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Infinite divisibility of solutions to some self-similar integro-differential equations and exponential functionals of Lévy processes

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Abstract. We first characterize the increasing eigenfunctions associated to the following family of integro-differential operators, for any $\alpha, x > 0, \gamma \geq 0$ and f a smooth function on $\mathfrak{R}^+$,

$$\mathbf{L}^{(\gamma)} f(x) = x^{-\alpha}\left(\frac{\sigma}{2}x^2 f''(x) + (\sigma\gamma + b)xf'(x) + \int_0^\infty \bigl(f\bigl(\mathrm{e}^{-r}x\bigr) - f(x)\bigr)\mathrm{e}^{-r\gamma} + xf'(x)r\mathbb{I}_{\{r\leq 1\}}\nu(\mathrm{d}r)\right), \tag{0.1}$$

where the coefficients $b \in \mathfrak{R}, \sigma \geq 0$ and the measure ν, which satisfies the integrability condition $\int_0^\infty (1 \wedge r^2)\nu(\mathrm{d}r) < +\infty$, are uniquely determined by the distribution of a spectrally negative, infinitely divisible random variable, with characteristic exponent ψ. $\mathbf{L}^{(\gamma)}$ is known to be the infinitesimal generator of a positive α-self-similar Feller process, which has been introduced by Lamperti [*Z. Wahrsch. Verw. Gebiete* **22** (1972) 205–225]. The eigenfunctions are expressed in terms of a new family of power series which includes, for instance, the modified Bessel functions of the first kind and some generalizations of the Mittag-Leffler function. Then, we show that some specific combinations of these functions are Laplace transforms of self-decomposable or infinitely divisible distributions concentrated on the positive line with respect to the main argument, and, more surprisingly, with respect to the parameter $\psi(\gamma)$. In particular, this generalizes a result of Hartman [*Ann. Sc. Norm. Super. Pisa Cl. Sci.* **IV-III** (1976) 267–287] for the increasing solution of the Bessel differential equation. Finally, we compute, for some cases, the associated decreasing eigenfunctions and derive the Laplace transform of the exponential functionals of some spectrally negative Lévy processes with a negative first moment.

Résumé. Nous commençons par caractériser les fonctions propres croissantes, au sens strict, de la famille d'opérateurs intégro-différentiels (0.1), pour tout $\alpha > 0, \gamma \geq 0$, f une function définie sur $\mathfrak{R}^+$ et suffissament régulière, et où les coefficients $b \in \mathfrak{R}, \sigma \geq 0$ et la mesure ν, qui satisfait la condition d'intégrabilité $\int_0^\infty (1 \wedge r^2)\nu(\mathrm{d}r) < +\infty$, sont données, de manière unique, par la distribution d'une variable aléatoire infiniment divisible et spectralement négative dont on écrit ψ son exposant caractéristique. $\mathbf{L}^{(\gamma)}$ est le générateur infinitésimal d'un processus positif Fellerien α-auto-similaire, introduit par Lamperti [*Z. Wahrsch. Verw. Gebiete* **22** (1972) 205–225]. Les fonctions propres sont définies en terme d'une nouvelle famille de séries entières qui contient, par exemple, les fonctions de Bessel modifiées du premier ordre et des généralisations des fonctions de Mittag-Leffler. Nous continuons par montrer que des combinaisons particulières de ces séries entières correspondent à des transformées de Laplace de variables aléatoires positives auto-décomposables ou infiniment divisibles, par rapport à la valeur propre associée mais aussi par rapport au paramètre $\psi(\gamma)$, ce qui est plus surprenant. En particulier, ceci généralise un résultat de Hartman [*Ann. Sc. Norm. Super. Pisa Cl. Sci.* **IV-III** (1976) 267–287] sur les fonctions de Bessel modifiées. Finalement, nous calculons, dans certains cas, les fonctions propres décroissantes, ce qui nous permet de caractériser la loi, par le biais de sa transformée de Laplace, de la fonctionnelle exponentielle de certains processus de Lévy spectralement négatifs ayant un premier moment négatif.

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1. Introduction

During the last decade, there has been a renewed interest in self-similar semigroups, something which seems to be attributed to their connections to several fields of mathematics and more generally to many area of the sciences. For instance, in probability theory, these semigroups arise in the study of important processes such as self-similar processes, branching processes and also in the investigation of self-similar fragmentation. Moreover, the Feller processes associated to self-similar semigroups are closely related, via the Lamperti's mapping, to the exponential functionals of Lévy processes which appear to be key objects in a variety of settings (random processes in random environments, mathematical finance, astrophysics...). We refer to Bertoin and Yor [6] for an interesting recent survey on this topic. Finally, we emphasize that they are also related to the theory of the fractional operator which is used intensively in many applied fields, see, e.g., the survey paper of Kilbas and Trujillo [21].

In this paper we provide, in terms of power series, the increasing eigenfunction associated to the linear operator $\mathbf{L}^{(\gamma)}$, given by (0.1), that is the increasing solution to the integro-differential equation, for $x, q \geq 0$,

$$\mathbf{L}^{(\gamma)} f_q(x) = q f_q(x).$$

As a byproduct, we compute the Laplace transform of the first passage times above for spectrally negative self-similar processes and some related quantities.

Moreover, when the spectrally negative random variable has a negative first moment, we provide, under an additional technical condition, the decreasing eigenfunctions associated with $\mathbf{L} = \mathbf{L}^{(0)}$. We deduce an explicit expression of the Laplace transform of the exponential functional of some spectrally negative Lévy processes with negative mean. This is a companion result of Bertoin and Yor [5] who characterized, in terms of its negative entire moments, the law of the exponential functional of spectrally positive Lévy processes which drift to $-\infty$.

Furthermore, it is plain that $\mathbf{L}^{(\gamma)}$ is a generalization of the infinitesimal generator of the (re-scaled) Bessel process ($\nu \equiv 0$ and $\alpha = 2$). In this specific case, Hartman [16], relying on purely analytical arguments, showed that the function

$$\gamma \mapsto \frac{\mathrm{I}_{\sqrt{2\gamma}}(a)\mathrm{I}_0(A)}{\mathrm{I}_{\sqrt{2\gamma}}(A)\mathrm{I}_0(a)}, \quad 0 < a < A < \infty, \tag{1.1}$$

is the Laplace transform of an infinitely divisible distribution concentrated on the positive line, where I_ν stands for the modified Bessel function of the first kind. We mention that in the limit case $A \to \infty$, the result above has been reproved, in an elegant fashion, by Pitman and Yor [30]. We shall provide a simple probabilistic explanation of Hartman's result (for any $0 < A < \infty$) and show that this property still holds for similar ratios of the increasing eigenfunctions associated with the non-local operator $\mathbf{L}^{(\gamma)}$.

The outline of the remainder of the paper is as follows. In the sequel, we set up the notation and provide some basic results. Section 2 is devoted to the statement of the main results. The proofs are given in Section 3. Finally, in the last section, we illustrate our approach by investigating some known and new examples.

1.1. *Notation and preliminaries*

1.1.1. *Some important sets of probability measures*

Let ξ_1 be a spectrally negative infinitely divisible random variable. It is well known that its characteristic exponent, ψ, admits the following Lévy–Khintchine representation

$$\psi(u) = bu + \frac{\sigma}{2}u^2 + \int_0^\infty \left(\mathrm{e}^{-ur} - 1 + ur\mathbb{I}_{\{r\leq 1\}}\right)\nu(\mathrm{d}r), \quad u \geq 0, \tag{1.2}$$

where the coefficients $b \in \mathfrak{R}, \sigma \geq 0$ and the measure $\tilde{\nu}$, image of ν by the mapping $x \to -x$, which satisfies the integrability condition $\int_{-\infty}^0 (1 \wedge r^2)\tilde{\nu}(\mathrm{d}r) < +\infty$, are uniquely determined by the distribution of ξ_1. We exclude the case where $b \leq 0$ and $\int_{-\infty}^0 (1 \wedge r)\tilde{\nu}(\mathrm{d}r) < +\infty$, i.e., when ψ is the Laplace exponent of the negative of a subordinator. It is plain that $\lim_{u\to\infty} \psi(u) = +\infty$ and by monotone convergence one gets $\mathbb{E}[\xi_1] = b - \int_1^\infty r\nu(\mathrm{d}r) \in [-\infty, \infty)$. Differentiating again, one observes that ψ is strictly convex unless ξ is degenerate, which we exclude. Note that 0 is always a root of the equation $\psi(u)=0$. However, in the case $\mathbb{E}[\xi_1]<0$, this equation admits another positive root, which we denote by θ. This yields the so-called Cramér condition

$$\mathbb{E}\left[e^{\theta\xi_1}\right]=1.$$

Then, for any $\mathbb{E}[\xi_1]\in[-\infty,\infty)$, the function $u\mapsto\psi(u)$ is continuous and increasing on $[\max(\theta,0),\infty)$ and thus it has a well-defined inverse function $\phi:[0,\infty)\to[\max(\theta,0),\infty)$ which is also continuous and increasing. We denote the totality of all functions ψ of the form (1.2) by $\mathcal{LK}$. Note, from the stability of infinitely divisible distributions under convolution and convolution powers to positive real numbers, that $\mathcal{LK}$ forms a convex cone in the space of real valued functions defined on $[0,\infty)$.

