{J}_j \le cn(\log n)^{-1/2}\right) \\
&+ \mathrm{O}\big((\log n)^{-1-\delta}\big) \\
&= \mathrm{O}\big((\log n)^{-1-\delta}\big).
\end{aligned}
$$
□

The following is an immediate corollary.

Corollary 2.2.4. *Let $d = 4$.* (a) *Recall $R_n = \sum_{j=0}^{n} J_j$ is the number of cut-points on the first n points of S. We have*

$$\mathrm{Var}(R_n) = E(R_n)^2 \mathrm{O}\left(\frac{\log\log n}{\log n}\right).$$

(b) *There exists $c > 0$ such that for each $\delta \in (0, 1)$,*

$$P\big(T_n \ge cn(\log n)^{1/2+3\delta}\big) = \mathrm{O}\big((\log n)^{-1-\delta}\big). \tag{2.17}$$

(c) *There exists $c > 0$ such that for each $\delta \in (0, 1)$,*

$$P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0, n)\big) \ge cn(\log n)^{1/2+3\delta}\big) = \mathrm{O}\big((\log n)^{-1-\delta}\big). \tag{2.18}$$

Proof. Part (a) can be proved in the same way as (2.14). Part (b) is readily obtained from Proposition 2.2.3. Since $\mu_{\mathcal{G}}(B_{\mathcal{G}}(0, n)) \le 8T_n$, part (b) implies part (c). □

Let

$$\tau_{\mathcal{G}}(0, n) = \inf\big\{n \ge 0\colon\ X_n \notin B_{\mathcal{G}}(0, n)\big\}$$

and the function $R_{\mathcal{G}}(\cdot, \cdot)$ be the effective resistance on $\mathcal{G}$ when we suppose that a unit resistor is placed along each edge. We are now in a position to prove the lower bound of Theorem 1.2.1. To establish it, we need the following proposition.

Proposition 2.2.5. *Let $d = 4$. There exist $c > 0$ and $\tilde{c} > 0$ such that for each $\delta \in (0, 1)$,*

$$
\begin{aligned}
&P\Big(cn^2(\log n)^{-6\delta} \le E_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0, n)\big) \le \sup_{x \in B_{\mathcal{G}}(0,n)} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0, n)\big) \le \tilde{c}n^2(\log n)^{1/2+3\delta}\Big) \\
&\quad = 1 - \mathrm{O}\big((\log n)^{-1-\delta/2}\big).
\end{aligned} \tag{2.19}
$$

Proof. We adapt the argument of [3], Lemma 4.3. It is easy to check that for $x \in B_{\mathcal{G}}(0, n)$,

$$
\begin{aligned}
E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0, n)\big) &\le R_{\mathcal{G}}\big(x, B_{\mathcal{G}}(0, n)^c\big)\mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0, n)\big) \\
&\le 2n\mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0, n)\big),
\end{aligned}
$$

and therefore the right-hand side inequality of (2.19) is a straightforward consequence of Corollary 2.2.4(c). For left-hand side inequality, we consider the following events

$$A_2 = \Big\{ \sup_{0\le m\le T_{2n}} d_{\mathcal{G}}(0, S_m) \le c_1 n (\log n)^{1/6+3\delta} \Big\}$$

$$A_3 = \big\{ \mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0,n)\big) \ge c_2 n (\log n)^{1/3} \big\}.$$

By Corollary 2.2.4(b), Propositions 2.1.1 and 2.1.2, it follows that for c_1 sufficiently large and c_2 sufficiently small,

$$P(A_2 \cap A_3) = 1 - \mathrm{O}\big((\log n)^{-1-\delta/2}\big).$$

But applying an argument from the proof of [3], Lemma 4.3, it is possible to deduce that on the set $A_2 \cap A_3$, we have

$$E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, c_1 n (\log n)^{1/6+3\delta}\big)\big) \ge c_3 n^2 (\log n)^{1/3},$$

for some $c_3 > 0$, which completes the proof. □

Proof of the lower bound of Theorem 1.2.1. Let

$$A_4 = \Big\{ 2c_3 n^2 (\log n)^{-2\delta} \le E_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n)\big) \le \sup_{x\in B_{\mathcal{G}}(0,n)} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n)\big) \le c_4 n^2 (\log n)^{1/2+\delta} \Big\}$$

$$A_5 = \big\{ \mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0,n)\big) \le c_5 n (\log n)^{1/2+\delta} \big\}.$$

By Corollary 2.2.4(c) and Proposition 2.2.5, we can choose $c_3 > 0$ sufficiently small, $c_4 > 0$ and $c_5 > 0$ sufficiently large so that

$$P(A_4 \cap A_5) = 1 - \mathrm{O}\big((\log n)^{-1-\delta/6}\big). \tag{2.20}$$

Assume that A_4 and A_5 hold. Then using the Markov property, we have

$$\begin{aligned} 2c_3 n^2 (\log n)^{-2\delta} &\le E_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n)\big) \\ &\le c_3 n^2 (\log n)^{-2\delta} + P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n) > c_3 n^2 (\log n)^{-2\delta}\big) \sup_{x\in B_{\mathcal{G}}(0,n)} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n)\big) \\ &\le c_3 n^2 (\log n)^{-2\delta} + P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n) > c_3 n^2 (\log n)^{-2\delta}\big) c_4 n^2 (\log n)^{1/2+\delta}. \end{aligned}$$

Therefore,

$$P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n) > c_3 n^2 (\log n)^{-2\delta}\big) \ge \frac{c_3}{c_4} \frac{n^2 (\log n)^{-2\delta}}{n^2 (\log n)^{1/2+\delta}} = c_6 (\log n)^{-1/2-3\delta},$$

where $c_6 = \frac{c_3}{c_4}$.

