

# Pointwise ergodic theorems with rate and application to the CLT for Markov chains

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Dedicated to Yves Derriennic on the occasion of his 60th birthday

**Abstract.** Let  $T$  be Dunford–Schwartz operator on a probability space  $(\Omega, \mu)$ . For  $f \in L^p(\mu)$ ,  $p > 1$ , we obtain growth conditions on  $\|\sum_{k=1}^n T^k f\|_p$  which imply that  $\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f \rightarrow 0$   $\mu$ -a.e. In the particular case that  $p = 2$  and  $T$  is the isometry induced by a probability preserving transformation we get better results than in the general case; these are used to obtain a quenched central limit theorem for additive functionals of stationary ergodic Markov chains, which improves those of Derriennic–Lin and Wu–Woodroffe.

**Résumé.** Soit  $T$  un opérateur de Dunford–Schwartz sur un espace de probabilité  $(\Omega, \mu)$ . Pour  $f \in L^p(\mu)$ ,  $p > 1$ , nous obtenons des théorèmes ergodiques du type  $\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f \rightarrow 0$   $\mu$ -p.s. sous des conditions portant sur la croissance de  $\|\sum_{k=1}^n T^k f\|_p$ . Lorsque  $T$  est induit par une transformation préservant la mesure et que  $p = 2$ , nous obtenons de meilleurs résultats. Ces derniers sont alors utilisés pour obtenir le théorème central limite “quenched” pour les sommes partielles associées aux fonctionnelles de chaînes de Markov stationnaires et ergodiques. Nous améliorons ainsi des résultats antérieurs de Derriennic–Lin et Wu–Woodroffe.

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## 1. Introduction

The motivation for this paper was the search for a *quenched* central limit theorem (CLT) for additive functionals of Markov chains which will include the results of [7] and [27]. We obtain the following:

**Theorem.** Let  $\{X_n\}_{n \geq 0}$  be a stationary ergodic Markov chain with state space  $(\mathbb{S}, \mathcal{S})$ , transition probability  $P$ , invariant initial distribution  $m$ , and corresponding Markov operator  $P$  on  $L^2(\mathbb{S}, m)$ . For  $x \in \mathbb{S}$  denote by  $\mathbb{P}_x$  the probability of the chain starting from  $x$ , defined on the product  $\sigma$ -algebra of  $\Omega := \mathbb{S}^{\mathbb{N}}$ .

Let  $f \in L^2(\mathbb{S}, m)$  with  $\int f \, dm = 0$ . If there exists  $\tau > 1$  such that

$$\sup_{n \geq 3} \frac{(\log n)^{5/2} (\log \log n)^\tau}{\sqrt{n}} \left\| \sum_{k=1}^n P^k f \right\|_2 < \infty \quad (1)$$

then for  $m$ -almost every point  $x \in \mathbb{S}$  the sequence  $\frac{1}{\sqrt{n}} \sum_{k=1}^n f(X_k)$  converges in distribution, in the space  $(\Omega, \mathbb{P}_x)$ , to a (possibly degenerate) Gaussian distribution  $\mathcal{N}(0, \sigma(f)^2)$  (with variance  $\sigma(f)^2$  independent of  $x$ ). Moreover, also the invariance principle holds.

The first general quenched CLT of this type seems to be that of Gordin and Lifshitz in Section IV.8 of [16], which assumed  $f \in (I - P)L^2(m)$ . Our theorem improves that of Derriennic and Lin [7], who assumed that  $\|\sum_{k=1}^n P^k f\| = O(n^\alpha)$  for some  $0 < \alpha < 1/2$ , and that of Wu and Woodroffe [27], proved for  $f$  satisfying  $\sup_{n \geq 2} \frac{(\log n)^\beta}{\sqrt{n}} \|\sum_{k=1}^n P^k f\|_2 < \infty$  for some  $\beta > 5/2$ , under the additional assumption that  $f \in L^p(m)$  for some  $p > 2$ . Both these results imply that of [23] (obtained independently).

During the preparation of the present manuscript, after completing our research, we discovered the preprint of Zhao and Woodroffe [28]; their main theorem implies the quenched CLT when (1) holds with  $\tau > 3/2$  (which also improves [7] and [27]); however, the result of [28] does not imply our result when  $1 < \tau \leq 3/2$  (see Chapter 5).

Our strategy follows that of Derriennic and Lin. We first prove some ergodic theorems with rates, then use them to show that the “error term” in the martingale approximation tends to 0; the CLT for the martingale follows (as shown in [6]) from Brown’s CLT.

Our main pointwise ergodic theorem with rate, used for proving the quenched CLT, may be of independent interest:

**Theorem.** *Let  $T$  be the isometry induced on  $L^2(\mu)$  by an ergodic probability preserving transformation. If  $f \in L^2(\mu)$  satisfies*

$$\sup_{n \geq 3} \frac{(\log n)^{3/2} (\log \log n)^\tau}{\sqrt{n}} \left\| \sum_{k=1}^n T^k f \right\|_2 < \infty$$

for some  $\tau > 1$ , then  $\frac{1}{\sqrt{n}} \sum_{k=1}^n T^k f \rightarrow 0$   $\mu$ -a.e.

The referee pointed out that a different sufficient condition was obtained by Wu [26]. Wu’s condition does not imply ours, and in Chapter 3 we will exhibit an example in which our condition holds while Wu’s does not.

## 2. On rates in the mean ergodic theorem

It is well known that in general there is no speed of convergence in the mean ergodic theorem for a power-bounded operator  $T$  on a reflexive Banach space  $X$ , not even for isometries of  $L^2$  induced by probability preserving transformations; a fixed rate for a given  $T$  implies that the averages converge in operator norm, and then  $(I - T)X$  is closed and we have a rate of  $1/n$  (e.g., see [4]). In general, only coboundaries (the elements of  $(I - T)X$ ) have convergence of the averages to 0 with rate of  $1/n$ .

For  $0 < \alpha < 1$ , convergence to 0 of the averages with rate of  $1/n^\alpha$  was obtained in [5] for  $\alpha$ -fractional coboundaries, which are the elements of  $(I - T)^\alpha X$ , with the operator  $(I - T)^\alpha$  defined there by  $(I - T)^\alpha = I - \sum_{j=1}^\infty a_j^{(\alpha)} T^j$ , where  $a_1^{(\alpha)} = \alpha$  and  $a_j^{(\alpha)} = \frac{1}{j!} \alpha \prod_{k=1}^{j-1} (k - \alpha)$  are the coefficients of the power-series expansion  $(1 - t)^\alpha = 1 - \sum_{j=1}^\infty a_j^{(\alpha)} t^j$  for  $|t| \leq 1$ .

It is shown in Corollary 2.15 and Theorem 2.17 of [5] that  $f \in (I - T)^\alpha X$  implies  $\|\frac{1}{n^{1-\alpha}} \sum_{k=1}^n T^k f\| \rightarrow 0$  (for  $X$  reflexive), but in general the latter convergence implies only  $f \in (I - T)^\gamma X$  for  $\gamma < \alpha$ . It is therefore of interest to find a growth condition on  $\|\sum_{k=1}^n T^k f\|$ , better (faster) than  $O(n^{1-\alpha-\epsilon})$ , which still yields  $f \in (I - T)^\alpha X$ ; we would like also to have in this case a rate in the ergodic theorem for  $h$  which satisfies  $f = (I - T)^\alpha h$ . Note also that since there is no rate in the mean ergodic theorem, knowing only  $f = (I - T)^\alpha h$  will not give any rate for  $h$ ; an exception is when in fact  $f \in (I - T)^{\alpha+\epsilon} X$ , which implies that we can take  $h \in (I - T)^\epsilon X$ .

It was shown in [5], Proposition 2.10, that convergence of  $\sum_{j=0}^\infty b_j^{(\alpha)} T^j f$ , with  $b_j^{(\alpha)} = (j + 1)a_{j+1}^{(1-\alpha)}/(1 - \alpha)$  the coefficients of  $(1 - t)^{-\alpha} = \sum_{j=0}^\infty b_j^{(\alpha)} t^j$  for  $|t| < 1$ , is sufficient (and necessary when  $X$  is reflexive) for  $f$  to be in  $(I - T)^\alpha X$ . We will use the asymptotic behavior ([29], vol. I, p. 77)

$$\left| b_n^{(\alpha)} - \frac{1}{\Gamma(\alpha)n^{1-\alpha}} \right| \leq \frac{C}{n^{2-\alpha}} \tag{2}$$

for some  $C > 0$ , where  $\Gamma$  is Euler’s function. When  $\alpha$  is understood, it will be convenient to denote  $b_n^{(\alpha)}$  simply by  $b_n$ .

In the paper we will make use of regularly varying functions. Following [11], p. 276, we say that a positive function  $L$ , defined on a half line  $[A, \infty)$ ,  $A \geq 0$ , is *slowly varying* (at infinity) if for every  $x > 0$ ,  $L(tx)/L(t) \xrightarrow{t \rightarrow +\infty} 1$ , and we say that a positive function  $\Phi$  is *regularly varying* with exponent  $\rho$  ( $-\infty < \rho < +\infty$ ) if  $\Phi(x) = x^\rho L(x)$  for some slowly varying function  $L$ . The regularly varying functions of particular interest in this paper are  $\Phi(x) = x^\rho |\log(x+c)|^\gamma$ .

**Lemma 2.1.** *Let  $T$  be a power-bounded operator on a Banach space  $X$  and let  $f \in X$ . If there exist  $0 < \alpha < 1$  and a non-decreasing regularly varying function  $\Phi$  with exponent  $\beta > 1$  such that*

$$\sup_{n \geq 2} \frac{\Phi(\log(n+1))}{n^{1-\alpha}} \left\| \sum_{k=1}^n T^k f \right\| < +\infty, \quad (3)$$

then  $\sum_{m \geq 0} b_m^{(\alpha)} T^m f$  converges in  $X$ , to an element  $h \in \overline{(I-T)X}$  which satisfies  $f = (I-T)^\alpha h$ , and for every  $n \geq 1$ ,

$$\left\| \sum_{m \geq n} b_m^{(\alpha)} T^m f \right\| \leq \frac{C}{\tilde{\Phi}(\log(n+1))},$$

where  $\tilde{\Phi}(x) := (\int_x^\infty \frac{du}{\Phi(u)})^{-1}$ , which is a non-decreasing regularly varying function with exponent  $\beta - 1$ .

**Proof.** For the given  $\alpha$ , we will denote  $b_m^{(\alpha)}$  by  $b_m$ . For  $n \geq 1$ , write  $S_n = \sum_{m=1}^n T^m f$ . For  $k > j > 1$  we have, by Abel's summation,

$$\sum_{n=j}^k \frac{1}{n^{1-\alpha}} T^n f = \sum_{n=j}^{k-1} S_n \left( \frac{1}{n^{1-\alpha}} - \frac{1}{(n+1)^{1-\alpha}} \right) + \frac{1}{k^{1-\alpha}} S_k - \frac{1}{j^{1-\alpha}} S_{j-1}.$$

By (3) and (2), there exists  $C_1$  such that

$$\begin{aligned} \left\| \sum_{n=j}^k b_n T^n f \right\| &\leq C_1 \left( \sum_{n=j}^k \frac{1}{n^{2-\alpha}} + \sum_{n=j}^k \frac{1}{n^{2-\alpha}} \frac{n^{1-\alpha}}{\Phi(\log(n+1))} + \frac{1}{\Phi(\log(j))} + \frac{1}{\Phi(\log(k+1))} \right) \\ &\leq C_2 \left( \frac{1}{j^{1-\alpha}} + \frac{1}{\Phi(\log(j))} + \int_{j-1}^{k-1} \frac{dx}{x \Phi(\log x)} \right) \\ &= C_2 \left( \frac{1}{j^{1-\alpha}} + \frac{1}{\Phi(\log(j))} + \int_{\log(j-1)}^{\log(k-1)} \frac{du}{\Phi(u)} \right). \end{aligned} \quad (4)$$

Since  $1/\Phi$  is regularly varying with exponent  $-\beta < -1$ , the lemma on p. 280 of [11] yields that  $\Phi^*(x) := \int_x^\infty \frac{dt}{\Phi(t)}$  defines a finite-valued regularly varying function of exponent  $-\beta + 1$ , so  $\tilde{\Phi}$  as defined in the theorem is regularly varying with exponent  $\beta - 1$ . The convergence of the integral defining  $\Phi^*$  show that the right-hand side of (4) converges to 0 as  $k > j \rightarrow \infty$ , so the series  $\sum_{n \geq 0} b_n T^n f$  converges.