We also mention that when a probability measure dm is supported on a subset of $[0,\infty)$, then dm is infinitely divisible if and only if its Laplace transform satisfies the conditions, for any $u\geq 0$,

$$\int_0^\infty e^{-ux}\,dm(x)=e^{-\phi(u)}$$

with $\phi(0)=0$ and $\phi'(u)$ is completely monotonic, i.e., ϕ' is infinitely differentiable on $(0,\infty)$ and for all $n=1,2,\ldots,$ $(-1)^{n-1}\phi^{(n)}(u)>0, u>0$. Moreover, the so-called Laplace exponent, ϕ, admits the following Lévy–Khintchine representation

$$\phi(u)=au+\int_0^\infty\left(1-e^{-ur}\right)\mu(dr),\quad u\geq 0,\tag{1.3}$$

for some $a\geq 0$ and some positive measure μ on $(0,\infty)$ satisfying $\int_0^\infty(1\wedge r)\mu(dr)<\infty$, see, e.g., Meyer [27].

Finally, we recall that a random variable H is self-decomposable (or of class L) if it is the solution to the random affine equation

$$H\stackrel{(d)}{=}cH+H_c,$$

where $\stackrel{(d)}{=}$ stands for the equality in distribution, $0<c<1$ and H_c is a random variable independent of H. It is well known that the law of these random variables is absolutely continuous, see, e.g., [32], Example 27.8. Moreover, Wolfe [35] showed that the density h of a positive self-decomposable random variable is unimodal, i.e., there exists $a\in\mathfrak{R}^+$ (the mode) such that h is increasing on $]0,a[$ and decreasing on $]a,\infty[$. The Laplace exponent, ϕ_s, of a self-decomposable distribution concentrated on $\mathfrak{R}^+$ is given by

$$\phi_s(u)=au+\int_0^\infty\left(1-e^{-ur}\right)\frac{k(r)}{r}\,dr,$$

where k is a positive decreasing function. We refer to the monographs of Sato [32] and Steutel and van Harn [33] for an excellent account on these sets of probability measures.

1.1.2. *Lévy and Lamperti*

Let $\mathbb{P}_x$ (we write simply $\mathbb{P}$ for $\mathbb{P}_0$) be the law of a spectrally negative Lévy process $\xi:=(\xi_t)_{t\geq0}$, starting at $x\in\mathfrak{R}$, with $(F_t)_{t\geq0}$ its natural filtration. This law is characterized by the characteristic exponent of ξ_1, which we assumed to belong to $\mathcal{LK}$, i.e., being of the form (1.2). We deduce, from the above discussion and the strong law of large numbers, that $\lim_{t\to+\infty}\xi_t=\mathrm{sgn}(\mathbb{E}[\xi_1])\infty$ a.s. and the process oscillates if $\mathbb{E}[\xi_1]=0$.

For any $\gamma\geq 0$, we write $\mathbb{P}^{(\gamma)}$ for the law of the Lévy process with characteristic exponent

$$\psi_\gamma(u)=\psi(u+\gamma)-\psi(\gamma),\quad u\geq 0.$$

The laws $\mathbb{P}^{(\gamma)}$ and $\mathbb{P}$ are connected via the following absolute continuity relationship, also known as the Esscher transform,

$$d\mathbb{P}^{(\gamma)}_{x}{}_{|F_t}=e^{\gamma(\xi_t-x)-\psi(\gamma)t}\,d\mathbb{P}_{x|F_t},\quad t>0, x\in\mathfrak{R}.\tag{1.4}$$

Lamperti [23] showed that there exists a one-to-one mapping between $\mathbb{P}_x$ and the law $\mathbb{Q}_{e^x}$ of a $\frac{1}{\alpha}$-self-similar Markov process X on $(0,\infty)$, i.e., a Feller process which enjoys the following α-self-similarity property for any $c>0$

$$\big((X_{c^\alpha t})_{t\geq 0}, \mathbb{Q}^{(\gamma)}_{ce^x}\big) \stackrel{(d)}{=} \big((cX_t)_{t\geq 0}, \mathbb{Q}^{(\gamma)}_{e^x}\big). \tag{1.5}$$

More precisely, Lamperti showed that X can be constructed from ξ as follows

$$\log(X_t) = \xi_{A_t}, \quad t\geq 0, \tag{1.6}$$

where

$$A_t = \inf\left\{s\geq 0; \Sigma_s := \int_0^s e^{\alpha\xi_u}\,du > t\right\}.$$

We write as $E^{(\gamma)}_x$ (resp. E_x) the expectation operator associated with $\mathbb{Q}^{(\gamma)}_x$ (resp. $\mathbb{Q}_x = \mathbb{Q}^{(0)}_x$). Moreover, for $\mathbb{E}[\xi_1]<0$, it is plain that X has an a.s. finite lifetime which is $\kappa_0 = \inf\{s\geq 0; X_{s^-}=0, X_s=0\}$. However, under the additional condition $0<\theta<\alpha$, where we recall that $\psi(\theta)=0$, Rivero [31] showed that the minimal process (X,κ_0) admits a *unique recurrent extension that hits and leaves* 0 *continuously a.s.* and which is an α-self-similar process on $[0,\infty)$. We simply write $(X,\mathbb{Q}_x)$ for the law of such a recurrent extension starting from $x\geq 0$. Furthermore, for $\mathbb{E}[\xi_1]\geq 0$, Bertoin and Yor [5], Proposition 1, showed that the family of probability measures $(\mathbb{Q}_x)_{x>0}$ converges in the sense of finite dimensional distribution to a probability measure $\mathbb{Q}_{0+}$ as $x\to 0+$, see also Caballero and Chaumont [10] for conditions for the weak convergence. Thus, for any $x\geq 0$, $(X,\mathbb{Q}_x)$ is also spectrally negative, in the sense that it has no positive jumps. Moreover, for any $x\geq 0$, $(X,\mathbb{Q}_x)$ is a Feller process on $[0,\infty)$ and we denote its semigroup (resp. its resolvent) by $(Q_t)_{t\geq 0}$ (resp. by $U^q, q>0$), i.e., for any $x,t\geq 0$ and $v\in B([0,\infty))$, the space of bounded Borelian functions on $[0,\infty)$,

$$Q_t v(x) = E_x\big[v(X_t)\big],$$
$$U^q v(x) = \int_0^\infty e^{-qt} Q_t v(x)\,dt.$$

We also introduce the semigroup and the resolvent of the minimal process (X,κ_0) for $x>0$,

$$Q^0_t v(x) = E_x\big[v(X_t), t<\kappa_0\big],$$
$$U^q_0 v(x) = \int_0^\infty e^{-qt} Q^0_t v(x)\,dt.$$

The strong Markov property yields the following expression for the resolvent of the recurrent extension

$$U^q v(x) = U^q_0 v(x) + \mathbb{E}_x\big[e^{-q\kappa_0}\big]U^q v(0), \quad x\geq 0. \tag{1.7}$$

Moreover, we recall that the Lamperti mapping reads in terms of the characteristic operator as follows.

Proposition 1.1. *Let* $f:\mathfrak{R}^+\to\mathfrak{R}$ *be such that* $f(x), xf'(x)$ *and* $x^2f''(x)$ *are continuous functions on* $\mathfrak{R}^+$, *then* f *belongs to the domain*, $\mathbb{D}(\mathbf{L})$, *of the characteristic operator* $\mathbf{L}$ *of* $(X,\mathbb{Q})$ *which is given, for* $x>0$, *by*

$$\mathbf{L}f(x) = x^{-\alpha}\left(\frac{\sigma}{2}x^2 f''(x) + bxf'(x) + \int_0^\infty \big(\big(f\big(e^{-r}x\big)-f(x)\big)+xf'(x)r\big)\nu(dr)\right). \tag{1.8}$$