By the Chapman–Kolmogorov equation and the Cauchy–Schwarz inequality,

$$\begin{aligned} P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0,n) > c_3 n^2 (\log n)^{-2\delta}\big)^2 &\le P_0^{\mathcal{G}}\big(X_{\lfloor c_3 n^2 (\log n)^{-2\delta}\rfloor} \in B_{\mathcal{G}}(0,n)\big)^2 \\ &\le \Big\{ \sum_{y\in B_{\mathcal{G}}(0,n)} h^{\mathcal{G}}_{\lfloor c_3 n^2 (\log n)^{-2\delta}\rfloor}(0,y) \mu_{\mathcal{G}}\big(\{y\}\big) \Big\}^2 \\ &\le \mu_{\mathcal{G}}\big(B_{\mathcal{G}}(0,n)\big) h^{\mathcal{G}}_{2\lfloor c_3 n^2 (\log n)^{-2\delta}\rfloor}(0,0) \\ &\le c_5 n (\log n)^{1/2+\delta} h^{\mathcal{G}}_{2\lfloor c_3 n^2 (\log n)^{-2\delta}\rfloor}(0,0), \end{aligned}$$

and therefore,

$$h^{\mathcal{G}}_{2\lfloor c_3 n^2 (\log n)^{-2\delta}\rfloor}(0,0) \geq \frac{c_6^2}{c_5} n^{-1} (\log n)^{-3/2-7\delta}.$$

With a simple reparameterisation we can conclude that on the set $A_4 \cap A_5$,

$$h^{\mathcal{G}}_{2n}(0,0) \geq c_7 n^{-1/2} (\log n)^{-3/2-8\delta},$$

for some $c_7 > 0$. So, using the Borel–Cantelli lemma first and then using the monotonicity of $h^{\mathcal{G}}_{2n}(0,0)$ as in the last part of the upper bound of Theorem 1.2.1, we deduce that there exists $N_1(\omega)$ with $P(N_1 < \infty)$ such that

$$n^{-1/2} (\log n)^{-3/2-\delta} \leq h^{\boldsymbol{\mathcal{G}(\omega)}}_{2n}(0,0)$$

for all $n \geq N_1(\omega)$. This completes the proof of the lower bound of Theorem 1.2.1. □

3. Proof of Theorem 1.2.2

3.1. *Lower bound*

Let $d = 4$. In this subsection we will show the lower bound of Theorem 1.2.2. Let $\delta \in (0,1)$ and recall that $\mathcal{T} = (T_n)_{n \geq 1}$ is the set of cut-times and $C_n = S_{T_n}$. We define the events

$$\tilde{A}_0 = \left\{\tau_{\mathcal{G}}(0,n) \leq n^2 (\log n)^{1/2+2\delta}\right\},$$
$$\tilde{A}_1 = \left\{\{S_m \colon 0 \leq m \leq T_{\lfloor n(\log n)^{-1/6-\delta}\rfloor}\} \subset B_{\mathcal{G}}(0,n)\right\},$$
$$\tilde{A}_2 = \left\{\max_{1 \leq k \leq \lfloor n(\log n)^{-1/6-\delta}\rfloor} |C_k| \geq n^{1/2} (\log n)^{1/6-4\delta}\right\},$$

where $|\cdot|$ in $\tilde{A}_2$ denotes Euclidean distance in $\mathbb{R}^4$. Then on the event $\tilde{A}_0 \cap \tilde{A}_1 \cap \tilde{A}_2$,

$$\begin{aligned}\max_{1 \leq m \leq n^2 (\log n)^{1/2+2\delta}} |X_m| &\geq \max_{1 \leq m \leq \tau_{\mathcal{G}}(0,n)} |X_m| \\ &\geq \max_{1 \leq k \leq \lfloor n(\log n)^{-1/6-\delta}\rfloor} |C_k| \\ &\geq n^{1/2} (\log n)^{1/6-4\delta},\end{aligned} \tag{3.1}$$

where we use the fact that any path from 0 to $B_{\mathcal{G}}(0,n)^c$ passes through all cut-points in $B_{\mathcal{G}}(0,n)$. Therefore, by a simple reparameterisation, we have

$$\max_{1 \leq m \leq n} |X_m| \geq n^{1/4} (\log n)^{1/24-9\delta/2}.$$

So all we need is to show that each $\tilde{A}_i$ occurs with probability which is high enough to apply the Borel–Cantelli lemma.

First, we consider $\tilde{A}_2$.

Lemma 3.1.1. *Let $d = 4$ and $\eta > 0$. Then it follows that*

$$P\left(T_n < n(\log n)^{1/2-\eta}\right) = \mathrm{O}\left((\log n)^{-1-\eta}\right).$$

Proof. Recall that R_n denotes the number of cut-times up to time n whose expectation is estimated as $E(R_n) \sim \tilde{c}n(\log n)^{-1/2}$ for some $\tilde{c} > 0$ and variance is estimated in Corollary 2.2.4(a). We have

$$\begin{aligned} P\big(T_n < n(\log n)^{1/2-\eta}\big) &\le P(R_{\lfloor n(\log n)^{1/2-\eta}\rfloor} \ge n) \\ &\le P\left(R_N \ge \frac{1}{2}N(\log N)^{-1/2+\eta}\right) \\ &\le P\left(R_N - E(R_N) \ge \frac{1}{4}N(\log N)^{-1/2+\eta}\right) \\ &\le 16\,\mathrm{Var}(R_N)\big(N^2(\log N)^{-1+2\eta}\big)^{-1} \\ &= \mathrm{O}\big((\log N)^{-1-2\eta}\log\log N\big), \end{aligned}$$

where $N = \lfloor n(\log n)^{1/2-\eta}\rfloor$. □

We have the following proposition for $\tilde{A}_2$.

Proposition 3.1.2. *Let $d = 4$ and $\delta \in (0, 1)$. Then, there exists $c_\delta > 0$ depending only on δ such that*

$$P\big(\tilde{A_2}^c\big) \le c_\delta(\log n)^{-1-\delta^2/9}.$$

Proof. Note that

$$\tilde{A_2}^c \subset \Big\{\max_{\lfloor n(\log n)^{-1/6-2\delta}\rfloor \le k \le \lfloor n(\log n)^{-1/6-\delta}\rfloor} |C_k| < n^{1/2}(\log n)^{1/6-4\delta}\Big\}. \tag{3.2}$$

Let $\eta := \frac{\delta^2}{3}$ and

$$k_i = \big\lfloor n(\log n)^{-1/6-2\delta+3(i-1)\eta}\big\rfloor, \quad i = 1, \dots, j,$$

where j is chosen as $j = 1 + \lfloor\frac{1}{\delta}\rfloor$. Note that

$$\delta - \delta^2 < 3(j-1)\eta \le \delta.$$

Notice that

$$\big[k_i(\log k_i)^{1/2-\eta}, k_i(\log k_i)^{1/2+\eta}\big] \cap \big[k_{i+1}(\log k_{i+1})^{1/2-\eta}, k_{i+1}(\log k_{i+1})^{1/2+\eta}\big] = \varnothing$$

for each i. By (3.2),

$$\begin{aligned} P\big(\tilde{A_2}^c\big) &\le P\Big(\max_{1\le i\le j}|C_{k_i}| < n^{1/2}(\log n)^{1/6-4\delta}\Big) \\ &\le P\big(T_{k_i} \notin \big[k_i(\log k_i)^{1/2-\eta}, k_i(\log k_i)^{1/2+\eta}\big],\ \text{for some } i = 1, \dots, j\big) \\ &\quad + P\Big(\max_{1\le i\le j}|C_{k_i}| < n^{1/2}(\log n)^{1/6-4\delta}, \\ &\qquad T_{k_i} \in \big[k_i(\log k_i)^{1/2-\eta}, k_i(\log k_i)^{1/2+\eta}\big],\ \text{for all } i = 1, \dots, j\Big). \end{aligned} \tag{3.3}$$