Letting  $k \rightarrow \infty$  in (4) gives an estimate for the tail of the series. By Lemma 2 in [11], p. 277,  $1/j^{1-\alpha} \leq C_3/\Phi(\log j)$  for large  $j$ ; by Theorem 1(a) in [11], p. 282, comparing  $1/\Phi$  and  $\Phi^*$  yields that the middle term in the estimate is bounded by a constant multiple of the last term. Hence the asserted estimate for the tail holds.  $\square$

**Remark.** For  $\Phi(x) = x^\beta$  we have  $\tilde{\Phi}(x) = (\beta - 1)x^{\beta-1}$ ; in this case the proof of the lemma is direct.

It is a natural question whether we must have  $\beta > 1$  (say in the case where  $\Phi(x) = x^\beta$ ). In the proof of Proposition 5.3(a) (see also (c)), we give a normal contraction on  $L^2$  and a function  $f$  for which the result holds for  $\alpha = 1/2$ , with  $\beta > 1/2$ . This motivates an improvement of the previous lemma when the operator is a normal contraction  $V$  in a complex Hilbert space  $\mathcal{H}$ , which is given below.

Let  $f \in \mathcal{H}$ . By the spectral theorem (e.g., [9,10,24]), there exists a unique positive measure  $\sigma_f$  on the Borel sets of the unit disk  $D$ , called the *spectral measure* of  $f$ , such that  $\langle V^n f, f \rangle = \int_D z^n \sigma_f(dz)$  for every  $n \geq 0$ , and

$$\left\| \sum_{k=1}^n V^k f \right\|^2 = \int_D \left| \sum_{k=1}^n z^k \right|^2 \sigma_f(dz) = \int_D |z|^2 \left| \frac{1-z^n}{1-z} \right|^2 \sigma_f(dz). \tag{5}$$

We will use the representation  $z = re^{2i\pi\theta}$  with  $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$ . For  $|z| \leq 1$ , we clearly have  $|\sum_{k=1}^n z^k| \leq \sum_{k=1}^n |z|^k \leq \min(n, 1/(1-r))$ .

Convexity of  $x \mapsto \sin(\pi x/2)$  on  $[0, 1]$  yields  $x \leq \sin(\pi x/2)$  for  $0 \leq x \leq 1$ , so  $|\sin(\pi\theta)| \geq 2|\theta|$  for  $|\theta| \leq \frac{1}{2}$ . Hence, for  $1/2 \leq r \leq 1$  we have

$$|1-z|^2 = 1 - 2r \cos(2\pi\theta) + r^2 = (1-r)^2 + 4r \sin^2(\pi\theta) \geq 8\theta^2.$$

For  $|z| \leq \frac{1}{2}$  we have  $|\sum_{k=1}^n z^k| < 1 < \frac{1}{|\theta|}$ , so we finally obtain

$$\left| \sum_{k=1}^n z^k \right| = |z| \left| \frac{1-z^n}{1-z} \right| \leq \min \left\{ n, \frac{1}{1-r}, \frac{1}{|\theta|} \right\} \quad (|z| \leq 1). \tag{6}$$

As in [1] and [17], we want to relate the rate  $\frac{1}{n} \|\sum_{k=1}^n V^k f\| = O(\frac{1}{n^\alpha \Phi(\log(n+1))})$ , to the concentration of  $\sigma_f$  at 1. We need some notation. For every  $n \geq 1$  define

$$D_n := \left\{ z = re^{2i\pi\theta} : 1 - \frac{1}{n} \leq r \leq 1, -\frac{1}{2n} \leq \theta \leq \frac{1}{2n} \right\}.$$

Notice that  $D_1$  is the unit disk  $D$ , and  $\forall m, n \geq 1$ , (6) yields

$$\left| \sum_{k=1}^n z^k \right| \leq \min(n, m) \quad \forall z \in D_1 - D_m. \tag{7}$$

We can now present the spectral characterization of condition (3).

**Theorem 2.2.** *Let  $V$  be a normal contraction on a complex Hilbert space  $\mathcal{H}$ , let  $\Phi$  be a monotone regularly varying function and  $0 \leq \alpha < 1$ . The following are equivalent for  $f \in \mathcal{H}$ .*

(i) *There exists  $C_1 > 0$ , such that,*

$$\frac{1}{n} \left\| \sum_{k=1}^n V^k f \right\| \leq \frac{C_1}{n^\alpha \Phi(\log(n+1))} \quad \forall n \geq 1, \tag{8}$$

(ii) *There exists  $C_2 > 0$ , such that,*

$$\sigma_f(D_n) \leq \frac{C_2}{n^{2\alpha} (\Phi(\log(n+1)))^2} \quad \forall n \geq 1. \tag{9}$$

**Proof.** (i)  $\Rightarrow$  (ii). Assume (8) holds. Let  $n \geq 2$ . Since  $(1 - \frac{1}{n})^{n-1}$  decreases to  $1/e$ , for  $1 - 1/n \leq r \leq 1$  we have

$$r^n \geq r \left( 1 - \frac{1}{n} \right)^{n-1} > r/3,$$

$$1 - r^n = (1-r) \sum_{k=0}^{n-1} r^k \geq (1-r)nr^{n-1} \geq n(1-r)/3.$$

On the other hand,  $|\sin(\pi n\theta)| \geq 2n|\theta| \geq \frac{2n}{\pi} |\sin(\pi\theta)|$  for  $|\theta| \leq \frac{1}{2n}$ .

For  $z = re^{2i\pi\theta} \in D_n$ ,  $n \geq 2$ , since  $r \geq \frac{1}{2}$ , we thus obtain

$$\begin{aligned} \left| \sum_{k=1}^n z^k \right|^2 &= |z|^2 \left| \frac{1-z^n}{1-z} \right|^2 = r^2 \frac{1-2r^n \cos(2\pi n\theta) + r^{2n}}{1-2r \cos(2\pi\theta) + r^2} \\ &\geq \frac{1}{4} \frac{(1-r^n)^2 + 4r^n \sin^2(\pi n\theta)}{(1-r)^2 + 4r \sin^2(\pi\theta)} \geq \frac{n^2}{36}, \end{aligned}$$

which yields, by (5),

$$\sigma_f(D_n) \leq \frac{36}{n^2} \int_{D_n} \left| \sum_{k=1}^n z^k \right|^2 \sigma_f(dz) \leq \frac{36}{n^2} \left\| \sum_{k=1}^n V^k f \right\|^2, \quad (10)$$

which proves (9), by (8).

(ii)  $\Rightarrow$  (i). Assume (9) holds. Since  $D = D_1$ , using (7) we obtain

$$\begin{aligned} \left\| \sum_{k=1}^n V^k f \right\|^2 &= \int_D \left| \sum_{k=1}^n z^k \right|^2 \sigma_f(dz) \\ &= \int_{D_n} \left| \sum_{k=1}^n z^k \right|^2 \sigma_f(dz) + \sum_{j=1}^{n-1} \int_{D_j - D_{j+1}} \left| \sum_{k=1}^n z^k \right|^2 \sigma_f(dz) \\ &\leq n^2 \sigma_f(D_n) + \sum_{j=1}^{n-1} (j+1)^2 (\sigma_f(D_j) - \sigma_f(D_{j+1})) \\ &\leq n^2 \sigma_f(D_n) + \sum_{j=2}^{n-1} \sigma_f(D_j) ((j+1)^2 - j^2) - n^2 \sigma_f(D_n) + 4\sigma_f(D_1) \\ &\leq C_2 \sum_{j=1}^{n-1} \frac{2j+1}{j^{2\alpha} (\Phi(\log(j+1)))^2} + \|f\|^2 \\ &\leq C_2 \left( \frac{3}{(\Phi(\log 2))^2} + \int_1^{n-1} \frac{3x}{x^{2\alpha} (\Phi(\log(x+1)))^2} dx \right) + \|f\|. \end{aligned} \quad (11)$$

Since  $1/\Phi^2$  is also a monotone regularly varying function,  $x \mapsto 1/(\Phi(\log(x+1)))^2$  is slowly varying. Since  $1-2\alpha > -1$ , it follows from [11], Theorem 1(b), p. 281 (with  $p = 1-2\alpha$  and  $\gamma = 0$ ), that the last integral is  $O(\frac{n^2}{n^{2\alpha} (\Phi(\log(n+1)))^2})$ . This concludes the proof of the theorem.  $\square$

**Lemma 2.3.** *Let  $T$  be an isometry or a normal contraction of a complex Hilbert space  $\mathcal{H}$  and let  $f \in \mathcal{H}$ . If there exist  $0 < \alpha < 1$  and a non-decreasing regularly varying function  $\Phi$  with exponent  $\beta > 1/2$  such that*

$$K := \sup_{n \geq 1} \frac{\Phi(\log(n+1))}{n^{1-\alpha}} \left\| \sum_{k=1}^n T^k f \right\| < +\infty, \quad (12)$$

then  $\sum_{m \geq 0} b_m^{(\alpha)} T^m f$  converges in  $\mathcal{H}$  to an element  $h \in \overline{(I-T)\mathcal{H}}$ , which satisfies  $f = (I-T)^\alpha h$ , and for every  $n \geq 1$

$$\left\| \sum_{m \geq n} b_m^{(\alpha)} T^m f \right\| \leq \frac{C}{\Phi(\log(n+1))}, \quad (13)$$

where  $\bar{\Phi} := (\int_x^\infty \frac{du}{\Phi(u)^2})^{-1/2}$ , which is a non-decreasing regularly varying function with exponent  $\beta - 1/2$ .

**Proof.** Assume first that  $T$  is a normal contraction. Let  $\sigma_f$  be the spectral measure of  $f$  as above. Assumption (12) implies  $\|\frac{1}{n} \sum_{k=1}^n T^k f\| \rightarrow 0$ , so  $f \in \overline{(I - T)\mathcal{H}}$  and  $\sigma_f\{1\} = 0$ . By (12) and the previous theorem there exists  $C > 0$  such that

$$\sigma_f(D_n) \leq \frac{C}{n^{2\alpha}(\Phi(\log(n+1)))^2} \quad \forall n \geq 1. \tag{14}$$

By Proposition 2.10 of [5], if we prove that the series  $\sum_{m \geq 0} b_m T^m f$  converges, then  $f \in (I - T)^\alpha \mathcal{H}$ . For  $k > j > 1$  we have

$$\left\| \sum_{n=j}^k b_n T^n f \right\|^2 = \int_{D_1} \left| \sum_{n=j}^k b_n z^n \right|^2 \sigma_f(dz).$$

Define the argument function on  $\mathbb{C} - \{0\}$  by  $\arg z = \theta$  for  $z = re^{2i\pi\theta}$ ,  $r > 0$ ,  $-1/2 < \theta \leq 1/2$ . It follows from (2) and [29], vol. I, p. 191, that there exists  $C_\alpha > 1$ , such that for every  $n \geq 1$  and  $|z| = 1$  with  $\theta = \arg z \neq 0$  we have

$$\left| \sum_{k=1}^n b_k e^{2i\pi k\theta} \right| \leq C_\alpha |\arg z|^{-\alpha}.$$

On the other hand, for every  $0 \leq |z| < 1$ ,  $|\sum_{m=0}^n b_m z^m| \leq \sum_{n \geq 0} b_n |z|^n = (1 - |z|)^{-\alpha}$ . Hence

$$\left| \sum_{n=j}^k b_n z^n \right| \leq 2C_\alpha \min\{|\arg z|^{-\alpha}, (1 - |z|)^{-\alpha}\} \quad \forall k > j \geq 1, 0 \leq |z| \leq 1. \tag{15}$$

Since  $b_n$  is decreasing (e.g., [5], Lemma 2.5), Abel summation and (6) yield

$$\left| \sum_{n=j}^k b_n z^n \right| \leq C b_j \min\left\{ \frac{1}{|\arg z|}, \frac{1}{1 - |z|} \right\}, \quad 0 < |z| \leq 1. \tag{16}$$

Using (16) and (15), and then (2) and the definition of  $D_r$ , we obtain

$$\begin{aligned} \int_{D_1} \left| \sum_{n=j}^k b_n z^n \right|^2 \sigma_f(dz) &= \sum_{r \geq 1} \int_{D_r - D_{r+1}} \left| \sum_{n=j}^k b_n z^n \right|^2 \sigma_f(dz) \\ &\leq C \sum_{r=1}^j b_j^2 \int_{D_r - D_{r+1}} \min\left\{ \frac{1}{|\arg z|}, \frac{1}{1 - |z|} \right\}^2 \sigma_f(dz) \\ &\quad + C \sum_{r \geq j+1} \int_{D_r - D_{r+1}} \min\left\{ \frac{1}{|\arg z|}, \frac{1}{1 - |z|} \right\}^{2\alpha} \sigma_f(dz) \\ &\leq C j^{2\alpha-2} \sum_{r=1}^j (r+1)^2 (\sigma_f(D_r) - \sigma_f(D_{r+1})) \\ &\quad + C \sum_{r \geq j+1} (r+1)^{2\alpha} (\sigma_f(D_r) - \sigma_f(D_{r+1})). \end{aligned}$$

The first term is  $O(1/(\Phi(\log n))^2)$ , by the estimate for the sum when it appeared in (11). For the series, using (14) and Abel summation by part (one can see that the “residual term” at infinity is 0 by (14)), we deduce

$$\begin{aligned} \sum_{r \geq j+1} (r+1)^{2\alpha} (\sigma_f(D_r) - \sigma_f(D_{r+1})) &\leq C_1 \sum_{r \geq j+1} \frac{1}{r(\Phi(\log r))^2} \\ &\leq C_1 \int_j^\infty \frac{dx}{x(\Phi(\log x))^2} = C_1 \int_{\log j}^\infty \frac{du}{(\Phi(u))^2}, \end{aligned}$$

where the last integral is convergent by the lemma in [11], p. 280, VIII.9, since  $2\beta > 1$ .

