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THE POLYTOPE OF m-SUBSPACES OF A FINITE AFFINE SPACE

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Abstract. The m-subspace polytope is defined as the convex hull of the characteristic vectors of all m-dimensional subspaces of a finite affine space. The particular case of the hyperplane polytope has been investigated by Maurras (1993) and Anglada and Maurras (2003), who gave a complete characterization of the facets. The general m-subspace polytope that we consider shows a much more involved structure, notably as regards facets. Nevertheless, several families of facets are established here. Then the group of automorphisms of the m-subspace polytope is completely described and the adjacency of vertices is fully characterized.

Keywords. Convex polytope, finite affine space, m-subspace polytope.

Mathematics Subject Classification. 51A30, 52B12, 90C27.

INTRODUCTION

Maurras [13] (for the plane) and Anglada and Maurras [2] (in general) investigate the polytope of hyperplanes of a finite affine space. Remarkably, they provide a complete description of all facets of the polytope. We extend their investigation by replacing hyperplanes with (affine) subspaces of a fixed dimension m. It turns out that the neat characterization of facets by Anglada and Maurras does not appear to be transposable, as we find wild families of facets for the polytope of m-subspaces (when $2 \leq m \leq n-2$). We establish several of these families, and

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then provide some additional results. Namely, we describe the group of automorphisms (a useful tool in the classification of facets) and characterize adjacency of vertices.

In the affine space $AG = AG(n, GF(q))$ of dimension n over the Galois field $GF(q)$, take for a given m all the m-dimensional subspaces (for an introduction to finite affine spaces, see for instance Artin [3], or Beutelspacher and Rosenbaum [4]). The m-subspaces are subsets of the set AG, and AG has q^n points. Hence the characteristic vectors of m-subspaces are (0/1)-vectors in $\mathbb{R}^{q^n}$. The *polytope of m-subspaces* $P(m, n, q)$ is the convex hull of all these characteristic vectors (classical books on [convex] polytopes include Brøndsted [6], Grünbaum [12], and Ziegler [15]). Thus the vertices of $P(m, n, q)$ correspond exactly to m-subspaces of $AG(n, GF(q))$. Our main goal is to produce facets of the polytope $P(m, n, q)$. In the particular case of the polytope $P(n-1, n, q)$, Anglada and Maurras [2] show that each facet-defining inequality (FDI) is built from a so-called "tangle", a result that we will recall in Section 3.

The `porta` [7] and `polymake` [11] softwares delivered us the list of FDIs for $P(m, n, q)$ in only one truly new case, namely the case of $P(2, 4, 2)$. A similar list appears in Anglada's thesis [1]. We mention in passing that Olivier Anglada has been working independently on the same polytope $P(m, n, q)$, although focusing mainly on the case $m = n - 2$ (see [1]). Running home-made programs on the output from `porta`, we derived a classification of the 16 400 FDIs of the polytope $P(2, 4, 2)$. Two facets are put in the same family if and only if they can be transformed one into the other by some automorphism of the polytope. The resulting eight families of affine inequalities are summarized in Table 1.

Our paper offers a generalization for each of the eight families of FDIs. Geometric interpretations in $AG(n, GF(2))$ are provided for both the generalized FDI and the set of vertices in the corresponding facet. (Many of our results about facets assume that the Galois field has only two elements; a few additional results are given in Christophe [8]). Some inequality families from Table 1 are easily handled, for example the "trivial inequalities" and the "plane-frame inequality" which keep the same form in the general case. On the other hand, some inequalities are more resistant to generalization (several possible interpretations having to be tested). Nevertheless, we provide in Section 2 an adequate generalization to the general case of $P(m, n, 2)$ with $1 \leq m \leq n-2$ for six of the eight families, and to $P(2, n, 2)$ for the remaining two families. Explicit proofs are included for three of the generalized families. The "trivial inequalities" family is a rather easy case. Two other families which require more work are also handled. The proofs for the remaining five families of inequalities can be based on similar arguments and are not given here (see Christophe [8] for details). In the two final sections, we describe the automorphism group of the polytope $P(m, n, q)$ and the adjacency relationship on its vertices.

