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On the volume of intersection of three independent Wiener sausages

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Abstract. Let K be a compact, non-polar set in $\mathbb{R}^m$, $m \geq 3$ and let $S^i_K(t) = \{B^i(s) + y\colon\ 0 \leq s \leq t,\ y \in K\}$ be Wiener sausages associated to independent Brownian motions B^i, $i = 1, 2, 3$ starting at 0. The expectation of volume of $\bigcap_{i=1}^3 S^i_K(t)$ with respect to product measure is obtained in terms of the equilibrium measure of K in the limit of large t.

Résumé. Soit K un ensemble compact, non-polaire dans $\mathbb{R}^m$ ($m \geq 3$) et soit $S^i_K(t) = \{B^i(s) + y\colon\ 0 \leq s \leq t,\ y \in K\}$ des saucisses de Wiener associées à des processus Browniens indépendants B^i, $i = 1, 2, 3$ initalisés à 0. L'espérance des volumes de $\bigcap_{i=1}^3 S^i_K(t)$ par rapport à la mesure produit est obtenue en termes de la mesure d'équilibre de K lorsque t tend vers l'infini.

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Keywords: Wiener sausage; Equilibrium measure

1. Introduction

Let K be a compact, non-polar set in Euclidean space $\mathbb{R}^m$ $(m = 2, 3, \ldots)$, and let $(B(s), s \geq 0; \mathbb{P}_x, x \in \mathbb{R}^m)$ be Brownian motion associated to the parabolic operator $-\Delta + \frac{\partial}{\partial t}$. The Wiener sausage associated to K, and generated by B up to time t is the random set $S_K(t)$ defined by

$$S_K(t) = \left\{B(s) + y\colon\ 0 \leq s \leq t,\ y \in K\right\}.$$

Its volume, denoted by $|S_K(t)|$, is a simple example of a non-Markovian functional of Brownian motion. It plays a key role in the study of stochastic phenomena like trapping in random media, random Schrödinger operators, and diffusion of matter [15]. The expectation of $|S_K(t)|$ has been the subject of extensive investigation. Spitzer [14], Le Gall [8–10] and Port [12] analyzed its asymptotic behaviour for large t, while van den Berg and Le Gall [17] initiated the study of the asymptotic behaviour for small t. See in particular Chapter 2 of [3] for an up to date account of the small t behaviour in the more general setting of a Riemannian manifold.

Let $S^i_{K_i}(t)$, $i = 1, \ldots, n$ denote Wiener sausages associated to compact, non-polar sets K_i, $i = 1, \ldots, n$, and generated by independent Brownian motions B^i, $i = 1, \ldots, n$ respectively. The random set $\bigcap_{i=1}^n S^i_{K_i}(t)$ shows up in numerous places in the physical sciences. In quantum field theory one is interested in estimates for the probability that this random set is empty, in particular in the case $n = 2$, $m = 4$, and $K_1 = \cdots = K_n = K$ [1,2]. The phenomenon of loop condensation [6] for the intersection of n independent random walks on $\mathbb{Z}^m$ (or Wiener sausages in $\mathbb{R}^m$) with

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$n > m/(m-2)$ initiated the study of the large deviations for the volume of intersection on the scale of their mean [18] in the case where the K_i's are balls with equal radius. In polymer physics one wishes to obtain properties of the volume of intersection in either the random walk approximation on $\mathbb{Z}^3$ [7,11], or in the Brownian motion approximation in $\mathbb{R}^3$. Below we calculate the precise expected volume of intersection in the physically relevant cases $m=3$, and $n=2$, or $n=3$. While we only consider identical polymers $K_1 = \cdots = K_n = K$, and of equal length t, extensions can easily be obtained to include expressions for the expected volume of $S^i_{K_i}(a_i t)$, $i = 1, \ldots, n$, where the a_i's are strictly positive and not all of them infinite. In the proofs of Theorems 1–3 below we will see that the physically relevant case $m=3$ is mathematically the most challenging yielding a non-trivial leading term.

Define for $t > 0$ the expectation with respect to the product law by

$$N_{n,m}(t) = \mathbb{E}^1_0 \otimes \cdots \otimes \mathbb{E}^n_0 \left[\left| \bigcap_{i=1}^n S^i_K(t) \right| \right].$$

Let $u : (\mathbb{R}^m - K) \times (0, \infty) \to \mathbb{R}$ be the solution of

$$\Delta u = \frac{\partial u}{\partial t}, \qquad x \in \mathbb{R}^m - K, t > 0, \tag{1}$$

with initial condition

$$u(x;0) = 0, \quad x \in \mathbb{R}^m - K, \tag{2}$$

and boundary condition

$$u(x;t) = 1, \quad x \in \partial K, t > 0, \tag{3}$$

where ∂K is the boundary of K. Equations (1)–(3) are to be understood in the weak sense. (The pointwise limit in (3) holds only at the regular points of ∂K.) It is well known that the solution of (1)–(3) is given by

$$u(x;t) = \mathbb{P}_x[T_K < t], \tag{4}$$

where T_K is the first hitting time of K,

$$T_K = \inf\{s \ge 0:\ B(s) \in K\}. \tag{5}$$

We adopt the usual convention in (5) that the infimum over the empty set equals $+\infty$. We extend u to all of $\mathbb{R}^m \times (0, \infty)$ by putting $u(x;t) = 1$ on $K \times (0, \infty)$. By Fubini's theorem we have that

$$N_{n,m}(t) = \int_{\mathbb{R}^m} \mathrm{d}x \big(\mathbb{P}_x[T_K < t]\big)^n. \tag{6}$$

In the last paragraph of this section we will show that

$$N_{2,m}(t) = 2N_{1,m}(t) - N_{1,m}(2t), \quad t > 0. \tag{7}$$

The asymptotic behaviour of $N_{2,m}(t)$ as $t \to \infty$ or $t \to 0$ can then be read-off from the results obtained in [3,8,9,12,14,17]. No such recursive formulae are known for $n = 3, 4, \ldots$. In this paper we obtain the asymptotic bahaviour in the case where $n = 3, m = 3, 4, \ldots,$ and $t \to \infty$. The strong dimension dependence in our main results, Theorems 1–3 below, is directly related to the integrability properties of $\mathbb{P}_x[T_K < \infty]$ for large $|x|$. These integrability properties improve as the dimension m increases. If $m = 3, 4, \ldots,$ then (Theorem 2 in [16])

$$\lim_{t\to\infty} N_{n,m}(t) = \int_{\mathbb{R}^m} \mathrm{d}x \left(c_m \int \mu_K(\mathrm{d}y)|x-y|^{2-m} \right)^n < \infty, \tag{8}$$

if and only if $n > m/(m-2)$, where μ_K denotes the equilibrium measure supported on K, and

$$c_m = 4^{-1}\pi^{-m/2}\Gamma\big((m-2)/2\big).$$

Throughout the paper we denote the Newtonian capacity of K by

$$C(K)=\mu_K(K).$$

Euler's constant is denoted by γ, Catalan's constant is denoted by G, and

$$H=\int_0^{1/\sqrt{3}}\left(1+\theta^2\right)^{-1}\log\theta\,\mathrm{d}\theta. \tag{9}$$

Theorem 1. *Let $m=3$. Then for $t\to\infty$*

$$\begin{aligned}N_{3,3}(t)&=2^{-1}(4\pi)^{-2}C(K)^3\log t\\&\quad-(4\pi)^{-2}C(K)^2\int\mu_K(\mathrm{d}y)\log|y|+2(4\pi)^{-3}(2\pi-\pi\gamma-12G-6H)C(K)^3\\&\quad+(4\pi)^{-3}\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\left(|x-y_1|^{-1}|x-y_2|^{-1}-|x|^{-2}\right)|x-y_3|^{-1}\\&\quad+3(4\pi)^{-7/2}C(K)^4t^{-1/2}\log t+\mathrm{O}\left(t^{-1/2}\right).\end{aligned}$$

Theorem 2. *Let $m=4$. Then for $t\to\infty$*

$$\begin{aligned}N_{3,4}(t)&=\int_{\mathbb{R}^4}\mathrm{d}x\left(c_4\int\mu_K(\mathrm{d}y)|x-y|^{-2}\right)^3-3(4\pi)^{-4}C(K)^3t^{-1}\log t\\&\quad+6(4\pi)^{-4}C(K)t^{-1}\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\log|y_1-y_2|\\&\quad+3(4\pi)^{-4}(\gamma-2-\log 4+\log 3)C(K)^3t^{-1}\\&\quad+3(4\pi)^{-2}\int_{\mathbb{R}^4}\mathrm{d}x\left(c_4\int\mu_K(\mathrm{d}y)|x-y|^{-2}\right)^3C(K)t^{-1}+\mathrm{O}\left(\left(t^{-1}\log t\right)^2\right).\end{aligned}$$

Theorem 3. *Let $m=5,6,7,\dots$. Then for $t\to\infty$*

$$\begin{aligned}N_{3,m}(t)&=\int_{\mathbb{R}^m}\mathrm{d}x\left(c_m\int\mu_K(\mathrm{d}y)|x-y|^{2-m}\right)^3\\&\quad-3(4\pi)^{-m}2^{m-3}(m-2)^{-1}\Gamma\left((m-4)/2\right)C(K)\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)|y_1-y_2|^{4-m}t^{(2-m)/2}\\&\quad+6(m-2)^{-1}(4\pi)^{-m/2}C(K)\int_{\mathbb{R}^m}\mathrm{d}x\left(c_m\int\mu_K(\mathrm{d}y)|x-y|^{2-m}\right)^3t^{(2-m)/2}\\&\quad+\begin{cases}3^{-1}(4\pi)^{-5}\left(12+\pi-2\sqrt{3}\right)C(K)^3t^{-2}+\mathrm{O}\left(t^{-5/2}\right), & m=5,\\ 2^{-1}(4\pi)^{-6}C(K)^3t^{-3}\log t+\mathrm{O}\left(t^{-3}\right), & m=6,\\ \mathrm{O}\left(t^{-m/2}\right), & m\geq 7.\end{cases}\end{aligned}$$

Define the last exit time of K by

$$L_K=\sup\left\{s\geq 0\colon\ B(s)\in K\right\}, \tag{10}$$

and $L_K=+\infty$ if the supremum is over the empty set. The law of L_K is given by (see [13,15])

$$\mathbb{P}_x[L_K<t]=\int_0^t\mathrm{d}s\int\mu_K(\mathrm{d}y)p(x,y;s), \tag{11}$$

where $p(x, y; s)$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, $s > 0$ is the transition density for Brownian motion (associated to $-\Delta + \frac{\partial}{\partial s}$) given by

$$p(x, y; s) = (4\pi s)^{-m/2} \mathrm{e}^{-|x-y|^2/(4s)}. \tag{12}$$

In particular

$$\mathbb{P}_x[L_K < \infty] = \mathbb{P}_x[T_K < \infty] = c_m \int \mu_K(\mathrm{d}y)|x-y|^{2-m}. \tag{13}$$

This reproves (8) by the monotone convergence theorem, (6) and (13).

The first step in the proofs of Theorems 1–3 is to substitute

$$\mathbb{P}_x[T_K < t] = \mathbb{P}_x[L_K < t] + \mathbb{P}_x[T_K < t < L_K] \tag{14}$$

in (6), and to bound higher order contributions from terms involving e.g. $(\mathbb{P}_x[T_K < t < L_K])^2$. These bounds are given in Proposition 4 below. The proof is deferred to Section 2. The asymptotic behaviour of $\int_{\mathbb{R}^m} \mathrm{d}x\, (\mathbb{P}_x[L_K < t])^3$ as $t \to \infty$ is given in Propositions 6–8 below for dimensions $m = 3$, $m = 4$, and $m \ge 5$ respectively. The corresponding calculations are deferred to Sections 4–6 respectively. Finally, the asymptotic behaviour of $\int_{\mathbb{R}^m} \mathrm{d}x\, (\mathbb{P}_x[L_K < \infty])^2 \mathbb{P}_x[T_K < t < L_K]$ as $t \to \infty$ is given in Proposition 5. The proof relies on an application of the strong Markov property at T_K and is deferred to Section 3.

