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POLYNOMIAL TIME ALGORITHMS FOR TWO CLASSES OF SUBGRAPH PROBLEM *

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Abstract. We design a $O(n^3)$ polynomial time algorithm for finding a $(k-1)$- regular subgraph in a k-regular graph without any induced star $K_{1,3}$(claw-free). A polynomial time algorithm for finding a cubic subgraph in a 4-regular locally connected graph is also given. A family of k-regular graphs with an induced star $K_{1,3}$ (k *even*, $k \geq 6$), not containing any $(k-1)$-regular subgraph is also constructed.

Keywords. Polynomial time algorithm, NP-complete, graph, star, regular graph, perfect marching.

Mathematics Subject Classification. 05C.

1. Introduction

Let G be a simple graph with vertex set X and edge set E. The number of vertices is denoted by n and the number of edges by m. A subgraph of a graph G is obtained by deleting a set of vertices(possibly empty) and a set of edges(possibly empty) of G. If the degree of each vertex is equal to the same integer k, then the graph is called a k-regular graph or simply a regular graph. The star $K_{1,3}$ with the vertex 1 as its *center* is the graph $G = (X, E)$ where $X = \{1, 2, 3, 4\}$ and $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$. A graph is *locally connected* if for every vertex v, the graph induced by the neighbor set of v, $\langle\Gamma(v)\rangle$ is connected. A connected component of G whose number of vertices is an odd integer is simply called an odd component of G. A graph G is h *edge-connected*, if at least h edges must be removed to disconnect G. The complete p-partite graph on n vertices is a graph

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* *Dedicated to the memory of Professor Claude Berge.*

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whose vertex set can be partitioned into p parts where each part contains exactly either $\lceil \frac{n}{p} \rceil$ or $\lfloor \frac{n}{p} \rfloor$ vertices. It is denoted by $K_{n_1,n_2,\ldots,n_p}$ where $|n_i - n_j| \leq 1$ for $1 \leq i, j \leq p$.

In this paper, we are interested in finding a $(k-1)$- regular subgraph in a k-regular graph without an induced star. The existence of such a graph is proved in [8]. In this paper, we derive a $O(n^3)$ polynomial algorithm for finding such a graph.

Berge [3] (p. 246) conjectured that every 4-regular simple graph contains a 3-regular subgraph. This conjecture was proved by Tashkinov [10]. In fact, he proved that every k-regular (≥ 4) simple graph contains a 3-regular graph.

Chvàtal *et al.* [4] proved that it is an NP-Complete problem to recognize those graphs having 3-regular subgraphs (see also Garey and Johnson [6], p. 198). So, all the known algorithms for finding a 3-regular subgraph in a given graph are of the type "try all possibilities". Evidently, the complexity of such an algorithm is exponential in the number of vertices of the given graph. Such algorithms are commonly called brute-force search. Polynomial time algorithms are generally obtained, through the gain of some deeper insight into the structure of the problem [6]. One of the approaches used to tackle an NP-Complete problem is to find a class of graphs for which a polynomial time algorithm can be designed. In this paper, we consider the classes of k-regular graph without an induced star $K_{1,3}$ and 4-regular locally connected graphs.

2. Notation and terminology

We follow the notation and terminology of Berge [2]. For disjoint subsets S and T of X, $m(S,T)$ denotes the number of edges having one end in S and the other end in T. In a connected graph, $d(x,y)$ denotes the length of a shortest *elementary path* connecting the vertices x and y. By a *path*, we always mean an elementary path. In a connected graph, *eccentricity* $e(x)$ of a vertex x, is defined as $\max(d(x,y))$ where y runs through all vertices of the graph. The *diameter* of a connected graph is the maximum eccentricity of any vertex and a vertex whose eccentricity coincides with the diameter is called a *peripheral* vertex of the graph. $\Gamma(x)$ denotes the neighbor set of the vertex x. For a subset S of the vertex set, $\Gamma(S) = \bigcup_{x \in S} \Gamma(x)$.

3. Main results

The following simple characterization is proved in [9].