Hence  $\|\sum_{n=j}^k b_n T^n f\| \leq C_2 (\int_{\log j}^\infty \frac{du}{(\Phi(u))^2})^{1/2} \rightarrow 0$ , so by Cauchy’s criterion  $\sum_{m \geq 0} b_m T^m f$  converges in  $\mathcal{H}$ . The estimation for the tail of the series follows from the last inequality, since by the lemma in [11], p. 280,  $\int_x^\infty \frac{du}{(\Phi(u))^2}$  is regularly varying with exponent  $-2\beta + 1$ ; hence  $\bar{\Phi}$  is regularly varying of exponent  $\beta - 1/2$ .

In case  $T$  is an isometry, we note that by the unitary dilation theorem [24], there exist a larger Hilbert space  $\mathcal{H}_1$  containing  $\mathcal{H}$  and a unitary operator on  $\mathcal{H}_1$  such that for  $f \in \mathcal{H}$  we have  $T^k f = EU^k f$  for every  $k \geq 0$ , where  $E$  is the orthogonal projection of  $\mathcal{H}_1$  onto  $\mathcal{H}$ . Since  $T$  is an isometry,  $\|EU^k f\| = \|f\| = \|U^k f\|$  shows that  $U^k f \in \mathcal{H}$ , so in fact  $T^k f = U^k f$ , and the result for the isometry follows from applying the above result to the unitary  $U$ .  $\square$

**Remarks.** 1. For  $\Phi(x) = x^\beta$  we have  $\bar{\Phi}(x) = \sqrt{2\beta - 1}x^{\beta-1/2}$ .

2. It is not possible to improve the condition on  $\beta$  in Lemma 2.3. Indeed, the example in [5], p. 127, has a symmetric contraction  $T$  on  $L^2[0, 1]$  and  $f \in L^2$  such that  $f \notin \sqrt{I - T}L^2$ , and checking the computations in the example we have that (3) holds with  $\Phi(x) = \sqrt{x}$  (so  $\beta = \frac{1}{2}$ ) and  $\alpha = \frac{1}{2}$ . The construction in the proof of Proposition 5.3 may be applied to show the same phenomenon with a Markovian symmetric contraction.

**Theorem 2.4.** Let  $T$  be a power-bounded operator on a Banach space  $X$  and let  $f \in X$ . If there exist  $0 < \alpha < 1$  and a non-decreasing regularly varying function  $\Phi$  with exponent  $\beta > 1$  such that

$$\sup_{n \geq 1} \frac{\Phi(\log(n+1))}{n^{1-\alpha}} \left\| \sum_{k=1}^n T^k f \right\| < +\infty,$$

then  $f \in (I - T)^\alpha X$ , there is a unique element  $h \in \overline{(I - T)X}$  such that  $f = (I - T)^\alpha h$ , and  $h$  satisfies

$$\sup_{n \geq 1} \frac{\tilde{\Phi}(\log(n+1))}{n} \left\| \sum_{k=1}^n T^k h \right\| < +\infty, \tag{17}$$

where  $\tilde{\Phi} := (\int_x^\infty \frac{du}{\Phi(u)})^{-1}$  is a non-decreasing regularly varying function with exponent  $\beta - 1$ .

**Theorem 2.5.** Let  $T$  be an isometry or a normal contraction of a complex Hilbert space  $\mathcal{H}$  and let  $f \in \mathcal{H}$ . If there exist  $0 < \alpha < 1$  and a non-decreasing regularly varying function  $\Phi$  with exponent  $\beta > 1/2$  such that

$$\sup_{n \geq 1} \frac{\Phi(\log(n+1))}{n^{1-\alpha}} \left\| \sum_{k=1}^n T^k f \right\| < +\infty,$$

then  $f \in (I - T)^\alpha \mathcal{H}$ , there is a unique element  $h \in \overline{(I - T)\mathcal{H}}$  such that  $f = (I - T)^\alpha h$ , and  $h$  satisfies

$$\sup_{n \geq 1} \frac{\bar{\Phi}(\log(n+1))}{n} \left\| \sum_{k=1}^n T^k h \right\| < +\infty, \tag{18}$$

where  $\bar{\Phi} := (\int_x^\infty \frac{du}{(\Phi(u))^2})^{-1/2}$  is a non-decreasing regularly varying function with exponent  $\beta - 1/2$ .

**Proof of Theorems 2.4 and 2.5.** We will prove both theorems together. In the proof  $X$  will stand for a Banach space or a Hilbert space. In either case, by Lemma 2.1 or 2.3, the series  $h := \sum_{n \geq 0} b_n T^n f$  converges. It follows from Proposition 2.10 and Theorem 2.11 of [5] that  $f \in (I - T)^\alpha X$ , and that  $h$  is the unique element of  $\overline{(I - T)X}$  satisfying  $f = (I - T)^\alpha h$ . Moreover, by Lemma 2.1 or 2.3,

$$\left\| \sum_{m \geq n} b_m T^m f \right\| \leq \frac{C}{\Phi^*(\log(n + 1))}, \tag{19}$$

where accordingly  $\Phi^*$  is a non-decreasing regularly varying function, either  $\tilde{\Phi}$  with exponent  $\beta - 1$ , or  $\bar{\Phi}$  with exponent  $\beta - 1/2$ .

Let us prove the estimates (17) and (18). For  $n \geq 1$  we have

$$\sum_{k=1}^n T^k h = \sum_{k=1}^n \sum_{m \geq 0} b_m T^{m+k} f = \sum_{k=1}^n \sum_{m=0}^{n-k} b_m T^{m+k} f + \sum_{k=1}^n \sum_{m \geq n+1-k} b_m T^{m+k} f. \tag{20}$$

Let us deal with the last sum. Using that  $T$  is a contraction and (19) we have

$$\begin{aligned} \left\| \sum_{k=1}^n \sum_{m \geq n+1-k} b_m T^{m+k} f \right\| &\leq \sum_{k=1}^n \left\| \sum_{m \geq n+1-k} b_m T^m f \right\| \\ &\leq \sum_{k=1}^n \frac{C}{\Phi^*(\log(n + 2 - k))} = \sum_{k=1}^n \frac{C}{\Phi^*(\log(k + 1))} \\ &\leq \frac{C}{\log 2} + C_2 \int_1^n \frac{dx}{\Phi^*(\log(x + 1))} \leq \frac{C_3 n}{\Phi^*(\log(n + 1))}, \end{aligned}$$

by Theorem 1(b), in [11], p. 281 (with  $p = \gamma = 0$ ), since  $1/(\Phi^* \circ \log)$  is slowly varying. It gives the desired bound for the second sum in (20).

Let us deal with the first sum in (20). Writing  $S_0 := 0$ , and using  $b_n \leq \frac{C}{n^{1-\alpha}}$  from (2), we obtain

$$\begin{aligned} \left\| \sum_{k=1}^n \sum_{m=0}^{n-k} b_m T^{m+k} f \right\| &= \left\| \sum_{k=1}^n \sum_{m=k}^n b_{m-k} T^m f \right\| \\ &= \left\| \sum_{m=1}^n \left( \sum_{k=1}^m b_{m-k} \right) T^m f \right\| = \left\| \sum_{m=1}^n \left( \sum_{k=0}^{m-1} b_k \right) T^m f \right\| \\ &= \left\| \sum_{m=1}^n \left( \sum_{k=0}^{m-1} b_k \right) (S_m - S_{m-1}) \right\| = \left\| \sum_{m=1}^{n-1} (-b_m S_m) + S_n \sum_{k=0}^{n-1} b_k \right\| \\ &\leq C \left[ \sum_{m=1}^n \frac{1}{\Phi(\log(m + 1))} + \frac{n^{1-\alpha}}{\Phi(\log(n + 1))} n^\alpha \right] \leq \frac{\tilde{C} n}{\Phi(\log(n + 1))}. \end{aligned}$$

Using the constructions of the corresponding  $\Phi^*$  and Theorem 1(a) in [11], p. 281, we obtain the  $\Phi(x)/\Phi^*(x) \rightarrow \infty$  as  $x \rightarrow \infty$ , which yields the desired bound also for the first sum of (20).  $\square$



### 3. Pointwise ergodic theorems with rates

Let  $T$  be a Dunford–Schwartz operator on a probability space  $(\Omega, \mu)$ . Theorem 3.2 of [5] shows that for  $p > 1$  with dual index  $q = p/(p - 1)$  and  $f \in (I - T)^{1/q} L^p(\mu)$  we have  $\frac{1}{n^{1/p}(\log n)^{1/q}} \sum_{k=0}^{n-1} T^k f \rightarrow 0$  a.s. We want to have  $\frac{1}{n^{1/p}} \sum_{k=0}^{n-1} T^k f \rightarrow 0$  a.s., so additional hypotheses are needed [5], Proposition 3.8, for example,

$$\sup_{n \geq 1} \frac{1}{n^{1/p-\epsilon}} \left\| \sum_{k=1}^n T^k f \right\|_p < +\infty.$$

In Theorem 3.2 we obtain the desired a.s. convergence under a weaker hypothesis; its proof uses the following proposition.

**Proposition 3.1.** *Fix  $p > 1$  and let  $T$  be a power-bounded operator on  $L^p(\mu)$  of a probability space  $(\Omega, \mu)$ . Let  $h \in L^p(\mu)$  and assume that there exist  $\delta \geq 1/p$  and  $\tau > 1/p$  such that*

$$\sup_{n \geq 1} \frac{(\log(n + 1))^\delta (\log \log(n + 1))^\tau}{n} \left\| \sum_{k=1}^n T^k h \right\|_p < +\infty.$$

Then for every  $\tau' < \tau - 1/p$  we have

$$\frac{(\log(n + 1))^{\delta-1/p} (\log \log(n + 1))^{\tau'}}{n} \sum_{k=1}^n T^k h \xrightarrow[n \rightarrow +\infty]{} 0 \quad a.s.$$

Moreover,  $\sup_{n \geq 1} \frac{(\log(n+1))^{\delta-1/p} (\log \log(n+1))^{\tau'}}{n} \left| \sum_{k=1}^n T^k h \right| \in L^p(\mu)$ .