Proposition 4. *Let $m = 3, 4, \dots$. Define $R_m(t)$ by*

$$N_{3,m}(t) = \int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < t]\right)^3 + 3\int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[T_K < t < L_K] + R_m(t).$$

Then for $t \to \infty$

$$R_3(t) = \mathrm{O}\big(t^{-1/2}\big),$$
$$R_4(t) = \mathrm{O}\big(t^{-2}\log t\big)$$

and for $m = 5, 6, \dots$

$$R_m(t) = \mathrm{O}\big(t^{2-m}\big).$$

Proposition 5. *Let $m = 3, 4, \dots$. Then for $t \to \infty$*

(i) $m = 3$

$$\int_{\mathbb{R}^3} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[T_K < t < L_K] = (4\pi)^{-7/2} C(K)^4 t^{-1/2} \log t + \mathrm{O}\big(t^{-1/2}\big),$$

(ii) $m = 4$

$$\int_{\mathbb{R}^4} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[T_K < t < L_K] = (4\pi)^{-2} \int_{\mathbb{R}^4} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^3 C(K) t^{-1} + \mathrm{O}\big(\big(t^{-1}\log t\big)^2\big),$$

(iii) $m = 5, 6, \dots$

$$\int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[T_K < t < L_K]$$
$$= (4\pi)^{-m/2} \left(\frac{m}{2} - 1\right)^{-1} \int_{\mathbb{R}^4} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^3 C(K) t^{(2-m)/2} + \mathrm{O}\big(t^{-m/2}\big).$$

Proposition 6. *Let $m = 3$. Then for $t \to \infty$*

$$\begin{aligned}
&\int_{\mathbb{R}^3} \mathrm{d}x \left(\mathbb{P}_x[L_K < t]\right)^3 \\
&\quad = 2^{-1}(4\pi)^{-2} C(K)^3 \log t - (4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \log|y| \\
&\qquad + 2(4\pi)^{-3}(2\pi - \pi\gamma - 12G - 6H) C(K)^3 \\
&\qquad + (4\pi)^{-3} \int_{\mathbb{R}^3} \mathrm{d}x \int\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\left(|x-y_1|^{-1}|x-y_2|^{-1} - |x|^{-2}\right)|x-y_3|^{-1} \\
&\qquad + \mathrm{O}\left(t^{-1/2}\right).
\end{aligned}$$

Proposition 7. *Let $m = 4$. Then for $t \to \infty$*

$$\begin{aligned}
\int_{\mathbb{R}^4} \mathrm{d}x \left(\mathbb{P}_x[L_K < t]\right)^3 &= \int_{\mathbb{R}^4} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^3 - 3(4\pi)^{-4} C(K)^3 t^{-1} \log t \\
&\quad + 6(4\pi)^{-4} C(K) t^{-1} \int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2) \log|y_1 - y_2| \\
&\quad + 3(4\pi)^{-4}(\gamma - 2 - \log 4 + \log 3) C(K)^3 t^{-1} + \mathrm{O}\left(t^{-2}\log t\right).
\end{aligned}$$

Proposition 8. *Let $m = 5, 6, \ldots$. Then for $t \to \infty$*

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < t]\right)^3 \\
&\quad = \int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^3 \\
&\qquad - 3(4\pi)^{-m} 2^{m-3} (m-2)^{-1} \Gamma\left((m-4)/2\right) C(K) \int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2) |y_1 - y_2|^{4-m} t^{(2-m)/2} \\
&\qquad + \begin{cases} 3^{-1}(4\pi)^{-5}\left(12 + \pi - 2\sqrt{3}\right) C(K)^3 t^{-2} + \mathrm{O}\left(t^{-5/2}\right), & m = 5, \\ 2^{-1}(4\pi)^{-6} C(K)^3 t^{-3} \log t + \mathrm{O}\left(t^{-3}\right), & m = 6, \\ \mathrm{O}\left(t^{-m/2}\right), & m \geq 7. \end{cases}
\end{aligned}$$

We conclude this Introduction with a short proof of (7). Let $t > 0$, and let $B'(s) = B(t+s) - B(t)$, and $B''(s) = B(t-s) - B(t)$ for every $s \in [0, t]$. Then B' and B'' are two independent Brownian motions on the time interval $[0, t]$. Let $S'_K(t)$ and $S''_K(t)$ respectively denote the corresponding Wiener sausages associated to K over the time interval $[0, t]$. Let $S_K(t, 2t) = \{B(s) + y\colon t \leq s \leq 2t, y \in K\}$ be the Wiener sausage generated by B over the time interval $[t, 2t]$. By translation invariance of Lebesgue measure $\mathbb{E}[|S_K(t) \cap S_K(t, 2t)|] = \mathbb{E}[|S'_K(t) \cap S''_K(t)|] = N_{2,m}(t)$. On the other hand, $N_{1,m}(2t) = \mathbb{E}[|S_K(2t)|] = \mathbb{E}[|S_K(t)|] + \mathbb{E}[|S_K(t, 2t)|] - \mathbb{E}[|S_K(t) \cap S_K(t, 2t)|] = 2N_{1,m}(t) - N_{2,m}(t)$, which is (7).

2. Proof of Proposition 4

To prove Proposition 4 we define L_K and T_K as in (10) and (5) respectively. Then

$$\left(\mathbb{P}_x[T_K < t]\right)^3 = \left(\mathbb{P}_x[L_K < t] + \mathbb{P}_x[T_K < t < L_K]\right)^3 \geq \left(\mathbb{P}_x[L_K < t]\right)^3 + 3\left(\mathbb{P}_x[L_K < t]\right)^2 \mathbb{P}_x[T_K < t < L_K].$$

On the other hand, since $\mathbb{P}_x[T_K < t < L_K] = \mathbb{P}_x[T_K < t < L_K < \infty] \leq \mathbb{P}_x[t < L_K < \infty]$, we have that

$$\mathbb{P}_x[L_K < t]\left(\mathbb{P}_x[T_K < t < L_K]\right)^2 \leq \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\mathbb{P}_x[T_K < t < L_K]$$

and

$$\big(\mathbb{P}_x[T_K < t < L_K]\big)^3 \le \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\mathbb{P}_x[T_K < t < L_K].$$

Hence

$$\begin{aligned}\big(\mathbb{P}_x[T_K < t]\big)^3 \le \big(\mathbb{P}_x[L_K < t]\big)^3 &+ 3\big(\mathbb{P}_x[L_K < t]\big)^2\mathbb{P}_x[T_K < t < L_K]\\ &+ 4\mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\mathbb{P}_x[T_K < t < L_K]\end{aligned}$$

and

$$0 \le R_m(t) \le 4\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\mathbb{P}_x[T_K < t < L_K]. \tag{15}$$

Lemma 9. *Let $m = 3, 4, \dots$. Then for $x \in \mathbb{R}^m$, $t > 0$*

$$\mathbb{P}_x[t < L_K < \infty] \le \left(\frac{m}{2} - 1\right)^{-1}(4\pi)^{-m/2}C(K)t^{(2-m)/2} \le C(K)t^{(2-m)/2}. \tag{16}$$

Proof. By (11) and (12)

$$\begin{aligned}\mathbb{P}_x[t < L_K < \infty] &= \int_t^\infty \mathrm{d}s \int \mu_K(\mathrm{d}y)p(x, y; s)\\ &\le \int_t^\infty \mathrm{d}s \int \mu_K(\mathrm{d}y)(4\pi s)^{-m/2}\\ &= \left(\frac{m}{2} - 1\right)^{-1}(4\pi)^{-m/2}C(K)t^{(2-m)/2}.\end{aligned} \tag{17}$$

□

By (15) and Lemma 9

$$0 \le R_m(t) \le C(K)t^{(2-m)/2}\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[T_K < t < L_K]. \tag{18}$$

Le Gall showed that for $m = 3$ and $t \to \infty$ (Lemma 2 in [8])

$$\int_{\mathbb{R}^3} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[T_K < t < L_K] = (4\pi)^{-2}C(K)^3 + \mathrm{o}(1), \tag{19}$$

and that for $m = 4$ and $t \to \infty$ ((9) in [8])

$$\int_{\mathbb{R}^4} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[T_K < t < L_K] = 2(4\pi)^{-4}C(K)^3t^{-1}(\log t)\big(1 + \mathrm{o}(1)\big). \tag{20}$$

Proposition 4 follows for $m = 3$ and $m = 4$ from (18)–(20). The proof of Proposition 4 for $m \ge 5$ relies on some independent estimates ((32), Lemmas 10 and 14) which will be proved in Sections 3 and 4 below. By (15) and (32) below we have that for $t \ge 2T$

$$\begin{aligned}R_m(t) \le 4C&\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\mathbb{P}_x[T_K < t]t^{(2-m)/2}\\ &+ 4C\left(\frac{m}{2} - 1\right)\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty]\int_0^{t-T} \mathrm{d}s\, \mathbb{P}_x[s < T_K < t](t - s)^{-m/2},\end{aligned} \tag{21}$$

where

$$C = \left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K) \tag{22}$$

and

$$T = C^{2/(m-2)}. \tag{23}$$

By Lemma 9 we have that the first term in the right-hand side of (21) is bounded by

$$4CC(K) \int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 t^{2-m}.$$

To estimate the contribution from the second term in the right-hand side of (21) we consider the contributions from $[0, T]$, $[T, t/2]$ and $[t/2, t-T]$ to the integral with respect to s. Since

$$\int_0^T \mathrm{d}s\, \mathbb{P}_x[s < T_K < t](t-s)^{-m/2} \le T(t-T)^{-m/2} \mathbb{P}_x[L_K < \infty]$$

we have by (17) that $[0, T]$ contributes at most

$$C(K)^2 \int_{\mathbb{R}^m} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 T(t-T)^{-m/2} t^{(2-m)/2} = \mathrm{O}\big(t^{1-m}\big).$$

To estimate the contribution from $[T, t/2]$ we note that

$$\mathbb{P}_x[s < T_K < t](t-s)^{-m/2} \le (t/2)^{-m/2} \mathbb{P}_x[s < L_K < \infty]. \tag{24}$$

Hence $[T, t/2]$ contributes, by (24), the first equality in (16) and Lemma 14 below, at most

$$\begin{aligned}
&4C\left(\frac{m}{2} - 1\right) 2^{m/2} t^{-m/2} \int_T^{t/2} \mathrm{d}s \int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty] \mathbb{P}_x[t < L_K < \infty] \mathbb{P}_x[s < L_K < \infty] \\
&\quad \le C(K)^4 t^{-m/2} \int_T^{t/2} \mathrm{d}s \int_0^\infty \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2 \int_s^\infty \mathrm{d}s_3\, (s_1 s_2 + s_2 s_3 + s_3 s_1)^{-m/2} \\
&\quad \le C(K)^4 t^{-m/2} \int_T^{t/2} \mathrm{d}s \int_t^\infty \mathrm{d}s_2 \int_s^\infty \mathrm{d}s_3\, (s_2 + s_3)^{-1} (s_2 s_3)^{(2-m)/2} \\
&\quad \le C(K)^4 t^{-m/2} \int_T^{t/2} \mathrm{d}s \int_t^\infty \mathrm{d}s_2\, s_2^{-m/2} \int_T^\infty \mathrm{d}s_3\, s_3^{(2-m)/2} \\
&\quad \le C(K)^4 T^{(4-m)/2} t^{2-m}.
\end{aligned}$$

To estimate the contribution from the interval $[t/2, t-T]$ we have, by Lemma 10 below, that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty] \mathbb{P}_x[t < L_K < \infty] \int_{t/2}^{t-T} \mathrm{d}s\, \mathbb{P}_x[s < T_K < t](t-s)^{-m/2} \\
&\quad \le C_1 \int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty] \mathbb{P}_x[t < L_K < \infty] \int_{t/2}^{t-T} \mathrm{d}s\, s^{-m/2} (t-s)^{(2-m)/2} \\
&\quad \le C_1 (t/2)^{-m/2} \int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty] \mathbb{P}_x[t < L_K < \infty] \int_{-\infty}^{t-T} \mathrm{d}s\, (t-s)^{(2-m)/2} \\
&\quad \le C_1 2^{(2+m)/2} T^{(4-m)/2} \int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty] \mathbb{P}_x[t < L_K < \infty] t^{-m/2}.
\end{aligned} \tag{25}$$

By the first equality in (16)

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\mathbb{P}_x[t < L_K < \infty] \\
&\quad = \int_{\mathbb{R}^m} \mathrm{d}x \int \mu_K(\mathrm{d}y_1) \int_0^\infty \mathrm{d}s_1\, p(x, y_1; s_1) \int \mu_K(\mathrm{d}y_2) \int_t^\infty \mathrm{d}s_2\, p(x, y_2; s_2) \\
&\quad = \int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2) \int_0^\infty \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2\, p(y_1, y_2; s_1 + s_2) \le C(K)^2 t^{(4-m)/2}.
\end{aligned}$$

Hence (25) is $\mathrm{O}(t^{2-m})$.