In the case $\mathbb{E}[\xi_1]<0$, $f\in\mathbb{D}(\mathbf{L})$ *if and only if it satisfies the boundary condition*

$$\lim_{x\to 0}\frac{f'(x)}{x^{\theta-1}} = 0. \tag{1.9}$$

Proof. The first part of the proposition follows from Lamperti [23], Theorem 6.1. We point out that, in the case $\mathbb{E}[\xi_1] < 0$, the characteristic operator of the minimal process and its recurrent extension coincide for $x > 0$. Indeed, from its Feller property, the semigroup $(Q_t, t > 0)$ corresponding to the recurrent extension leaves invariant $C_0([0,\infty))$, the space of continuous functions vanishing at infinity. Therefore, if we let (1.8) stand for the characteristic operator of the process, then we can conclude, see Gikhman and Skorokhod [15], p. 130, Theorem 1, that the domain $D(L)$ of the strong infinitesimal generator, L, of the process $(X, \mathbb{Q})$ consists of all the functions $f \in C_0([0,\infty)) \cap \mathbb{D}(\mathbf{L})$ such that $\mathbf{L}f \in C_0([0,\infty))$. However, it is plain that the analytic form of the function $\mathbf{L}f(x)$ for $x > 0$ and for any function f, such that $f(x), xf'(x), x^2 f''(x)$ are continuous, does not depend on the method of extension and is given by the expression above. Let us now turn to the boundary condition in the case $\mathbb{E}[\xi_1] < 0$. We first deal with the necessary condition. From the discussion above, it is clear that it is enough to characterize the boundary condition for the strong infinitesimal generator. To this end, we recall that U^q is a Fellerian resolvent, see Rivero [31], Theorem 2. Thus, let $f \in C_0([0,\infty)) \cap \mathbb{D}(\mathbf{L})$; then there exists $v \in C_0([0,\infty))$ such that $\mathbf{L}f - qf = v$, i.e., for $x \geq 0$

$$U^q v(x) = f(x).$$

Then, using the expression of the resolvent of the recurrent extension (1.7) and the continuity of f, we get

$$f(x) - f(0) = U_0^q v(x) - \mathbb{E}_x\bigl[1 - \mathrm{e}^{-q\kappa_0}\bigr] U^q v(0). \tag{1.10}$$

Let us now consider $v_0 \in C_0((0,\infty))$, the space of continuous function vanishing at 0 and ∞. Then, from [31], Lemma 1, we have $U^q v_0(x) \in C_0((0,\infty)) \cap \mathbb{D}(\mathbf{L})$. Moreover, from [31], Theorem 2, we know that a necessary condition for the existence of a unique recurrent extension which hits and leaves 0 continuously a.s. is that both limits

$$\lim_{x\to 0} \frac{U_0^q v_0(x)}{x^\theta} \quad \text{and} \quad \lim_{x\to 0} \frac{\mathbb{E}_x[1 - \mathrm{e}^{-q\kappa_0}]}{x^\theta}$$

exist for any $v_0 \in C_0((0,\infty))$. Moreover, in this case, the identity

$$\lim_{x\to 0} \frac{U_0^q v_0(x)}{\mathbb{E}_x[1 - \mathrm{e}^{-q\kappa_0}]} = U^q v_0(0) \tag{1.11}$$

holds for any $v_0 \in C_0((0,\infty))$. Hence, for $v_0 \in C_0((0,\infty))$, the necessary condition (1.9) follows by dividing both sides of (1.10) by x^θ, by taking the limit $x \to 0$ and by invoking the uniqueness of the limit. The general case, i.e., for any $v \in C_0([0,\infty))$, follows from the fact that the Lebesgue measure of the set $\{t \geq 0; X_t = 0\}$ is, by construction, 0 with probability 1, and from Blumenthal [8], Section 4. The sufficient part is readily obtained from the uniqueness of the recurrent extension that hits and leaves 0 continuously a.s. □

1.1.3. *The family of power series*

Let $\psi \in \mathcal{LK}$ and for $\gamma \geq 0$ and $\alpha > 0$, set

$$a_n(\psi_\gamma; \alpha) = \left(\prod_{k=1}^{n} \psi_\gamma(\alpha k)\right)^{-1}, \qquad a_0 = 1,$$

where we recall that $\psi_\gamma(u) = \psi(u+\gamma) - \psi(\gamma)$, $u \geq 0$. Then, we introduce the function $\mathcal{I}_{\alpha,\psi_\gamma}$ which admits the series representation

$$\mathcal{I}_{\alpha,\psi_\gamma}(z) = \sum_{n=0}^{\infty} a_n(\psi_\gamma; \alpha) z^n, \quad z \in \mathfrak{C}.$$

We simply write $\mathcal{I}_{\alpha,\psi}$ when $\gamma = 0$. We gather some basic properties of this family of power series which will be useful for the sequel.

Proposition 1.2. *For any $\psi \in \mathcal{LK}$, $\gamma \in \mathfrak{C}$, $\mathfrak{Re}(\gamma) \geq 0$ and $\alpha > 0$, $\mathcal{I}_{\alpha,\psi_\gamma}$ is an entire function. Moreover, if $\mathbb{E}[\xi_1] \geq 0$ or if $\mathbb{E}[\xi_1] < 0$ and $\theta < \alpha$, $\mathcal{I}_{\alpha,\psi}$ is positive and increasing on $[0, \infty)$.*

Proof. Observe that

$$\frac{|\, a_{n+1} \,|}{|\, a_n \,|} = \frac{1}{|\, \psi(\alpha n + \gamma) - \psi(\gamma) \,|}.$$

The analyticity of $\mathcal{I}_{\alpha,\psi_\gamma}$ follows from the fact that $\lim_{u\to\infty} \psi_\gamma(u) = +\infty$. The positivity and the monotonicity is secured by observing that under the condition $\theta < \alpha$ in the case $\mathbb{E}[\xi_1] < 0$, we have $\psi(\alpha) > 0$ and ψ is increasing on $[\max(\theta, 0), \infty)$. □

Remark 1.3. *Note that Rivero's condition, $0 < \theta < \alpha$ for $\mathbb{E}[\xi_1] < 0$, arises naturally in the previous proposition to ensure that the associated functions are positive and increasing.*

2. Main results

Let $\psi \in \mathcal{LK}$. Moreover, if $\mathbb{E}[\xi_1] < 0$, we assume that $\theta < \alpha$, recalling that $\psi(\theta) = 0$. Next, for $a \in \mathfrak{R}$, we introduce the stopping times

$$\tau_a = \inf\{s \geq 0; \xi_s = a\} \quad \text{and} \quad \kappa_{e^a} = \inf\{s \geq 0; X_s = e^a\}$$

with the convention that $\inf\{\emptyset\} = \infty$. For any $\lambda \geq 0$, we denote $\rho = \phi(\lambda)$ where $\phi : [0, \infty) \to [\max(\theta, 0), \infty)$ is the increasing and continuous inverse function of ψ.

Theorem 2.1. *Let $q \geq 0$ and $0 \leq x \leq a$. Then, we have*

$$\mathrm{E}_x\big[e^{-q\kappa_a}\big] = \frac{\mathcal{I}_{\alpha,\psi}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi}(qa^\alpha)}. \tag{2.1}$$

Moreover, for $\lambda \geq 0$,

$$\mathrm{E}_x\big[e^{-q\kappa_a - \lambda A_{\kappa_a}} \mathbb{I}_{\{\kappa_a < \kappa_0\}}\big] = \left(\frac{x}{a}\right)^\rho \frac{\mathcal{I}_{\alpha,\psi_\rho}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi_\rho}(qa^\alpha)} \tag{2.2}$$

and

$$\mathbb{E}_x\big[e^{-\lambda\tau_a - q\Sigma_{\tau_a}} \mathbb{I}_{\{\tau_a < +\infty\}}\big] = e^{\rho(x-a)} \frac{\mathcal{I}_{\alpha,\psi_\rho}(qe^{\alpha x})}{\mathcal{I}_{\alpha,\psi_\rho}(qe^{\alpha a})}. \tag{2.3}$$

Remark 2.2. (1) *Note, by letting $q \to 0$ in* (2.1), *that $\mathbb{Q}_x[\kappa_a < +\infty] = 1$ for any $0 \leq x \leq a$. Thus, the points above the starting point are recurrent states for $(X, \mathbb{Q})$. Moreover, for $\mathbb{E}[\xi_1] \geq 0$ and $x < a$, we recall that $\mathbb{P}_x[\tau_a < +\infty] = 1$ and $\mathbb{Q}_{e^x}[\kappa_{e^a} < \kappa_0] = 1$. Thus, under such a condition, the indicator functions in* (2.2) *and* (2.3) *can be omitted.*

(2) *Consider the case $\psi(u) = \frac{1}{2}u^2 + \gamma u$, $\gamma > -1$, i.e., $(X, \mathbb{Q})$ is a Bessel process of index γ. Then, the expression of the Laplace transform of κ_a, which is well known to be expressed in terms of the modified Bessel functions of the first kind, dates back to Ciesielski and Taylor* [11] *and Kent* [20]. *We refer to Section* 4.1 *for more detailed computations related to the modified Bessel functions.*

We proceed by characterizing, through its Laplace transform, the law of the exponential functional, Σ_∞, of spectrally negative Lévy processes satisfying Rivero's condition. We emphasize that it is a companion result of Bertoin and Yor [5], Proposition 2, who computed the negative entire moments of the exponential functional of spectrally positive Lévy processes when they drift to $-\infty$.

Theorem 2.3. *Assume $\mathbb{E}[\xi_1] < 0$ and $0 < \theta < \alpha$. Then, there exists a positive constant C_θ such that*

$$\mathcal{I}_{\alpha,\psi}\big(x^\alpha\big) \sim C_\theta x^\theta \mathcal{I}_{\alpha,\psi_\theta}\big(x^\alpha\big) \quad \text{as } x \to \infty.$$

Moreover, if we assume that there exists $\beta \in [0, 1]$ such that $\lim_{u\to\infty} \psi(u)/u^{1+\beta} = l_\beta$ then we have

$$C_\theta = \frac{\Gamma(1-\theta/\alpha)}{\alpha} l_\beta^{-\theta_\alpha} \mathrm{e}^{M_\gamma \beta\theta_\alpha} \prod_{k=1}^{\infty} \mathrm{e}^{-\beta\theta_\alpha/k} \frac{(k+\theta_\alpha)\psi(\alpha k)}{k\psi(\alpha k+\theta_\alpha)},$$

where $M_\gamma = 0.577\ldots$ stands for the Euler–Mascheroni constant and $\theta_\alpha = \frac{\theta}{\alpha} < 1$.