**Proof.** For any natural number  $n$  define

$$\Psi(n) := \frac{n^{(p-1)/p}}{(\log(n + \kappa))^\delta (\log \log(n + \kappa))^\tau},$$

where  $\kappa$  is large enough, so  $\Psi$  is non-decreasing. Since  $T$  is power-bounded, the hypothesis on  $h$  yields

$$\left\| \sum_{j=k+1}^{k+n} T^j h \right\|_p^p \leq Cn\Psi^p(n).$$

Using the definition  $\Lambda(n) = \sum_{k=0}^{\lceil \log_2 n \rceil} \Psi(\lceil \frac{n}{2^{k+1}} \rceil)$ , with  $\log_2 x$  being the logarithm to base 2 and  $\lceil x \rceil$  the upper integral part of  $x$ , we can compute for our  $\Psi$  that

$$\Lambda(n) = \sum_{k=0}^{\lceil \log_2 n \rceil} \Psi\left(\left\lceil \frac{n}{2^{k+1}} \right\rceil\right) \leq \sum_{k=0}^{\lceil 1/2 \log_2 n \rceil} \Psi\left(\left\lceil \frac{n}{2^{k+1}} \right\rceil\right) + C'n^{(p-1)/2p} \leq \tilde{C}\Psi(n).$$

It follows from [21], Theorem 4, that there exists  $C_1$  such that for  $k \geq 0$  and  $n \geq 1$ ,

$$\begin{aligned} \left\| \max_{1 \leq l \leq n} \left\| \sum_{j=k+1}^{k+l} T^j h \right\|_p \right\|_p^p &\leq C\Lambda^p(n)n \leq C_1n\Psi^p(n) \\ &\leq C_1 \frac{n^p}{(\log(n + 1))^{p\delta} (\log \log(n + 1))^{\tau p}}. \end{aligned}$$

Hence for “binary blocks” we have

$$\left\| \max_{2^m \leq l \leq 2^{m+1}} \left\| \sum_{j=2^m}^l T^j h \right\|_p \right\|^p \leq C_1 \frac{2^{mp}}{(\log(2^m + 1))^{p\delta} (\log \log(2^m + 1))^{\tau p}}.$$

Now let  $0 < \tau' < \tau - \frac{1}{p}$ . Then

$$\begin{aligned} & \sum_{m \geq 1} \left\| \frac{m^{\delta-1/p} (\log m)^{\tau'}}{2^m} \max_{2^m \leq l \leq 2^{m+1}} \left\| \sum_{j=2^m}^l T^j h \right\|_p \right\|^p \\ & \leq C_1 \sum_{m \geq 1} \frac{m^{p\delta-1} (\log m)^{p\tau'}}{2^{mp}} \frac{2^{mp}}{(\log(1 + 2^m))^{p\delta} (\log \log(2^m + 1))^{\tau p}} < +\infty. \end{aligned}$$

The assertions of the theorem now follow easily, since also

$$\begin{aligned} & \sum_{m \geq 1} \left\| \frac{m^{\delta-1/p} (\log m)^{\tau'}}{2^m} \sum_{j=1}^{2^m} T^j h \right\|^p \\ & \leq C_1 \sum_{m \geq 1} \frac{m^{p\delta-1} (\log m)^{p\tau'}}{2^{mp}} \frac{2^{mp}}{(\log(1 + 2^m))^{p\delta} (\log \log(2^m + 1))^{\tau p}} < +\infty. \end{aligned} \quad \square$$

**Remark.** Pointwise ergodic theorems with rates as consequence of rates in the mean ergodic theorem were obtained by Gaposhkin [12] for general unitary operators in a complex Hilbert space, by Derriennic–Lin [5] for Dunford–Schwartz operators, and by Weber [25] for power-bounded operators on  $L^p$ . A theorem of this type for the isometries induced in  $L^2$  by probability preserving transformations is in fact proved in [28]. Related results are in Assani–Lin [1]. For more results and references see [2].

**Theorem 3.2.** Let  $T$  be a Dunford–Schwartz operator on a probability space  $(\Omega, \mu)$ . Let  $f \in L^p(\mu)$ ,  $p > 1$ . Assume that there exists  $\tau > 1$ , such that

$$\sup_{n \geq 2} \frac{(\log(n + 1))^2 (\log \log(n + 1))^\tau}{n^{1/p}} \left\| \sum_{k=1}^n T^k f \right\|_p < +\infty. \tag{21}$$

Then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f \xrightarrow{n \rightarrow +\infty} 0 \text{ a.s. and } \sum_{k=1}^{\infty} \frac{T^k f}{k^{1/p}} \text{ converges a.s.}$$

Moreover, there exists  $K_f > 0$  such that for every  $\lambda > 0$ ,

$$\lambda^p \mu \left\{ \sup_{n \geq 1} \frac{|\sum_{k=1}^n T^k f|}{n^{1/p}} > \lambda \right\} \leq K_f \text{ and } \lambda^p \mu \left\{ \sup_n \left| \sum_{k=1}^n \frac{T^k f}{k^{1/p}} \right| > \lambda \right\} \leq K_f.$$

**Proof.** Let  $\Phi(x) = x^2(\log x)^\tau$ , which is non decreasing on  $[1, +\infty)$  and regularly varying with exponent 2. Estimating  $\int_x^\infty \frac{du}{u^2(\log u)^\tau} \geq C \frac{1}{x(\log x)^\tau}$ , we obtain from Theorem 2.4, with  $\alpha = 1 - 1/p$  and  $X = L^p(\mu)$ , that there exists  $h \in L^p(\mu)$  with  $f = (I - T)^\alpha h$  such that

$$\sup_{n \geq 2} \frac{\log(n + 1) (\log \log(n + 1))^\tau}{n} \left\| \sum_{k=1}^n T^k h \right\|_p < +\infty. \tag{22}$$

Hence  $f = h - \sum_{n \geq 1} a_n T^n h$  converges in  $L^p(\mu)$ , where  $a_n = a_n^{(\alpha)}$  are the coefficients in the expansion  $(1 - t)^\alpha = 1 - \sum_{n \geq 1} a_n t^n$ . Since  $\sum_{n \geq 1} a_n$  is absolutely convergent, the series  $\sum_{n \geq 1} a_n T^n h$  is  $\mu$ -almost everywhere absolutely convergent.

By [5] (proof of Theorem 3.2), we have  $\sum_{k=0}^{n-1} T^k f = A_n + B_n + C_n$ , where

$$A_n := h + \sum_{j=1}^{n-1} \left( \sum_{k \geq j+1} a_k \right) T^j h,$$

$$B_n := - \sum_{j=1}^n T^{n-1} \left( \sum_{k=j}^n a_k \right) T^j h,$$

$$C_n := - \sum_{j \geq n+1} a_j T^j \left( \sum_{k=0}^{n-1} T^k h \right).$$

It is proved in steps I and II of the proof of Theorem 3.2 in [5], p. 111, that  $A_n/n^{1/p} \rightarrow 0$  and  $C_n/n^{1/p} \rightarrow 0$  a.s. under the only condition  $f = (I - T)^\alpha h$ , since  $\alpha = 1 - \frac{1}{p}$ . Moreover, it is proved in [5], p. 113, that  $\sup_{n \geq 1} |\frac{C_n}{n^{1/p}}| \in L^p(\mu)$ , for every  $h \in L^p(\mu)$ . The corresponding statement for  $A_n/n^{1/p}$  follows by Abel summation and the classical inequality  $\sup_{n \geq 1} |\frac{1}{n} \sum_{k=1}^n T^k h| \in L^p(\mu)$  for every  $h \in L^p(\mu)$ .

Let us prove that  $B_n/n^{1/p} \rightarrow 0$  a.s. and that  $\lambda^p \mu\{\sup_{n \geq 1} |\frac{\sum_{k=1}^n T^k f|}{n^{1/p}} > \lambda\} \leq K_f$  for every  $\lambda > 0$ .

By [5], Lemma 2.5,  $\alpha \sum_{k \geq j} a_k = j a_j$ , so we have (see (3.6) in [5])

$$B_n = -T^{n-1} \sum_{j=1}^n \frac{1}{\alpha} j a_j T^j h + \frac{1}{\alpha} (n+1) a_{n+1} \sum_{k=n}^{2n-1} T^k h. \tag{23}$$

For the last term in (23), we use  $\frac{1}{\alpha} (n+1) a_{n+1} = c_n n^{-\alpha}$  with  $c_n$  bounded to obtain

$$\frac{1}{n^{1/p}} \left( \frac{1}{\alpha} (n+1) a_{n+1} \sum_{k=n}^{2n-1} T^k h \right) = \frac{c_n}{n} \sum_{k=n}^{2n-1} T^k h \rightarrow 0 \quad \text{a.s.}$$

By the classical maximal inequality, we also have a maximal inequality for this term.

It remains to prove that

$$\frac{1}{n^{1/p}} T^{n-1} \sum_{j=1}^n j a_j T^j h \rightarrow 0 \quad \text{a.s.}$$

Let  $q := \frac{p}{p-1}$  be the dual index, and let  $\frac{1}{q} < \tau' < \tau - \frac{1}{p}$  (here we use  $\tau > 1$ ). Applying Proposition 3.1 with  $\delta = 1 = \frac{1}{p} + \frac{1}{q}$  to (22) we obtain

$$h^* := \sup_{n \geq 2} \frac{(\log(n+1))^{1/q} (\log \log(n+1))^{\tau'}}{n} \left| \sum_{k=1}^n T^k h \right| \in L^p(\mu).$$

For  $k \geq 1$  put  $S_k := \sum_{j=1}^k T^j h$ , and define  $S_0 := 0$ . Notice that for every  $k \geq 1$ ,

$$\begin{aligned} |T^k S_k| &= |S_{2k} - S_k| \\ &\leq \frac{2k}{(\log(2k+1))^{1/q} (\log \log(2k+1))^{\tau'}} h^* + \frac{k}{(\log(k+1))^{1/q} (\log \log(k+1))^{\tau'}} h^* \\ &\leq \frac{3k}{(\log(k+1))^{1/q} (\log \log(k+1))^{\tau'}} h^*. \end{aligned}$$

Denote by  $\mathbf{T}$  the linear modulus of  $T$ . For  $n \geq 1$  we have

$$\begin{aligned} \left| T^{n-1} \sum_{j=1}^n j a_j T^j h \right| &= \left| T^{n-1} \sum_{j=1}^n j a_j (S_j - S_{j-1}) \right| \\ &\leq \left| \sum_{j=1}^{n-1} T^{n-1-j} T^j S_j (j a_j - (j+1) a_{j+1}) \right| + n a_n |T^{n-1} S_n| \\ &\leq \sum_{j=1}^{n-1} (j a_j - (j+1) a_{j+1}) \frac{3j}{(\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} T^{n-1-j} h^* + n a_n |T^{n-1} S_n|. \end{aligned}$$

We have already proved that  $\frac{n a_n}{n^{1/p}} T^{n-1} S_n \rightarrow 0$  a.s., with a maximal inequality, when dealing with the last term of (23). Since  $j a_j - (j+1) a_{j+1} = \alpha a_j = 0(1/(j+1)^{1+\alpha})$ , the proof will be finished if we show that for every  $g \in L^p(\mu)$ ,

$$\frac{1}{n^{1/p}} \sum_{j=1}^{n-1} \frac{1}{j^\alpha (\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} T^{n-1-j} g \rightarrow 0 \quad \text{a.s.} \tag{24}$$

We proceed as in [5], with the required slight modifications. Let  $g \in L^p(\mu)$ . Using Hölder's inequality (with  $1/p + 1/q = 1$ ) and  $|\mathbf{T}g|^p \leq \mathbf{T}(|g|^p)$  ([19], p. 65), we obtain

$$\begin{aligned} &\frac{1}{n^{1/p}} \left| \sum_{j=1}^{n-1} \frac{1}{j^\alpha (\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} T^{n-1-j} g \right| \\ &\leq \frac{1}{n^{1/p}} \left( \sum_{k=1}^{n-1} \frac{1}{j^{q\alpha} (\log(j+1)) (\log \log(j+1))^{q\tau'}} \right)^{1/q} \left| \sum_{j=1}^{n-1} |\mathbf{T}^{n-1-j} g|^p \right|^{1/p} \\ &\leq \left( \sum_{j=1}^{n-1} \frac{1}{j (\log(j+1)) (\log \log(j+1))^{q\tau'}} \right)^{1/q} \left| \frac{1}{n} \sum_{j=0}^{n-2} \mathbf{T}^j (|g|^p) \right|^{1/p}. \end{aligned} \tag{25}$$

The series is convergent by our choice of  $\tau'$ , and the second term is bounded a.s., by the ergodic theorem for  $|g|^p \in L^1(\mu)$ . Hence, for every  $g \in L^p(\mu)$  we have

$$\sup_{n \geq 1} \frac{1}{n^{1/p}} \left| \sum_{j=1}^{n-1} \frac{1}{j^\alpha (\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} T^{n-1-j} g \right| < +\infty \quad \text{a.s.} \tag{26}$$

By the Banach principle it suffices to prove (24) for  $g$  in a dense subset of  $L^p(\mu)$ . We prove it for every  $g \in L^\infty(\mu)$ :

$$\begin{aligned} &\left| \frac{1}{n^{1/p}} \sum_{j=1}^{n-1} \frac{1}{j^\alpha (\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} T^{n-1-j} g \right| \\ &\leq \frac{1}{n^{1/p}} \sum_{j=1}^{n-1} \frac{1}{j^\alpha (\log(j+1))^{1/q} (\log \log(j+1))^{\tau'}} \|g\|_\infty \\ &\leq C \frac{1}{n^{1/p}} \frac{n^{1-\alpha}}{(\log(n+1))^{1/q} (\log \log(n+1))^{\tau'}} \|g\|_\infty \rightarrow 0. \end{aligned}$$

Hence (24) holds for every  $g \in L^p(\mu)$ .