TABLE 1. The eight families of FDIs for $P(2,4,2)$, where "Quantity" refers to the number of facets in the family and "Vertices" to the number of vertices on any facet of the family. Explanations for the inequalities are provided in the corresponding subsections of Section 2.

	Inequality	Name	Quantity	Vertices
1	$x_i \geq 0$	Trivial I	16	105
2	$x_i \leq 1$	Trivial II	16	35
3	$x_i \leq x_j + x_r + x_s + x_t$ $+ x_u + x_v + x_w$	Point-hyperplane	240	42
4	$x_i + x_j + x_k - 2 \leq x_l$	Plane-frame	560	19
5	$x_a + x_b \leq 2x_i + 2x_j + 2x_k + 2x_l$ $+ x_r + x_s + x_t + x_u + x_v + x_w$		6 720	19
6	$x_i + x_j + x_k + x_l - 2$ $\leq x_s + x_t + x_u + x_v$	3D-frame	1 680	28
7	$x_s + x_t + x_u + x_v + x_w + x_a \leq 3$	4D-frame	448	20
8	$2x_a + x_i + x_j - 2 \leq 2x_b$ $+ x_r + x_s + x_t + x_u + x_v + x_w$		6 720	20

1. The polytope $P(m, n, q)$ and some geometrical facts

The *n-dimensional affine space* $AG(n, GF(q))$ over the Galois field $GF(q)$ of q elements has q^n points. When one of these points is selected as the origin, $AG(n, GF(q))$ becomes a vector space of dimension n over the field $GF(q)$. The *(affine) subspaces* of $AG(n, GF(q))$ are the empty set (of dimension -1) and all translates of the vector subspaces (keeping the same dimension). Thus an m-subspace, that is a subspace of dimension m, is a particular subset of $AG(n, GF(q))$; when $m \geq 0$, it is formed by q^m points. A *frame* of an m-subspace T of $AG(n, GF(q))$ is a subset of $m+1$ points of T consisting of an origin o and the m vectors of a basis for the resulting vector subspace T. We often abbreviate $AG(n, GF(q))$ into AG.

For terminology and notation about polytopes, we generally follow Ziegler [15]. By definition, the vertices of the *m-subspace polytope* $P(m, n, q)$ are the characteristic vectors of all m-subspaces of the affine space $AG(n, GF(q))$. Let us first indicate the dimension of $P(m, n, q)$. For $m \geq 0$, the polytope $P(m, n, q)$ lies in the affine hyperplane defined in the real vector space $\mathbb{R}^{q^n}$ by the affine equation $\sum_{i \in AG} x_i = q^m$. Consequently, the dimension of (the affine subspace spanned by) a set S of vertices of $P(m, n, q)$ is one less than the rank of the set S of vertices seen as vectors of $\mathbb{R}^{q^n}$. Also, any facet is defined by more than one affine inequality. The following result can be easily established:

Proposition 1.1. *In $\mathbb{R}^{q^n}$, we have $\dim(P(m, n, q)) = q^n - 1$ for all $0 \leq m \leq n-1$ and $q \geq 2$.*

Table 2. Values of the subspace dimension m, the dimension n of the finite affine space and the cardinality q of the Galois field for which `porta` and `polymake` are able to provide a linear description of the polytope $P(m, n, q)$, together with the numbers of vertices and facets of the polytope.

m	n	q	Vertices	Facets
1	3	2	28	16
2	3	2	14	128
1	4	2	120	32
2	4	2	140	16 400
1	5	2	496	64