Next, introduce the function

$$\mathcal{N}_{\alpha,\psi,\theta}(x) = \mathcal{I}_{\alpha,\psi}(x) - C_\theta x^{\theta/\alpha} \mathcal{I}_{\alpha,\psi_\theta}(x), \quad x \geq 0.$$

$\mathcal{N}_{\alpha,\psi,\theta}$ is analytical on the right-half plane and decreasing on $\mathfrak{R}^+$. Finally, the positive random variable Σ_∞ has the following Laplace transform

$$\mathbb{E}\big[\mathrm{e}^{-q\Sigma_\infty}\big] = \mathcal{N}_{\alpha,\psi,\theta}(q). \tag{2.4}$$

Remark 2.4. *Note that the random variable Σ_∞ is the solution to the random equation, for any $a > 0$,*

$$\Sigma_\infty \stackrel{(\mathrm{d})}{=} \Sigma_{\tau_{-a}} + \mathrm{e}^{-\xi_{\tau_{-a}}} \Sigma'_\infty,$$

where Σ'_∞ is an independent copy of Σ_∞. Indeed, together, the strong Markov property, the stationarity and independence of the increments of the Lévy process ξ entail that, for any $a \in \mathfrak{R}$, the shifted process $(\xi_{t+\tau_a} - \xi_{\tau_a})_{t\geq 0}$ is distributed as $(\xi_t)_{t\geq 0}$ and is independent of $(\xi_t, t \leq \tau_a)$. Finally, we get the equation by noting that, since $\mathbb{E}[\xi_1] < 0$, we have for any $a > 0$, $\mathbb{P}(\tau_{-a} < \infty)$.

Bertoin and Yor [5] determined, in terms of their positive entire moments, the entrance law of spectrally negative self-similar positive Markov processes when $\mathbb{E}[\xi_1] \geq 0$. In the sequel, we characterize the entrance law of the dual process of $(X, \mathbb{Q})$ when $-\infty < \mathbb{E}[\xi_1] < 0$, i.e., of spectrally positive self-similar positive Markov processes when the underlying Lévy process, in the Lamperti mapping, has a finite positive mean. To this end, let $(\widehat{X}, \mathbb{Q})$ be the self-similar process associated with the Lévy process $(\widehat{\xi}, \mathbb{P})$, the dual of $(\xi, \mathbb{P})$ with respect to the Lebesgue measure. We recall that for $-\infty < \mathbb{E}[\xi_1] < 0$, Bertoin and Yor [4], Lemma 2, showed that, for $x > 0$, $(\widehat{X}, \mathbb{Q}_x)$ is in weak duality with respect to the reference measure $m(\mathrm{d}y) = \alpha y^{\alpha-1}\,\mathrm{d}y$ with the minimal process (X, κ_0). In the same vein, Rivero [31], Lemma 7, proved that in the cases $\mathbb{E}[\xi_1] < 0$ and $\theta < \alpha$, $(X, \mathbb{Q})$ is in weak duality with respect to the measure $m^\theta(\mathrm{d}y) = y^{\alpha-\theta-1}\,\mathrm{d}y$, with $(\widehat{X}, \mathbb{Q}^{(\theta)})$, the unique recurrent extension which hits and leaves 0 continuously a.s., of the self-similar process associated via the Lamperti's mapping with $(\widehat{\xi^\theta}, \mathbb{P})$, the dual of the θ-Esscher transform of $(\xi, \mathbb{P})$. Before stating the next result, we recall that an entrance law $\{\eta_s; s > 0\}$ for the semi-group Q_t is a family of finite measures on the Borel sets of $(0, \infty)$ such that $\eta_s Q_t = \eta_{s+t}$ for all strictly positive s and t and such that $\mathrm{E}^{\eta_s}[1 - \mathrm{e}^{-\kappa_0}]$ remains bounded as s approaches 0.

Corollary 2.5. *If $-\infty < \mathbb{E}[\xi_1] < 0$, then $(\widehat{X}, \mathbb{Q})$ admits an entrance law which is absolutely continuous with respect to the reference measure $m(\mathrm{d}y)$. Its Laplace transform with respect to the time variable is given, for $y, q > 0$, by*

$$\widehat{n}^q(y) = \frac{1}{|\mathbb{E}[\xi_1]|} \mathcal{N}_{\alpha,\psi,\theta}\big(qy^\alpha\big).$$

Moreover, assume $\mathbb{E}[\xi_1] < 0$ *and* $0 < \theta < \alpha$. *Then*, $(\widehat{X}, \mathbb{Q}^{(\theta)})$ *admits an entrance law which is absolutely continuous with respect to the reference measure* $m^\theta(\mathrm{d}y)$. *Its Laplace transform with respect to the time variable is given*, *for* y, $q > 0$, *by*

$$\widehat{n}_\theta^q(y) = \frac{1}{\psi'(\theta) C_\theta} \mathcal{N}_{\alpha,\psi,\theta}\big(q y^\alpha\big).$$

Finally, we show that some specific combinations of the functions $\mathcal{I}_{\alpha,\psi}$ define some mappings from the convex cone $\mathcal{LK}$ into the convex cone of positive self-decomposable distributions or into the convex cone of positive infinitely divisible distributions.

Theorem 2.6. *Let* $q \geq 0$. *Then*, *the mapping*

$$q \mapsto \frac{1}{\mathcal{I}_{\alpha,\psi}(q)} \tag{2.5}$$

is the Laplace transform of a positive self-decomposable distribution.

The mappings

$$q \mapsto \exp\left(-q \frac{\mathrm{d}\mathcal{I}_{\alpha,\psi}(q)/\mathrm{d}q}{\mathcal{I}_{\alpha,\psi}(q)}\right) \quad \text{and} \quad q \mapsto \frac{1}{\mathcal{I}_{\alpha,\psi}(q)} \exp\left(-q \frac{\mathrm{d}\mathcal{I}_{\alpha,\psi}(q)/\mathrm{d}q}{\mathcal{I}_{\alpha,\psi}(q)}\right) \tag{2.6}$$

are the Laplace transforms of positive infinitely divisible distributions.

Finally, *for* $\mathbb{E}[\xi_1] \geq 0$, $\lambda \geq 0$, $0 < a < A < \infty$ *and recall that* $\rho = \phi(\lambda)$, *the mapping*

$$\lambda \mapsto \left(\frac{a}{A}\right)^\rho \frac{\mathcal{I}_{\alpha,\psi_\rho}(a) \mathcal{I}_{\alpha,\psi}(A)}{\mathcal{I}_{\alpha,\psi_\rho}(A) \mathcal{I}_{\alpha,\psi}(a)} \tag{2.7}$$

is the Laplace transform of an infinitely divisible distribution on the positive line.

Remark 2.7. *The random variable—the Laplace transform of which is given in* (2.7)—*is characterized below in* (3.7). *Moreover, consider again the case* $\psi(u) = \frac{1}{2}u^2 + \gamma u$, $\gamma \geq 0$, *i.e.*, $(X, \mathbb{Q})$ *is a Bessel process of index* γ:

(1) *Then*, (2.7) *corresponds to Hartman's result* (1.1). *Moreover*, *by letting* $A \to \infty$ *and using the asymptotic behavior of the modified Bessel function of the first kind*, *see* (4.1), *we get the Laplace transform of the so-called Hartman–Watson law* [17], *the density of which has been characterized by Yor* [37]. *Note also that this law is the mixture distribution in the representation of the von Mises distribution as a mixture of wrapped normal distributions*, *see* [17].

(2) *Moreover, by choosing* $\gamma = \frac{1}{2}$, *the mapping on the right-hand side of* (2.6) *corresponds to the Laplace transform of the Lévy stochastic area integral. Indeed, for* $B_t = (B_t^1, B_t^2)$, *a Brownian motion on* $\Re^2$, *Lévy* [25] *computed the Laplace transform of the process* $L_t = \int_0^t B_s^1 \, \mathrm{d}B_s^2 - B_s^2 \, \mathrm{d}B_s^1$, $t > 0$, *for fixed* $u > 0$, *and* $a = (\sqrt{u}, \sqrt{u}) \in \Re^2$, *as follows*

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}tL_u} | B_u = a\big] = \frac{tu}{\sinh(tu)} \exp -\big(tu \coth(tu) - 1\big), \quad t \in \Re.$$

This has been generalized by Biane and Yor [7] *to the Lévy stochastic area integral associated to some planar Gaussian Markov processes*, *in terms of the modified Bessel functions of the first kind of any index* $\gamma > 0$.