By Corollary 2.2.4(b) and Lemma 3.1.1, the first term of the right-hand side of (3.3) can be bounded above by

$$\begin{aligned}\sum_{i=1}^{j} P\bigl(T_{k_i} \notin \bigl[k_i(\log k_i)^{1/2-\eta}, k_i(\log k_i)^{1/2+\eta}\bigr]\bigr) &\le c\sum_{i=1}^{j}(\log k_i)^{-1-\eta/3}\\ &\le cj(\log n)^{-1-\eta/3}\\ &\le c'\frac{2}{\delta}(\log n)^{-1-\delta^2/9},\end{aligned}$$

where $c>0$ and $c'>0$ do not depend on δ. We will explain how to bound the second term.

First by the definition of k_i, this term is bounded above by

$$\begin{aligned}&P\Bigl(\inf_{k_i(\log k_i)^{1/2-\eta}\le m\le k_i(\log k_i)^{1/2+\eta}}|S_m|<n^{1/2}(\log n)^{1/6-4\delta},\ \text{for all } i=1,\dots,j\Bigr)\\ &\quad\le P\Bigl(\inf_{n(\log n)^{1/3-2\delta+(3i-4)\eta}\le m\le n(\log n)^{1/3-2\delta+(3i-2)\eta}}|S_m|<n^{1/2}(\log n)^{1/6-4\delta},\ \text{for all } i=1,\dots,j\Bigr).\end{aligned}\tag{3.4}$$

By the Markov property, for each $1\le i\le j$,

$$\begin{aligned}&P\Bigl(\inf_{(1/2)n(\log n)^{1/3-2\delta+(3i-4)\eta}\le m\le 2n(\log n)^{1/3-2\delta+(3i-2)\eta}}|S_m|\ge n^{1/2}(\log n)^{1/6-3\delta}\Bigr)\\ &\quad\ge P\Bigl(\bigl|S_{\lfloor(1/2)n(\log n)^{1/3-2\delta+(3i-4)\eta}\rfloor}\bigr|\ge n^{1/2}(\log n)^{1/6-2\delta}\Bigr)\\ &\qquad\times\inf_{|x|\ge n^{1/2}(\log n)^{1/6-2\delta}}P^x\bigl(|S_l|\ge n^{1/2}(\log n)^{1/6-3\delta},\ \text{for all } l\ge 0\bigr).\end{aligned}\tag{3.5}$$

By using the local central limit theorem of [9], Theorem 1.2.1 and gambler's ruin estimate, (see, for example, [9], Proposition 1.5.10), the right-hand side of (3.5) is bounded below by

$$\bigl(1-\mathrm{O}\bigl((\log n)^{-(3/2)\delta}\bigr)\bigr)\bigl(1-\mathrm{O}\bigl((\log n)^{-2\delta}\bigr)\bigr)=1-\mathrm{O}\bigl((\log n)^{-(3/2)\delta}\bigr).\tag{3.6}$$

Now we estimate (3.4) by using (3.6). Let

$$\tau_i:=\inf\bigl\{m\in\bigl[n(\log n)^{1/3-2\delta+(3i-4)\eta},n(\log n)^{1/3-2\delta+(3i-2)\eta}\bigr]:\ |S_m|<n^{1/2}(\log n)^{1/6-4\delta}\bigr\}$$

for each $i=1,\dots,j$. Here we use the convention $\inf\varnothing=+\infty$. Then by the strong Markov property at τ_{j-1},

$$\begin{aligned}(3.4)&=P\bigl(\tau_i\in\bigl[n(\log n)^{1/3-2\delta+(3i-4)\eta},n(\log n)^{1/3-2\delta+(3i-2)\eta}\bigr],\ \text{for all } i=1,\dots,j\bigr)\\ &\le P\bigl(\tau_i\in\bigl[n(\log n)^{1/3-2\delta+(3i-4)\eta},n(\log n)^{1/3-2\delta+(3i-2)\eta}\bigr],\ \text{for all } i=1,\dots,j-1\bigr)\\ &\quad\times\max_{|x|<n^{1/2}(\log n)^{1/6-4\delta}}P^x\Bigl(\inf_{(1/2)n(\log n)^{1/3-2\delta+(3j-4)\eta}\le m\le 2n(\log n)^{1/3-2\delta+(3j-2)\eta}}|S_m|<n^{1/2}(\log n)^{1/6-4\delta}\Bigr).\end{aligned}\tag{3.7}$$

By (3.6), if $|x|<n^{1/2}(\log n)^{1/6-4\delta}$, then

$$\begin{aligned}&P^x\Bigl(\inf_{(1/2)n(\log n)^{1/3-2\delta+(3j-4)\eta}\le m\le 2n(\log n)^{1/3-2\delta+(3j-2)\eta}}|S_m|\ge n^{1/2}(\log n)^{1/6-4\delta}\Bigr)\\ &\quad\ge P^x\Bigl(\inf_{(1/2)n(\log n)^{1/3-2\delta+(3j-4)\eta}\le m\le 2n(\log n)^{1/3-2\delta+(3j-2)\eta}}|S_m|\ge \frac{1}{2}n^{1/2}(\log n)^{1/6-3\delta}\Bigr)\\ &\quad\ge P^x\Bigl(\inf_{(1/2)n(\log n)^{1/3-2\delta+(3j-4)\eta}\le m\le 2n(\log n)^{1/3-2\delta+(3j-2)\eta}}|S_m-x|\ge n^{1/2}(\log n)^{1/6-3\delta}\Bigr)\\ &\quad=1-\mathrm{O}\bigl((\log n)^{-(3/2)\delta}\bigr).\end{aligned}$$

Therefore (3.7) is bounded above by

$$P\big(\tau_i \in \big[n(\log n)^{1/3-2\delta+(3i-4)\eta}, n(\log n)^{1/3-2\delta+(3i-2)\eta}\big], \text{ for all } i = 1, \dots, j-1\big) \times \mathrm{O}\big((\log n)^{-3/2\delta}\big).$$

So, by iterating this argument, we can deduce that (3.4) is bounded above by $c_\delta(\log n)^{-3/2}$. □

Next we consider $\tilde{A}_1$.