Next, we would like to find facets of $P(m, n, q)$. Table 2 lists the values of m, n and q for which `porta` and `polymake` are able to produce the linear description of $P(m, n, q)$, and also the number of vertices and facets of the polytope. The most interesting case for us in Table 2 is the one of $P(2, 4, 2)$. Indeed, in the cases of polytopes $P(1, n, 2)$, vertices encode lines of a finite affine space $AG(n, GF(2))$ of dimension n on $GF(2)$, that is they encoded unordered pairs of points of $AG(n, GF(2))$. So the polytope is just a hypersimplex (see *e.g.* Ziegler [15]); the trivial inequalities $x_i \geq 0$ and $x_i \leq 1$ are its only FDIs, and moreover the polytope satisfies the single affine equation $\sum_{i \in AG} x_i = 2$. For another case, the polytope $P(2, 3, 2)$ of planes of a 3-dimensional affine space is a hyperplane polytope. Thus we know its facets from Anglada and Maurras [2]. As a conclusion, the only case in Table 2 that is really instructive for us is the one of $P(2, 4, 2)$, namely the polytope of planes of $AG(4, GF(2))$. The FDI data for this case were sorted out into families by our programs. The resulting eight families are provided in Table 1. The next section will list generalizations of all of these eight families to polytopes $P(m, n, q)$.

Let us now recall some known geometrical facts, first about finite affine spaces. For $0 \leq m \leq n$, the number of m-dimensional subspaces of the affine space $AG = AG(n, GF(q))$ is $q^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q$, where we make use of the Gaussian number (cf. van Lint and Wilson [14])

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q^n - 1)(q^{n-1} - 1) \cdots (q^{n-m+1} - 1)}{(q^m - 1)(q^{m-1} - 1) \cdots (q - 1)}. \tag{1}$$

More generally, we set $\begin{bmatrix} n \\ m \end{bmatrix}_q^t = \prod_{x=t}^{m-1} \frac{q^n - q^x}{q^m - q^x}$ for the number of m-subspaces of $AG(n, GF(q))$ containing a given subspace of dimension t. Thus, $\begin{bmatrix} n \\ m \end{bmatrix}_q^m = 1$ and $\begin{bmatrix} n \\ m \end{bmatrix}_q^t = 0$ for $t > m$.

Here are two easy lemmas from real linear algebra (for a proof of the second one, see Anglada and Maurras [2]; another proof can be based on the eigenvalues of the matrix, which are easily obtained).

Lemma 1.2. *For A and B two matrices respectively of size $r \times s$ and $s \times t$,*

$$\operatorname{rank}(A \cdot B) \quad \leq \quad \min\{\operatorname{rank}(A), \operatorname{rank}(B)\}.$$

Lemma 1.3. *If S is a real, symmetric matrix of the form*

$$S = \left(\begin{array}{c|c|c|c|c}
\begin{matrix}\alpha & & \beta\\ & \ddots & \\ \beta & & \alpha\end{matrix} & \gamma & \cdots & \gamma & \gamma \\ \hline
\gamma & \begin{matrix}\alpha & & \beta\\ & \ddots & \\ \beta & & \alpha\end{matrix} & \cdots & \gamma & \gamma \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots \\ \hline
\gamma & \gamma & \cdots & \begin{matrix}\alpha & & \beta\\ & \ddots & \\ \beta & & \alpha\end{matrix} & \gamma \\ \hline
\gamma & \gamma & \cdots & \gamma & \begin{matrix}\alpha & & \beta\\ & \ddots & \\ \beta & & \alpha\end{matrix}
\end{array}\right),$$

composed of r^2 square submatrices of size t, with $r \geq 1$ and $t \geq 1$, then

$$\begin{aligned} \mathit{rank}(S) = r \cdot t \quad \Longleftrightarrow \quad & \alpha \neq \beta, \\ & (\alpha - \beta) + t \cdot (\beta - \gamma) \neq 0, \text{ and} \\ & (\alpha - \beta) + t \cdot (\beta - \gamma) + r \cdot t \cdot \gamma \neq 0. \end{aligned}$$

2. Facets

We now present a generalization for each of the eight families of facets of $P(2,4,2)$ from Table 1. In the sequel, we assimilate the characteristic vector of an m-subspace of the affine space $AG(n, GF(q))$ to the m-subspace itself. Otherwise said, a vertex of $P(m,n,q)$ is identified with an m-subspace of $AG(n, GF(q))$. We will always assume $0 \leq m \leq n-1$ (and, of course, that q is a prime power — a basic property of Galois fields).