Theorem 3.1. *Let $G = (X,E)$ be a k-regular simple graph. Then G contains a regular subgraph of degree $k-1$ if and only if there is a subset $S \neq X$ of the vertex set X satisfying the following two conditions:*

1. *If x is a vertex of $X - S$ then we have $m(x,S) \leq 1$.*

2. *The induced subgraph generated by $X - S - \Gamma(S)$ contains a perfect matching M.*

In fact, if such a set S exists as in the above theorem, then $G - S - M$ is a $(k-1)$-regular subgraph of G.

We shall show that if G is a k-regular simple graph on an odd number of vertices without an induced star, then there is a set S consisting of only one vertex satisfying the two properties of the above theorem. Let us note that the above characterization gives an exponential time algorithm for the following decision problem:

Does there exist a $(k-1)$-regular subgraph in a given k-regular graph?

If $k = 4$ then we have always an affirmative answer [10] (conjectured by Berge) and if $k \geq 6$ there are k-regular graphs without containing a $(k-1)$-regular subgraph.

The Turan graph $K_{3,3,3}$, the complete tripartite graph is 6-regular which does not contain a 5-regular subgraph. The problem is, however, open for $k = 5$.

Theorem 3.1 is not a good characterization of k-regular graphs without a $(k-1)$-regular subgraph. A good characterization is one which establishes an equivalence between an NP-property and a negation of another NP-property. For example, Tutte's [11] perfect matching theorem is a good characterization of graphs *not* containing a perfect matching. Our algorithm hinges on the following simple lemma:

Lemma 3.1.1. *Let x be a peripheral vertex of a connected graph without an induced star $K_{1,3}$. Then $G - x - \Gamma(x)$ is a connected graph.*

Proof. If not, consider a connected component C_1 of $G - x - \Gamma(x)$ containing a vertex y such that the distance $d(x, y)$ is the diameter d of the graph. Now take any other component C_2 of $G - x - \Gamma(x)$ and a vertex z of C_2. We claim that $d(y, z) = d$. Since $d(y, x) = d$, there is a vertex $t \in \Gamma(x)$ such that $d(y, t) = d - 1$ and the vertex t lies in the distance path between y and x. Let S be the set of all vertices of $\Gamma(x)$ at distance $d - 1$ from y. Then at least one of the vertices of S is adjacent to z. (In fact, for every vertex v of C_2, there is a vertes s of S such that vs is an edge of G.) Otherwise, $d(y, z) > d$ which is impossible. Hence the claim. Since $d(y, z) = d$, there must exist an elementary path $(y, x_1, x_2, \dots x_{d-2}, s)$ of length $d-1$ where s is in $\Gamma(x) \cap S = S$ and sz is an edge of the graph. But then the set $\{x_{d-2}, s, x, z\}$ induces a star with the vertex s as its center, a contradiction, to the fact that the graph G is claw-free. This completes the proof of the lemma. □

Now, we are ready to prove the main result.

Theorem 3.2. *Let G be k-regular graph on n vertices without any induced star $K_{1,3}$. Then G contains a $(k-1)$-regular subgraph on n or $n-1$ vertices. In other words, the set S of theorem 3.1 satisfies the inequality $|S| \leq 1$.*

Proof. We may assume without loss of generality that the graph G is connected. We distinguish two cases.

Case 1: n is an odd integer.

Since n is odd, the integer k should be even. Let x be a peripheral vertex of G. Then by lemma 3.1.1, $G - x - \Gamma(x)$ is a connected subgraph. The induced subgraph, $G - x - \Gamma(x)$ is still without an induced star $K_{1,3}$.

Since $n - k - 1$ is even, the graph $G - x - \Gamma(x)$ is a connected graph on an even number of vertices. By a result of Las Vergnas [7], $G - x - \Gamma(x)$ contains a perfect matching M. But then, $G - x - M$ is a $(k-1)$-regular subgraph of G.

Case 2: n is an even integer.

Then by [7], the graph has a perfect matching M and $G - M$ is a $(k-1)$-regular spanning subgraph of G. This finishes the proof of the theorem. □

We shall now present a polynomial time algorithm for finding a $(k-1)$-regular subgraph in a k-regular graph which does not contain any induced graph $K_{1,3}$.

Algorithm

Input: A k-regular graph without any induced star.

Output: A $(k-1)$-regular subgraph of G.