3. Proofs

3.1. *Proof of Theorem* 2.1

First, since the mapping $x \mapsto \mathcal{I}_{\alpha,\psi}(x^\alpha)$ is analytic on the right-half plane, it is plain that $\mathcal{I}_{\alpha,\psi} \in \mathbb{D}(\mathbf{L})$. Observe also that, for any $\beta > 0$, $x^\beta \in \mathbb{D}(\mathbf{L})$ and

$$\mathbf{L} x^\beta = x^{\beta - \alpha} \psi(\beta). \tag{3.1}$$

Then, for any positive integer N, using the linearity of the operator $\mathbf{L}$ and (3.1), we get, for any $x \geq 0$,

$$\begin{aligned}\mathbf{L}\sum_{n=0}^{N} a_n(\psi;\alpha)q^n x^{\alpha n} &= \sum_{n=1}^{N} a_n(\psi;\alpha)q^n \mathbf{L}x^{\alpha n}\\ &= \sum_{n=1}^{N} a_n(\psi;\alpha)q^n \psi(\alpha n)x^{\alpha(n-1)}\\ &= q\sum_{n=0}^{N-1} a_n(\psi;\alpha)q^n x^{\alpha n}.\end{aligned}$$

The series being analytic on the right-half plane, then the right-hand side of the previous line convergences as $N \to \infty$. Hence, we get by monotone convergence (the series has only positive terms) that

$$\mathbf{L}\mathcal{I}_{\alpha,\psi}\big(qx^{\alpha}\big) = q\mathcal{I}_{\alpha,\psi}\big(qx^{\alpha}\big), \quad x \geq 0.$$

Moreover, recalling that for $\mathbb{E}[\xi_1] < 0$, $\theta < \alpha$ we derive, in this case, that

$$\begin{aligned}\lim_{x\to 0} x^{-\theta+1}\frac{\partial}{\partial x}\mathcal{I}_{\alpha,\psi}\big(x^{\alpha}\big) &= \lim_{x\to 0}\sum_{n=0}^{\infty}\alpha(n+1)a_{n+1}(\psi,\alpha)x^{\alpha(n+1)-\theta}\\ &= 0.\end{aligned}$$

Hence, for $\mathbb{E}[\xi_1] < 0$, $\mathcal{I}_{\alpha,\psi}$ satisfy the condition (1.9). Thus, it is an eigenfunction for the recurrent extension $(X, \mathbb{Q})$. Next, applying Dynkin's formula [12] with the bounded stopping time $t \wedge \kappa_a$, we get, for any $t \geq 0$,

$$\mathrm{E}_x\big[\mathrm{e}^{-q(t\wedge\kappa_a)}\mathcal{I}_{\alpha,\psi}\big(qX^{\alpha}_{t\wedge\kappa_a}\big)\big] = \mathcal{I}_{\alpha,\psi}\big(qx^{\alpha}\big).$$

Since the mapping $a \mapsto \mathcal{I}_{\alpha,\psi}(qa)$ is increasing on $[0,\infty)$, we obtain, by dominated convergence and by using the fact that the process has no positive jumps, that, for any $0 \leq x \leq a$,

$$\mathrm{E}_x\big[\mathrm{e}^{-q\kappa_a}\big] = \frac{\mathcal{I}_{\alpha,\psi}(qx^{\alpha})}{\mathcal{I}_{\alpha,\psi}(qa^{\alpha})}.$$

The proof of (2.1) is completed. Next, the Esscher transform (1.4) combined with the Doob optional stopping theorem yields

$$\mathbb{E}_x\big[\mathrm{e}^{-\lambda\tau_a - q\Sigma_{\tau_a}}\mathbb{I}_{\{\tau_a<+\infty\}}\big] = \mathrm{e}^{\rho(x-a)}\mathbb{E}^{(\rho)}_x\big[\mathrm{e}^{-q\Sigma_{\tau_a}}\mathbb{I}_{\{\tau_a<+\infty\}}\big].$$

The proof of Theorem 2.1 is completed by invoking the obvious identity $(\kappa_{\mathrm{e}^a}, \mathbb{Q}_{\mathrm{e}^x}) \overset{(\mathrm{d})}{=} (\Sigma_{\tau_a}, \mathbb{P}_x)$ on $\{\kappa_{\mathrm{e}^a} < \kappa_0\}$ and by using (2.1). □

We end this part by providing an interesting absolute continuity relationship between self-similar processes, which will be useful for the sequel. We recall from the discussion above that for $\mathbb{E}[\xi_1] < 0$, the self-similar process associated with $(\xi, \mathbb{P})$ has an a.s. finite lifetime κ_0. Then, by using Lamperti's mapping (1.6) and applying the chain rule, it follows that the Esscher transform (1.4) reads for $\gamma, \delta \geq 0$ and $x > 0$, as follows

$$\mathrm{d}\mathbb{Q}^{(\gamma)}_{x|F_{A_t}} = \left(\frac{X_t}{x}\right)^{\gamma-\delta}\mathrm{e}^{-(\psi(\gamma)-\psi(\delta))A_t}\,\mathrm{d}\mathbb{Q}^{(\delta)}_{x|F_{A_t}} \quad \text{on } \{t < \kappa_0\}, \tag{3.2}$$

where

$$A_t = \int_0^t X_u^{-\alpha}\,\mathrm{d}u.$$

Since for every F_{A_t}-stopping time T, $\Sigma(T)$ is an F_t-stopping time, the absolute continuity relationship (3.2) holds for every F_{A_t}-stopping time on $F_{A_{T+}} \cup \{T < \kappa_0\}$. We mention that, for $\mathbb{E}[\xi_1] > 0$, where the condition "on $\{t < \kappa_0\}$" can be omitted, the relationship (3.2) was already established by Carmona et al. [9], Proposition 2.1, under the name of the Girsanov power transformation. Finally, if we set $\gamma = \theta$ and $\delta = 0$ in the case $\mathbb{E}[\xi_1] < 0$, then (3.2) simplifies to the following Doob's h-transform, for $x > 0$,

$$\mathrm{d}\mathbb{Q}^{(\theta)}_{x|F_{A_t}} = \left(\frac{X_t}{x}\right)^{\theta} \mathrm{d}\mathbb{Q}_{x|F_{A_t}} \quad \text{on } \{t < \kappa_0\}. \tag{3.3}$$

3.2. *Proof of Theorem* 2.3

We assume that $\mathbb{E}[\xi_1] < 0$ and $\theta < \alpha$. Then, the identity $(x^\alpha \Sigma_\infty, \mathbb{P}) \stackrel{(\mathrm{d})}{=} (\kappa_0, \mathbb{Q}_x)$ yields

$$\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\right] = \mathbb{E}\left[\mathrm{e}^{-qx^\alpha \Sigma_\infty}\right].$$

Hence, the mapping $x \mapsto \mathrm{E}_x[\mathrm{e}^{-q\kappa_0}]$ is decreasing on $[0, \infty)$, and by dominated convergence we have, see also Vuolle-Apiala [34],

$$\begin{aligned}\lim_{x\to\infty} \mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\right] &= \lim_{x\to\infty} \mathbb{E}\left[\mathrm{e}^{-qx^\alpha \Sigma_\infty}\right] \\ &= 0 \end{aligned} \tag{3.4}$$

and

$$\lim_{x\to 0} \mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\right] = 1.$$

We now compute the Laplace transform of κ_0. From the Doob's h-transform (3.3), we deduce that, for any x, a such that $0 < x \le a$,

$$\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_a}\mathbb{I}_{\{\kappa_a < \kappa_0\}}\right] = \left(\frac{x}{a}\right)^{\theta} \frac{\mathcal{I}_{\alpha,\psi_\theta}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}.$$

Then, the strong Markov property and the absence of positive jumps yield

$$\begin{aligned}\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\mathbb{I}_{\{\kappa_0 < \kappa_a\}}\right] &= \frac{1}{\mathrm{E}_0[\mathrm{e}^{-q\kappa_a}]}\left(\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_a}\right] - \mathrm{E}_x\left[\mathrm{e}^{-q\kappa_a}\mathbb{I}_{\{\kappa_a < \kappa_0\}}\right]\right) \\ &= \mathcal{I}_{\alpha,\psi}\left(qx^\alpha\right) - \mathcal{I}_{\alpha,\psi}\left(qa^\alpha\right)\left(\frac{x}{a}\right)^{\theta} \frac{\mathcal{I}_{\alpha,\psi_\theta}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\right] &= \mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\mathbb{I}_{\{\kappa_0 < \kappa_a\}}\right] + \mathrm{E}_x\left[\mathrm{e}^{-q\kappa_a}\mathbb{I}_{\{\kappa_a < \kappa_0\}}\right]\mathrm{E}_a\left[\mathrm{e}^{-q\kappa_0}\right] \\ &= \mathcal{I}_{\alpha,\psi}\left(qx^\alpha\right) - \mathcal{I}_{\alpha,\psi}\left(qa^\alpha\right)\left(\frac{x}{a}\right)^{\theta} \frac{\mathcal{I}_{\alpha,\psi_\theta}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)} \\ &\quad + \left(\frac{x}{a}\right)^{\theta} \frac{\mathcal{I}_{\alpha,\psi_\theta}(qx^\alpha)}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}\mathrm{E}_a\left[\mathrm{e}^{-q\kappa_0}\right] \\ &= \mathcal{I}_{\alpha,\psi}\left(qx^\alpha\right) - x^\theta \mathcal{I}_{\alpha,\psi_\theta}\left(qx^\alpha\right)\frac{a^{-\theta}}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}\left(\mathcal{I}_{\alpha,\psi}\left(qa^\alpha\right) - \mathrm{E}_a\left[\mathrm{e}^{-q\kappa_0}\right]\right).\end{aligned}$$