2.1. Trivial inequality of type I

The *Trivial Inequality of type I* is defined from the choice of one point i in the affine space $AG = AG(n, GF(q))$:

$$x_i \quad \geq \quad 0. \tag{2}$$

There are thus q^n Trivial Inequalities of type I. The vertices of the face defined by inequality (2) are the (characteristic vectors of the) m-dimensional subspaces which do not contain the point i, so their number is equal to $(q^{n-m} - 1) \begin{bmatrix} n \\ m \end{bmatrix}_q$. In all cases, a trivial inequality of type I defines a facet.

Proposition 2.1. *For $0 \leq m \leq n-1$ and for all $q \geq 2$, the Trivial Inequality from Equation (2) defines a facet of $P(m, n, q)$.*

Proof. We want to prove that the face defined by $x_i \geq 0$ has dimension $q^n - 2$. Let M be the incidence matrix of points of $AG \setminus \{i\}$ (corresponding to rows) versus m-subspaces which do not contain the point i (corresponding to columns). So M is a $(q^n - 1) \times s$ matrix where $s = (q^{n-m} - 1) \begin{bmatrix} n \\ m \end{bmatrix}_q$. Our goal is to prove $\operatorname{rank}(M) = q^n - 1$. By Lemma 1.2, it suffices to show $\operatorname{rank}(M \cdot M^t) = q^n - 1$.

Consider first the case $q = 2$. Then

$$M \cdot M^t = \begin{pmatrix} \alpha & & \beta \\ & \ddots & \\ \beta & & \alpha \end{pmatrix}$$

with α equal to the number of vertices of the face which contain a given point p distinct from i, and β equal to the number of vertices of the face containing two given points p and p' distinct from i. The numbers α and β are independent of the choice of p and p' because the stabilizer of i in the affine group of AG acts doubly transitively on $AG \setminus \{i\}$ (remember $q = 2$; we recall the definition of the affine group in Section 4). Then Lemma 1.3 gives $\operatorname{rank}(M \cdot M^t) = q^n - 1$ (indeed $\alpha \neq \beta$ is easily verified while $\alpha \neq -(n-1) \cdot \beta$ is obvious). Thus by Lemma 1.2 $\operatorname{rank}(M) = q^n - 1$, which completes the proof in case $q = 2$.

Consider now the case $q > 2$. To form the rows of the incidence matrix M, we sort the points of $AG \setminus \{i\}$ into classes of $q - 1$ points, where two points are in the same class if they belong to the same line through i. We then obtain

$$M \cdot M^t = \left(\begin{array}{ccc|ccc|ccc} \alpha & & 0 & & & & & & \\ & \ddots & & & \gamma & & & \gamma & \\ 0 & & \alpha & & & & & & \\ \hline & & & \alpha & & 0 & & & \\ & \gamma & & & \ddots & & & \gamma & \\ & & & 0 & & \alpha & & & \\ \hline & & & & & & \alpha & & 0 \\ & \gamma & & & \gamma & & & \ddots & \\ & & & & & & 0 & & \alpha \end{array}\right). \tag{3}$$

Indeed, there is no subspace containing two points aligned with i and avoiding i; this explains the zeroes in the matrix. That other values repeat themselves is easily established. Moreover, we have

$$\begin{aligned} \alpha &> 0, && (4) \\ \alpha &\neq (q-1)\gamma, && (5) \\ \alpha &\neq (q-q^n)\gamma && (6) \end{aligned}$$

because $\alpha > 0$ is clear, and together with $\gamma \geq 0$ and $q - q^n \leq 0$, the latter immediately gives Inequality (6). To check (5), we compute explicitly the values of α and γ.

If $m = 0$, then $\alpha = 1$ and $\gamma = 0$.