Algorithm: If G has an even number of vertices, we find a perfect matching M of G using Edmonds's [5] matching algorithm. Then $G - M$ is the desired graph.

If the number of vertices is odd, then we find a peripheral vertex x of G and construct the connected graph $G - (x \bigcup \Gamma(x))$. Then, by applying Edmonds's matching algorithm again, we find a perfect matching M of $G - x - \Gamma(x)$. But then, $G - x - M$ is a $(k-1)$-regular spanning subgraph of $G - x$.

Hence the algorithm.

(see the graph G of figure 1: the graph G is without any induced star $K_{1,3}$. The graph is 4-regular, the vertex 1 is a peripheral vertex, and the matching M of the induced subgraph $G - 1 - \Gamma(1)$ is $M = \{\{6,7\},\{8,9\}\}$. $G - 1 - M$ is a 3-regular subgraph of G.)

In fact the graph of figure 1 can be used as the basis to generate a family of 4-regular graphs without any induced star $K_{1,3}$.

The complexity of the algorithm

We use the Floyd's all-pair shortest path algorithm [1] to find the distance matrix of the given k-regular graph. The time needed for the Floyd's algorithm is $O(n^3)$.

The cost of finding a "peripheral' vertex is only $O(n^2)$ as every entry of the distance matrix is examined. Therefore, the complexity of the implementation is $O(n^3)$, because it is known that Edmonds' matching algorithm can be implemented in $O(n^3)$.

Remark: The complexity of the implementation may be improved by taking into account the fact that the graph does not have any induced star.

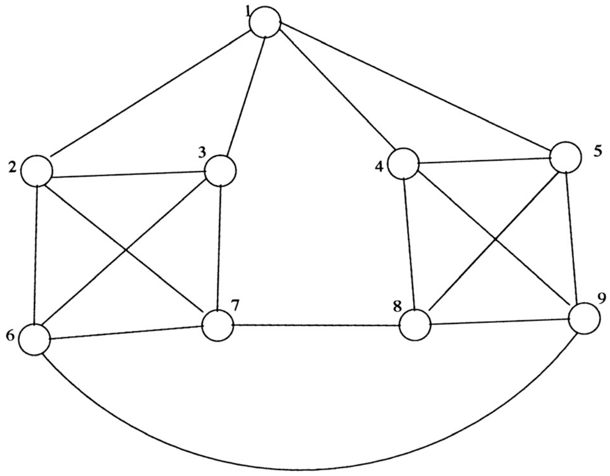

Figure 1. A 4-regular graph without any induced star.

4. Cubic subgraphs in a locally connected 4-regular graphs

In this section, we give a polynomial time algorithm for finding a 3-regular subgraph in a locally connected 4-regular graph and prove the correctness of our result.

Algorithm:
Input: A 4-regular locally connected graph G.
Output: A cubic subgraph of G.
Algorithm: Consider any connected component C of G. Connected components can be found by using depth-first search or breadth-first search in polynomial time. We distinguish two cases:
Case 1: The graph C has an even number of vertices.
We now find a perfect matching M of C. The existence of such a matching is proved below. The graph $C - M$ is a required cubic subgraph of G.
Case 2: The graph C has an odd number of vertices.
Consider any vertex v of C. We now find a perfect matching M of the subgraph $C - v - \Gamma(v)$. The existence of such a matching is proved below. The graph $C - v - M$ is a desired cubic subgraph of C.
Hence the algorithm.

The complexity of the Algorithm: It can be seen that the complexity of the above algorithm is $O(n^3)$.
We now give a **proof of the above algorithm**:
We use the following two theorems of Tutte.

Theorem 4.1. *A graph G has a perfect matching if and only if there is no set S of vertices such that the number of odd components of $G-S$ exceeds the number of vertices in S.*

The following theorem is also due to Tutte [3] (p. 80, Ex. 5.3.2).

Theorem 4.2. *Every $(k-1)$ edge-connected k-regular graph on even number of vertices contains a perfect matching.*

Now we are ready to prove the following theorem which gives a proof of the algorithm.
We need the following lemma.