To derive an expression of the sought quantity, we first differentiate with respect to a the previous equation, the function involved being smooth, to get the Riccati equation

$$\frac{a^{-\theta}}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}\left(\frac{\partial}{\partial a}\mathrm{E}_a\left[\mathrm{e}^{-q\kappa_0}\right]-\frac{\partial}{\partial a}\mathcal{I}_{\alpha,\psi}\left(qa^\alpha\right)\right)+\left(\mathrm{E}_a\left[\mathrm{e}^{-q\kappa_0}\right]-\mathcal{I}_{\alpha,\psi}\left(qa^\alpha\right)\right)\frac{\partial}{\partial a}\frac{a^{-\theta}}{\mathcal{I}_{\alpha,\psi_\theta}(qa^\alpha)}=0,$$

where we have used the fact that $x^\theta\mathcal{I}_{\alpha,\psi_\theta}(qx^\alpha)>0$ for any $x>0$. By solving this Riccati equation, we observe that the solution has the following form

$$\mathrm{E}_x\left[\mathrm{e}^{-q\kappa_0}\right]=A\mathcal{I}_{\alpha,\psi}\left(qx^\alpha\right)-Cx^\theta\mathcal{I}_{\alpha,\psi_\theta}\left(qx^\alpha\right) \tag{3.5}$$

for some constants A, C. It is immediate that $A=1$ since $\mathrm{E}_0[\mathrm{e}^{-q\kappa_0}]=1$. Then, the self-similarity property yields $C=q^{1/\alpha}c$, for some real constant c. Furthermore, (3.4) ensures the existence of a constant $C_\theta>0$ such that

$$\mathcal{I}_{\alpha,\psi}\left(x^\alpha\right)\sim C_\theta x^\theta\mathcal{I}_{\alpha,\psi_\theta}\left(x^\alpha\right)\quad\text{as } x\to\infty.$$

Moreover, observe from (3.5) that

$$\lim_{x\to 0}\frac{\mathrm{E}_x[1-\mathrm{e}^{-\kappa_0}]}{x^\theta}=C_\theta.$$

Next, we recall the following identities (see [31], (11) and Remark 1, p. 489):

$$\lim_{x\to 0}\frac{\mathrm{E}_x[1-\mathrm{e}^{-\kappa_0}]}{x^\theta}=\frac{\Gamma(1-\theta/\alpha)}{\alpha\psi'(\theta)}\mathbb{E}\left[\Sigma_\infty^{\theta/\alpha-1}\right]$$

and

$$\mathbb{E}\left[\Sigma_\infty^{\theta/\alpha-1}\right]=\mathbb{E}^{(\theta)}\left[\left(\int_0^\infty \mathrm{e}^{-\alpha\xi_s}\,\mathrm{d}s\right)^{\theta/\alpha-1}\right].$$

Since $(\xi,\mathbb{P}^{(\theta)})$ has a positive mean, the proof of the Theorem 2.3 is completed by using Proposition 2.3 in Maulik and Zwart [26] and by choosing $c=C_\theta$. □

3.3. *Proof of Corollary* 2.5

We characterize the entrance laws of $(\widehat{X},\mathbb{Q})$ and $(\widehat{X},\mathbb{Q}^{(\theta)})$. To this end, we state the following easy result.

Lemma 3.1. *Let us assume that* $-\infty<\mathbb{E}[\xi_1]<0$. *Then, the entrance law of* $(\widehat{X},\mathbb{Q})$ *admits a density* $\eta_t(y)$ *with respect to the reference measure* $m(\mathrm{d}y)$, $y>0$. *Moreover, the Laplace transform in time of* $\eta_t(y)$, *denoted by* $n^q(y)$, $q\geq 0$, *is characterized by the identity*

$$n^q(y)=\frac{1}{|\mathbb{E}[\xi_1]|}\mathbb{E}\left[\mathrm{e}^{-qy^\alpha\Sigma_\infty}\right].$$

Proof. The claims follow readily from Bertoin and Yor [5], p. 396. Indeed, they characterize the q-potential of the entrance law of a self-similar process associated via Lamperti's mapping to a Lévy process with a positive and finite mean, as follows. For any measurable function $f:\mathfrak{R}^+\to\mathfrak{R}^+$, we have

$$n^q f=\frac{1}{|\mathbb{E}[\xi_1]|}\int_0^\infty f(y)\mathbb{E}\left[\mathrm{e}^{-qy^\alpha\Sigma_\infty}\right]m(\mathrm{d}y),$$

where we have used the identity $\mathbb{E}[\hat{\xi}_1]=-\mathbb{E}[\xi_1]$. □

The first part of the corollary follows from (2.4). For the second statement, we use [31], Proposition 3, where the q-potential, n^q, of the entrance law is given, for a bounded continuous function f, by

$$n^q f = \frac{1}{\psi'(\theta)C_\theta} \int_0^\infty f(y)\mathbb{E}\big[\mathrm{e}^{-qy^\alpha \Sigma_\infty}\big] y^{\alpha-1-\theta} \,\mathrm{d}y.$$

The proof of the corollary is completed by means of the identity (2.4).

Note the following result regarding the resolvent of the minimal process (X, κ_0).

Lemma 3.2. *For $\mathbb{E}[\xi_1] < 0$, the resolvent, U_0^q, of the minimal process (X, κ_0) admits a density with respect to the reference measure $m(\mathrm{d}y)$, which is jointly continuous and bounded. Thus, the processes (X, κ_0) and $(\widehat{X}, \mathbb{Q})$ are in classical duality.*

Proof. We recall that in the case $\mathbb{E}[\xi_1] < 0$, the process $(\xi, \mathbb{P})$ is necessarily of unbounded variation since we have excluded the case of negative subordinators. Thus, each point of the real line is regular for itself, see [3], VII, Corollary 5. Moreover, since the q-capacity of $\{0\}$, which is $\phi'(q)$, is positive for any $q > 0$ (see [3], Chapter VII.5.2), and the resolvent of $(\xi, \mathbb{P})$ is absolutely continuous, we deduce that points are not polar for $(\xi, \mathbb{P})$, i.e., $\mathbb{P}_x(\tau_y < +\infty) > 0$ for any $x, y \in \Re$. It is not difficult to see that these two properties are left invariant by time change with a continuous additive functional, see Bally and Stoica [2], Proposition 4.1. The assertions follow from [2], Proposition 3.1. □

3.4. *Proof of Theorem* 2.6

The claim (2.5) is contained in the following lemma.

Lemma 3.3. *The process $(\kappa_a)_{a\geq 0}$ is under $\mathbb{Q}_{0^+}$ an α-self-similar additive process, i.e., a process with independent increments which enjoy the scaling property* (1.5). *Hence κ_1 is under $\mathbb{Q}_{0^+}$ a positive self-decomposable random variable.*

Proof. The first assertion follows from the absence of positive jumps, the strong Markov property and the self-similarity of $(X, \mathbb{Q})$. The last statement is a straightforward consequence of the property of the law of additive processes, see Sato [32], Chapter 3.16. □

Moreover, it is well known (see Wolfe [36] and also Jeanblanc et al. [19] for related results) that if the mapping $q \mapsto f(q)$ is the Laplace transform of a positive self-decomposable random variable, then there exists a unique, in distribution, (increasing) Lévy process L such that $\mathbb{E}[\log(1 + L_1)] < +\infty$ and its Laplace exponent, ϕ_{L}, is given by

$$\phi_{\mathrm{L}}(q) = q\frac{\mathrm{d}f(q)/\mathrm{d}q}{f(q)}, \quad q \geq 0.$$

Thus, from Theorem 2.3 and Lemma 3.3, we deduce the statement (2.6), after recalling that the infinite divisibility property is stable under convolution.