Proposition 3.1.3. *Let $d = 4$ and $\delta \in (0, 1)$. We have*

$$P(\tilde{A}_1^{\,c}) = \mathrm{O}\big((\log n)^{-1-\delta/6}\big).$$

Proof. By Proposition 2.1.1 and Corollary 2.2.4(b), there exists $c > 0$ such that

$$\begin{aligned} P\big(\tilde{A}_1^{\,c}\big) &\le P\big(T_{\lfloor n(\log n)^{-1/6-\delta}\rfloor} \ge cn(\log n)^{1/3-\delta/2}\big) \\ &\quad + P\Big(\max_{0\le m\le cn(\log n)^{1/3-\delta/2}} d_{\mathcal{G}}(0, S_m) \ge n\Big) \\ &= \mathrm{O}\big((\log n)^{-1-\delta/6}\big). \end{aligned}$$

□

Finally, we will consider $\tilde{A}_0$. Denote

$$F := \Big\{\max_{x\in B_{\mathcal{G}}(0,n)} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0, n)\big) \le n^2(\log n)^{1/2+\delta}\Big\}.$$

Note that by (2.20),

$$P(F) = 1 - \mathrm{O}\big((\log n)^{-1-\delta/7}\big). \tag{3.8}$$

Proposition 3.1.4. *Let $d = 4$. For $\omega \in F$, we have*

$$P_0^{\mathcal{G}(\omega)}\big(\tilde{A}_0^{\,c}\big) \le c_\delta(\log n)^{-2},$$

where $c_\delta > 0$ is a constant depending only on δ.

Proof. By applying the Markov property at $\lfloor \frac{\delta}{2}(\frac{2}{\delta} - 1)n^2(\log n)^{1/2+2\delta}\rfloor$,

$$\begin{aligned} P_0^{\mathcal{G}}\big(\tilde{A}_0^{\,c}\big) &= P_0^{\mathcal{G}}\bigg(\tau_{\mathcal{G}}(0, n) > \bigg\lfloor \frac{\delta}{2}\bigg(\frac{2}{\delta} - 1\bigg)n^2(\log n)^{1/2+2\delta}\bigg\rfloor, \tau_{\mathcal{G}}(0, n) > n^2(\log n)^{1/2+2\delta}\bigg) \\ &\le P_0^{\mathcal{G}}\bigg(\tau_{\mathcal{G}}(0, n) > \bigg\lfloor \frac{\delta}{2}\bigg(\frac{2}{\delta} - 1\bigg)n^2(\log n)^{1/2+2\delta}\bigg\rfloor\bigg) \\ &\quad \times \max_{x\in B_{\mathcal{G}}(0,n)} P_x^{\mathcal{G}}\bigg(\tau_{\mathcal{G}}(0, n) > \frac{\delta}{2}n^2(\log n)^{1/2+2\delta}\bigg). \end{aligned}$$

On the set F, we have

$$P_x^{\mathcal{G}}\bigg(\tau_{\mathcal{G}}(0, n) > \frac{\delta}{2}n^2(\log n)^{1/2+2\delta}\bigg) \le \bigg(\frac{\delta}{2}n^2(\log n)^{1/2+2\delta}\bigg)^{-1} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}(0, n)\big) \le \frac{2}{\delta}(\log n)^{-\delta},$$

for each $x \in B_{\mathcal{G}}(0, n)$. So, by iterating this argument, we deduce the claim. □

Now we finish the proof of the lower bound of Theorem 1.2.2.

Proof of the lower bound of Theorem 1.2.2. Let $\delta \in (0,1)$ and $c_\delta > 0$ be the constant depending only on δ as described in the above argument. We sketch the proof (see the proof of [8], Theorem 1.5 (I)(c) for details). By Propositions 3.1.2, 3.1.3, 3.1.4, (3.8), and the Borel–Cantelli lemma, we see that there exists $N'(\omega)$ with $P(N' < \infty) = 1$ such that

$$\text{(a)} \quad P_0^{\mathcal{G}(\omega)}\bigl(\tau_{\mathcal{G}(\omega)}(0,n) \le n^2(\log n)^{1/2+2\delta}\bigr) \ge 1 - c_\delta(\log n)^{-2},$$

$$\text{(b)} \quad \bigl\{S_m(\omega)\colon\ 0 \le m \le T_{\lfloor n(\log n)^{-1/6-\delta}\rfloor}(\omega)\bigr\} \subset B_{\mathcal{G}(\omega)}(0,n),$$

$$\text{(c)} \quad \max_{1 \le k \le \lfloor n(\log n)^{-1/6-\delta}\rfloor} \bigl|C_k(\omega)\bigr| \ge n^{1/2}(\log n)^{1/6-4\delta}$$

for all $n \ge N'(\omega)$. Then, by the Borel–Cantelli lemma (with respect to the law $P_0^{\mathcal{G}(\omega)}$), there exists $N'(\omega, \overline{\omega})$ with

$$P_0^{\mathcal{G}(\omega)}\bigl(\bigl\{\overline{\omega}\colon\ N'(\omega,\overline{\omega}) < \infty\bigr\}\bigr) = 1$$

such that

$$\text{(d)} \quad \tau_{\mathcal{G}(\omega)}(0,n)(\overline{\omega}) \le n^2(\log n)^{1/2+2\delta},$$

(b) and (c) hold for all $n \ge N'(\omega,\overline{\omega})$. Therefore, it follows from (3.1) that the desired lower bound holds for each $\omega \in \{\omega\colon\ N'(\omega) < \infty\}$ and $\overline{\omega} \in \{\overline{\omega}\colon\ N'(\omega,\overline{\omega}) < \infty\}$. □

3.2. *Upper bound*

In this subsection, we will show the upper bound of Theorem 1.2.2. Let $\delta \in (0,1)$. We define the events B_i as follows,

$$B_0 = \bigl\{\tau_{\mathcal{G}}(0,n) \ge n^2(\log n)^{-10/3-\delta}\bigr\},$$
$$B_1 = \bigl\{T_{n+1} \le cn(\log n)^{1/2+\delta}\bigr\},$$
$$B_2 = \Bigl\{\sup_{0 \le m \le cn(\log n)^{1/2+\delta}} |S_m| \le n^{1/2}(\log n)^{1/4+\delta}\Bigr\}.$$

On the event $B_0 \cap B_1 \cap B_2$, we have

$$\begin{aligned}
\max_{0 \le m \le n^2(\log n)^{-10/3-\delta}} |X_m| &\le \max_{0 \le m \le \tau_{\mathcal{G}}(0,n)} |X_m| \\
&\le \sup_{x \in B_{\mathcal{G}}(0,n+1)} |x| \\
&\le \sup_{0 \le m \le T_{n+1}} |S_m| \\
&\le \sup_{0 \le m \le cn(\log n)^{1/2+\delta}} |S_m| \\
&\le n^{1/2}(\log n)^{1/4+\delta}.
\end{aligned} \tag{3.9}$$

Therefore, by a simple reparameterisation, we have

$$\max_{0 \le m \le n} |X_m| \le n^{1/4}(\log n)^{13/12+(5/4)\delta}.$$

So, once we have good estimates for each $P(B_i^c)$ then, by a similar argument to that of Section 3.1, we can deduce the upper bound of Theorem 1.2.2.