3. Proof of Proposition 5

Let $c \in \mathbb{R}^m$, $r > 0$, and let $B(c; r) = \{x\colon\ |x - c| \le r\}$, $R_c = \inf\{r > 0\colon\ K \subset B(c; r)\}$, and let $R = \inf\{R_c\colon\ c \in \mathbb{R}^m\}$. Without loss of generality we may assume that the infimum in the latter is attained at the origin.

The main ingredient in the proof of Proposition 5 is to use the strong Markov property for $\mathbb{P}_x[T_K < t < L_K]$ at the stopping time T_K. So

$$\mathbb{P}_x[T_K < t < L_K] = \mathbb{E}_x\bigl\{1_{\{T_K \le t\}}\mathbb{P}_{B(T_K)}[t - T_K < L_K < \infty]\bigr\}. \tag{26}$$

For any $z \in K$ and $y \in K$, we have that $|y - z| \le 2R$. Hence

$$\begin{aligned}
\mathbb{P}_z[t - s < L_K < \infty] &= \int_{t-s}^\infty \mathrm{d}\tau\, (4\pi\tau)^{-m/2} \int \mu_K(\mathrm{d}y) \mathrm{e}^{-|y-z|^2/(4\tau)} \\
&\ge \int_t^\infty \mathrm{d}\tau\, (4\pi\tau)^{-m/2} \int \mu_K(\mathrm{d}y) \mathrm{e}^{-R^2/t} \\
&= \left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K) \mathrm{e}^{-R^2/t} t^{(2-m)/2}.
\end{aligned} \tag{27}$$

It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \mathbb{P}_x[T_K < t < L_K] \\
&\quad \ge \left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K) \mathrm{e}^{-R^2/t} t^{(2-m)/2} \int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \mathbb{P}_x[T_K < t].
\end{aligned} \tag{28}$$

To prove the lower bound in Proposition 5 for $m \ge 5$ we note that

$$\mathbb{P}_x[T_K < t] \ge \mathbb{P}_x[L_K < \infty] - \mathbb{P}_x[t < L_K < \infty]. \tag{29}$$

Hence for $m \ge 5$ we have, by Lemma 9, that

$$\int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \mathbb{P}_x[T_K < t] \ge \int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^3 - C(K) t^{(2-m)/2} \int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2.$$

We conclude that for $m \ge 5$ the left-hand side of (28) is bounded from below by

$$\left(\frac{m}{2} - 1\right)^{-1} (4\pi)^{-m/2} C(K) \int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^3 t^{(2-m)/2} + \mathrm{O}\bigl(t^{-m/2}\bigr).$$

To prove the lower bound in Proposition 5 for $m=4$ we note that by (28), (29)

$$\int_{\mathbb{R}^4} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2 \mathbb{P}_x[T_K<t<L_K]$$
$$\geq (4\pi)^{-2}C(K)\mathrm{e}^{-R^2/t}t^{-1}\int_{\mathbb{R}^4} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^3 - C(K)t^{-1}\int_{\mathbb{R}^4} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[t<L_K<\infty].$$

Lemma 15 in Section 5 implies that

$$\int_{\mathbb{R}^4} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[t<L_K<\infty] = \mathrm{O}\big(t^{-1}\log t\big).$$

We conclude that for $m=4$ the left-hand side of (28) is bounded from below by

$$(4\pi)^{-2}C(K)\int_{\mathbb{R}^4} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^3 t^{-1} + \mathrm{O}\big(t^{-2}\log t\big).$$

Finally for $m=3$ we have that the left-hand side of (28) is bounded from below by

$$2(4\pi)^{-3/2}C(K)\mathrm{e}^{-R^2/t}t^{-1/2}\int_{\mathbb{R}^3} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[L_K<t].$$

Lemma 12 in Section 4 implies that

$$\int_{\mathbb{R}^3} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[L_K<t] = 2^{-1}(4\pi)^{-2}C(K)^3\log t + \mathrm{O}(1). \tag{30}$$

We conclude that for $m=3$ the left-hand side of (28) is bounded from below by

$$(4\pi)^{-7/2}C(K)^4t^{-1/2}\log t + \mathrm{O}\big(t^{-1/2}\big).$$

This completes the proof of the lower bound in Proposition 5.

To prove the upper bound in Proposition 5 we note that by the first equality in (27)

$$\mathbb{P}_z[t-s<L_K<\infty] \leq \left(\frac{m}{2}-1\right)^{-1}(4\pi)^{-m/2}\big(C(K)(t-s)^{(2-m)/2}\wedge 1\big). \tag{31}$$

By (26), (31) and the identity $(t-T_K)^{(2-m)/2} = t^{(2-m)/2} + 2^{-1}(m-2)\int_0^{T_K}\mathrm{d}s\,(t-s)^{-m/2}$ it follows that

$$\mathbb{P}_x[T_K<t<L_K] \leq C\mathbb{P}_x[T_K<t]t^{(2-m)/2} + C\left(\frac{m}{2}-1\right)\int_0^{t-T}\mathrm{d}s\,\mathbb{P}_x[s<T_K<t](t-s)^{-m/2}, \tag{32}$$

where C and T are given by (22) and (23) respectively. For $m\geq 4$ we have, by (32), that

$$\begin{aligned}&\int_{\mathbb{R}^m} \mathrm{d}x\, \big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[T_K<t<L_K]\\ &\quad\leq C\int_{\mathbb{R}^m}\mathrm{d}x\,\big(\mathbb{P}_x[L_K<\infty]\big)^3t^{(2-m)/2}\\ &\qquad + C(K)\int_{\mathbb{R}^m}\mathrm{d}x\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\int_0^{t-T}\mathrm{d}s\,\mathbb{P}_x[s<T_K<t](t-s)^{-m/2}.\end{aligned} \tag{33}$$

The first term in the right-hand side of (33) jibes with the leading term in Proposition 5 (for $m\geq 4$). To estimate the second term in the right-hand side of (33) we need the following lemma.

Lemma 10. *Let $m \geq 3$, and let T be given by (23). There exists a constant C_1 depending on K only such that for all $s \leq t - T$ and $x \in \mathbb{R}^m$*

$$\mathbb{P}_x[s < T_K < t] \leq C_1 s^{-m/2}(t-s).$$

Proof. By the Markov property at s we have that

$$\begin{aligned}\mathbb{P}_x[s < T_K < t] &= \int_{\mathbb{R}^m} \mathrm{d}y\, p_{\mathbb{R}^m - K}(x, y; s)\mathbb{P}_y[T_K < t-s] \\ &\leq \int_{\mathbb{R}^m} \mathrm{d}y\, p(x, y; s)\mathbb{P}_y[T_K < t-s] \leq (4\pi s)^{-m/2} N_{1,m}(t-s),\end{aligned} \tag{34}$$

where $p_{\mathbb{R}^m - K}(x, y; s)$, $x \in \mathbb{R}^m$, $y \in R^m$, $s > 0$ is the transition density with killing on K. Since $N_{1,m}(t) = C(K)t + o(t)$ as $t \to \infty$ we have that there exists T_1 such that $t \geq T_1$ implies $N_{1,m}(t) \leq 2C(K)t$. Let $T > 0$ be arbitrary. If $T \geq T_1$ then $N_{1,m}(t) < 2C(K)t$ for all $t \geq T$. If $T < T_1$ and $T \leq t \leq T_1$ then we have by monotonicity of $t \to N_{1,m}(t)$ that $N_{1,m}(t) \leq N_{1,m}(T_1) \leq 2C(K)T_1 \leq (2C(K)T_1/T)t$. Hence for all $t \geq T$

$$N_{1,m}(t) \leq 2C(K)\left(\frac{T_1}{T} \vee 1\right)t, \tag{35}$$

and the lemma holds with

$$C_1 = 2(4\pi)^{-m/2}C(K)\left(\frac{T_1}{T} \vee 1\right). \tag{36}$$

□

To complete the proof of Proposition 5 for $m \geq 5$ we estimate the second term in the right-hand side of (33) as follows. The contribution from $s \in [0, T]$ to the integral is bounded by $T(t-T)^{-m/2}C(K)\int_{\mathbb{R}^m} \mathrm{d}x\, (\mathbb{P}_x[L_K < \infty])^2 = O(t^{-m/2})$. The contribution from $s \in [T, t/2]$ is bounded, using (16), by

$$\begin{aligned}&C(K)\int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \int_T^{t/2} \mathrm{d}s\, \mathbb{P}_x[s < L_K < \infty](t-s)^{-m/2} \\ &\quad\leq C(K)^2\int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \int_T^{t/2} \mathrm{d}s\, s^{(2-m)/2}(t-s)^{-m/2} = O\bigl(t^{-m/2}\bigr),\end{aligned}$$

while the contribution from $s \in [t/2, t-T]$ is bounded, using Lemma 10, by

$$C(K)C_1\int_{\mathbb{R}^m} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \int_{t/2}^{t-T} \mathrm{d}s\, s^{-m/2}(t-s)^{(2-m)/2} = O\bigl(t^{-m/2}\bigr).$$

This completes the proof of Proposition 5 for $m \geq 5$.

To complete the proof of Proposition 5 for $m = 4$ we note that Lemma 10 cannot be used to estimate the integral in (33) since $\int_{\mathbb{R}^4} \mathrm{d}x\, (\mathbb{P}_x[L_K < \infty])^2$ is divergent. Instead we will use the first inequality in (34). First we note that by (16), $\mathbb{P}_x[s < T_K < t] \leq \mathbb{P}_x[s < L_K < \infty] \leq C(K)s^{-1}$. Hence

$$\int_{B(0;2R)} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \mathbb{P}_x[s < T_K < t] \leq \bigl|B(0;2R)\bigr| C(K)s^{-1}. \tag{37}$$

Furthermore for $|x| \geq 2R$ and $y \in K$ we have that $|x-y|^2 \geq (|x|-R)^2 \geq |x|^2/4$. Since $K \subset B(0;R)$ we have that

$$\begin{aligned}&\int_{\{|x|>2R\}} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_K < \infty]\bigr)^2 \mathbb{P}_x[s < L_K < \infty] \\ &\quad\leq \int_{\{|x|>2R\}} \mathrm{d}x\, \bigl(\mathbb{P}_x[L_{B(0;R) < \infty}]\bigr)^2 \int_s^\infty \mathrm{d}\tau \int \mu_K(\mathrm{d}y)p(x, y; \tau)\end{aligned} \tag{38}$$

$$\le C(K)\int_{\{|x|>2R\}}\mathrm{d}x\left(1\wedge\frac{R}{|x|}\right)^4\int_s^\infty \mathrm{d}\tau\,(4\pi\tau)^{-2}\mathrm{e}^{-|x|^2/(16\tau)}$$

$$\le C(K)R^4(8s)^{-1}\int_{R/(2\sqrt{s})}^\infty \mathrm{d}\rho\,\rho^{-3}\bigl(1-\mathrm{e}^{-\rho^2}\bigr)\le C(K)R^4s^{-1}\bigl(\log\bigl(R^{-2}s\bigr)\vee 1\bigr).$$

By (37) and (38)

$$\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^2\mathbb{P}_x[s<T_K<t]\le C_2s^{-1}\left(\log\left(\frac{s}{R^2}\right)\vee 1\right)\tag{39}$$

for some C_2 depending on K only. First of all

$$\int_0^T\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^2\mathbb{P}_x[s<T_K<t](t-s)^{-2}\le T(t-T)^{-2}\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^3=\mathrm{O}\bigl(t^{-2}\bigr).$$

By (39)

$$\int_T^{t/2}\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^2\mathbb{P}_x[s<T_K<t](t-s)^{-2}$$

$$\le 4t^{-2}C_2\int_T^{t/2}\mathrm{d}s\,s^{-1}\left(\log\left(\frac{s}{R^2}\right)\vee 1\right)=\mathrm{O}\bigl(\bigl(t^{-1}\log t\bigr)^2\bigr).$$

To estimate the contribution from $s\in[t/2,t-T]$ we have by the first inequality in (34) and Fubini's theorem that

$$\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^2\int_{t/2}^{t-T}\mathrm{d}s\,\mathbb{P}_x[s<T_K<t](t-s)^{-2}$$

$$\le\int_{t/2}^{t-T}\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{d}y\,\mathbb{P}_y[T_K<t-s](t-s)^{-2}\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_K<\infty]\bigr)^2(4\pi s)^{-2}\mathrm{e}^{-|x-y|^2/(4s)}$$

$$\le(2\pi t)^{-2}\int_{t/2}^{t-T}\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{d}y\,\mathbb{P}_y[T_K<t-s](t-s)^{-2}\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_{B(0;R)}<\infty]\bigr)^2\mathrm{e}^{-|x-y|^2/(4t)}.\tag{40}$$

By the Hardy–Littlewood rearrangement inequality [5] and (35), (36) we have that the right-hand side of (40) is bounded by

$$(2\pi t)^{-2}\int_{t/2}^{t-T}\mathrm{d}s\int_{\mathbb{R}^4}\mathrm{d}y\,\mathbb{P}_y[T_K<t-s](t-s)^{-2}\int_{\mathbb{R}^4}\mathrm{d}x\,\bigl(\mathbb{P}_x[L_{B(0;R)}<\infty]\bigr)^2\mathrm{e}^{-|x|^2/(4t)}$$

$$\le\frac{C_1}{2t^2}\int_{t/2}^{t-T}\mathrm{d}s\,(t-s)^{-1}\int_0^\infty\mathrm{d}\rho\,\rho^3\left(1\wedge\frac{R}{\rho}\right)^4\mathrm{e}^{-\rho^2/(4t)}=\mathrm{O}\bigl(\bigl(t^{-1}\log t\bigr)^2\bigr).$$

This completes the proof of Proposition 5 for $m=4$.