Before stating the second Lemma, we assume $\mathbb{E}[\xi_1] \geq 0$ and we introduce some notation. Let $(P_t)_{t\geq 0}$ be the semigroup of the Lévy process ξ. We denote by $(P_t^\Sigma)_{t\geq 0}$ the subordinate semigroup of $(P_t)_{t\geq 0}$ by the continuous decreasing multiplicative functional $(\mathrm{e}^{-q\Sigma_t})_{t\geq 0}$. That is for $f \in B(\Re)$, we have, for any $t \geq 0$,

$$P_t^\Sigma f(x) = \mathbb{E}_x\big[\mathrm{e}^{-q\Sigma_t} f(\xi_t)\big], \quad x \in \Re.$$

Next, by choosing $\lambda = 0$ in (2.3), we deduce that the function $x \mapsto \mathcal{I}_{\alpha,\psi}(q\mathrm{e}^{\alpha x})$ is excessive for the semigroup $(P_t^\Sigma)_{t\geq 0}$. Moreover, it is plain that, for any $x \in \Re$, $0 < \mathcal{I}_{\alpha,\psi}(q\mathrm{e}^{\alpha x}) < \infty$. Thus, one can define a new real-valued (sub)-Markov process with semigroup (resp. law) denoted by $(P_t^{\mathcal{I}})_{t\geq 0}$ (resp. $\mathbb{P}^{\mathcal{I}}$), as Doob's h-transform of $(P_t^\Sigma)_{t\geq 0}$, as follows, for any $f \in B(\Re)$ and $t \geq 0$,

$$P_t^{\mathcal{I}} f(x) = \frac{1}{\mathcal{I}_{\alpha,\psi}(q\mathrm{e}^{\alpha x})} P_t^\Sigma\big(f\mathcal{I}_{\alpha,\psi}\big(q\mathrm{e}^{\alpha\cdot}\big)\big)(x), \quad x \in \Re. \tag{3.6}$$

We are now ready to state the following result which characterizes the random variable associated with the Laplace transform (2.7).

Lemma 3.4. *For any* $0 \le x \le a$, $\lambda \ge 0$ *and recalling that* $\rho = \phi(\lambda)$, *we have*

$$\mathbb{E}_x^{\mathcal{I}}\left[e^{-\lambda \tau_a}\right] = e^{\rho(x-a)} \frac{\mathcal{I}_{\alpha,\psi_\rho}(q e^{\alpha x}) \mathcal{I}_{\alpha,\psi}(q e^{\alpha a})}{\mathcal{I}_{\alpha,\psi_\rho}(q e^{\alpha a}) \mathcal{I}_{\alpha,\psi}(q e^{\alpha x})}. \tag{3.7}$$

Proof. We deduce from (3.6) that the following absolute continuity relationship

$$d\mathbb{P}_{x|F_t}^{\mathcal{I}} = \frac{e^{-q\Sigma_t} \mathcal{I}_{\alpha,\psi}(q e^{\alpha \xi_t})}{\mathcal{I}_{\alpha,\psi}(q e^{\alpha x})} \, d\mathbb{P}_{x|F_t} \tag{3.8}$$

holds for any $t > 0$ and $x \in \mathfrak{R}$. It is plain that this relationship remains valid on $F_{T^+} \cap \{T^+ < \infty\}$ for any F_∞-stopping time T. Recalling that $\mathbb{P}_x[\tau_a < \infty] = 1$ for $\mathbb{E}[\xi_1] \ge 0$, we get that

$$\begin{aligned}
\mathbb{E}_x^{\mathcal{I}}\left[e^{-\lambda \tau_a}\right] &= \frac{\mathcal{I}_{\alpha,\psi}(q e^{\alpha a})}{\mathcal{I}_{\alpha,\psi}(q e^{\alpha x})} \mathbb{E}_x\left[e^{-\lambda \tau_a - q\Sigma_{\tau_a}}\right] \\
&= e^{\rho(x-a)} \frac{\mathcal{I}_{\alpha,\psi_\rho}(q e^{\alpha x}) \mathcal{I}_{\alpha,\psi}(q e^{\alpha a})}{\mathcal{I}_{\alpha,\psi_\rho}(q e^{\alpha a}) \mathcal{I}_{\alpha,\psi}(q e^{\alpha x})},
\end{aligned}$$

where the last line follows from (2.3). □

Finally, it is plain from the absolute continuity (3.8) that the process under $\mathbb{P}^{\mathcal{I}}$ is also spectrally negative. Then, from the strong Markov and the absence of positive jumps, we have for any $x < c < a$,

$$\left(\tau_a, \mathbb{P}_x^{\mathcal{I}}\right) \stackrel{(d)}{=} \left(\tau_c, \mathbb{P}_x^{\mathcal{I}}\right) + \left(\tau_a, \mathbb{P}_c^{\mathcal{I}}\right),$$

where the random variables on the right-hand side are independent. Hence, $(\tau_a, \mathbb{P}_x^{\mathcal{I}})$ is infinitely divisible. The proof of Theorem 2.6 is then completed. □

4. Some illustrative examples

We end by investigating some well-known and new examples in more detail.

4.1. *The modified Bessel functions*

We consider ξ to be a Brownian motion with drift $\gamma \in \mathfrak{R}$, i.e., $\psi(u) = \frac{1}{2}u^2 + \gamma u$ and we set $\alpha = 2$. In the case $\gamma < 0$, we have $\theta = 2\gamma$ and therefore we assume $\gamma > -1$. Its associated self-similar process is well known to be a Bessel process of index γ. In the sequel, we simply indicate the connections between the power series $\mathcal{I}_{2,\psi}$ and the modified Bessel functions since the results of this paper are well known and can be found, for instance, in Hartman [16], Pitman and Yor [30] and Yor [38]. We have

$$\begin{aligned}
a_n(\psi, \gamma; 2)^{-1} &= 2^n n! \prod_{k=1}^{n} (k + \gamma) \\
&= 2^n n! \frac{\Gamma(n + \gamma + 1)}{\Gamma(\gamma + 1)}, \qquad a_0 = 1.
\end{aligned}$$

Thus, we get

$$\mathcal{I}_{2,\psi}(x) = (x/2)^{-\gamma/2} \Gamma(\gamma + 1) \mathrm{I}_\gamma\left(\sqrt{2x}\right),$$

where

$$\mathrm{I}_\gamma(x)=\sum_{n=0}^{\infty}\frac{(x/2)^{\gamma+2n}}{n!\Gamma(\gamma+n+1)}$$

stands for the modified Bessel function of index γ, see e.g., [24], Chapter 5. The asymptotic behavior of this function is well known to

$$\mathrm{I}_\gamma(x)\sim\frac{\mathrm{e}^x}{\sqrt{2\pi x}}\quad\text{as } x\to\infty. \tag{4.1}$$

Thus, we obtain that

$$\begin{aligned}\mathcal{N}_{2,\psi_{2\gamma}}(x)&=(x/2)^{\gamma/2}\Gamma(-\gamma+1)\big(\mathrm{I}_{-\gamma}\big(\sqrt{2x}\big)-\mathrm{I}_\gamma\big(\sqrt{2x}\big)\big)\\&=(x/2)^{\gamma/2}\frac{2}{\Gamma(\gamma)}\mathrm{K}_\gamma\big(\sqrt{2x}\big),\end{aligned}$$

where $2\mathrm{K}_\gamma(x)=\Gamma(1-\gamma)\Gamma(\gamma)(\mathrm{I}_{-\gamma}(x)-\mathrm{I}_\gamma(x))$ is the MacDonald function of index γ.

4.2. *Some generalizations of the Mittag-Leffler function*

In [29], the author introduced a new parametric family of one-sided Lévy processes which are characterized by the following Laplace exponent, for any $1<\varrho<2$, $\beta\geq 0$ and $\gamma>1-\varrho$,

$$\psi(\beta u+\gamma)-\psi(\gamma)=\frac{1}{\varrho}\big((\beta u+\gamma-1)_\varrho-(\gamma-1)_\varrho\big), \tag{4.2}$$

where $(k)_\varrho=\frac{\Gamma(k+\varrho)}{\Gamma(k)}$ stands for the Pochhammer symbol. Its characteristic triplet is $\sigma=0$,

$$\tilde{\nu}(\mathrm{d}y)=\frac{\varrho(\varrho-1)}{\beta\Gamma(2-\varrho)}\frac{\mathrm{e}^{(\varrho+\gamma-1)y/\beta}}{(1-\mathrm{e}^{y/\beta})^{\varrho+1}}\,\mathrm{d}y,\quad y<0$$

and

$$b_\gamma=\beta(\gamma)_\varrho\big(\Psi(\gamma-1+\varrho)-\Psi(\gamma-1)\big),$$

where $\Psi(\lambda)=\frac{\Gamma'(\lambda)}{\Gamma(\lambda)}$ is the digamma function. In particular, if γ_0 denotes the zero of the function $\gamma\to b_\gamma$, then for $\gamma\geq\gamma_0\in(1-\varrho,0)$, $\mathbb{E}[\xi_1]\geq 0$.