If $m \geq 1$, then α denotes the number of m-subspaces containing p but not i, for some point p in AG, and

$$\alpha = \prod_{x=0}^{m-1} \frac{q^n - q^x}{q^m - q^x} - \prod_{x=1}^{m-1} \frac{q^n - q^x}{q^m - q^x}.$$

On the other hand, γ gives the number of m-subspaces containing two given points p and p' not aligned with i, so

$$\gamma = \prod_{x=1}^{m-1} \frac{q^n - q^x}{q^m - q^x} - \prod_{x=2}^{m-1} \frac{q^n - q^x}{q^m - q^x}.$$

Equation (5) becomes

$$\left(\frac{q^n - 1}{q^m - 1} - 1\right) \cdot \prod_{x=1}^{m-1} \frac{q^n - q^x}{q^m - q^x} \neq (q-1)\left(\frac{q^n - q}{q^m - q} - 1\right) \cdot \prod_{x=2}^{m-1} \frac{q^n - q^x}{q^m - q^x},$$

and after simplification

$$q \cdot \left(q^{n-1} - 1\right) \neq (q-1) \cdot (q^m - 1).$$

As $q > q-1$ and $q^{n-1} \geq q^m$, we conclude that equation (5) is satisfied. Applying Lemma 1.2 gives $\operatorname{rank}(M \cdot M^t) = q^n - 1$. □

2.2. Trivial inequality of type II

As for the other trivial inequality (see previous subsection), the present inequality is defined from the choice of one point i in the affine space $AG(n, GF(q))$:

$$x_i \quad \leq \quad 1. \tag{7}$$

There are q^n *Trivial inequalities of type II.* The vertices of the corresponding face of $P(m, n, q)$ are the $\begin{bmatrix} n \\ m \end{bmatrix}_q$ m-dimensional subspaces which contain the point i.

Proposition 2.2. *The Trivial Inequality of Type II from Equation* (7) *defines a facet if and only if* $q = 2$ *and* $1 \leq m \leq n-1$.

Proof. The sufficiency part can be established similarly as it was for Proposition 2.1 (details are given in [8]). The necessity of $q = 2$ derives from the following remark. If $q > 2$, all vertices satisfying $x_i = 1$ also satisfy $x_j = x_k$ for any two points j and k aligned with i; thus they belong to more than one proper face and cannot span a facet. □

2.3. Point-hyperplane inequality

Select one hyperplane H in the affine space $AG(n, GF(q))$ and then choose one point i in H. The resulting *Point-Hyperplane Inequality* is

$$(q^{m-1} - 1)\, x_i \quad \leq \quad \sum_{j \in H \setminus \{i\}} x_j. \tag{8}$$

Proposition 2.3. *Let* H *be an hyperplane of* $AG(n, GF(q))$ *and* i *be a point of* H. *Inequality* (8) *defines a facet if and only if* $q = 2$ *and* $2 \leq m \leq n-1$.

For the proof of Proposition 2.3 as well as of the next Proposition 2.4, the reader is referred to Christophe [8]. There are $2^n \begin{bmatrix} n \\ n-1 \end{bmatrix}_2$ facets defined by a Point-Hyperplane Inequality as in Proposition 2.3. Anyone of them has $2^{n-1-m} \begin{bmatrix} n-1 \\ m \end{bmatrix}_2$ $+ 2^{n-m} \begin{bmatrix} n-1 \\ m-1 \end{bmatrix}_2$ vertices.

Because of the present lack of satisfactory generalizations to other values of q, we will assume $q = 2$ for generalizing the remaining inequalities in Table 1. A few more results for the case $q > 2$ are collected in Christophe [8].

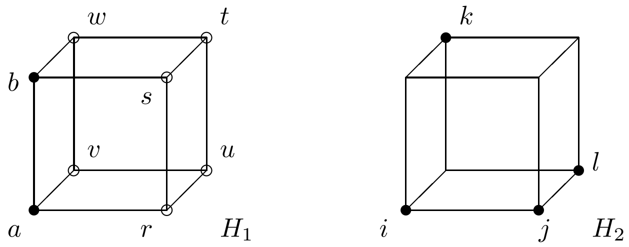

FIGURE 1. Illustration for the inequality of the fifth type in the case of $P(2, 4, 2)$.