Finally we prove the upper bound in Proposition 5 for $m=3$. By (22)

$$\mathbb{P}_x[T_K<t<L_K]\le C\bigl(\mathbb{P}_x[L_K<t]+\mathbb{P}_x[T_K<t<L_K]\bigr)t^{-1/2}+\frac{C}{2}\int_0^{t-T}\mathrm{d}s\,\mathbb{P}_x[s<T_K<t](t-s)^{-3/2}.$$

It follows that for $t\ge(2T)\vee(4C^2)$

$$\mathbb{P}_x[T_K<t<L_K]\le Ct^{-1/2}\bigl(1+2Ct^{-1/2}\bigr)\mathbb{P}_x[L_K<t]+C\int_0^{t-T}\mathrm{d}s\,\mathbb{P}_x[s<T_K<t](t-s)^{-3/2}.$$

By (30) we conclude that

$$\begin{aligned}&\int_{\mathbb{R}^3} \mathrm{d}x \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[T_K < t < L_K]\\
&\quad \le (4\pi)^{-7/2} C(K)^4 t^{-1/2} \log t + \mathrm{O}\big(t^{-1/2}\big)\\
&\qquad + C \int_{\mathbb{R}^3} \mathrm{d}x \int_0^{t-T} \mathrm{d}s \left(\mathbb{P}_x[L_K < \infty]\right)^2 \mathbb{P}_x[s < T_K < t](t-s)^{-3/2}. \end{aligned} \tag{41}$$

By the first inequality in (34) we have that the second term in the right-hand side of (41) is bounded by

$$C \int_{\mathbb{R}^3} \mathrm{d}x \int_0^{t-T} \mathrm{d}s \left(\mathbb{P}_x[L_{B(0;R)} < \infty]\right)^2 \int_{\mathbb{R}^3} \mathrm{d}y\, p(x,y;s) \mathbb{P}_y[T_{B(0;R)} < t-s](t-s)^{-3/2}. \tag{42}$$

Let $1/2 < \alpha < 1$. Since $\mathbb{P}_x[L_{B(0;R)} < \infty] \le \frac{R}{|x|}$ and $p(x;y;s) \le s^{(2\alpha-3)/2}|x-y|^{-2\alpha}$ we have that

$$\int_{\mathbb{R}^3} \mathrm{d}x \left(\mathbb{P}_x[L_{B(0;R)} < \infty]\right)^2 p(x,y;s) \le R^2 s^{(2\alpha-3)/2} \int_{\mathbb{R}^3} \mathrm{d}x\, |x|^{-2}|x-y|^{-2\alpha}. \tag{43}$$

The integral in the right-hand side of (43) converges for $1/2 < \alpha < 3/2$. By scaling there exists C_3 depending on α only such that

$$\int_{\mathbb{R}^3} \mathrm{d}x\, |x|^{-2}|x-y|^{-2\alpha} = C_3 |y|^{1-2\alpha}. \tag{44}$$

By (43), (44) and Fubini's theorem we obtain that (42) is bounded by

$$C C_3 R^2 \int_0^{t-T} \mathrm{d}s\, s^{(2\alpha-3)/2}(t-s)^{-3/2} \int_{\mathbb{R}^3} \mathrm{d}y\, |y|^{1-2\alpha} \mathbb{P}_y[T_{B(0;R)} < t-s]. \tag{45}$$

Lemma 11. *Let $1/2 < \alpha < 1$. There exists $C_4 < \infty$ depending on α and R such that for $s \le t-T$*

$$\int_{\mathbb{R}^3} \mathrm{d}y\, |y|^{1-2\alpha} \mathbb{P}_y[T_{B(0;R)} < t-s] \le C_4 (t-s)^{(3-2\alpha)/2}.$$

Proof. By p. 392 in [8], we have that for $|y| > R$

$$\mathbb{P}_y[T_{B(0;R)} < t] = \int_0^t \mathrm{d}\tau \left(4\pi\tau^3\right)^{-1/2} R\big(|y| - R\big)|y|^{-1} \mathrm{e}^{-(|y|-R)^2/(4\tau)}.$$

Hence for $|y| > 2R$

$$\mathbb{P}_y[T_{B(0;R)} < t] \le R \int_0^t \mathrm{d}\tau \left(4\pi\tau^3\right)^{-1/2} \mathrm{e}^{-|y|^2/(16\tau)}.$$

It follows that

$$\begin{aligned}\int_{\{|y|>2R\}} \mathrm{d}y\, |y|^{1-2\alpha} \mathbb{P}_y[T_{B(0;R)} < t-s] &\le R \int_0^{t-s} \mathrm{d}\tau \left(4\pi\tau^3\right)^{-1/2} \int_{\mathbb{R}^3} \mathrm{d}y\, |y|^{1-2\alpha} \mathrm{e}^{-|y|^2/(16\tau)}\\
&\le R C_5 (t-s)^{(3-2\alpha)/2} \end{aligned} \tag{46}$$

for some constant C_5 depending on α only. Furthermore for $1/2 < \alpha < 1$ and $s < t-T$

$$\int_{\{|y|<2R\}} \mathrm{d}y\, |y|^{1-2\alpha} \mathbb{P}_y[T_{B(0;R)} < t-s] \le \int_{\{|y|<2R\}} \mathrm{d}y\, |y|^{1-2\alpha} \le 2\pi(2R)^{4-2\alpha} T^{(2\alpha-3)/2}(t-s)^{(3-2\alpha)/2}. \tag{47}$$

The lemma follows by (46) and (47). □

Finally, by Lemma 11 we have that (45) is bounded by

$$CC_3C_4R^2\int_0^{t-T}\mathrm{d}s\, s^{(2\alpha-3)/2}(t-s)^{-\alpha}=\mathrm{O}\big(t^{-1/2}\big).$$

This completes the proof of Proposition 5.

4. Proof of Proposition 6

To prove Proposition 6 we write

$$\begin{aligned}\big(\mathbb{P}_x[L_K<t]\big)^3&=\big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[L_K<t]\\&\quad-2\mathbb{P}_x[L_K<\infty]\mathbb{P}_x[L_K<t]\mathbb{P}_x[t<L_K<\infty]\\&\quad+\big(\mathbb{P}_x[t<L_K<\infty]\big)^2\mathbb{P}_x[L_K<t].\end{aligned}$$

Lemma 12. *Let $m=3$. Then for $t\to\infty$*

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathrm{d}x\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[L_K<t]\\&\quad=2^{-1}(4\pi)^{-2}C(K)^3\log t-(4\pi)^{-2}C(K)^2\int\mu_K(\mathrm{d}y)\log|y|+2^{-1}(4\pi)^{-2}(2-\gamma)C(K)^3\\&\qquad+(4\pi)^{-3}\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\big(|x-y_1|^{-1}|x-y_2|^{-1}-|x|^{-2}\big)|x-y_3|^{-1}\\&\qquad+\mathrm{O}\big(t^{-1/2}\big).\end{aligned}$$

Proof.

$$\begin{aligned}&\int_{\mathbb{R}^3}\mathrm{d}x\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[L_K<t]\\&\quad=\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\big(4\pi|x-y_1|\big)^{-1}\big(4\pi|x-y_2|\big)^{-1}\int_0^t\mathrm{d}s\,(4\pi s)^{-3/2}\mathrm{e}^{-|x-y_3|^2/(4s)}\\&\quad=\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)(4\pi)^{-3}\big(|x-y_1|^{-1}|x-y_2|^{-1}-|x|^{-2}\big)|x-y_3|^{-1}\\&\qquad-\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)(4\pi)^{-2}\big(|x-y_1|^{-1}|x-y_2|^{-1}-|x|^{-2}\big)\\&\qquad\times\int_t^\infty\mathrm{d}s\,(4\pi s)^{-3/2}\mathrm{e}^{-|x-y_3|^2/(4s)}\\&\qquad+(4\pi)^{-2}C(K)^2\int\mu_K(\mathrm{d}y_3)\int_{\mathbb{R}^3}\mathrm{d}x\,|x|^{-2}\int_0^t\mathrm{d}s\,(4\pi s)^{-3/2}\mathrm{e}^{-|x-y_3|^2/(4s)}:=A_1+A_2+A_3.\end{aligned}\tag{48}$$

By a change of variables we have that A_2 equals

$$\begin{aligned}&-(4\pi)^{-2}\int_{\mathbb{R}^3}\mathrm{d}x\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\big(|x+y_3-y_1|^{-1}|x+y_3-y_2|^{-1}-|x+y_3|^{-2}\big)\\&\quad\times\int_t^\infty\mathrm{d}s\,(4\pi s)^{-3/2}\mathrm{e}^{-|x|^2/(4s)}.\end{aligned}\tag{49}$$

The contribution from the set $\{|x| < 3R\}$ to the integral in (49) is bounded by

$$\int_{\{|x|<3R\}} \mathrm{d}x \int \mu_K(\mathrm{d}y_3)\big(\mathbb{P}_{x+y_3}[L_K < \infty]\big)^2 \int_t^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \mathrm{e}^{-|x|^2/(4s)}$$
$$+ (4\pi)^{-2} C(K)^2 \int_{\{|x|<3R\}} \mathrm{d}x \int \mu_K(\mathrm{d}y_3)|x+y_3|^{-2} \int_t^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \mathrm{e}^{-|x|^2/(4s)}. \tag{50}$$

The first term in (50) is bounded by $C(K)|B(0;3R)|t^{-1/2}$. The second term in (50) is bounded, using (16), by

$$(4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y_3) \int_{\{|x+y_3|<4R\}} \mathrm{d}x\, |x+y_3|^{-2} \int_t^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \mathrm{e}^{-|x|^2/(4s)}$$
$$\le C(K)^3 R t^{-1/2}. \tag{51}$$

To estimate the contribution from the set $\{|x| > 3R\}$ to the integral in (49) we note that for $y_i \in \mathbb{R}^3$, $|y_i| \le R$, $i = 1, 2, 3$ and $|x| > 3R$

$$|x+y_3-y_1|^{-1}|x+y_3-y_2|^{-1} - |x+y_3|^{-2} = \frac{x\cdot(y_1+y_2)}{|x|^4} + \mathrm{O}\big(|x|^{-4}\big), \tag{52}$$

with uniform remainder. The contribution from the $\mathrm{O}(|x|^{-4})$ term in (52) is, using (16), $\mathrm{O}(t^{-1/2})$. By spherical symmetry we have that for all $y_1, y_2 \in K$

$$\int_{\{|x|>3R\}} \mathrm{d}x \frac{x\cdot(y_1+y_2)}{|x|^4} \int_t^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \mathrm{e}^{-|x|^2/(4s)} = 0. \tag{53}$$

Putting (49)–(53) together we obtain that $A_2 = \mathrm{O}(t^{-1/2})$.