Since  $|g|^p \in L^1(\mu)$ , the classical inequality  $\lambda \mu\{\sup_{n \geq 1} \frac{|\sum_{k=1}^n T^k |g|^p}{n} > \lambda\} \leq \| |g|^p \|_1 = \|g\|_p^p$  for every  $\lambda > 0$  yields the asserted weak-type maximal inequality by the estimate (25).

Now denote  $R_0 = 0$  and  $R_k := \sum_{j=1}^k T^j f$  for  $k \geq 1$ . Then

$$\sum_{k=1}^n \frac{T^k f}{k^{1/p}} = \sum_{k=1}^n \frac{R_k - R_{k-1}}{k^{1/p}} = \sum_{k=1}^{n-1} \left( \frac{1}{k^{1/p}} - \frac{1}{(k+1)^{1/p}} \right) R_k + \frac{R_n}{n^{1/p}}. \tag{27}$$

The last term tends to 0 as  $n \rightarrow \infty$  as we have seen, and

$$\sum_{k=1}^n \left| \left( \frac{1}{k^{1/p}} - \frac{1}{(k+1)^{1/p}} \right) R_k \right| \leq \sum_{k=1}^n \frac{|R_k|}{p k^{1+1/p}}$$

converges a.s. by Beppo Levi, since by (21) we have

$$\left\| \sum_{k=2}^{\infty} \frac{|R_k|}{k^{(p+1)/p}} \right\|_p \leq \sum_{k=2}^{\infty} \frac{\|R_k\|_p}{k^{(p+1)/p}} \leq \sum_{k=2}^{\infty} \frac{C k^{1/p}}{k^{(p+1)/p} (\log(k+1))^2 (\log \log(k+1))^\tau} < \infty.$$

The above and weak maximal inequality for the last term in (27) yield the a.s. convergence of  $\sum_{n \geq 1} \frac{T^n f}{n^{1/p}}$  and the existence of  $K_f > 0$  such that  $\lambda^p \mu\{\sup_n |\sum_{k=1}^n \frac{T^k f}{k^{1/p}}| > \lambda\} \leq K_f$  for every  $\lambda > 0$ . □

**Remarks.** 1. The series  $\sum_{k=1}^{\infty} |b_k - \frac{1}{\Gamma(\frac{1}{q})k^{1/p}}| |T^k f|$ , with the coefficients  $b_k = b_k^{(1/p)}$ , converges a.s. by (2) and Beppo Levi's theorem, so the theorem yields that  $h = \sum_{k=0}^{\infty} b_k T^k f$  also with a.s. convergence.

2. Wu ([26], Proposition 1(iii)) showed that for  $T$  induced by a probability preserving transformation and  $f \in L_p$  (i.e.,  $\{T^n f\} \subset L_p$  is strictly stationary), the condition

$$\sum_{n \geq 1} \left( \frac{\|\sum_{k=1}^{2^n} T^k f\|_p^p}{2^n} \right)^{1/(p+1)} < \infty \tag{28}$$

is sufficient to obtain  $\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f \xrightarrow[n \rightarrow +\infty]{} 0$  a.s. Actually it can be shown that Wu's condition yields also a.s. convergence of the series  $\sum_{k=1}^{\infty} \frac{T^k f}{k^{1/p}}$ . Hence, for strictly stationary sequences our result follows from Wu's, since (21) implies (28). Thus, the novelty in our theorem is its application to all Dunford–Schwartz operators.

We can improve the theorem, by weakening the assumption (21), when  $T$  is an isometry of  $L^2(\mu)$  induced by a probability preserving transformation.

**Theorem 3.3.** Let  $\vartheta$  be a measure preserving transformation of  $(\Omega, \mathcal{F}, \mu)$ . Let  $f \in L^2(\mu)$ . Assume that there exists  $\tau > 1$ , such that

$$\sup_{n \geq 2} \frac{(\log(n+1))^{3/2} (\log \log(n+1))^\tau}{\sqrt{n}} \left\| \sum_{k=1}^n f \circ \vartheta^k \right\|_2 < +\infty. \tag{29}$$

Then

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n f \circ \vartheta^k \xrightarrow[n \rightarrow +\infty]{} 0 \text{ a.s. and } \sum_{k=1}^{\infty} \frac{f \circ \vartheta^k}{\sqrt{k}} \text{ converges a.s.}$$

Moreover, there exists  $K_f > 0$  such that for every  $\lambda > 0$ ,

$$\lambda^2 \mu \left\{ \sup_{n \geq 1} \frac{|\sum_{k=1}^n f \circ \vartheta^k|}{\sqrt{n}} > \lambda \right\} \leq K_f \text{ and } \lambda^2 \mu \left\{ \sup_{n \geq 1} \left| \sum_{k=1}^n \frac{f \circ \vartheta^k}{\sqrt{k}} \right| > \lambda \right\} \leq K_f.$$

**Proof.** Note that  $\vartheta$  induces an isometry  $T$  of  $L^2(\mu)$ , hence we can use Theorem 2.5 instead of Theorem 2.4 at the beginning of the proof of Theorem 3.2, this time with  $\Phi(x) = x^{3/2}(\log x)^\tau$ , to obtain (22). From that point on the proof is exactly the same, taking  $p = q = 2$  and using (29) instead of (21).  $\square$

**Remarks.** 1. One can see that Theorem 3.3 is valid for any Dunford–Schwartz operator  $T$  that is an isometry of  $L^2(\mu)$ . However, there is an example in [13], p. 258, showing that Theorem 3.3 is not true if one assumes only that  $T$  is unitary.

2. As mentioned above, Wu [26] obtained a different sufficient condition, namely (28), implying the conclusion of Theorem 3.3. When  $\tau > 3/2$ , (29) implies (28). In the next proposition we show an example satisfying condition (29), with  $1 < \tau < 3/2$ , but not Wu’s (28).

Let  $\{\varepsilon_n\}_{n \in \mathbb{Z}} \subset L^2(\Omega, \mu)$  be a sequence of strictly stationary martingale differences (e.g., i.i.d. centered random variables with finite variance). For a sequence  $\{a_n\} \in \ell^2(\mathbb{N})$ , we define the moving average sequence  $X_n := \sum_{k \geq 0} a_k \varepsilon_{n-k}$ , which is strictly stationary, i.e.,  $X_n = X_0 \circ \vartheta^n$ , with  $\vartheta$  the shift associated with  $\{\varepsilon_n\}$ .

**Proposition 3.4.** *There exists a moving average  $\{X_n\}$  such that*

$$\sup_{n \geq 2} \frac{(\log(n+1))^{3/2} (\log \log(n+1))^{5/4}}{\sqrt{n}} \left\| \sum_{k=1}^n X_0 \circ \vartheta^k \right\|_2 < \infty \tag{30}$$

and

$$\sum_{n \geq 1} \left( \frac{\| \sum_{k=1}^{2^n} X_0 \circ \vartheta^k \|_2^2}{2^n} \right)^{1/3} = \infty. \tag{31}$$

In particular, Theorem 3.3 applies while Wu’s condition does not hold.

**Proof.** Let  $a_1 = a_2 = 0$ , for every  $n \geq 3$  put  $a_n = \frac{1}{n(\log n)^{5/2} (\log \log n)^{5/4}}$ , and let  $a_0 = -\sum_{k \geq 1} a_k$ , so  $\sum_{k \geq 0} a_k = 0$ . Fix a stationary martingale difference sequence with unit variance  $\{\varepsilon_n\}_{n \in \mathbb{Z}}$ , and define a moving average as above.

By orthonormality of  $\{\varepsilon_n\}$  we have

$$\begin{aligned} \left\| \sum_{k=1}^n X_k \right\|_2^2 &= \left\| \sum_{k=1}^n \sum_{m \geq 0} a_m \varepsilon_{k-m} \right\|_2^2 = \left\| \sum_{k=1}^n \sum_{m \leq k} a_{k-m} \varepsilon_m \right\|_2^2 \\ &= \left\| \sum_{m=1}^n \left( \sum_{k=m}^n a_{k-m} \right) \varepsilon_m + \sum_{m=-\infty}^0 \left( \sum_{k=1}^n a_{k-m} \right) \varepsilon_m \right\|_2^2 \\ &= \sum_{m=1}^n \left( \sum_{k=0}^{n-m} a_k \right)^2 + \sum_{m \geq 0} \left( \sum_{k=m+1}^{m+n} a_k \right)^2. \end{aligned}$$

Using  $\sum_{k \geq 0} a_k = 0$  we obtain

$$\left\| \sum_{k=1}^n X_k \right\|_2^2 = \sum_{m=0}^{n-1} \left( \sum_{k \geq n-m} a_k \right)^2 + \sum_{m \geq 0} \left( \sum_{k=m+1}^{m+n} a_k \right)^2. \tag{32}$$

We will use the estimate  $\sum_{k \geq j+1} a_k \sim \frac{C}{(\log j)^{3/2} (\log \log j)^{5/4}}$  for  $j \geq 3$ .

For the first sum on the right-hand side of (32) we have, for large  $n$ ,

$$\sum_{m=0}^{n-1} \left( \sum_{k \geq n-m} a_k \right)^2 \sim c + \sum_{m=0}^{n-4} \frac{C}{(\log(n-m-1))^3 (\log \log(n-m-1))^{5/2}} \sim c + \frac{C'n}{(\log n)^3 (\log \log n)^{5/2}}.$$

For the second sum, we obtain (splitting the sum, according to  $m \leq n/\log n$ )

$$\begin{aligned} \sum_{m \geq 0} \left( \sum_{k=m+1}^{m+n} a_k \right)^2 &\leq \sum_{0 \leq m \leq n/\log n} \left( \sum_{k \geq m+1} a_k \right)^2 + \sum_{m > n/\log n} \frac{1}{(\log(m+1))^5 (\log \log(m+1))^{5/2}} \left( \log \frac{n+m}{m} \right)^2 \\ &\leq C \left( \frac{n}{(\log n)^4 (\log \log n)^{5/2}} + \frac{n^2}{(\log(n+1))^5 (\log \log(n+1))^{5/2}} \sum_{m > n/\log n} \frac{1}{m^2} \right) \\ &= o \left( \frac{n}{(\log n)^3 (\log \log n)^{5/2}} \right). \end{aligned}$$

Hence,  $\| \sum_{k=1}^n X_k \|_2 \sim \frac{\sqrt{n}}{(\log n)^{3/2} (\log \log n)^{5/4}}$ . In particular, (30) holds, while (28) does not. □

Theorem 3.3 raises the question whether the assumption on the power of the logarithm in Theorem 3.2 can be improved in general. Indeed, if we know that the function  $f$  is also in some  $L^r$ ,  $r > p$ , we can assume a smaller power of the logarithm in (21), and the proof is also somewhat simpler. The result below is inspired by the arguments of Wu and Woodroffe [27] (when  $p = 2$ ) in the proof of their quenched CLT.

**Theorem 3.5.** *Let  $T$  be a Dunford–Schwartz operator on a probability space  $(\Omega, \mu)$ . Let  $f \in L^p(\mu)$ ,  $p > 1$ . Assume that there exists  $\tau > 1/p$ , such that*

$$\sup_{n \geq 1} \frac{(\log(n+1))^{1+1/p} (\log \log(n+1))^\tau}{n^{1/p}} \left\| \sum_{k=1}^n T^k f \right\|_p < +\infty. \tag{33}$$

If in addition  $f \in L^r(\mu)$  for some  $r > p$ , then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f \xrightarrow{n \rightarrow +\infty} 0 \text{ a.s. and } \sum_{k=1}^\infty \frac{T^k f}{k^{1/p}} \text{ converges a.s.}$$

Moreover,  $\sup_n | \sum_{k=1}^n \frac{T^k f}{k^{1/p}} |$  and  $\sup_{n \geq 1} | \frac{1}{n^{1/p}} \sum_{k=1}^n T^k f |$  are in  $L^p(\mu)$ .