By Corollary 2.2.4(b), there exists $c > 0$ such that

$$P\bigl(B_1^c\bigr) = \mathrm{O}\bigl((\log n)^{-1-\delta/3}\bigr),$$

and it is a straightforward consequence of the large deviation estimate (see, for example, [9], Lemma 1.5.1) that

$$P\big(B_2^c\big) = \mathrm{O}\big(\mathrm{e}^{-(\log n)^{\delta/2}}\big).$$

Therefore, all we need is to estimate $P_0^{\mathcal{G}(\omega)}(B_0^c)$ on a good subset of Ω.

Let $\alpha = 3 + 11\delta$ and $\beta = \frac{3}{2} + 4\delta$. We define $\tilde{F}_1$ and $\tilde{F}_2$ as follows,

$$\tilde{F}_1 := \Big\{ \sup_{0\le m\le T_{2n}} d_{\mathcal{G}}(0, S_m) \le n(\log n)^{1/6+\delta} \Big\}$$

$$\begin{aligned}\tilde{F}_2 := \Big\{ & n^2(\log n)^{-2\beta-2\delta} \le E_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big)\big) \\ & \le \sup_{x\in B_{\mathcal{G}}(0,n(\log n)^{-\beta})} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big)\big) \le n^2(\log n)^{1/2-2\beta+\delta}\Big\}.\end{aligned}$$

Note that by Proposition 2.1.1 and Corollary 2.2.4(b),

$$P(\tilde{F}_1) = 1 - \mathrm{O}\big((\log n)^{-1-\delta/6}\big), \tag{3.10}$$

and by Proposition 2.2.5,

$$P(\tilde{F}_2) = 1 - \mathrm{O}\big((\log n)^{-1-\delta/7}\big). \tag{3.11}$$

The following proposition gives that on the event $\tilde{F}_1 \cap \tilde{F}_2$, $P_0^{\mathcal{G}(\omega)}(B_0^c)$ is small enough to apply the Borel–Cantelli lemma. Therefore, we complete the proof of Theorem 1.2.2 by showing this proposition.

Proposition 3.2.1. *Let $d = 4$ and $\delta \in (0, 1)$. For $\omega \in \tilde{F}_1 \cap \tilde{F}_2$, we have*

$$P_0^{\mathcal{G}(\omega)}\big(\tau_{\mathcal{G}}(0, n) \le n^2(\log n)^{-10/3-13\delta}\big) = \mathrm{O}\big((\log n)^{-1-\delta}\big).$$

Proof. Our argument is similar to that in [3], the proof of Lemma 4.3, so we only sketch the proof. By the strong Markov property,

$$\begin{aligned}& P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) \le n^2(\log n)^{-\alpha}\big) \\ &\quad \le P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big) < n^2(\log n)^{-\alpha}\big) \\ &\qquad \times \sup_{x\in B_{\mathcal{G}}(0,n(\log n)^{-\beta})} P_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) \le n^2(\log n)^{-\alpha}\big).\end{aligned}$$

Let $T_0 := \inf\{m\colon X_m = 0\}$. Then, as in the proof of Lemma 4.3 in [3], we have

$$\begin{aligned}P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) \le n^2(\log n)^{-\alpha}\big) &\le P_0^{\mathcal{G}}\Big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big) \ge n^2(\log n)^{-\alpha}\Big)^{-1} \\ &\quad \times \sup_{x\in B_{\mathcal{G}}(0,n(\log n)^{-\beta})} P_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) < T_0\big).\end{aligned} \tag{3.12}$$

But by the Markov property,

$$\begin{aligned}E_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big)\big) &\le n^2(\log n)^{-\alpha} \\ &\quad + P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big) \ge n^2(\log n)^{-\alpha}\big) \\ &\quad \times \sup_{x\in B_{\mathcal{G}}(0,n(\log n)^{-\beta})} E_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big)\big),\end{aligned}$$

and therefore, we have

$$
\begin{aligned}
&P_0^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{-\beta}\big) \geq n^2(\log n)^{-\alpha}\big) \\
&\quad \geq \frac{E_0^{\mathcal{G}}(\tau_{\mathcal{G}}(0, n(\log n)^{-\beta})) - n^2(\log n)^{-\alpha}}{\sup_{x \in B_{\mathcal{G}}(0, n(\log n)^{-\beta})} E_x^{\mathcal{G}}(\tau_{\mathcal{G}}(0, n(\log n)^{-\beta}))}.
\end{aligned} \tag{3.13}
$$

On the set $\tilde{F}_1 \cap \tilde{F}_2$, since any path from $x \in B_{\mathcal{G}}(0, n(\log n)^{-\beta})$ to $B_{\mathcal{G}}(0, n(\log n)^{1/6+\delta})^c$ passes through C_n and C_{2n}, we have

$$
\begin{aligned}
&\sup_{x \in B_{\mathcal{G}}(0, n(\log n)^{-\beta})} P_x^{\mathcal{G}}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) < T_0\big) \\
&\quad \leq \sup_{x \in B_{\mathcal{G}}(0, n(\log n)^{-\beta})} \frac{R_{\mathcal{G}}(0, x)}{R_{\mathcal{G}}(x, B_{\mathcal{G}}(0, n(\log n)^{1/6+\delta})^c)} \\
&\quad \leq \sup_{x \in B_{\mathcal{G}}(0, n(\log n)^{-\beta})} \frac{R_{\mathcal{G}}(0, x)}{R_{\mathcal{G}}(C_n, C_{2n})} \\
&\quad \leq \frac{n(\log n)^{-\beta}}{n} = (\log n)^{-\beta}.
\end{aligned}
$$

On the set $\tilde{F}_1 \cap \tilde{F}_2$, the right-hand side of (3.13) is bounded below by

$$
\begin{aligned}
\frac{n^2(\log n)^{-2\beta-2\delta} - n^2(\log n)^{-\alpha}}{n^2(\log n)^{1/2-2\beta+\delta}} &\geq \frac{1}{2} \frac{n^2(\log n)^{-2\beta-2\delta}}{n^2(\log n)^{1/2-2\beta+\delta}} \\
&= \frac{1}{2}(\log n)^{-1/2-3\delta},
\end{aligned}
$$

where we use $\alpha = 2\beta + 3\delta$ in the first inequality. Putting these estimates into (3.12), for $\omega \in \tilde{F}_1 \cap \tilde{F}_2$, we have

$$
P_0^{\mathcal{G}(\omega)}\big(\tau_{\mathcal{G}}\big(0, n(\log n)^{1/6+\delta}\big) \leq n^2(\log n)^{-\alpha}\big) \leq 2(\log n)^{1/2+3\delta-\beta} = \mathrm{O}\big((\log n)^{-1-\delta}\big). \qquad \square
$$

4. Proof of Theorem 1.2.3

In this section we will prove Theorem 1.2.3. As in the proof of the upper bound of Theorem 1.2.1, we need to estimate the lower bound of $\mu_{\mathcal{G}}(B_{\mathcal{G}}(0, n))$, and do so by considering the loop-erased random walk.