In order to compute the asymptotic behaviour of A_3 as $t \to \infty$ we use spherical coordinates to calculate that

$$\int_{\mathbb{R}^3} \mathrm{d}x\, |x|^{-2} \mathrm{e}^{-|x-y_3|^2/(4s)} = \frac{8\pi s}{|y_3|} \mathrm{e}^{-|y_3|^2/(4s)} \int_0^\infty \mathrm{d}\rho\, \frac{\sinh\rho}{\rho} \mathrm{e}^{-s\rho^2/|y_3|^2}. \tag{54}$$

Hence (putting $y_3 = y$)

$$\begin{aligned} A_3 = {} & (4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \int_0^t \mathrm{d}s\, (4\pi s)^{-3/2} \frac{8\pi s}{|y|} \mathrm{e}^{-|y|^2/(4s)} \int_0^\infty \mathrm{d}\rho\, \mathrm{e}^{-s\rho^2/|y|^2} \\ & + (4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \int_0^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \frac{8\pi s}{|y|} \mathrm{e}^{-|y|^2/(4s)} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1\right) \mathrm{e}^{-s\rho^2/|y|^2} \\ & - (4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \int_t^\infty \mathrm{d}s\, (4\pi s)^{-3/2} \frac{8\pi s}{|y|} \mathrm{e}^{-|y|^2/(4s)} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1\right) \mathrm{e}^{-s\rho^2/|y|^2} \\ & := A_4 + A_5 + A_6. \end{aligned} \tag{55}$$

The absolute value of A_6 is bounded, for $t \ge R^2$, by

$$\begin{aligned} & C(K)^2 \int \mu_K(\mathrm{d}y)|y|^{-1} \int_t^\infty \mathrm{d}s\, s^{-1/2} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1\right) \mathrm{e}^{-s\rho^2/|y|^2} \\ & \le 2C(K)^2 \int \mu_K(\mathrm{d}y)|y|^{-1} t^{1/2} \int_1^\infty \mathrm{d}\sigma \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1\right) \mathrm{e}^{-t\sigma^2\rho^2/|y|^2} \\ & \le 2C(K)^2 \int \mu_K(\mathrm{d}y)|y| t^{1/2} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1\right) \frac{1}{t\rho^2} \mathrm{e}^{-t\rho^2/|y|^2} \end{aligned}$$

$$\le 2C(K)^3 R t^{-1/2} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1 \right) \rho^{-2} \mathrm{e}^{-t\rho^2/R^2}$$
$$\le 2C(K)^3 R t^{-1/2} \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1 \right) \rho^{-2} \mathrm{e}^{-\rho^2}. \tag{56}$$

To calculate A_5 we use 3.471.15 in [4] to obtain that

$$\int_0^\infty \mathrm{d}s\, s^{-1/2} \mathrm{e}^{-|y|^2/(4s) - s\rho^2/|y|^2} = \pi^{1/2} \frac{|y|}{\rho} \mathrm{e}^{-\rho}. \tag{57}$$

Hence, by the expression for A_5 in (55), and (56)–(57) we conclude that

$$A_5 = (4\pi)^{-2} C(K)^3 \int_0^\infty \mathrm{d}\rho \left(\frac{\sinh\rho}{\rho} - 1 \right) \rho^{-1} \mathrm{e}^{-\rho} = (4\pi)^{-2}(1 - \log 2) C(K)^3. \tag{58}$$

For A_4, defined in (55), we use 8.211.1 and 8.214.1 in [4] to obtain that

$$A_4 = 2^{-1}(4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \int_{|y|^2/(4t)}^\infty \mathrm{d}s\, s^{-1} \mathrm{e}^{-s}$$
$$= 2^{-1}(4\pi)^{-2} C(K)^2 \int \mu_K(\mathrm{d}y) \log\left(\frac{t}{|y|^2} \right) + 2^{-1}(4\pi)^{-2} C(K)^3 (\log 4 - \gamma) + \mathrm{O}\big(t^{-1}\big), \tag{59}$$

where γ is Euler's constant. The lemma follows by (48)–(59). □

Lemma 13. *Let $m = 3$. Then for $t \to \infty$*

$$\int_{\mathbb{R}^3} \mathrm{d}x\, \mathbb{P}_x[L_K < t] \mathbb{P}_x[t < L_K < \infty] \big(\mathbb{P}_x[t < L_K < \infty] - 2\mathbb{P}_x[L_K < \infty] \big)$$
$$= -3\pi^{-1}(4\pi)^{-2}(2G + H) C(K)^3 + \mathrm{O}\big(t^{-1/2}\big), \tag{60}$$

where G is Catalan's constant, and H is the constant defined in (9).

Before we give the proof of (60) we state the following.

Lemma 14. *For $y_i \in \mathbb{R}^m$, $s_i > 0$, $i = 1, 2, 3$,*

$$\int_{\mathbb{R}^m} \mathrm{d}x\, p(x, y_1; s_1) p(x, y_2; s_2) p(x, y_3; s_3) = (4\pi)^{-m} (s_1 s_2 + s_2 s_3 + s_3 s_1)^{-m/2} \mathrm{e}^{Q(y_1, y_2, y_3; s_1, s_2, s_3)}, \tag{61}$$

where the quadratic form Q is given by

$$Q(y_1, y_2, y_3; s_1, s_2, s_3) = \frac{1}{4} \left(\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} \right)^{-1} \left| \frac{y_1}{s_1} + \frac{y_2}{s_2} + \frac{y_3}{s_3} \right|^2 - \frac{|y_1|^2}{4s_1} - \frac{|y_2|^2}{4s_2} - \frac{|y_3|^2}{4s_3}. \tag{62}$$

Moreover, Q is negative semi-definite.

Proof. A straightforward calculation of the Gaussian integral in the left-hand side of (61) yields (61) and (62). $Q(y_1, y_2, y_3; s_1, s_2, s_3) \le 0$ by Jensen's inequality. □

Proof of Lemma 13. By Lemma 14 and (11) we have that the left-hand side of (60) equals

$$-(4\pi)^{-3} \int_0^t \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2 \left(\int_0^\infty \mathrm{d}s_3 + \int_0^t \mathrm{d}s_3 \right) (s_1 s_2 + s_2 s_3 + s_3 s_1)^{-3/2}$$
$$\times \int\int\int \mu_K(\mathrm{d}y_1) \mu_K(\mathrm{d}y_2) \mu_K(\mathrm{d}y_3) \mathrm{e}^{Q(y_1, y_2, y_3; s_1, s_2, s_3)}. \tag{63}$$

Since $Q \le 0$ we have that (63) is bounded from below by

$$-(4\pi)^{-3}C(K)^3 \int_0^t \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2 \left(\int_0^\infty \mathrm{d}s_3 + \int_0^t \mathrm{d}s_3\right)(s_1s_2+s_2s_3+s_3s_1)^{-3/2}. \tag{64}$$

On the other hand, for $y_1, y_2, y_3 \in K$

$$Q(y_1, y_2, y_3; s_1, s_2, s_3) \ge -R^2\left(\frac{1}{4s_1}+\frac{1}{4s_2}+\frac{1}{4s_3}\right), \tag{65}$$

and so (63) is bounded from above by

$$\begin{aligned}&-(4\pi)^{-3}C(K)^3 \int_0^t \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2 \left(\int_0^\infty \mathrm{d}s_3 + \int_0^t \mathrm{d}s_3\right)(s_1s_2+s_2s_3+s_3s_1)^{-3/2}\\ &\quad + 2(4\pi)^{-3}C(K)^3 \int_0^t \mathrm{d}s_1 \int_t^\infty \mathrm{d}s_2 \int_0^\infty \mathrm{d}s_3 \left(1-\mathrm{e}^{-R^2(1/(4s_1)+1/(4s_2)+1/(4s_3))}\right)(s_1s_2+s_2s_3+s_3s_1)^{-3/2}.\end{aligned} \tag{66}$$

Changing variables $s_i \to t/s_i^2, i = 1, 2, 3$ we obtain that the second term in (66) is bounded by

$$\begin{aligned}&2^4(4\pi)^{-3}C(K)^3 \int_1^\infty \mathrm{d}s_1 \int_0^1 \mathrm{d}s_2 \int_0^\infty \mathrm{d}s_3 \left(1-\mathrm{e}^{-R^2(s_1^2+s_2^2+s_3^2)/(4t)}\right)\left(s_1^2+s_2^2+s_3^2\right)^{-3/2}\\ &\quad \le 2^4(4\pi)^{-3}C(K)^3 \int_0^\infty \int_0^\infty \mathrm{d}s_1\,\mathrm{d}s_3 \int_0^1 \mathrm{d}s_2 \left(1-\mathrm{e}^{-R^2(s_1^2+s_2^2+s_3^2)/(4t)}\right)\left(s_1^2+s_2^2+s_3^2\right)^{-3/2}\\ &\quad = (4\pi)^{-2}C(K)^3 \int_0^1 \mathrm{d}s \int_{s^2}^\infty \mathrm{d}\rho\,\rho^{-3/2}\left(1-\mathrm{e}^{-R^2\rho/(4t)}\right)\\ &\quad \le (4\pi)^{-2}C(K)^3 \int_0^\infty \mathrm{d}\rho\,\rho^{-3/2}\left(1-\mathrm{e}^{-R^2\rho/(4t)}\right) = \mathrm{O}\left(t^{-1/2}\right).\end{aligned}$$

It remains to calculate the multiple integral (64). Changing variables $s_i \to ts_i, i = 1, 2, 3$, and integrating with respect to s_3 gives that (64) equals

$$-(4\pi)^{-3}C(K)^3 \int_0^1 \mathrm{d}s_1 \int_1^\infty \mathrm{d}s_2 \frac{2}{s_1+s_2}\left(\frac{2}{(s_1s_2)^{1/2}} - \frac{1}{(s_1s_2+s_1+s_2)^{1/2}}\right).$$

Furthermore,

$$\int_1^\infty \mathrm{d}s_2 \int_0^1 \mathrm{d}s_1 \frac{4}{s_1+s_2}\frac{1}{(s_1s_2)^{1/2}} = 8\int_1^\infty \mathrm{d}s_2\, s_2^{-1}\arctan s_2^{-1/2} = 16\int_0^1 \mathrm{d}s\, s^{-1}\arctan s = 16G,$$

by 4.531.1 in [4], and

$$\int_0^1 \mathrm{d}s_1 \int_1^\infty \mathrm{d}s_2 \frac{1}{s_1+s_2}\frac{1}{(s_1s_2+s_1+s_2)^{1/2}} = 2\int_0^1 \frac{\mathrm{d}s}{s}\arctan\frac{s}{(2s+1)^{1/2}},$$

by 2.224.5 in [4]. A change of variables $s = (2s+1)^{1/2}\tan\theta$ gives that

$$2\int_0^1 \frac{\mathrm{d}s}{s}\arctan\frac{s}{(2s+1)^{1/2}} = \pi\int_{\pi/3}^{\pi/2}\frac{\mathrm{d}\theta}{\sin\theta} - 2\int_0^{\pi/2}\frac{\theta}{\sin\theta}\,\mathrm{d}\theta + 3\int_0^{\pi/3}\frac{\theta}{\sin\theta}\,\mathrm{d}\theta := J_1+J_2+J_3.$$

Finally,

$$J_1 = \frac{\pi}{2}\log 3, \qquad J_2 = 2\int_0^{\pi/2}\log\left(\tan\frac{\theta}{2}\right)\mathrm{d}\theta = 4\int_0^{\pi/4}\log(\tan\theta)\,\mathrm{d}\theta = -4G,$$

by 4.227.2 in [4], and

$$J_3 = -\frac{\pi}{2}\log 3 - 6H.$$

We conclude that the multiple integral in (64) equals $-3\pi^{-1}(4\pi)^{-2}(H+2G)C(K)^3$. □

The proof of Proposition 6 follows by Lemmas 12 and 13.