2.4. Plane-frame inequality

Take three non aligned points i, j and k of the affine space $AG(n, GF(2))$ over $GF(2)$, and let l be the fourth point in their plane. The resulting *Plane-Frame Inequality* reads:

$$x_i + x_j + x_k - 2 \quad \leq \quad x_l. \tag{9}$$

Thus, Plane-Frame Inequalities are based on the choice of one pair of incident point and plane. They are in number $2^n \begin{bmatrix} n \\ 2 \end{bmatrix}_2$, each one having $\left(3\left(\frac{2^n-2}{2^m-2}\right)-2\right) \cdot \begin{bmatrix} n \\ m \end{bmatrix}_2^2$ vertices.

Proposition 2.4. *The Plane-Frame Inequality* (9) *is a FDI for* $P(m, n, 2)$ *if and only if* $2 \leq m \leq n-1$ *or (*$m = 1$ *and* $n = 2$*).*

2.5. The fifth type of inequality

Obtaining a generalization of Case 5 in Table 2 that would deliver FDIs was found more difficult than for other cases. Here is such a generalization, although under the restrictions $q = 2$ and $m = 2$. Assuming $n \geq 4$, consider a partition of the affine space $AG = AG(n, GF(2))$ into two parallel hyperplanes H_1 and H_2 (see Fig. 1 for an illustration with $n = 4$). Then in H_1 take two points a and b; they will be seen to correspond to the terms in the left-hand-side of Inequality (10) we are constructing. The points of $H_1 \setminus \{a, b\}$ give terms in the right-hand side of the inequality with a coefficient 1. Next, the hyperplane H_2 is partitioned into lines parallel to the line $\{a, b\}$. Construct some subset S of H_2 under the following restriction:

The subset S contains one point on each line of the hyperplane H_2 which is parallel to $\{a, b\}$. Moreover, the resulting 2^{n-2} points do not form an $(n-2)$-dimensional subspace of AG.

The points of S appear in the right-hand side of the inequality with a coefficient 2. When $m = 2$ and $n = 4$, such a set S is necessary an affine frame. In the general case, the inequality is of the following form:

$$x_a + x_b \quad \leq \quad \sum_{t \in H_1 \setminus \{a,b\}} x_t + 2 \sum_{s \in S} x_s. \tag{10}$$

This inequality is valid for the polytope $P(2,n,2)$: If the left-hand side is zero, the inequality is trivially valid. At a plane, that is a vertex of the polytope, the only two other cases are those where the left-hand-side takes value 1 or 2, depending on wether the plane contains one or two of the points a, b. Notice that in $AG(n, GF(2))$, a plane not parallel to a hyperplane intersects this hyperplane in exactly two points. If a plane contains the point a but not the point b, then it contains a point of $H_1 \setminus \{a,b\}$. If a plane contains both points a and b, then either the plane is included in H_1 and then it contains two points of $H_1 \setminus \{a,b\}$, or the plane is not parallel to H_1 and thus contains a point of S.

Proposition 2.5. *Inequality* (10) *is facet defining for the polytope* $P(m,n,2)$ *if and only if* $m = 2$ *and* $n \geq 4$.

Proof. Necessity is left to the reader. To prove sufficiency, let F be the face of the polytope $P(2,n,2)$ defined by Inequality (10). The vertices of F are the planes in $AG = AG(n, GF(2))$ which

(1) are entirely contained in $H_2 \setminus S$ (the number of vertices of this type varies with the choice of S);
(2) are entirely contained in H_1 and are then formed with a, b and two points in $H_1 \setminus \{a,b\}$ (the number of these vertices is $2^{n-2} - 1$);
(3) have half of their points in each of the two hyperplanes H_1 and H_2 and necessarily contain
    (a) a and b, one point in S and the last one in $H_2 \setminus S$ (there are 2^{n-2} vertices of this type), or
    (b) one point in $\{a,b\}$, one point in $H_1 \setminus \{a,b\}$ and two points in $H_2 \setminus S$. The number of these vertices equals $\left(2^{n-1} - 2\right) 2^{n-3}$.