Lemma 4.2.1. *The edge-connectivity of a 4-regular connected, locally connected graph is 4.*

Proof. Since G is 4-regular, its edge connectivity is even, because if $E' \subset E$ is a cut-set of G with components C_1 and C_2 of $G-E'$, then $|E'| = \sum_{v \in C_1} d(v) - 2m(C_1) = 4|C_1| - 2m(C_1)$ which is even. Here, $|C_1|$ is the number of vertices of the induced subgraph C_1 and $m(C_1)$ is the number of edges of the induced subgraph C_1.

Let $E' = \{e_1, e_2\}$ be a cut-set with $e_1 = xy$ and $e_2 = st$. If $x = s$, then the induced subgraph $< \Gamma(x) >$ is disconnected, a contradiction to the locally connectedness of G. If e_1 and e_2 form a matching, then again the induced subgraph $\langle \Gamma(x) \rangle$ is disconnected, a contradiction. Hence the edge-connectivity of G is 4. Hence the lemma. □

Theorem 4.3. *Let $G = (X, E)$ be a connected, locally connected 4-regular graph on n vertices. Then G contains a 3-regular subgraph on n or $n-1$ vertices.*

Proof. Two cases can arise.
Case 1: n is even.
By lemma 4.2.1, G is a 4-regular 4 edge-connected graph. Hence by Theorem 4.2, the graph G has a perfect matching M. But then $G-M$ is a cubic subgraph on n vertices.
Case 2: n is odd.

Consider any vertex v of G. We shall prove that the induced subgraph $G' = G - v - \Gamma(v)$ has a perfect matching. Note that the number of vertices of G' is even. If G' does not contain a perfect matching, then by theorem 4.1 there is a set S of vertices of G' such that $G'-S$ possesses more odd components than $|S|$. Since the number of vertices of G' is even, $G'-S$ has at least $|S|+2$ odd components. Hence $G-S-\Gamma(v)$ contains at least $|S|+3$ odd components including the singleton component v. Since G is 4 edge-connected, the number of edges between the odd components of $G-S-\Gamma(v)$ and $S \cup \Gamma(v)$ is at least $4(|S|+3) = 4|S|+12$. But then at most $4|S|+10$ edges can be incident with $S \cup \Gamma(v)$, since $\langle \Gamma(v) \rangle$ is a connected graph. This contradiction proves that the graph G' contains a perfect matching M. But then, $G-v-M$ is a cubic subgraph of $G-v$. This completes the proof. □

5. A family of k-regular graphs (k *even*, $k \geq 6$) with an induced star $K_{1,3}$ not containing any $(k-1)$-regular subgraph

We distinguish two cases:
Case 1: k is not divisible by 8.

The complete p-partite graph $G = K_{q,q,\ldots,q}$ in which each part contains exactly q vertices where p and q are odd integers ≥ 3, is a regular graph of degree $k = (p-1)q$.

We shall prove that this graph does not contain any $(k-1)$-regular subgraph. For $p = 3$ and $q = 3$, we have the complete tripartite graph $K_{3,3,3}$ where each part contains exactly 3 vertices. This graph is 6-regular and does not contain any 5-regular subgraph.

To prove this, we shall use theorem 3.1. Since the number of vertices of the graph is odd, the set S of theorem 3.1 satisfies $S \neq \emptyset$. In other words, the graph does not contain a perfect matching.

If $|S| = 1$, then $G - S - \Gamma(S)$ contains an isolated vertex, a vertex of degree zero, and hence does not contain any perfect matching.

If $|S| \geq 2$, then it can be verified that the first condition of theorem 3.1 will not be satisfied.

Thus no subset of vertices exists satisfying simultaneously the conditions of Theorem 3.1. Therefore the graph G does not contain any $k-1$-regular subgraph. Hence the proof of this case.

Case 2: k is a multiple of 8.

Consider a matrix $A = (a_{ij})_{((2p+1)\times 3)}$ where p is an even integer ≥ 2.
Let us define a graph G in terms of the matrix A.
The vertices are the different entries of the matrix and two vertices are joined by an edge if and only if the corresponding entries are neither in the same line nor in the same column of the matrix. Clearly, G is a regular of degree $k = 4p$. By a similar argument used in case 1, we can prove that the graph G does not have any $(k-1)$-regular subgraph.
This completes the construction.

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