4.1. *Setting and notation*

Let $d = 3$ and $\alpha \in (0, 1)$. We choose

$$
0 = j_0 < j_1 < \cdots < j_{m-1} \leq n
$$

such that $j_i = i\lfloor n^\alpha \rfloor$ for each i, where $m = \inf\{k \in \mathbb{N}\colon k\lfloor n^\alpha \rfloor > n\}$. Define Y_k as in (2.1), namely,

$$
Y_k = \mathbf{1}\big\{L\big(S[j_{i-1}, k]\big) \cap S(k, j_i] = \varnothing\big\} \quad \text{for } j_{i-1} \leq k < j_i.
$$

Using (2.2), we see that

$$
\max_{1 \leq l \leq n} d_{\mathcal{G}}(0, S_l) \leq \sum_{k=0}^{j_{m-1}} Y_k + n^\alpha. \tag{4.1}
$$

So we need to estimate $E(\sum_{k=0}^{j_{m-1}} Y_k) \vee n^\alpha$. In the next subsection we will show the following for a suitable choice of α.

$$E\left(\sum_{k=0}^{j_{m-1}} Y_k\right) \vee n^\alpha \le c n^{9/10} (\log n)^a, \tag{4.2}$$

where c and a are some positive constants. By definition of Y_k, we have

$$E\left(\sum_{k=j_{i-1}}^{j_i} Y_k\right) = E\left(\sum_{k=0}^{j_1} Y_k\right) \quad \forall i = 1, 2, \dots, m-1. \tag{4.3}$$

Therefore, we will consider $E(\sum_{k=0}^{j_1} Y_k)$.

We give some notations. Let $0 \le k \le j_1$ and define $\omega_k = L(S[0,k])$, that is the path obtained by erasing loops on $S[0,k]$. We define random variable Z_k by

$$Z_k = P^{S_k}\big(\omega_k \cap S[1, j_1 - k] = \varnothing\big). \tag{4.4}$$

Then we have $E(Y_k) = E(Z_k)$. Let Q_k be the nearest neighbor of S_k with $Q_k \notin \omega_k$ which maximizes $P^{\cdot}(\omega_k \cap S[1, j_1 - k - 1] = \varnothing)$. If there is more than one such points choose one arbitrarily. If each nearest neighbor of S_k is in ω_k, choose Q_k arbitrarily. (Note that this Q_k is denoted as T_k in the proof of Theorem 7.5.1 in [9].) Define

$$\begin{aligned} \sigma(0) &= \sup\big\{ j \le j_1 : \ S_j = 0 \big\}, \\ \sigma(i) &= \sup\big\{ j \le j_1 : \ S_j = S\big(\sigma(i-1)+1\big) \big\}, \end{aligned}$$

for $1 \le i \le i_0$ where $\sigma(i_0) = j_1$.

Let

$$W_k = \begin{cases} 0 & \text{if } \ Y_k = 0, \\ \sigma(i+1) - \sigma(i) & \text{if } \ Y_k = 1 \ \text{ and } \ k = \sigma(i) \ \text{ for some } \ i. \end{cases}$$

Then

$$\sum_{k=0}^{j_1 - 1} W_k \le n^\alpha. \tag{4.5}$$

4.2. *Estimating the amount erased*

By applying the argument used in the proof of [9], Theorem 7.5.1, we have the following proposition.

Proposition 4.2.1. *Let $d = 3$ and $0 \le 2r < j_1 - k$. Then*

$$P\big(W_k = 2r + 1 | S_j, 0 \le j \le k\big) \ge \frac{1}{6} q_{2r}^{\omega_k}(Q_k, Q_k) P^{Q_k}\big(S[1, j_1 - k - 2r - 1] \cap \big(\omega_k \cup \{Q_k\}\big) = \varnothing\big), \tag{4.6}$$

where

$$q_l^{\omega_k}(x, y) = P^x\big(S[0, l] \cap \omega_k = \varnothing, S_l = y\big).$$

Moreover, on the event $\{Z_k \ge n^{-4}\}$, we have

$$q_{2r}^{\omega_k}(Q_k, Q_k) \ge c Z_k^2 r^{-3/2} (\log n)^{-3}, \tag{4.7}$$

for some $c > 0$.

Proof. We will sketch the proof (see [9], the proof of Theorem 7.5.1). Let $0 \le 2r < j_1 - k$. By definition of W_k and the Markov property, we have

$$\begin{aligned} P(W_k = 2r+1 | S_j, 0 \le j \le k) &= P\big(Y_k = 1, k = \sigma(i), \sigma(i+1) - \sigma(i) = 2r+1, \text{ for some } i | \\ &\quad S_j, 0 \le j \le k\big) \\ &\ge P\big(S_{k+1} = Q_k, L\big(S[0,k]\big) \cap S(k, j_1] = \varnothing, \\ &\quad S_{k+1} = S_{k+2r+1}, S_{k+1} \notin S[k+2r+2, j_1] | S_j, 0 \le j \le k\big) \\ &\ge \frac{1}{6} P^{Q_k}\big(S[0,2r] \cap \omega_k = \varnothing, S_{2r} = Q_k\big) \\ &\quad \times P^{Q_k}\big(S[1, j_1 - k - 2r - 1] \cap \big(\omega_k \cup \{Q_k\}\big) = \varnothing\big). \end{aligned}$$

For (4.7), we assume $Z_k \ge n^{-4}$. By the large deviation estimate (see, for example, [9], Lemma 1.5.1)

$$P^{Q_k}\big(|S_r - Q_k| \ge 5r^{1/2}(\log n)\big) \le c\mathrm{e}^{-5(\log n)} = cn^{-5}.$$