We will now prove that among the vertices of types 2 and 3 there are $2^n - 1$ affinely independent points in $\mathbb{R}^{2^n}$. To this aim, we consider the matrix M whose columns are the (characteristic vectors of) vertices of types 2 and 3 of the face F. In the remaining part of the proof, the term "vertex" will mean "vertex of type 2 or 3 lying on the face F". It is sufficient to show $\operatorname{rank}(M) \geq 2^n - 1$, which in turn follows from $\operatorname{rank}\left(M \cdot M^t\right) \geq 2^n - 1$. The rows and columns of $M \cdot M^t$ are indexed by the points of AG. We group the rows, and accordingly the columns, as follows: first, we take the pairs of points in $H_1 \setminus \{a,b\}$ forming a line parallel to $\{a,b\}$, then we take a and b, next the points of $H_2 \setminus S$ and finally the points of S with the restriction that the first listed point of S forms with the first listed point of

$H_2 \setminus S$ a line parallel to $\{a, b\}$, similarly for the second points and so on. Then

$$M \cdot M^t = \begin{array}{c} \begin{array}{cccc} H_1 \setminus \{a,b\} & a \quad b & H_2 \setminus S & S \end{array} \\ \left(\begin{array}{c|c|c|c} A & B & 1 & 0 \\ \hline B^t & \begin{matrix} \beta & \gamma \\ \gamma & \beta \end{matrix} & 2^{n-2} & 1 \\ \hline 1 & 2^{n-2} & \begin{matrix} \gamma & & 2 \\ & \ddots & \\ 2 & & \gamma \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \\ \hline 0 & 1 & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} & \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \end{array}\right) \end{array} \tag{11}$$

with

$$A = \begin{pmatrix} \begin{matrix} \alpha & 1 \\ 1 & \alpha \end{matrix} & & 0 \\ & \ddots & \\ 0 & & \begin{matrix} \alpha & 1 \\ 1 & \alpha \end{matrix} \end{pmatrix} \quad \text{and} \quad B = \left(\begin{array}{cc} C_1 & C_2 \\ C_2 & C_1 \\ \hline & \vdots \\ \hline C_k & C_{k+1} \\ C_{k+1} & C_k \end{array}\right), \tag{12}$$

where

- α is the number of vertices that contain a given point of $H_1 \setminus \{a, b\}$ and takes the value $1 + 2^{n-3}$;
- $k = 2^{n-1} - 3$;
- C_i gives the number of vertices containing the point a (resp. b) and one point of $H_1 \setminus \{a, b\}$. The values of the C_i's depend on the choice of S, but we always have, for i odd, $C_i + C_{i+1} = 2 + 2^{n-3}$;
- β is the number of vertices containing the point a (resp. b). Thus β is equal to $2^{n-2} + (2^{n-3} + 1) \cdot (2^{n-2} - 1)$;
- γ gives the number of vertices containing the points a and b. Thus γ equals $2^{n-1} - 1$. Moreover, γ gives the number of vertices whose associated subspaces contain a given point in $H_2 \setminus S$.

We skip the arguments for the other values in Equation (11) (details are provided in [8]). Let us prove that the rank of the matrix $M \cdot M^t$ in Equation (11) is equal to $2^n - 1$. Remember that the matrix $M \cdot M^t$ has both its rows and columns indexed by points of AG. After adding the row indexed by the point b to the row indexed by a, and making the similar operation on the columns, we obtain the new matrix