5. Proof of Proposition 7

To prove Proposition 7 we write

$$\begin{aligned}\big(\mathbb{P}_x[L_K < t]\big)^3 &= \big(\mathbb{P}_x[L_K < \infty]\big)^3 - 3\big(\mathbb{P}_x[L_K < \infty]\big)^2\mathbb{P}_x[t < L_K < \infty] \\ &\quad + 3\mathbb{P}_x[L_K < \infty]\big(\mathbb{P}_x[t < L_K < \infty]\big)^2 - \big(\mathbb{P}_x[t < L_K < \infty]\big)^3. \end{aligned}\tag{67}$$

Lemma 15. *Let $m = 4$. Then for $t \to \infty$*

$$\begin{aligned}&\int_{\mathbb{R}^4} \mathrm{d}x\,\big(\mathbb{P}_x[L_K < \infty]\big)^2\mathbb{P}_x[t < L_K < \infty] \\ &\quad = (4\pi)^{-4}C(K)^3\frac{\log t}{t} + (4\pi)^{-4}C(K)\frac{1}{t}\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\log\frac{1}{|y_1-y_2|^2} \\ &\qquad + (2+\log 4-\gamma)(4\pi)^{-4}C(K)^3t^{-1} + \mathrm{O}\big(t^{-2}\log t\big). \end{aligned}\tag{68}$$

Proof. Denote the left-hand side of (68) by I. Then

$$I = \int_{\mathbb{R}^4}\mathrm{d}x\,\big(\mathbb{P}_x[L_K < \infty]\big)^2\int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x-y|^2/(4s)}.\tag{69}$$

We will show that we may replace $\mathrm{e}^{-|x-y|^2/(4s)}$ by $\mathrm{e}^{-|x|^2/(4s)}$ in (69) at a cost of a remainder $\mathrm{O}(t^{-2}\log t)$. Since

$$\mathrm{e}^{-|x|^2/(4s)}\bigg(1+\frac{x\cdot y}{2s}-\frac{|y|^2}{4s}\bigg) \le \mathrm{e}^{-|x-y|^2/(4s)} \le \mathrm{e}^{-|x|^2/(4s)}\bigg(1+\frac{x\cdot y}{2s}+\frac{|x|^2|y|^2}{8s^2}\mathrm{e}^{|x||y|/(2s)}\bigg),$$

we have to show that $R_i = \mathrm{O}(t^{-2}\log t)$, $i = 1, 2, 3,$ where

$$\begin{aligned} R_1 &= \int_{\mathbb{R}^4}\mathrm{d}x\,\big(\mathbb{P}_x[L_K < \infty]\big)^2\int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)}\frac{x\cdot y}{2s}, \\ R_2 &= -\int_{\mathbb{R}^4}\mathrm{d}x\,\big(\mathbb{P}_x[L_K < \infty]\big)^2\int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)}\frac{|y|^2}{4s}, \\ R_3 &= \int_{\mathbb{R}^4}\mathrm{d}x\,\big(\mathbb{P}_x[L_K < \infty]\big)^2\int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)+|x||y|/(2s)}\frac{|x|^2|y|^2}{8s^2}. \end{aligned}\tag{70}$$

First we note that

$$R_1 = \int_{\mathbb{R}^4}\mathrm{d}x\,\bigg(\big(\mathbb{P}_x[L_K < \infty]\big)^2 - \bigg(\frac{C(K)}{4\pi^2|x|^2}\bigg)^2\bigg)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)}\int \mu_K(\mathrm{d}y)\frac{x\cdot y}{2s},\tag{71}$$

since

$$\int_{\mathbb{R}^4}\mathrm{d}x\,\bigg(\frac{C(K)}{4\pi^2|x|^2}\bigg)^2\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)}\int \mu_K(\mathrm{d}y)\frac{x\cdot y}{2s} = 0,\tag{72}$$

by spherical symmetry. The absolute value of contribution from $\{|x|<2R\}$ to the integral in (71) is bounded by

$$\int_{\{|x|<2R\}} \mathrm{d}x \left(1+\left(\frac{C(K)}{4\pi^2|x|^2}\right)^2\right) \int_t^\infty \mathrm{d}s\, s^{-3}|x| \int \mu_K(\mathrm{d}y)|y| = \mathrm{O}\big(t^{-2}\big). \tag{73}$$

In order to estimate the absolute value of the contribution of $\{|x|>2R\}$ we need some bounds on $\mathbb{P}_x[L_K<\infty]$. We state these for $m\geq 3$. Since

$$\mathbb{P}_x[L_K<\infty]-c_m C(K)|x|^{2-m} = c_m \int \mu_K(\mathrm{d}y)\big(|x-y|^{2-m}-|x|^{2-m}\big), \tag{74}$$

and $||x-y|^{2-m}-|x|^{2-m}| \leq (m-2)|y|\max\{|x-y|^{1-m},|x|^{1-m}\}$ we have that

$$\big|\mathbb{P}_x[L_K<\infty]-c_m C(K)|x|^{2-m}\big| \leq c_m(m-2)\int \mu_K(\mathrm{d}y)|y|\max\big\{|x-y|^{1-m},|x|^{1-m}\big\}. \tag{75}$$

On the set $\{|x|\geq 2R\geq 2|y|\}$ we have that $|x-y|\geq |x|/2$. Hence by (75)

$$\big|\mathbb{P}_x[L_K<\infty]-c_m C(K)|x|^{2-m}\big| \leq c_m 2^{m-1}(m-2)C(K)R|x|^{1-m}. \tag{76}$$

Similarly for $|x|>2R$

$$\mathbb{P}_x[L_K<\infty] \leq c_m 2^{m-2} C(K)|x|^{2-m}. \tag{77}$$

The contribution from $\{|x|>2R\}$ to the absolute value of the integral in (71) is bounded, up to a numerical factor, by

$$\begin{aligned} &C(K)^3R^2\int_{\{|x|>2R\}} \mathrm{d}x\,|x|^{-4}\int_t^\infty \mathrm{d}s\,s^{-3}\mathrm{e}^{-|x|^2/(4s)} \\ &\quad\leq C(K)^3R^2\int_{\{|x|>2R\}} \mathrm{d}x\,|x|^{-4}\big(t^{-2}\wedge 16|x|^{-4}\big) = \mathrm{O}\big(t^{-2}\log t\big). \end{aligned} \tag{78}$$

By (71)–(78) we conclude that $R_1=\mathrm{O}(t^{-2}\log t)$. To bound the absolute value of R_2 we have that the contribution of $\{|x|<2R\}$ is bounded by

$$\int_{\{|x|<2R\}} \mathrm{d}x \int \mu_K(\mathrm{d}y)|y|^2 \int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}(4s)^{-1} = \mathrm{O}\big(t^{-2}\big). \tag{79}$$

The contribution from $\{|x|>2R\}$ is bounded, using (77), up to a numerical factor, by

$$\begin{aligned} &\int_{\{|x|>2R\}} \mathrm{d}x \left(\frac{C(K)}{|x|^2}\right)^2 \int \mu_K(\mathrm{d}y)|y|^2 \int_t^\infty \mathrm{d}s\,s^{-3}\mathrm{e}^{-|x|^2/(4s)} \\ &\quad\leq C(K)^3R^2\int_{\{|x|>2R\}} \mathrm{d}x\,|x|^{-4}\int_t^\infty \mathrm{d}s\,s^{-3}\mathrm{e}^{-|x|^2/(4s)} = \mathrm{O}\big(t^{-2}\log t\big), \end{aligned} \tag{80}$$

where we have used (78). So $R_2=\mathrm{O}(t^{-2}\log t)$ by (79) and (80). The contribution from $\{|x|<3R\}$ to the integral in (70) is bounded by $\mathrm{O}(t^{-3})$. On $\{|x|>3R\}$ we have that $|y|\leq R\leq |x|/3$. Hence $\mathrm{e}^{-|x|^2/(4s)+|x||y|/(2s)} \leq \mathrm{e}^{-|x|^2/(12s)}$. Hence

$$\begin{aligned} &\int_{\{|x|>3R\}} \mathrm{d}x\,\big(\mathbb{P}_x[L_K<\infty]\big)^2 \int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,(4\pi s)^{-2}\mathrm{e}^{-|x|^2/(4s)+|x||y|/(2s)}\frac{|x|^2|y|^2}{8s^2} \\ &\quad\leq \int_{\{|x|>3R\}} \mathrm{d}x \left(\frac{C(K)}{|x|^2}\right)^2 \int \mu_K(\mathrm{d}y)\int_t^\infty \mathrm{d}s\,s^{-4}\mathrm{e}^{-|x|^2/(12s)}|x|^2|y|^2 \\ &\quad\leq C(K)^3R^2\int_t^\infty \mathrm{d}s\,s^{-4}\int_{\{|x|>3R\}} \mathrm{d}x\,|x|^{-2}\mathrm{e}^{-|x|^2/(12s)} = \mathrm{O}\big(t^{-2}\big). \end{aligned}$$

We conclude that $R_3 = \mathrm{O}(t^{-2})$, and that for $t \to \infty$

$$
\begin{aligned}
I &= \int_{\mathbb{R}^4} \mathrm{d}x \, \big(\mathbb{P}_x[L_K < \infty]\big)^2 \int \mu_K(\mathrm{d}y) \int_t^\infty \mathrm{d}s \, (4\pi s)^{-2} \mathrm{e}^{-|x|^2/(4s)} + \mathrm{O}\big(t^{-2}\log t\big) \\
&= \left(\frac{C(K)}{4\pi^2}\right)^2 \int_{\mathbb{R}^4} \mathrm{d}x \, \mathbb{P}_x[L_K < \infty] |x|^{-4}\big(1 - \mathrm{e}^{-|x|^2/(4t)}\big) \\
&\quad + \frac{C(K)}{4\pi^2} \int_{\mathbb{R}^4} \mathrm{d}x \, \mathbb{P}_x[L_x < \infty] \left(\mathbb{P}_x[L_K < \infty] - \frac{C(K)}{4\pi^2 |x|^2}\right) |x|^{-2}\big(1 - \mathrm{e}^{-|x|^2/(4t)}\big) + \mathrm{O}\big(t^{-2}\log t\big) \\
&:= I_1 + I_2 + \mathrm{O}\big(t^{-2}\log t\big).
\end{aligned}
$$

By (11) and Lemma 14 we have that

$$
\begin{aligned}
I_1 &= \int_{\mathbb{R}^4} \mathrm{d}x \int\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3) \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_t^\infty \mathrm{d}s_3 \, p(x, y_1; s_1) p(x, 0; s_2) p(x, 0; s_3) \\
&= (4\pi)^{-4} \int\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3) \\
&\quad \times \int_0^\infty \mathrm{d}s_1 \int_0^\infty \mathrm{d}s_2 \int_t^\infty \mathrm{d}s_3 \, (s_1 s_2 + s_2 s_3 + s_3 s_1)^{-2} \mathrm{e}^{Q(y_1, 0, 0; s_1, s_2, s_3)},
\end{aligned}
\tag{81}
$$

where we have used Fubini's theorem and the definition of Q in (62). In order to evaluate the multiple integral we note that

$$
(s_1 s_2 + s_2 s_3 + s_3 s_1)^{-2} \mathrm{e}^{Q(y_1, 0, 0; s_1, s_2, s_3)} = 4|y_1|^{-2}(s_2 + s_3)^{-2} \frac{\partial}{\partial s_1} \mathrm{e}^{-|y_1|^2(s_2+s_3)/(4(s_1 s_2 + s_2 s_3 + s_3 s_1))}. \tag{82}
$$

Putting (81) and (82) together we obtain that

$$
I_1 = 4(4\pi)^{-4} C(K)^2 \int \mu_K(\mathrm{d}y_1)|y_1|^{-2} \int_0^\infty \mathrm{d}s_2 \int_t^\infty \mathrm{d}s_3 \, (s_2 + s_3)^{-2}\big(1 - \mathrm{e}^{-|y_1|^2(1/(4s_2) + 1/(4s_3))}\big). \tag{83}
$$

An integration by parts with respect to s_2 yields

$$
\begin{aligned}
&\int_0^\infty \mathrm{d}s_2 \, (s_2 + s_3)^{-2}\big(1 - \mathrm{e}^{-(|y_1|^2/4)(1/s_2 + 1/s_3)}\big) \\
&\quad = s_3^{-1}\big(1 - \mathrm{e}^{-|y_1|^2/(4s_3)}\big) + \int_0^\infty \mathrm{d}s_2 \, s_2^{-1} s_3^{-1} (s_2 + s_3)^{-1} \mathrm{e}^{-|y_1|^2(1/(4s_2) + 1/(4s_3))} |y_1|^2/4.
\end{aligned}
\tag{84}
$$