**Proof.** For the claims about  $\frac{1}{n^{1/p}} \sum_{k=1}^n T^k f$  it suffices to prove (the stronger results) that

$$\frac{1}{2^{m/p}} \max_{1 \leq n \leq 2^m} \left| \sum_{k=1}^n T^k f \right| \xrightarrow{m \rightarrow +\infty} 0 \text{ and } \sup_{m \geq 1} \frac{1}{2^{m/p}} \max_{1 \leq n \leq 2^m} \left| \sum_{k=1}^n T^k f \right| \in L^p(\mu).$$

Let  $0 < \gamma < 1/p - 1/r$ . Define  $u_m := \lceil 2^{\gamma m} \rceil + 1$  (block size) and  $v_m := \lceil 2^{(1-\gamma)m} \rceil + 1$  (upper bound on number of blocks). Since  $\gamma < 1$ , we have

$$\max_{1 \leq n \leq 2^m} \left| \sum_{k=1}^n T^k f \right| \leq \max_{1 \leq l \leq v_m} \left| \sum_{k=1}^{lu_m} T^k f \right| + \max_{1 \leq j \leq u_m} \left| \sum_{k=(l-1)u_m+1}^{(l-1)u_m+j} T^k f \right|.$$

Since for every  $1 \leq l \leq v_m$ ,

$$\max_{1 \leq j \leq u_m} \left| \sum_{k=(l-1)u_m+1}^{(l-1)u_m+j} T^k f \right| \leq u_m \max_{(l-1)u_m+1 \leq k \leq lu_m} |T^k f| \leq u_m \max_{1 \leq k \leq u_m v_m} |T^k f|,$$

we obtain

$$\max_{1 \leq n \leq 2^m} \left| \sum_{k=1}^n T^k f \right| \leq \max_{1 \leq l \leq v_m} \left| \sum_{k=1}^{lu_m} T^l f \right| + u_m \max_{1 \leq k \leq u_m v_m} |T^k f|. \tag{34}$$

Let  $\mathbf{T}$  be the linear modulus of  $T$ , which is also a Dunford–Schwartz operator. Since  $(\mathbf{T}^k|f|)^r \leq \mathbf{T}^k(|f|^r)$  ([19], p. 65) and  $u_m v_m < 3 \cdot 2^m$ , we obtain

$$\left(u_m \max_{1 \leq k \leq u_m v_m} |T^k f|\right)^r \leq u_m^r \max_{1 \leq k \leq u_m v_m} \mathbf{T}^k(|f|^r) \leq u_m^r \sum_{k=1}^{3 \cdot 2^m} \mathbf{T}^k(|f|^r).$$

This yields

$$\begin{aligned} \int \sum_{m \geq 1} \left(\frac{u_m}{2^{m/p}} \max_{1 \leq k \leq u_m v_m} |T^k f|\right)^r d\mu &\leq \sum_{m \geq 1} \int \frac{u_m^r}{2^{rm/p}} \sum_{k=1}^{3 \cdot 2^m} \mathbf{T}^k(|f|^r) d\mu \\ &\leq 3 \sum_{m \geq 1} 2^m \frac{u_m^r \|f\|_r^r}{2^{mr/p}} \leq 3 \cdot 2^r \|f\|_r^r \sum_{m \geq 1} 2^{m(1+\gamma r-r/p)}, \end{aligned}$$

which converges by our choice of  $\gamma$ . Hence

$$\sup_{m \geq 1} \frac{u_m}{2^{m/p}} \max_{1 \leq k \leq u_m v_m} |T^k f| \in L^r \subset L^p \quad \text{and} \quad \frac{u_m}{2^{m/p}} \max_{1 \leq k \leq u_m v_m} |T^k f| \xrightarrow{m \rightarrow \infty} 0 \quad \text{a.s.}$$

We now deal with the first term on the right-hand side of (34). Fix  $m$ , and for  $1 \leq k \leq v_m$  define  $R_k = \sum_{i=(k-1)u_m+1}^{ku_m} T^i f$ . For  $0 \leq j < l \leq v_m$ , (33) yields

$$\begin{aligned} \left\| \sum_{k=j+1}^l R_k \right\|_p^p &= \left\| \sum_{i=ju_m+1}^{lu_m} T^i f \right\|_p^p \leq \left\| \sum_{i=1}^{(l-j)u_m} T^i f \right\|_p^p \\ &\leq C \frac{(l-j)u_m}{(\log((l-j)u_m+1))^{p+1} (\log \log((l-j)u_m+1))^{p\tau}} \\ &\leq \frac{C u_m}{(\log(u_m+1))^{p+1} (\log \log(u_m+1))^{p\tau}} (l-j). \end{aligned}$$

Since we have a linear bound, we can use [21], Theorem 3 (see also [2], Proposition 2.3), which yields the maximal inequality

$$\left\| \max_{1 \leq j \leq v_m} \left\| \sum_{k=1}^j R_k \right\|_p \right\|_p^p \leq \frac{C_1 u_m}{(\log(u_m+1))^{p+1} (\log \log(u_m+1))^{p\tau}} (\log v_m)^p v_m.$$

Hence

$$\left\| \frac{1}{2^{m/p}} \max_{1 \leq j \leq v_m} \left\| \sum_{i=1}^{ju_m} T^i f \right\|_p \right\|_p^p = \frac{1}{2^m} \left\| \max_{1 \leq j \leq v_m} \left\| \sum_{k=1}^j R_k \right\|_p \right\|_p^p \leq \frac{C_2}{m(\log m)^{p\tau}},$$

which is the term of a convergent series by our assumption on  $\tau$ .

The assertions concerning the series  $\sum_{k=1}^{\infty} \frac{T^k f}{k^{1/p}}$  are proved as in Theorem 3.2, using (33) instead of (21). □

**Remarks.** 1. In view of the previous theorem, it would be interesting to know whether Theorem 3.3 remains true if we take only  $\tau > \frac{1}{2}$ , without assuming  $f \in L^r$  for some  $r > 2$ .

2. When  $f \in L^r$ ,  $r > p$ , the previous theorem requires a smaller power of the logarithm than in (21), namely  $1 + \frac{1}{p}$  instead of 2. For  $T$  induced by a measure-preserving transformation and  $p \neq 2$ , is the  $L^p$  analogue of Theorem 3.3, with the power  $1 + \frac{1}{p}$  in the logarithm, true (without the additional condition  $f \in L^r$  for some  $r > p$ )?



#### 4. A quenched CLT for Markov chains

We now use our results to obtain a quenched central limit theorem for additive functionals of stationary ergodic Markov chains.

Let  $P(x, A)$  be a transition probability on  $(\mathbb{S}, \mathcal{S})$  with Markov operator  $Pg(x) = \int g(y)P(x, dy)$  defined on bounded measurable functions, and let  $m$  be an invariant probability for  $P$ , assumed ergodic. The invariance of  $m$  and the inequality  $|Pg(x)|^2 \leq P(g^2)(x)$  yield that  $P$  extends to a contraction of  $L^2(m)$ . Let  $\Omega := \mathbb{S}^{\mathbb{N}}$  be the space of trajectories and  $\{X_n\}_{n \geq 0}$  the corresponding Markov chain with transition probability  $P$ . The probability law of the chain is denoted by  $\mathbb{P}_m$  when the initial distribution is  $m$ , and by  $\mathbb{P}_x$  when the chain starts at the point  $x \in \mathbb{S}$ . We denote by  $\vartheta$  the shift on  $\Omega$ , which is measure preserving and ergodic in  $(\Omega, \mathbb{P}_m)$ .

For  $f \in L^2(m)$ , consider  $S_n(f) = \sum_{k=0}^{n-1} f(X_k)$ .

**Theorem 4.1.** *Let  $f \in L^2(m)$ , with  $\int f(x) dm(x) = 0$ . If there exists  $\tau > 1$  such that*

$$\sup_{n \geq 2} \frac{(\log(n+1))^{5/2} (\log \log(n+1))^\tau}{\sqrt{n}} \left\| \sum_{k=0}^{n-1} P^k f \right\|_2 < +\infty, \quad (35)$$

then  $\sigma^2(f) := \lim_n \frac{1}{n} \mathbb{E}_m(S_n(f)^2)$  exists and is finite, and for  $m$ -almost every point  $x \in \mathbb{S}$ , the sequence  $n^{-1/2} S_n(f)$  converges in distribution, under the probability measure  $\mathbb{P}_x$ , to the Gaussian distribution  $\mathcal{N}(0, \sigma^2(f))$  (if  $\sigma^2(f) = 0$ , it is the Dirac measure at 0); furthermore, also the invariance principle holds.

**Proof.** We basically follow the proof of [7] with the corresponding modifications. Define  $\Phi(x) := x^{5/2}(\log x)^\tau$ ,  $x > 1$ . For  $0 < t < 1$  define the Green kernel  $G_t = \sum_{k \geq 0} t^k P^k$ . By Abel summation,  $G_t = (1-t) \sum_{k \geq 0} t^k \sum_{j=0}^k P^j$ , and by assumption

$$\|G_t f\|_2 \leq C(1-t) \sum_{k \geq 0} \frac{\sqrt{k}}{\Phi(\log(k+2))} t^k.$$

By a Tauberian theorem (see Theorem 5 in Section XIII.5 of [11]), with the slowly varying  $L(x) := (\Phi(\log x))^{-1}$ , for every  $\frac{2}{3} \leq t < 1$

$$\|G_t f\|_2 \leq C_1 \frac{\pi}{(1-t)^{1/2} \Phi(|\log(1-t)|)}. \quad (36)$$

Write  $\varphi_t(X_0, X_1) := G_t f(X_1) - P G_t f(X_0)$ . We have

$$\mathbb{E}_m[(\varphi_s - \varphi_t)^2] = \int \{[(I+P)(G_s f - G_t f)][(I-P)(G_s f - G_t f)]\} dm.$$

Since  $(I-P)G_t = I - (1-t)P G_t$  for  $0 \leq t < 1$ , we have

$$(I-P)(G_s f - G_t f) = (1-t)(P G_t f - P G_s f) + (s-t)P G_s f.$$

By the Cauchy–Schwarz inequality and the triangle inequality, we obtain

$$\begin{aligned} \mathbb{E}_m[(\varphi_s - \varphi_t)^2] &\leq \|(I+P)(G_s f - G_t f)\|_2 \cdot \|(I-P)(G_s f - G_t f)\|_2 \\ &\leq 2(\|G_s f\|_2 + \|G_t f\|_2) \times [(1-t)(\|G_s f\|_2 + \|G_t f\|_2) + |s-t|\|G_s f\|_2]. \end{aligned}$$

Hence, using (36) to estimate  $\|G_t f\|_2$  for  $t \geq \frac{1}{2}$ , we have

$$\sup_{s \in [t, (1+t)/2]} \mathbb{E}_m[(\varphi_s - \varphi_t)^2]^{1/2} \leq \frac{C_2}{\Phi(|\log(1-t)|)}.$$

Fix  $\frac{1}{2} \leq t < 1$  and apply the above inequality with  $t_n = (2^n + t - 1)/2^n$  instead of  $t$  to obtain

$$\begin{aligned} \sup_{s \in [t, 1)} \mathbb{E}_m [(\varphi_s - \varphi_t)^2]^{1/2} &\leq \sum_{n \geq 0} \sup_{s \in [t_n, t_{n+1})} \mathbb{E}_m [(\varphi_s - \varphi_{t_n})^2]^{1/2} \\ &\leq C_2 \sum_{n \geq 0} \frac{1}{\Phi(|\log(1 - t_n)|)} = C_2 \sum_{n \geq 0} \frac{1}{\Phi(|\log((1 - t)/2^n)|)} \\ &\leq \frac{C_2}{\Phi(|\log(1 - t)|)} + C_2 \int_0^\infty \frac{dx}{\Phi(|\log(1 - t)| + x \log 2)} \\ &\leq \frac{C_3}{|\log(1 - t)|^{3/2} (\log |\log(1 - t)|)^\tau}. \end{aligned}$$

By Cauchy's criterion, there exists  $M \in L^2(\mathbb{P}_m)$  such that  $\lim_{t \rightarrow 1^-} \|\varphi_t - M\|_2 = 0$ . The sums  $M_n := \sum_{k=0}^{n-1} M \circ \vartheta^k$  define a martingale with stationary increments, since  $M_n(t) := \sum_{k=0}^{n-1} \varphi_t(X_0, X_1) \circ \vartheta^k$  is a martingale and  $\mathbb{P}_m$  is  $\vartheta$ -invariant. By orthogonality and stationarity of the martingale differences,

$$\mathbb{E}_m [(M_n(t) - M_n)^2] = n \mathbb{E}_m [(\varphi_t - M)^2] \leq D \frac{n}{\tilde{\Phi}(|\log(1 - t)|)}, \quad (37)$$

where,  $\tilde{\Phi}(x) := x^{3/2} (\log x)^\tau$ ,  $x > 1$ .