Therefore, it follows that

$$\begin{aligned} &P^{Q_k}\big(|S_r - Q_k| \le 5r^{1/2}(\log n), S[0,r] \cap \omega_k = \varnothing\big) \\ &\quad \ge P^{Q_k}\big(S[0,r] \cap \omega_k = \varnothing\big) - P^{Q_k}\big(|S_r - Q_k| \ge 5r^{1/2}(\log n)\big) \\ &\quad \ge \frac{1}{2} Z_k, \end{aligned}$$

so

$$\sum_{|x - Q_k| \le 5r^{1/2}(\log n)} P^{Q_k}\big(S[0,r] \cap \omega_k = \varnothing, S_r = x\big) \ge \frac{1}{2} Z_k.$$

Hence, by the Cauchy–Schwarz inequality, it follows that on the set $\{Z_k \ge n^{-4}\}$,

$$\begin{aligned} q_{2r}^{\omega_k}(Q_k, Q_k) &\ge \sum_{|x - Q_k| \le 5r^{1/2}(\log n)} q_r^{\omega_k}(Q_k, x)^2 \\ &\ge cr^{-3/2}(\log n)^{-3} Z_k^2. \end{aligned}$$

□

In order to establish (4.2), we need to estimate

$$P^{Q_k}\big(S[1, j_1 - k - 2r - 1] \cap \big(\omega_k \cup \{Q_k\}\big) = \varnothing\big) \tag{4.8}$$

by using Z_k. For this purpose, we now make some preparations. Let $0 < \beta < \alpha < 1$ and $\gamma \ge 1$ (the exact values of these numbers will be determined later). We estimate (4.8) for $0 \le k \le j_1 - \lfloor n^\beta \rfloor$. Let $N = (j_1 - k)^{1/2}(\log n)^{-\gamma}$ and $\xi_N = \inf\{j \ge 0\colon |S_j - Q_k| > N\}$.

First, we will bound this hitting time as follows.

Lemma 4.2.2. *There exists* $c \in (0,1)$ *such that*

$$P^{Q_k}(\xi_N > j_1 - k) \le c^{(\log n)^{2\gamma}}.$$

Proof. We have

$$\begin{aligned} P^{Q_k}(\xi_N > j_1 - k) &= P^{Q_k}\big(|S_j - Q_k| \le N \, \forall j = 1, \dots, j_1 - k\big) \\ &= P\big(|S_j| \le N \, \forall j = 1, \dots, j_1 - k\big). \end{aligned}$$

By the large deviation estimate (see, for example, [9], Lemma 1.5.1), the right-hand side can be bounded above by

$$P\bigl(|S_{\lfloor lN^2\rfloor}-S_{\lfloor (l-1)N^2\rfloor}|\le 2N\ \forall l=1,\dots,(\log n)^{2\gamma}\bigr)\le c^{(\log n)^{2\gamma}},$$

for some $c\in(0,1)$. □

Now we define $\gamma\ge 1$. Let $c\in(0,1)$ be the constant in Lemma 4.2.2. Define $\rho>0$ such that $c=\mathrm{e}^{-\rho}$ and write $\gamma:=\frac{5}{2\rho}\vee 1$. Then since $2\gamma>1$,

$$P^{Q_k}(\xi_N>j_1-k)\le \mathrm{e}^{-\rho(\log n)^{2\gamma}}\le \mathrm{e}^{-2\rho\gamma(\log n)}\le n^{-5}. \tag{4.9}$$

We are now in a position to estimate (4.8) by using Z_k and with these preparations in place, it is easy to conclude the following proposition.

Proposition 4.2.3. *Let $d=3$ and*

$$\begin{aligned}&0\le k\le j_1-\lfloor n^\beta\rfloor,\\&(j_1-k)\bigl(1-(\log n)^{-2\gamma-10}\bigr)\le 2r+1\le j_1-k.\end{aligned} \tag{4.10}$$

Then, on the event $\{Z_k\ge n^{-4}\}$, we have

$$P^{Q_k}\bigl(S[1,j_1-k-2r-1]\cap\bigl(\omega_k\cup\{Q_k\}\bigr)=\varnothing\bigr)\ge\tilde c Z_k,$$

for some $\tilde c>0$.

Proof. Assume that k and r satisfy (4.10). We have

$$\begin{aligned}&P^{Q_k}\bigl(S[1,j_1-k-2r-1]\cap\bigl(\omega_k\cup\{Q_k\}\bigr)=\varnothing\bigr)\\&\quad\ge P^{Q_k}\bigl(S[1,j_1-k-2r-1]\cap\bigl(\omega_k\cup\{Q_k\}\bigr)=\varnothing,\xi_N\ge j_1-k-2r-1\bigr)\\&\quad\ge P^{Q_k}\bigl(S[1,\xi_N]\cap\bigl(\omega_k\cup\{Q_k\}\bigr)=\varnothing\bigr)-P^{Q_k}(\xi_N<j_1-k-2r-1).\end{aligned} \tag{4.11}$$

Since $j_1-k-2r-1\le(j_1-k)(\log n)^{-2\gamma-10}$, a straightforward large deviation estimate gives

$$P^{Q_k}(\xi_N<j_1-k-2r-1)=\mathrm{O}\bigl(n^{-5}\bigr).$$

So on the event $\{Z_k\ge n^{-4}\}$, we have only to consider the first term of (4.11). By using the strong Markov property, we have

$$P^{Q_k}\bigl(S[1,\xi_N]\cap\bigl(\omega_k\cup\{Q_k\}\bigr)=\varnothing\bigr)\ge cP^{Q_k}\bigl(S[1,\xi_N]\cap\omega_k=\varnothing\bigr),$$

for some $c>0$, cf. [12], Lemma 2.1, for example. By Lemma 4.2.2, it follows that on the event $\{Z_k\ge n^{-4}\}$,

$$\begin{aligned}&P^{Q_k}\bigl(S[1,\xi_N]\cap\omega_k=\varnothing\bigr)\\&\quad\ge P^{Q_k}\bigl(S[1,\xi_N]\cap\omega_k=\varnothing,\xi_N\le j_1-k\bigr)\\&\quad\ge P^{Q_k}\bigl(S[1,j_1-k]\cap\omega_k=\varnothing\bigr)-P^{Q_k}(\xi_N>j_1-k)\\&\quad\ge Z_k-\mathrm{O}\bigl(n^{-5}\bigr)\\&\quad\ge\tilde cZ_k,\end{aligned}$$

for some $\tilde c>0$. □

We can combine Proposition 4.2.3 with Proposition 4.2.1 to conclude the following; if k and r satisfy (4.10), then on the event $\{Z_k \geq n^{-4}\}$, we have

$$P(W_k = 2r+1 | S_j, 0 \leq j \leq k) \geq c Z_k^3 r^{-3/2} (\log n)^{-3}. \tag{4.12}$$

We finish this subsection by showing (4.2).