Hence $I_1 = I_3 + I_4$, where

$$
\begin{aligned}
I_3 &:= 4(4\pi)^{-4} C(K)^2 \int \mu_K(\mathrm{d}y_1)|y_1|^{-2} \int_t^\infty \mathrm{d}s_3 \, s_3^{-1}\big(1 - \mathrm{e}^{-|y_1|^2/(4s_3)}\big) \\
&= (4\pi)^{-4} C(K)^3 t^{-1} + 4(4\pi)^{-4} C(K)^2 \int \mu_K(\mathrm{d}y)|y|^{-2} \int_{4t/|y|^2}^\infty \mathrm{d}\theta \, \theta^{-1}\big(1 - \mathrm{e}^{-\theta^{-1}} - \theta^{-1}\big).
\end{aligned}
\tag{85}
$$

Since $|y| \le R$, $4t/|y|^2 \ge 4t/R^2$. Moreover, $-\theta^{-2} \le 1 - \mathrm{e}^{-\theta^{-1}} - \theta^{-1} \le 0$. Hence

$$
I_3 = (4\pi)^{-4} C(K)^3 t^{-1} + \mathrm{O}\big(t^{-2}\big). \tag{86}
$$

By (83)–(85) we have that I_4 is implicitly defined by

$$
I_4 = (4\pi)^{-4} C(K)^2 \int \mu_K(\mathrm{d}y) \int_t^\infty \mathrm{d}s_3 \, s_3^{-1} \int_0^\infty \mathrm{d}s_2 \, s_2^{-1} (s_2 + s_3)^{-1} \mathrm{e}^{-|y_1|^2(1/(4s_2) + 1/(4s_3))}. \tag{87}
$$

By first changing the variable $s_2 = s_3/\theta$ and then performing the integration with respect to s_3 we obtain, by 8.211.1 and 8.214.1 in [4], that

$$
\begin{aligned}
I_4 &= 4(4\pi)^{-4}C(K)^2\int \mu_K(\mathrm{d}y)|y|^{-2}\int_0^\infty \mathrm{d}\theta\,(\theta+1)^{-2}\big(1-\mathrm{e}^{-|y|^2(1+\theta)/(4t)}\big)\\
&= 4(4\pi)^{-4}C(K)^2\int \mu_K(\mathrm{d}y)|y|^{-2}\big(1-\mathrm{e}^{-|y|^2/(4t)}\big)+(4\pi)^{-4}C(K)^2\int \mu_K(\mathrm{d}y)t^{-1}\int_1^\infty \mathrm{d}\theta\,\theta^{-1}\mathrm{e}^{-|y|^2\theta/(4t)}\\
&= (4\pi)^{-4}(1-\gamma)C(K)^3t^{-1}+(4\pi)^{-4}C(K)^2t^{-1}\int \mu_K(\mathrm{d}y)\int_1^\infty \log\bigg(\frac{4t}{|y|^2}\bigg)+\mathrm{O}\big(t^{-2}\big).
\end{aligned}
\tag{88}
$$

Putting (85)–(88) together we obtain that

$$
I_1 = (4\pi)^{-4}C(K)^2t^{-1}\int \mu_K(\mathrm{d}y)\log\frac{t}{|y|^2}+(4\pi)^{-4}(2+\log 4-\gamma)C(K)^3t^{-1}+\mathrm{O}\big(t^{-2}\big).
\tag{89}
$$

It remains to find the asymptotic behaviour of $I_2, t\to\infty$. By definition

$$
\begin{aligned}
I_2 \;&=\; \frac{C(K)}{4\pi^2}(4t)^{-1}\int_{\mathbb{R}^4}\mathrm{d}x\,\mathbb{P}_x[L_K<\infty]\bigg(\mathbb{P}_x[L_K<\infty]-\frac{C(K)}{4\pi^2|x|^2}\bigg)\\
&\quad+\frac{C(K)}{4\pi^2}\int_{\mathbb{R}^4}\mathrm{d}x\,\mathbb{P}_x[L_K<\infty]\bigg(\mathbb{P}_x[L_K<\infty]-\frac{C(K)}{4\pi^2|x|^2}\bigg)\frac{1}{|x|^2}\bigg(1-\mathrm{e}^{-|x|^2/(4t)}-\frac{|x|^2}{4t}\bigg)\\
&:= I_5+I_6.
\end{aligned}
$$

It is easy seen that

$$
\begin{aligned}
I_5 &= \frac{C(K)}{16\pi^2t}\int_{\mathbb{R}^4}\mathrm{d}x\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\big(p(x,y_1;s_1)p(x,y_2;s_2)-p(x,y_1;s_1)p(x,0;s_2)\big)\\
&= \frac{C(K)}{16\pi^2t}\int_0^\infty \mathrm{d}s_1\int_0^\infty \mathrm{d}s_2\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\big(p(y_1,y_2;s_1+s_2)-p(y_1,0;s_1+s_2)\big)\\
&= (4\pi)^{-4}C(K)t^{-1}\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\int_0^\infty \mathrm{d}s\,s^{-1}\big(\mathrm{e}^{-|y_1-y_2|^2/(4s)}-\mathrm{e}^{-|y_1|^2/(4s)}\big)\\
&= (4\pi)^{-4}C(K)t^{-1}\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\log\frac{|y_1|^2}{|y_1-y_2|^2}.
\end{aligned}
$$

By expanding $\mathrm{e}^{-|x|^2/(4t)}$ we see that the contribution from $\{|x|<2R\}$ to the integral defining I_6 is $\mathrm{O}(t^{-2})$. The contribution from $\{|x|>2R\}$ to the integral defining I_6 can be written as

$$
\begin{aligned}
&\frac{C(K)}{4\pi^2}\int_{\{|x|>2R\}}\mathrm{d}x\,\bigg(\mathbb{P}_x[L_K<\infty]-\frac{C(K)}{4\pi^2|x|^2}\bigg)^2\frac{1}{|x|^2}\bigg(1-\mathrm{e}^{-|x|^2/(4t)}-\frac{|x|^2}{4t}\bigg)\\
&\quad+\bigg(\frac{C(K)}{4\pi^2}\bigg)^2\int_{\{|x|>2R\}}\mathrm{d}x\,\bigg(\mathbb{P}_x[L_K<\infty]-\frac{C(K)}{4\pi^2|x|^2}\bigg)\frac{1}{|x|^4}\bigg(1-\mathrm{e}^{-|x|^2/(4t)}-\frac{|x|^2}{4t}\bigg):= I_7+I_8.
\end{aligned}
$$

By (76) we see that by expanding $\mathrm{e}^{-|x|^2/(4t)}$

$$
\begin{aligned}
|I_7| &\le C(K)^3R^2\int_{\{|x|>2R\}}\mathrm{d}x\,|x|^{-8}\bigg(\mathrm{e}^{-|x|^2/(4t)}+\frac{|x|^2}{4t}-1\bigg)\\
&\le C(K)^3R^2\int_{\{|x|>2R\}}\mathrm{d}x\,|x|^{-8}\bigg(\frac{|x|^2}{4t}\wedge\frac{|x|^4}{32t^2}\bigg)=\mathrm{O}\big(t^{-2}\log t\big).
\end{aligned}
\tag{90}
$$

By expanding $\mathbb{P}_x[L_K<\infty]$ in x we have that

$$\mathbb{P}_x[L_K<\infty]=\frac{C(K)}{4\pi^2|x|^2}+\frac{1}{2\pi^2}\int \mu_K(\mathrm{d}y)\frac{x\cdot y}{|x|^4}+\mathrm{O}\big(|x|^{-4}\big), \tag{91}$$

where the remainder is uniform on $\{|x|>2R\}$. The term with $\frac{x\cdot y}{|x|^4}$ in (91) does not contribute to I_8. Hence $|I_8|$ is bounded, up to a numerical factor, by the right-hand side of (90). So $I_6=\mathrm{O}(t^{-2}\log t)$. We conclude that

$$I_2=(4\pi)^{-4}C(K)t^{-1}\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\log\frac{|y_1|^2}{|y_1-y_2|^2}+\mathrm{O}\big(t^{-2}\log t\big), \tag{92}$$

and the lemma follows by (89) and (92). □

Lemma 16. *Let $m=4$. Then for $t\to\infty$*

$$\int_{\mathbb{R}^4}\mathrm{d}x\,\mathbb{P}_x[L_K<\infty]\big(\mathbb{P}_x[t<L_K<\infty]\big)^2=(4\pi)^{-4}(\log 4)C(K)^3t^{-1}+\mathrm{O}\big(t^{-2}\log t\big),$$
$$\int_{\mathbb{R}^4}\mathrm{d}x\,\big(\mathbb{P}_x[t<L_K<\infty]\big)^3=3(4\pi)^{-4}(\log 4-\log 3)C(K)^3t^{-1}+\mathrm{O}\big(t^{-2}\big). \tag{93}$$

Proof. By Lemma 14

$$\begin{aligned}&\int_{\mathbb{R}^4}\mathrm{d}x\,\mathbb{P}_x[L_K<\infty]\big(\mathbb{P}_x[t<L_K<\infty]\big)^2\\&\quad=(4\pi)^{-4}\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\\&\qquad\times\int_0^\infty\int_t^\infty\int_t^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,(s_1s_2+s_2s_3+s_3s_1)^{-2}\mathrm{e}^{Q(y_1,y_2,y_3;s_1,s_2,s_3)}.\end{aligned} \tag{94}$$

Since $Q\le 0$ we have that the right-hand side of (94) is bounded from above by

$$\begin{aligned}&(4\pi)^{-4}\int\int\int\mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3)\int_0^\infty\int_t^\infty\int_t^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,(s_1s_2+s_2s_3+s_3s_1)^{-2}\\&\quad=(4\pi)^{-4}(\log 4)C(K)^3t^{-1}.\end{aligned}$$

To obtain a lower bound we note that for $y_1,y_2,y_3\in K$ and $s_2,s_3\in[t,\infty)$

$$Q(y_1,y_2,y_3;s_1,s_2,s_3)\ge-\frac{R^2}{4s_1}-\frac{R^2}{2t}.$$

Hence the left-hand side of (94) is bounded from below by

$$\begin{aligned}&(4\pi)^{-4}C(K)^3\int_0^\infty\int_t^\infty\int_t^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,(s_1s_2+s_2s_3+s_3s_1)^{-2}\mathrm{e}^{-R^2(1/(4s_1)+1/(2t))}\\&\quad=(4\pi)^{-4}C(K)^3t^{-1}\int_0^\infty\mathrm{d}s_1\int_1^\infty\mathrm{d}s_2\,(s_1+s_2)^{-1}(s_1s_2+s_1+s_2)^{-1}\mathrm{e}^{-R^2/(4s_1t)}+\mathrm{O}\big(t^{-2}\big)\\&\quad=(4\pi)^{-4}C(K)^3t^{-1}\int_0^\infty\mathrm{d}s_1\,\mathrm{e}^{-R^2/(4s_1t)}s_1^{-2}\log\bigg(\frac{(s_1+1)^2}{2s_1+1}\bigg)+\mathrm{O}\big(t^{-2}\big)\\&\quad=(4\pi)^{-4}(\log 4)C(K)^3t^{-1}+(4\pi)^{-4}C(K)^3t^{-1}\int_0^\infty\mathrm{d}s\,s^{-2}\big(\mathrm{e}^{-R^2s/(4t)}-1\big)\log\frac{(s+1)^2}{s(s+2)}+\mathrm{O}\big(t^{-2}\big)\\&\quad=(4\pi)^{-4}(\log 4)C(K)^3t^{-1}+\mathrm{O}\big(t^{-2}\log t\big).\end{aligned}$$

To prove (93) we have that its left-hand side is bounded from above by

$$(4\pi)^{-4}\int\int\int \mu_K(dy_1)\mu_K(dy_2)\mu_K(dy_3)\int_t^\infty\int_t^\infty\int_t^\infty ds_1\,ds_2\,ds_3\,(s_1s_2+s_2s_3+s_3s_1)^{-2}$$
$$= 3(4\pi)^{-4}(\log 4-\log 3)C(K)^3t^{-1}. \tag{95}$$

For $y_1, y_2, y_3 \in K$ and $s_1, s_2, s_3 \in [t,\infty)$ we have by (65) that

$$e^{Q(y_1,y_2,y_3;s_1,s_2,s_3)} \geq 1-\frac{3R^2}{4t}. \tag{96}$$

We conclude that the left-hand side of (93) is bounded from below by (95) up to $O(t^{-2})$. □

Proposition 7 follows from (67), Lemmas 15 and 16.