To obtain the CLT we need to estimate the residual term  $W_n := S_n(f) - M_n$ . By construction,  $(I - tP)G_t f = f$  for  $0 \leq t < 1$ . Hence, for every  $0 \leq t < 1$  we have

$$\begin{aligned} W_n &= \sum_{k=0}^{n-1} (G_t f(X_k) - tP G_t f(X_k)) - \sum_{k=0}^{n-1} \varphi_t(X_k, X_{k+1}) + M_n(t) - M_n \\ &= (M_n(t) - M_n) + G_t f(X_0) - G_t f(X_n) + (1 - t) \sum_{k=0}^{n-1} P G_t f(X_k). \end{aligned}$$

Hence, using (37), estimating  $\|G_t f\|_2$  by (36), and then taking  $t = 1 - 1/n$ , we obtain

$$\begin{aligned} (\mathbb{E}_m(W_n^2))^{1/2} &\leq D^{1/2} \frac{n^{1/2}}{\tilde{\Phi}(|\log(1 - t)|)} + 2C \frac{\pi}{(1 - t)^{1/2} \Phi(|\log(1 - t)|)} + (1 - t)nC \frac{\pi}{(1 - t)^{1/2} \Phi(|\log(1 - t)|)} \\ &\leq \frac{\tilde{C}n^{1/2}}{\tilde{\Phi}(\log n)} = \frac{\tilde{C}n^{1/2}}{(\log n)^{3/2} (\log \log n)^\tau}. \end{aligned}$$

Since  $W_n = \sum_{k=0}^{n-1} (f(X_0) - M) \circ \vartheta^k$  and  $\tau > 1$ , we apply Theorem 3.3 to the function  $f(X_0) - M \in L^2(\Omega, \mathbb{P}_m)$  and obtain that  $\frac{W_n}{\sqrt{n}} \rightarrow 0$   $\mathbb{P}_m$ -a.s., so for  $m$ -a.e.  $x$  we have

$$\frac{W_n}{\sqrt{n}} \rightarrow 0 \quad \mathbb{P}_x\text{-a.s.}$$

The end of the proof is now similar to [7], p. 75. □

**Remarks.** 1. Since  $\mathbb{P}_m = \int \mathbb{P}_x dm(x)$ , the quenched CLT, with the variances of the limiting Gaussian equal a.s. to  $\sigma^2$  (independently of  $x$ ), implies the annealed CLT for  $\{f(X_n)\}$ : In the space  $(\Omega, \mathbb{P}_m)$ , the sequence  $n^{-1/2} S_n(f)$  converges in distribution to the Gaussian distribution  $\mathcal{N}(0, \sigma^2)$  (if  $\sigma^2 = 0$ , it is the Dirac measure at 0); furthermore, also the invariance principle holds.

2. Improving the result of [14], Maxwell and Woodroffe [20] proved the annealed CLT, with variance of the limit  $\sigma^2(f) := \lim_n \frac{1}{n} \mathbb{E}_m(S_n(f)^2)$ , under the assumption that

$$\sum_{n=1}^{\infty} n^{-3/2} \left\| \sum_{k=1}^n P^k f \right\|_2 < \infty. \quad (38)$$

The main question is whether this condition is sufficient for the quenched CLT.

3. Since (35) implies (38),  $\sigma^2(f)$  is the variance in  $(\Omega, \mathbb{P}_m)$  of the stationary martingale differences  $M \circ \vartheta^k$ . Corollary 1 of [28] and its proof show that (35) implies also

$$\limsup_{n \rightarrow \infty} \frac{S_n(f)}{\sqrt{2n \log \log n}} = \sigma(f) \quad \mathbb{P}_m\text{-a.s.}$$

## 5. On conditions for the CLT for Markov chains

In this section we compare some of the conditions for the CLT. We use the notations of the previous section:  $P(x, A)$  is a transition probability on  $(\mathbb{S}, \mathcal{S})$  with invariant probability  $m$ , assumed ergodic. The Markov operator  $P$  then extends to a contraction of  $L^2(m)$ . We denote by  $\{X_n\}_{n \geq 0}$  the corresponding Markov chain on the space of trajectories.

For  $f \in L^2(m)$  we define  $S_n(f) = \sum_{k=0}^{n-1} f(X_k)$ . Let us recall:

**Proposition 5.1.** *Let  $P$  be a Markov operator as described. Let  $f \in L^2(m)$ . Assume that one of the following conditions is satisfied*

- (i)  $P$  is normal and  $f \in \sqrt{T - P}L^2(m)$  [15,16].
- (ii)  $\sum_{n \geq 1} n^{-3/2} \left\| \sum_{k=1}^n P^k f \right\|_2 < +\infty$  [20].
- (iii)  $\sum_{n \geq 1} f P^n f$  converges in  $L^1(m)$  [3].

Then  $\{f(X_n)\}$  satisfies the annealed Central Limit Theorem.

**Remarks.** 1. It is known that (for  $P$  normal) (ii) implies (i); in fact, (ii) always implies  $f \in \sqrt{T - P}L^2$  [8]. We will answer the question asked in [8], whether there exists a normal Markov operator  $P$  and  $f \in L^2$  such that (i) is satisfied but not (ii).

2. Condition (ii) was introduced by Maxwell–Woodroffe [20]. For general strictly stationary processes, the condition reads  $\sum_{n \geq 1} n^{-3/2} \mathbb{E}(S_n | X_0) \|_2 < +\infty$ , were  $S_n := \sum_{k=0}^{n-1} f(X_k)$ . It was proved by Peligrad–Utev [22] that it is sufficient for the functional CLT in that case.

3. Condition (iii) is due to Dedecker–Rio [3], and also has an analogous sufficient condition for the general stationary case which ensures the functional CLT. Of course, (iii) implies  $\langle P^n f, f \rangle \rightarrow 0$ , so if  $P$  has eigenfunctions with unimodular eigenvalues, they are coboundaries which do not satisfy (iii).

4. Conditions (ii) and (iii) look different in nature. We will provide an example of  $P$  mixing where (ii) is satisfied but (iii) is not.

**Proposition 5.2.** *Let  $P$  be a Markov operator as above. Let  $f \in L^2(m)$ . Assume that one of the following conditions is satisfied:*

- (i')  $\sup_{n \geq 1} \frac{(\log n)^{5/2} (\log \log n)^\tau}{n^{1/2}} \left\| \sum_{k=1}^n P^k f \right\|_2 < +\infty$ , for some  $\tau > 1$ .
- (ii')  $\sup_{n \geq 1} \frac{1}{n^\alpha} \left\| \sum_{k=1}^n P^k f \right\|_2 < +\infty$ , for some  $\alpha < 1/2$ .
- (iii') There exists a positive non-decreasing slowly varying function  $\ell$  such that

$$\sum_{n \geq 1} \frac{1}{n \ell(n)} < +\infty \quad \text{and} \quad \sum_{n \geq 1} \frac{\sqrt{\ell(n)} \log(n) \left\| \sum_{k=1}^n P^k f \right\|}{n^{3/2}} < +\infty.$$

Then  $\{f(X_n)\}$  satisfies the quenched Central Limit Theorem.

**Remark.** Condition (ii') obviously implies condition (i'), which clearly implies (ii). The quenched CLT under (iii') was obtained by Zhao and Woodroffe [28]. It is probably not comparable to our condition (i'). We will provide an example where (i') is satisfied but (iii') is not. Note that convergence of the second series of (iii') and monotonicity of  $\ell$  imply (ii). Condition (ii') implies (iii') with  $\ell(n) = (\log(n + 1))^{1+\varepsilon}$ .

In order to compare the previous conditions we will use the same symmetric Markov operator  $P$  on  $L^2[0, 1) := L^2([0, 1), \lambda)$ , where  $\lambda$  denotes the Lebesgue measure (which generates a reversible chain).

**Proposition 5.3.** *There exists a symmetric positive definite Markov operator  $P$  on  $L^2[0, 1)$  such that:*

- (a) *There exists  $f \in L^2$  such that (i) is satisfied but (ii) and (iii) are not.*
- (b) *There exists  $f \in L^2$  satisfying (ii) but not (iii).*
- (c) *There exists  $f \in L^2$  such that (i') is satisfied but (ii') and (iii') are not.*

**Proof.** We first construct  $P$ . Let  $\alpha \in \mathbb{R} - \mathbb{Q}$  and take  $P = \frac{1}{4}(2I + R_\alpha + R_{-\alpha})$ , where  $R_\alpha$  denotes the rotation of the unit circle by the angle  $\alpha$ . The irrationality of  $\alpha$  makes  $P$  ergodic.

Let  $f \in L^2[0, 1]$ , with Fourier expansion  $f(x) = \sum_{n \in \mathbb{Z}} c_n e^{2i\pi nx}$ . Then for  $0 \leq x \leq 1$  we have

$$P^k f(x) = \sum_{n \in \mathbb{Z}} c_n \left( \frac{2 + e^{2i\pi n\alpha} + e^{-2i\pi n\alpha}}{4} \right)^k e^{2i\pi nx} = \sum_{n \in \mathbb{Z}} c_n \cos^{2k}(\pi n\alpha) e^{2i\pi nx}. \tag{39}$$

We will take  $\alpha := 2e$ . In the proof of each part of the proposition, the appropriate function  $f$  will be defined by its Fourier coefficients  $\{c_n\}_{n \in \mathbb{Z}}$ . In all these definitions we take  $c_n = 0$  if there is no  $k \geq 2$  with  $|n| = k!$ , and  $c_{-k!} = c_{k!} \in \mathbb{R}$ , which makes  $f$  real valued. We will need the following lemma.

**Lemma 5.4.** *For every  $k \geq 2$ , there exists  $l_k \in \mathbb{N}$  such that*

$$\frac{1}{2k} \leq ek! - l_k \leq \frac{1}{k}$$

and for every  $k \geq 2\pi$ ,

$$1 - \frac{2\pi^2}{k^2} \leq \cos(2\pi ek!) \leq 1 - \frac{\pi^2}{3k^2}.$$

**Proof.** For every  $k \geq 2$ , define  $l_k := k! \sum_{j=0}^k \frac{1}{j!} \in \mathbb{N}$ . Since  $e = \sum_{j=0}^{+\infty} \frac{1}{j!}$ , we have

$$\frac{1}{k+1} \leq ek! - l_k \leq \frac{k!}{(k+1)!} \sum_{j=k+1}^{+\infty} \frac{(k+1)!}{j!} = \frac{1}{k+1} \sum_{s=0}^{+\infty} \left( \frac{1}{k+1} \right)^s.$$

Hence

$$\frac{1}{2k} \leq ek! - l_k \leq \frac{1}{(k+1)(1 - 1/(k+1))} = \frac{1}{k}.$$

Hence the first estimation is true.