Proposition 4.2.4. *Let $d = 3$ and $0 < \beta < \alpha < 1$. Recall $\gamma = \frac{5}{2\rho} \vee 1$. Then there exists $c > 0$ such that the following holds*

$$E\left(\sum_{k=0}^{j_{m-1}} Z_k\right) \leq c n^{1-\beta/6} (\log n)^{(2\gamma+13)/3} + n^{1-\alpha/3+\beta/3}. \tag{4.13}$$

Further, the right-hand side of (4.13) *attains the minimum $cn^{9/10}(\log n)^{(2\gamma+13)/3}$ when $\alpha = \frac{9}{10}$ and $\beta = \frac{3}{5}$, so that* (4.2) *holds.*

Proof. Let $0 \leq k \leq j_1 - \lfloor n^\beta \rfloor$. Then by (4.12),

$$\begin{aligned} E(W_k) &\geq {\sum_r}' (2r+1) E\big(P\big(W_k = 2r+1 | S_j, 0 \leq j \leq k\big)\big) \\ &\geq c {\sum_r}' E\big(Z_k^3 r^{-1/2} (\log n)^{-3}; Z_k \geq n^{-4}\big) \\ &= c (\log n)^{-3} E\big(Z_k^3; Z_k \geq n^{-4}\big) {\sum_r}' r^{-1/2}, \end{aligned}$$

for some $c > 0$, where the summation ${\sum_r}'$ is over all r with $2r+1$ satisfies (4.10). By a simple calculation, the last expression is bounded below by

$$\begin{aligned} &\tilde{c} E\big(Z_k^3; Z_k \geq n^{-4}\big) (j_1 - k)^{1/2} (\log n)^{-2\gamma-13} \\ &\quad \geq \tilde{c} E\big(Z_k^3; Z_k \geq n^{-4}\big) n^{\beta/2} (\log n)^{-2\gamma-13}, \end{aligned}$$

for some $\tilde{c} > 0$. Therefore, by (4.5),

$$\begin{aligned} n^\alpha &\geq \sum_{k=0}^{j_1-1} E(W_k) \\ &\geq \sum_{k=0}^{j_1-\lfloor n^\beta \rfloor} E(W_k) \\ &\geq \tilde{c} n^{\beta/2} (\log n)^{-2\gamma-13} \sum_{k=0}^{j_1-\lfloor n^\beta \rfloor} E\big(Z_k^3; Z_k \geq n^{-4}\big). \end{aligned}$$

This implies

$$\begin{aligned} \sum_{k=0}^{j_1} E\big(Z_k^3\big) &\leq \sum_{k=0}^{j_1-\lfloor n^\beta \rfloor} E\big(Z_k^3\big) + n^\beta \\ &\leq \sum_{k=0}^{j_1-\lfloor n^\beta \rfloor} E\big(Z_k^3; Z_k \geq n^{-4}\big) + n^{-12+\alpha} + n^\beta \\ &\leq c n^{\alpha-\beta/2} (\log n)^{2\gamma+13} + n^\beta, \end{aligned} \tag{4.14}$$

for some $c > 0$. So by (),

$$\begin{aligned}\sum_{k=0}^{j_{m-1}} E\big(Z_k^3\big) &\le \sum_{i=1}^{m-1} \sum_{k=j_{i-1}}^{j_i} E\big(Z_k^3\big)\\ &= \sum_{i=1}^{m-1} \sum_{k=0}^{j_1} E\big(Z_k^3\big)\\ &= (m-1) \sum_{k=0}^{j_1} E\big(Z_k^3\big)\\ &\le cn^{1-\beta/2} (\log n)^{2\gamma+13} + n^{1-\alpha+\beta}, \end{aligned}\tag{4.15}$$

and by using Hölder's inequality we can deduce (4.13). It is easy to check that (4.2) holds when $\alpha = \frac{9}{10}$ and $\beta = \frac{3}{5}$. □

Remark 4.2.5. *It is quite likely that the estimate* (4.14) *is not sharp. We expect that the following holds for each* $0 < \beta < \alpha < 1$,

$$\sum_{k=0}^{j_1} E\big(Z_k^3\big) \le cn^{\alpha-\beta/2} (\log n)^{2\gamma+13}. \tag{4.16}$$

4.3. *Proof of Theorem* 1.2.3

In order to complete the proof of Theorem 1.2.3, it is enough to prove the following proposition. Indeed, once the proposition is proved, we can apply similar argument as in the proof of the upper bound of Theorem 1.2.1.

Proposition 4.3.1. *Let* $d = 3$ *and* $\varepsilon > 0$. *Then we have*

$$P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}\big(0, n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\big)\big) \le cn\big) = \mathrm{O}\big((\log n)^{-1-\varepsilon}\big),$$

for some $c > 0$.

Proof. By (4.1), Proposition 4.2.4, and Chebyshev's inequality,

$$\begin{aligned} &P\Big(\max_{1\le l\le n} d_{\mathcal{G}}(0, S_l) \ge n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\Big)\\ &\quad\le P\left(\sum_{k=0}^{j_{m-1}} Y_k + n^{9/10} \ge n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\right)\\ &\quad\le E\left(\sum_{k=0}^{j_{m-1}} Y_k + n^{9/10}\right) n^{-9/10} (\log n)^{-(2\gamma+13)/3-(1+\varepsilon)}\\ &\quad= \mathrm{O}\big((\log n)^{-1-\varepsilon}\big). \end{aligned}$$

Therefore,

$$\begin{aligned} &P\big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}\big(0, n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\big)\big) \le cn\big)\\ &\quad\le P\Big(\mu_{\mathcal{G}}\big(B_{\mathcal{G}}\big(0, n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\big)\big) \le cn,\\ &\qquad \max_{1\le l\le n} d_{\mathcal{G}}(0, S_l) < n^{9/10} (\log n)^{(2\gamma+13)/3+(1+\varepsilon)}\Big) + \mathrm{O}\big((\log n)^{-1-\varepsilon}\big)\\ &\quad\le P\big(\sharp\{S_k\colon 0 \le k \le n\} \le cn\big) + \mathrm{O}\big((\log n)^{-1-\varepsilon}\big). \end{aligned}$$

But by [7], there exist $\tilde{c} > 0$ and $c' > 0$ such that

$$E\big(\sharp\{S_k \colon 0 \le k \le n\}\big) \sim \tilde{c}n,$$
$$\mathrm{Var}\big(\sharp\big\{S_k \colon 0 \le k \le n\big\}\big) \sim c' n \log n.$$

So we can complete the proof by taking $c = \frac{\tilde{c}}{2}$. □

Acknowledgments

I would like to thank Professor Takashi Kumagai for a lot of helpful discussions and careful reading of the early version of the manuscript. I also thank Dr. David A. Croydon for English corrections.

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