6. Proof of Proposition 8

Lemma 17. *Let $m \geq 5$. Then for $t \to \infty$*

$$\int_{\mathbb{R}^m} dx\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[t<L_K<\infty]$$
$$= (4\pi)^{-m}2^{m-3}(m-2)^{-1}\Gamma\big((m-4)/2\big)\int\int \mu_K(dy_1)\mu_K(dy_2)|y_1-y_2|^{4-m}C(K)t^{(2-m)/2}$$
$$-\begin{cases} 12^{-1}(4\pi)^{-4}C(K)^3t^{-2}+O\big(t^{-5/2}\big), & m=5,\\ 6^{-1}(4\pi)^{-6}C(K)^3t^{-3}\log t+O\big(t^{-3}\big), & m=6,\\ O\big(t^{-m/2}\big), & m\geq 7.\end{cases}$$

Proof. By (11)

$$\int_{\mathbb{R}^m} dx\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\mathbb{P}_x[t<L_K<\infty]$$
$$= \int_{\mathbb{R}^m} dx\,\big(\mathbb{P}_x[L_K<\infty]\big)^2Ct^{(2-m)/2}$$
$$+\int_{\mathbb{R}^m} dx\,\big(\mathbb{P}_x[L_K<\infty]\big)^2\int \mu_K(dy)\int_t^\infty ds\,(4\pi s)^{-m/2}\big(e^{-|x-y|^2/(4s)}-1\big), \tag{97}$$

where C is given by (22). A straightforward calculation, using (11) and the semigroup property of the heat kernel, gives that for $m \geq 5$

$$\int_{\mathbb{R}^m} dx\,\big(\mathbb{P}_x[L_K<\infty]\big)^2 = (4\pi)^{-m/2}2^{m-4}\Gamma\big((m-4)/2\big)\int\int \mu_K(dy_1)\mu_K(dy_2)|y_1-y_2|^{4-m}.$$

It remains to obtain the asymptotic behaviour of the second term in the right-hand side of (97). For $|x| \leq 3R$, $y \in K, s \geq t$ we have that $(1-e^{-|x-y|^2/(4s)}) \leq 4R^2/t$. Hence the contribution from $\{|x| \leq 3R\}$ to the integral with respect to x is bounded by $O(t^{-m/2})$. For $|x| > 3R$, $y \in K, s \geq t$ we have that $1-e^{-|x-y|^2/(4s)} \leq 4|x|^2/s$. By (77) we have that $\int_{\{|x|\geq 2R\}} dx\,\mathbb{P}_x[L_K<\infty]^2|x|^2<\infty$ for $m \geq 7$. Hence the contribution from $\{|x| > 3R\}$ to the integral with respect to x is $O(t^{-m/2})$ for $m \geq 7$. Next we consider the cases $m = 5, m = 6$. By a change of variable we have that the contribution from $\{|x| > 3R\}$ to the integral equals J, where

$$J := \int \mu_K(dy)\int_{\{|x+y|>3R\}} dx\,\big(\mathbb{P}_{x+y}[L_K<\infty]\big)^2\int_t^\infty ds\,(4\pi s)^{-m/2}\big(e^{-|x|^2/(4s)}-1\big).$$

Moreover, $J_\ell \le J \le J_u$, where

$$J_\ell = \int \mu_K(\mathrm{d}y) \int_{\{|x|>4R\}} \mathrm{d}x\, \big(\mathbb{P}_{x+y}[L_K < \infty]\big)^2 \int_t^\infty \mathrm{d}s\, \big(\mathrm{e}^{-|x|^2/(4s)} - 1\big)(4\pi s)^{-m/2},$$

$$J_u = \int \mu_K(\mathrm{d}y) \int_{\{|x|>2R\}} \mathrm{d}x\, \big(\mathbb{P}_{x+y}[L_K < \infty]\big)^2 \int_t^\infty \mathrm{d}s\, \big(\mathrm{e}^{-|x|^2/(4s)} - 1\big)(4\pi s)^{-m/2}.$$

It is easily seen that

$$J_\ell = J_u + \mathrm{O}\big(t^{-m/2}\big). \tag{98}$$

Furthermore,

$$\mathbb{P}_{x+y}[L_K < \infty] = c_m C(K)|x|^{2-m} + c_m(m-2)\int \mu_K(\mathrm{d}y_1)|x|^{-m} x\cdot(y_1 - y) + \mathrm{O}\big(|x|^{-m}\big), \tag{99}$$

where the remainder is uniform on $\{|x| > 4R\}$. Hence

$$\begin{aligned}&\big(\mathbb{P}_{x+y}[L_K < \infty]\big)^2 \\ &\quad = c_m^2 C(K)^2 |x|^{4-2m} + 2c_m^2(m-2)|x|^{2-2m} C(K) \int \mu_K(\mathrm{d}y_1) x\cdot(y_1 - y) + \mathrm{O}\big(|x|^{2-2m}\big),\end{aligned} \tag{100}$$

where the remainder is uniform on $\{|x| > 4R\}$. By spherical symmetry

$$\int_{\{|x|>4R\}} \mathrm{d}x\, |x|^{2-2m} \int \mu_K(\mathrm{d}y_1) x\cdot(y - y_1)\big(\mathrm{e}^{-|x|^2/(4s)} - 1\big) = 0. \tag{101}$$

Also

$$\int_{\{|x|>4R\}} \mathrm{d}x\, |x|^{2-2m} \int_t^\infty \mathrm{d}s\, (4\pi s)^{-m/2}\big(\mathrm{e}^{-|x|^2/(4s)} - 1\big) = \mathrm{O}\big(t^{-m/2}\big). \tag{102}$$

We conclude that

$$J_\ell = c_m^2 C(K)^3 \int_{\{|x|>4R\}} \mathrm{d}x\, |x|^{4-2m} \int_t^\infty \mathrm{d}s\, (4\pi s)^{-m/2}\big(\mathrm{e}^{-|x|^2/(4s)} - 1\big) + \mathrm{O}\big(t^{-m/2}\big). \tag{103}$$

Straightforward calculations show that

$$J_\ell = -12^{-1}(4\pi)^{-4} C(K)^3 t^{-2} + \mathrm{O}\big(t^{-5/2}\big), \quad m = 5, \tag{104}$$

$$J_\ell = -6^{-1}(4\pi)^{-6} C(K)^3 t^{-3}\log t + \mathrm{O}\big(t^{-3}\big), \quad m = 6. \tag{105}$$

The lemma follows by (98)–(105). □

Lemma 18. *Let $m = 5$. Then for $t \to \infty$*

$$\int_{\mathbb{R}^5} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\big(\mathbb{P}_x[t < L_K < \infty]\big)^2 = 3^{-1}(4\pi)^{-5}(4-\pi)C(K)^3 t^{-2} + \mathrm{O}\big(t^{-5/2}\big), \tag{106}$$

$$\int_{\mathbb{R}^5} \mathrm{d}x\, \big(\mathbb{P}_x[t < L_K < \infty]\big)^3 = 3^{-1}(4\pi)^{-5}\big(2\sqrt{3} - \pi\big)C(K)^3 t^{-2} + \mathrm{O}\big(t^{-3}\big). \tag{107}$$

Let $m \ge 6$. Then for $t \to \infty$

$$\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\big(\mathbb{P}_x[t < L_K < \infty]\big)^2 = \mathrm{O}\big(t^{3-m}\big), \tag{108}$$

$$\int_{\mathbb{R}^m} \mathrm{d}x\, \big(\mathbb{P}_x[t < L_K < \infty]\big)^3 = \mathrm{O}\big(t^{3-m}\big). \tag{109}$$

Proof. By Lemma 14

$$
\begin{aligned}
&\int_{\mathbb{R}^m} \mathrm{d}x\, \mathbb{P}_x[L_K < \infty]\big(\mathbb{P}_x[t < L_K < \infty]\big)^2 \\
&\quad = (4\pi)^{-m} \int\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3) \\
&\qquad \times \int_0^\infty \int_t^\infty \int_t^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-m/2} \mathrm{e}^{Q(y_1,y_2,y_3;s_1,s_2,s_3)}.
\end{aligned} \tag{110}
$$

Since $Q \le 0$ we have that the left-hand side of (110) is for $m = 5$ bounded from above by

$$
(4\pi)^{-5} C(K)^3 \int_0^\infty \int_t^\infty \int_t^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-5/2} = 3^{-1}(4\pi)^{-5}(4-\pi)C(K)^3 t^{-2}. \tag{111}
$$

For $y_1, y_2, y_3 \in K$ and $s_2, s_3 \in [t, \infty)$ we have that

$$
\mathrm{e}^{Q(y_1,y_2,y_3;s_1,s_2,s_3)} \ge \mathrm{e}^{-R^2(1/(4s_1)+1/(2t))}.
$$

Hence (110) is for $m = 5$ bounded from below by

$$
\begin{aligned}
&(4\pi)^{-5} C(K)^3 t^{-2} \int_0^\infty \int_1^\infty \int_1^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-5/2} \mathrm{e}^{-R^2(1/(4s_1t)+1/(2t))} \\
&\quad = 3^{-1}(4\pi)^{-5}(4-\pi)C(K)^3 t^{-2} + (4\pi)^{-5} C(K)^3 t^{-2} \\
&\qquad \times \int_0^\infty \int_1^\infty \int_1^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-5/2} \big(\mathrm{e}^{-R^2(1/(4s_1t)+1/(2t))} - 1\big) \\
&\quad := E_1 + E_2.
\end{aligned}
$$

Furthermore

$$
\begin{aligned}
|E_2| &\le C(K)^3 t^{-2} \int_0^\infty \mathrm{d}s_1 \int_1^\infty \mathrm{d}s_2\, (s_1 + s_2)^{-1}(s_1s_2 + s_1 + s_2)^{-3/2} \big(1 - \mathrm{e}^{-R^2(1/(4s_1t)+1/(2t))}\big) \\
&\le C(K)^3 t^{-2} \int_0^\infty \mathrm{d}s_1\, (s_1 + 1)^{-1} \int_1^\infty \mathrm{d}s_2\, (s_1s_2 + s_1 + s_2)^{-3/2} \big(1 - \mathrm{e}^{-R^2(1/(4s_1t)+1/(2t))}\big) \\
&\le 2C(K)^3 t^{-2} \int_0^\infty \mathrm{d}s_1\, (s_1 + 1)^{-5/2} \big(1 - \mathrm{e}^{-R^2(1/(4s_1t)+1/(2t))}\big) \\
&\le 2C(K)^3 t^{-2} \int_0^\infty \mathrm{d}s\, (s + 1)^{-5/2} \left(\frac{R^2}{4st} + \frac{R^2}{2t}\right)^{1/2} = \mathrm{O}\big(t^{-5/2}\big).
\end{aligned}
$$

This proves (106). Estimates (108) and (109) follow immediately from (110) and the fact that $Q \le 0$.

To prove (107) we note that by Lemma 14

$$
\begin{aligned}
\int_{\mathbb{R}^m} \mathrm{d}x\, \big(\mathbb{P}_x[t < L_K < \infty]\big)^3 &= (4\pi)^{-m} \int\int\int \mu_K(\mathrm{d}y_1)\mu_K(\mathrm{d}y_2)\mu_K(\mathrm{d}y_3) \\
&\quad \times \int_t^\infty \int_t^\infty \int_t^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-m/2} \mathrm{e}^{Q(y_1,y_2,y_3;s_1,s_2,s_3)}.
\end{aligned} \tag{112}
$$

By (96) and (112) we have that

$$
\int_{\mathbb{R}^5} \mathrm{d}x\, \big(\mathbb{P}_x[t < L_K < \infty]\big)^3 = (4\pi)^{-5} C(K)^3 t^{-2} \int_1^\infty \int_1^\infty \int_1^\infty \mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3\, (s_1s_2 + s_2s_3 + s_3s_1)^{-5/2} + \mathrm{O}\big(t^{-3}\big).
$$

A tedious calculation shows that

$$\int_1^\infty \int_1^\infty \int_1^\infty \mathrm{d}s_1\,\mathrm{d}s_2\,\mathrm{d}s_3\,(s_1s_2+s_2s_3+s_3s_1)^{-5/2} = 3^{-1}\bigl(2\sqrt{3}-\pi\bigr). \qquad \square$$

The proof of Proposition 8 follows by (67), and Lemmas 17 and 18.

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