Let  $k \geq 2\pi$ . By the above, we have  $\frac{\pi}{k} \leq 2\pi(k!e - l_k) \leq \frac{2\pi}{k} \leq 1$ . Using the fact that  $\cos$  is decreasing on  $[0, 1]$  and that for every  $x \in [0, 1]$ ,

$$1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24} \leq 1 - \frac{x^2}{2} + \frac{x^2}{24} \leq 1 - \frac{x^2}{3},$$

we obtain the second estimation. □

**Lemma 5.5.** *Let  $P$  be the above Markov operator. Let  $f \in L^2[0, 1)$  with Fourier expansion  $f(x) := \sum_{k \in \mathbb{Z}} c_k e^{2i\pi kx}$ . Assume that  $c_k = 0$  if there is no  $n \in \mathbb{N}$  such that  $|k| = n!$  and  $c_{-n!} = c_{n!}$ . Then there exists  $K > 0$ , such that for every non-decreasing sequence  $\{u_l\}$  with  $u_1 \geq 1$ , and every  $m \geq 1$ , we have*

$$\frac{m^2}{8} \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2 \leq \left\| \sum_{k=1}^m P^k f \right\|_2^2 \leq K + \frac{18}{\pi^4} \sum_{7 \leq n \leq u_m} n^4 |c_{n!}|^2 + 2m^2 \sum_{n > u_m} |c_{n!}|^2.$$

**Proof.** Let  $m \geq 1$ . By (39) we have

$$\left\| \sum_{k=1}^m P^k f \right\|_2^2 = \sum_{n \in \mathbb{Z}} |c_n|^2 \left( \sum_{k=1}^m \cos^{2k}(2\pi n e) \right)^2.$$

Hence, by the symmetry and using Lemma 5.4 for the non-zero coefficients,

$$\begin{aligned} \left\| \sum_{k=1}^m P^k f \right\|_2^2 &= 2 \sum_{n \geq 1} |c_{n!}|^2 \left( \sum_{k=1}^m \cos^{2k}(2\pi n! e) \right)^2 \\ &\geq 2 \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2 m^2 \cos^{4m}(2\pi n! e) \\ &\geq 2m^2 \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2 \left( 1 - \frac{1}{2m} \right)^{4m}. \end{aligned}$$

Using Bernoulli’s inequality:  $(1 + u)^m \geq (1 + mu)$  for  $u \geq -1$ , we obtain

$$\left\| \sum_{k=1}^m P^k f \right\|_2^2 \geq \frac{m^2}{2^3} \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2.$$

Let us prove the second inequality. By (39), the symmetry, and Lemma 5.4,

$$\begin{aligned} \frac{1}{2} \left\| \sum_{k=1}^m P^k f \right\|_2^2 &= \sum_{1 \leq n \leq 2\pi} |c_{n!}|^2 \left( \sum_{k=1}^m \cos^{2k}(2\pi n! e) \right)^2 + \sum_{n > 2\pi} |c_{n!}|^2 \left( \sum_{k=1}^m \cos^{2k}(2\pi n! e) \right)^2 \\ &\leq K' + \sum_{7 \leq n \leq u_m} |c_{n!}|^2 \frac{1}{(1 - \cos^2(2\pi n! e))^2} + \sum_{n > u_m} |c_{n!}|^2 m^2 \\ &\leq K' + \frac{9}{\pi^4} \sum_{7 \leq n \leq u_m} n^4 |c_{n!}|^2 + m^2 \sum_{n > u_m} |c_{n!}|^2. \end{aligned}$$

□

*Proof of Proposition 5.3(a).* Fix  $1/2 < \beta \leq 1$ . For  $n \geq 2$  put  $c_{-n!} = c_{n!} := \frac{1}{n^{3/2}(\log n)^\beta}$ , and define  $f(x) := \sum_{n \in \mathbb{Z}} c_{n!} e^{2i\pi n!x}$ . Using Lemma 5.5 we obtain

$$\left\| \sum_{k=1}^m P^k f \right\|_2^2 \geq \frac{m^2}{8} \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2 \geq \frac{m^2}{8} \int_{2\pi\sqrt{m}}^\infty \frac{dt}{t^3(\log t)^{2\beta}} \geq C \frac{m}{(\log m)^{2\beta}},$$

which proves that (ii) is not satisfied.

Let us prove that  $f$  satisfies (i). Since  $P$  is positive definite, by [5] it suffices to prove the convergence of the series  $\sum_{m \geq 0} \langle P^m f, f \rangle$ . Using (39), the definition of  $\{c_n\}$  and Lemma 5.4, we obtain

$$\begin{aligned} \sum_{m \geq 0} \langle P^m f, f \rangle &= \sum_{m \geq 0} \sum_{n \in \mathbb{Z}} |c_n|^2 \cos^{2m}(2en\pi) = 2 \sum_{k \geq 2} \frac{1}{(k^3 \log^{2\beta} k)(1 - \cos^2(2ek!\pi))} \\ &\leq 2 \sum_{2 \leq k \leq 2\pi} \frac{1}{(k^3 \log^{2\beta} k)(1 - \cos^2(2ek!\pi))} + \frac{6}{\pi^2} \sum_{k \geq 2\pi} \frac{1}{k \log^{2\beta} k} < +\infty. \end{aligned}$$

The proof that (iii) is not satisfied follows from the proof of part (b) below, since  $c_{n!} > \frac{1}{n^2}$ .

*Proof of Proposition 5.3(b).* Take  $c_{-n!} = c_{n!} := \frac{1}{n^2}$ ,  $n \geq 1$ , and put  $f(x) := \sum_{n \in \mathbb{Z}} c_{n!} e^{2i\pi n!x}$ . Apply Lemma 5.5 to  $f$  with  $u_m := m^{1/4}$ . We have

$$\left\| \sum_{k=1}^m P^k f \right\|_2^2 \leq K + \frac{9}{\pi^4} \sum_{7 \leq n \leq u_m} n^4 |c_{n!}|^2 + m^2 \sum_{n > u_m} |c_{n!}|^2 \leq K + \frac{9}{\pi^4} u_m + \frac{m^2}{3u_m^3} \leq C m^{1/4},$$

which proves (ii).

Let us prove that (iii) is not satisfied. Assume that  $\{g_n = \sum_{m=1}^n f P^m f\}_{n \geq 1}$  converges in  $L^1[0, 1)$  to a function  $g \in L^1[0, 1)$ . For  $l \in \mathbb{Z}$  and  $h \in L^1[0, 1)$  define the Fourier coefficient  $\gamma_l(h) := 2 \int_0^1 h(t) e^{-2\pi i l t} dt$ . Then,  $\gamma_l(g_n) \rightarrow_{n \rightarrow +\infty} \gamma_l(g)$  for every  $l \in \mathbb{Z}$ .

On the other hand, for every  $n \geq 1$  and  $x \in [0, 1)$  (39) yields

$$\sum_{k=1}^n P^k f(x) = \sum_{j \in \mathbb{Z}} c_j \left( \sum_{k=1}^n \cos^{2k}(2\pi j e) \right) e^{2i\pi j x}.$$

Hence, since by the choice of  $\{c_n\}$  the previous series are absolutely summable,

$$g_n(x) = \sum_{j \in \mathbb{Z}} \sum_{m \in \mathbb{Z}} c_j c_m \left( \sum_{k=1}^n \cos^{2k}(2\pi j e) \right) e^{2i\pi(j+m)x}.$$

Let  $r \geq 2$ . By positivity of the coefficients,  $\gamma_{r!+2!}(g_n) \geq \frac{1}{4r^2} \sum_{m=1}^n \cos^{2m}(2\pi r!e)$ . Hence

$$\gamma_{r!+2}(g) = \lim_{n \rightarrow +\infty} \gamma_{r!+2}(g_n) \geq \lim_{n \rightarrow +\infty} \frac{1}{4r^2} \sum_{m=1}^n \cos^{2m}(2\pi r!e) = \frac{\cos^2(2\pi r!e)}{4r^2(1 - \cos^2(2\pi r!e))}.$$

By Lemma 5.4, we obtain

$$\gamma_{r!+2}(g) \geq \frac{(1 - 2\pi^2/r^2)^2}{4r^2(2\pi^2/r^2)(2 - \pi^2/3r^2)} \xrightarrow{r \rightarrow \infty} \frac{1}{16\pi^2}.$$

Hence  $\{\gamma_{r!+2}\}$  does not converge to zero when  $r$  tends to infinity, which contradicts the fact that  $g$  belongs to  $L^1[0, 1)$ , by the Riemann–Lebesgue lemma.

*Proof of Proposition 5.3(c).* Take  $c_{-n!} = c_{n!} := \frac{1}{n^{3/2}(\log n)^{5/2}(\log \log n)^{3/2}}$  for  $n \geq 3$ , and put  $f(x) := \sum_{n \in \mathbb{Z}} c_j e^{2\pi j x}$ . By Lemma 5.5, we have

$$\left\| \sum_{k=1}^m P^k f \right\|_2^2 \geq \frac{m^2}{8} \sum_{n \geq 2\pi\sqrt{m}} |c_{n!}|^2 \geq C \frac{m}{(\log m)^5 (\log \log m)^3}, \tag{40}$$

which proves that (ii') is not satisfied.

Let us prove that (iii') is not satisfied either. Let  $\ell$  be any positive function of the integers. By Hölder's inequality with conjugate exponents 3 and 3/2, and (40), we have, for every  $n \geq 3$

$$\begin{aligned} \sum_{m=3}^n \frac{1}{m \log m \log \log m} &= \sum_{m=3}^n \left( \frac{1}{m \ell(m)} \right)^{1/3} \left( \frac{(\ell(m))^{1/3}}{m^{2/3} \log m \log \log m} \right) \\ &\leq \left( \sum_{m=3}^n \frac{1}{m \ell(m)} \right)^{1/3} \left( \sum_{m=3}^n \frac{\sqrt{\ell(m)}}{m (\log m \log \log m)^{3/2}} \right)^{2/3} \\ &\leq \left( \sum_{m=3}^n \frac{1}{m \ell(m)} \right)^{1/3} \left( \sum_{m=3}^n \frac{\sqrt{\ell(m)} \log m \|\sum_{k=1}^m P^k f\|}{\sqrt{C} m^{3/2}} \right)^{2/3}. \end{aligned}$$

Hence one of the series on the right must diverge and (iii') cannot be satisfied.

However, taking  $u_m = \sqrt{m}$ , in Lemma 5.5, we obtain

$$\begin{aligned} \left\| \sum_{k=1}^m P^k f \right\|_2^2 &\leq K + \frac{18}{\pi^4} \sum_{7 \leq n \leq u_m} n^4 |c_n|^2 + 2m^2 \sum_{n > u_m} |c_n|^2 \\ &\leq K + C \left( \frac{u_m^2}{(\log u_m)^5 (\log \log u_m)^3} + \frac{2m^2}{u_m^2 (\log u_m)^5 (\log \log u_m)^3} \right) \leq C_1 \frac{m}{(\log u_m)^5 (\log \log u_m)^3}, \end{aligned}$$

which proves (i') with  $\tau = 3/2$ . □

**Remark.** The example of Proposition 5.3(a) presents  $P$  symmetric with a function  $f \in \sqrt{I - P}L^2(m)$  which does not satisfy any of the other conditions, in particular none of the conditions for the quenched CLT. However, since this is an example of a (symmetric) random walk on orbits of a rotation, the quenched CLT holds for  $\{f(X_n)\}$  by [8].

We mention that the general question of Kipnis and Varadhan [18], whether for every  $P$  symmetric and  $f \in \sqrt{I - P}L^2(m)$  the quenched CLT holds for  $\{f(X_n)\}$ , is still open.

By looking at the two-sided Markov shift, we see that the annealed CLT holds for  $f$  in the “forward” chain, governed by  $P$ , if and only if it does for  $f$  in the “backward” chain, which is governed by  $P^*$ . When  $P$  is normal, all the conditions on  $f \in L^2$ , except for condition (iii) of Proposition 5.1, hold with respect to  $P$  if and only if they hold with respect to  $P^*$ . We now show that in general this is not so.

Let  $\vartheta$  be the transformation of  $[0, 1]$ , defined by  $\vartheta(x) = 2x \bmod 1$  for  $x \in [0, 1)$ , which preserves Lebesgue's measure. Let  $P$  be defined by  $Pg = g \circ \vartheta$ , for every measurable  $g$ ; then  $P$  is a Markov operator, and  $P^*$  is given by  $P^*g(x) = \frac{1}{2}(g(\frac{x}{2}) + g(\frac{x+1}{2}))$ . Let  $f$  be defined by  $f(x) := 2\mathbf{1}_{[0, 1/2)}(x) - 1$ .

**Proposition 5.6.** *Let  $\vartheta$  and  $f$  as above. We have:*

- (i)  $\sum_{m \geq 1} m^{-3/2} \left\| \sum_{k=1}^m (P^*)^k f \right\| < \infty$ , so  $f \in \sqrt{I - P^*}L^2$ .
- (ii)  $f \notin \sqrt{I - P}L^2$ , so  $\sum_{m \geq 1} m^{-3/2} \left\| \sum_{k=1}^m P^k f \right\| = \infty$ .
- (iii)  $\sum_{n \geq 0} f(P^*)^n f$  converges in  $L^1(m)$ , but  $\sum_{n \geq 0} f P^n f$  does not.

**Proof.** One can see that  $P^*f = 0$ , hence the series in (i) converges in  $L^2[0, 1)$  and  $f \in \sqrt{I - P^*}L^2$ . To prove (ii) just note that the process  $\{f \circ \vartheta^n\}_{n \geq 0}$  is the Rademacher sequence, hence the series  $\sum_{m \geq 1} \frac{P^m f}{\sqrt{m}}$  does not converge in  $L^2[0, 1)$ , so  $f \notin \text{Im} \sqrt{I - P}$ . Obviously  $\sum_{n \geq 0} f(P^*)^n f$  converges in  $L^1(m)$ , since  $P^*f = 0$ . Since  $P$  is an isometry of  $L^1$  and  $|f| \equiv 1$ , we have  $\|f P^n f\|_1 = \|f\|_1 = 1$ , so  $\sum_{n \geq 0} f P^n f$  does not converge in  $L^1$ . □

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