4.2.1. *The case* $\gamma=0$

Equation (4.2) reduces to $\psi(u)=\frac{1}{\varrho}(u-1)_\varrho$. Observe that $\theta=1$ and $\psi'(1)=\frac{\Gamma(\varrho)}{\varrho}$. Moreover, setting $\alpha=\varrho$, we get

$$a_n(\varrho,0;\varrho)^{-1}=\frac{\Gamma(\varrho(n+1)-1)}{\Gamma(\varrho-1)},\qquad a_0=1.$$

The series, in this case, can be written as follows

$$\begin{aligned}&\mathcal{I}_{\varrho,\psi_\theta}(x)=\Gamma(\varrho)\mathcal{E}_{\varrho,\varrho}(\varrho x),\\&\mathcal{I}_{\varrho,\psi}(x)=\Gamma(\varrho-1)\mathcal{E}_{\varrho,\varrho-1}(\varrho x),\end{aligned}$$

where

$$\mathcal{E}_{\varrho,\beta}(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\varrho n+\beta)},\quad z\in\mathfrak{C},$$

stands for the Mittag-Leffler function of parameter $\varrho, \beta > 0$. The function $\mathcal{E}_{\varrho,0}(z)$ was defined and studied by Mittag-Leffler [28]. It is a direct generalization of the exponential function. The generalization $\mathcal{E}_{\varrho,\beta}(z)$ was given by Agarwal [1] following the work of Humbert [18]. A detailed account of these functions is available from the monograph of Erdélyi et al. [13]. Next, we recall the following Mellin–Laplace transform of the generalized Mittag-Leffler function, for $\lambda, \beta \geq 0$, we have

$$\int_0^\infty \mathrm{e}^{-(\lambda+1)x} x^{\beta-1} \mathcal{E}_{\varrho,\beta}\big(x^\varrho\big)\,\mathrm{d}x = \frac{(\lambda+1)^{\varrho-\beta}}{(\lambda+1)^\varrho - 1}.$$

We deduce by invoking a Tauberian theorem, see Feller [14], Chapter XIII.5, the following asymptotic behavior

$$\mathcal{E}_{\varrho,\beta}\big(x^\varrho\big) \sim \frac{1}{\varrho}\mathrm{e}^x x^{1-\beta} l\big(x^\varrho\big) \quad \text{as } x \to \infty,$$

with l as a slowly varying function at infinity. Thus, $C_1 = \frac{\varrho}{\varrho-1}$, $\mathbb{E}[\Sigma_\infty^{1/\varrho-1}] = \frac{\varrho\Gamma(\varrho-1)}{\Gamma(1-1/\varrho)}$ and

$$\mathcal{N}_{\varrho,\psi_1}\big(x^\varrho\big) = \mathcal{E}_{\varrho,\varrho-1}\big(x^\varrho\big) - \frac{\varrho x}{\varrho-1}\mathcal{E}_{\varrho,\varrho}\big(x^\varrho\big).$$

Finally, we can state the following properties of the Mittag-Leffler function.

Corollary 4.1. *Let* $1 < \varrho < 2$. *The mappings*

$$q \mapsto \frac{1}{\mathcal{E}_{\varrho,\varrho}(q)} \quad \textit{and} \quad q \mapsto \mathcal{E}_{\varrho,\varrho-1}(q) - \frac{\varrho q^{1/\varrho}}{\varrho-1}\mathcal{E}_{\varrho,\varrho}(q)$$

are the Laplace transforms of positive self-decomposable distributions. The mapping

$$q \mapsto \exp\left(\frac{q\mathcal{E}'_{\varrho,\varrho}(q)}{\mathcal{E}_{\varrho,\varrho}(q)}\right)$$

is the Laplace transform of a positive infinitely divisible distribution.

4.2.2. *The general case*
In this case,

$$a_n(\beta,\gamma;\varrho)^{-1} = \prod_{k=1}^n \big((\beta k+\gamma-1)_\varrho - (\gamma-1)_\varrho\big), \qquad a_0 = 1$$

and write $\mathcal{E}_{\varrho,\beta,\gamma}(x) = \sum_{n=0}^\infty a_n(\beta,\gamma;\varrho)x^n$. We point out that this function is closely related to the power series introduced by Kilbas and Saigo [22] which has coefficients of the following form

$$\tilde{a}_n(\beta,\gamma;\varrho)^{-1} = \prod_{k=1}^n \big(\varrho(\beta k+\gamma)+1\big)_\varrho, \qquad a_0 = 1.$$

From Theorem 2.6, we deduce the following properties.

Corollary 4.2. *Let* $1 < \varrho < 2$, $\beta \geq 0$ *and* $\gamma > 1-\varrho$. *Then, the mapping*

$$q \mapsto \frac{1}{\mathcal{E}_{\varrho,\varrho\beta,\gamma}(q)}$$

is the Laplace transform of a positive self-decomposable distribution. The mapping

$$q \mapsto \exp\left(\frac{q\mathcal{E}'_{\varrho,\varrho\beta,\gamma}(q)}{\mathcal{E}_{\varrho,\varrho\beta,\gamma}(q)}\right)$$

is the Laplace transform of a positive infinitely divisible distribution. Finally, for $0 < a < A$ *and* $\gamma \geq \gamma_0$, *recalling that* $\rho = \phi(\lambda)$, *the mapping*

$$\lambda \mapsto \left(\frac{a}{A}\right)^{\rho} \frac{\mathcal{E}_{\varrho,\varrho\beta,\gamma+\rho}(a)\mathcal{E}_{\varrho,\varrho\beta,\gamma}(A)}{\mathcal{E}_{\varrho,\varrho\beta,\gamma+\rho}(A)\mathcal{E}_{\varrho,\varrho\beta,\gamma}(a)}$$

is the Laplace transform of a positive infinitely divisible distribution.

4.3. *The power series associated to stable processes and a new generalization of the exponential function*

Finally, we consider the Esscher transform of a spectrally negative stable process, i.e., $\psi_\gamma(u) = c_\varrho((u+\gamma)^\varrho - (\gamma)^\varrho)$, $\gamma \geq 0$, $1 < \varrho < 2$ and $c_\varrho > 0$. Its characteristic triplet is $\sigma = 0$,

$$\tilde{\nu}(\mathrm{d}y) = \frac{c_\varrho \varrho(\varrho - 1)}{\Gamma(2-\varrho)} \frac{\mathrm{e}^{\gamma y}}{|y|^{\varrho+1}}\,\mathrm{d}y, \quad y < 0,$$

and $b = c_\varrho \varrho \gamma^{\varrho-1} \geq 0$. The inverse function of ψ is $\phi(u) = (c_\varrho u + \gamma^\varrho)^{1/\varrho} - \gamma$ and

$$a_n(\alpha, \varrho, \gamma)^{-1} = \prod_{k=1}^{n} \big((\alpha k + \gamma)^\varrho - \gamma^\varrho\big), \qquad a_0 = 1.$$

Such formulation motivates us to introduce a generalization of the factorial symbol, which we defined, for $n \in \mathfrak{N}$, $\alpha \in \mathfrak{C}$, $\mathfrak{Re}(\alpha) \geq 0$ and $\gamma \in \mathfrak{C}$, $\mathfrak{Re}(\gamma) \geq 0$, by

$$(\alpha, \gamma)_{\varrho,n} = \prod_{k=1}^{n} \big((k\alpha + \gamma)^\varrho - \gamma^\varrho\big) \quad \text{and} \quad (\alpha, \gamma)_{\varrho,0} = 1.$$

Note the obvious identities

$$\begin{aligned}
&(\alpha, \gamma)_{0,n} = 0, \\
&(\alpha, \gamma)_{1,n} = \alpha^n n!, \\
&(\alpha, 0)_{\varrho,n} = \alpha^n n!(\alpha, 0)_{\varrho-1,n}, \\
&(\alpha, \gamma)_{\varrho,n} = \alpha^{\varrho n} \left(1, \frac{\gamma}{\alpha}\right)_{\varrho,n}.
\end{aligned}$$

Moreover, we have

$$(\alpha, \gamma)_{\varrho,n} = \alpha^{\varrho n} \sum_{k=0}^{n} (-1)^{n-k} \left(\frac{\gamma}{\alpha}\right)^{\varrho(n-k)} \left(1, \frac{\gamma}{\alpha}\right)_{\varrho,k}.$$

We write simply

$$\mathcal{I}_{\alpha,\gamma,\varrho}(c_\varrho x) = \sum_{n=0}^{\infty} \frac{x^n}{(\alpha, \gamma)_{\varrho,n}}.$$

Observe that

$$\lim_{\varrho \downarrow 1} \mathcal{I}_{\alpha,\gamma,\varrho}(c_\varrho x) = \mathrm{e}^{x/\alpha}.$$

Corollary 4.3. *Let* $1 < \varrho < 2$, $\alpha > 0$ *and* $\gamma \geq 0$. *Then*, *the mapping*

$$q \mapsto \frac{1}{\mathcal{I}_{\alpha,\gamma,\varrho}(q)}$$

is the Laplace transform of a positive self-decomposable distribution. *The mapping*

$$q \mapsto \exp\left(\frac{q\mathcal{I}'_{\alpha,\gamma,\varrho}(q)}{\mathcal{I}_{\alpha,\gamma,\varrho}(q)}\right)$$

is the Laplace transform of a positive infinitely divisible distribution. *Finally*, *for* $0 < a < A$ *and writing* $\rho = \phi(\lambda)$, *the mapping*

$$\lambda \mapsto \left(\frac{a}{A}\right)^{\rho} \frac{\mathcal{I}_{\alpha,\rho,\varrho}(a)\mathcal{I}_{\alpha,0,\varrho}(A)}{\mathcal{I}_{\alpha,\rho,\varrho}(A)\mathcal{I}_{\alpha,0,\varrho}(a)}$$

is the Laplace transform of a positive infinitely divisible distribution.

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