$$
\left(\begin{array}{c|c|c|cc|cc}
H_1\setminus\{a,b\} & a & b & \multicolumn{2}{c|}{H_2\setminus S} & \multicolumn{2}{c}{S}\\
\hline\hline
A & k & C & \multicolumn{2}{c|}{1} & \multicolumn{2}{c}{0}\\ \hline
k & \epsilon & \eta & \multicolumn{2}{c|}{2^{n-1}} & \multicolumn{2}{c}{2} \\ \hline
C^t & \eta & \beta & \multicolumn{2}{c|}{2^{n-2}} & \multicolumn{2}{c}{1}\\ \hline
 & & & 2^{n-1}-1 & 2 & 1 & 0\\
1 & 2^{n-1} & 2^{n-2} & \ddots & & \ddots & \\
 & & & 2 & 2^{n-1}-1 & 0 & 1 \\ \hline
 & & & 1 & 0 & 1 & 0\\
0 & 2 & 1 & \ddots & & \ddots & \\
 & & & 0 & 1 & 0 & 1
\end{array}\right) \tag{13}
$$

where C is the column matrix containing the second column of matrix B from Equation (12), $k = 2^{n-3} + 2$, $\epsilon = 2\beta + 2\gamma$ and $\eta = \beta + \gamma$. Next, execute the following operations on the matrix from equation (13):

- subtract one (resp. two) time(s) each row indexed by the points of S to the rows indexed by b (resp. a);
- subtract to the rows indexed by the points of $H_2 \setminus S$ a well chosen row of S.

We get then the following matrix (still having the same rank as $M \cdot M^t$):

$$
\left(\begin{array}{c|c|c|cc||cc}
H_1\setminus\{a,b\} & a & b & \multicolumn{2}{c||}{H_2\setminus S} & \multicolumn{2}{c}{S}\\
\hline\hline
A & k & C & \multicolumn{2}{c||}{1} & \multicolumn{2}{c}{0}\\ \hline
k & \epsilon' & \eta' & \multicolumn{2}{c||}{2^{n-1}-2} & \multicolumn{2}{c}{0} \\ \hline
C^t & \eta' & \beta' & \multicolumn{2}{c||}{2^{n-2}-1} & \multicolumn{2}{c}{0}\\ \hline
 & & & 2^{n-1}-2 & 2 & & \\
1 & 2^{n-1}-2 & 2^{n-2}-1 & \ddots & & \multicolumn{2}{c}{0} \\
 & & & 2 & 2^{n-1}-2 & & \\ \hline\hline
 & & & 1 & 0 & 1 & 0\\
0 & 2 & 1 & \ddots & & \ddots & \\
 & & & 0 & 1 & 0 & 1
\end{array}\right) \tag{14}
$$

where $\beta' = \beta - 2^{n-2}$, $\epsilon' = 2\,\beta + 2\,\gamma - 2^n$ and $\eta' = \beta + \gamma - 2^{n-1}$. The rank of the submatrix in the lower right part trivially equals $|S| = 2^{n-2}$. Because the matrix in the upper right part is null, we now work with the submatrix in the top left part.

Subtract $1/k$ times the row indexed by a to each of the rows indexed by the points of $H_2 \setminus S$. We thus get the matrix

$$
\begin{array}{c}
\begin{array}{c|c|c|c}
H_1 \setminus \{a,b\} & a & b & H_2 \setminus S
\end{array} \\
\left(\begin{array}{c|c|c|ccc}
A & k & C & & 1 & \\
\hline
k & \epsilon' & \eta' & & 2^{n-1}-2 & \\
\hline
C^t & \eta' & \beta' & & 2^{n-2}-1 & \\
\hline
 & & & \mu & & \nu \\
0 & \tau & \rho & & \ddots & \\
 & & & \nu & & \mu
\end{array}\right)
\end{array}
\tag{15}
$$

where

$$
\begin{aligned}
\tau &= 2^{n-1} - 2 - \frac{\epsilon'}{k}, \\
\rho &= 2^{n-2} - 1 - \frac{\eta'}{k}, \\
\mu &= 2^{n-1} - 2 - \frac{(2^{n-1}-2)}{k}, \\
\nu &= 2 - \frac{(2^{n-1}-2)}{k}.
\end{aligned}
$$

Letting $\delta = \mu + (2^{n-2}-1)\nu$, execute the following operations:

- to each row indexed by the points of $H_1 \setminus \{a,b\}$ subtract the sum of the rows indexed by the points of $H_2 \setminus S$ divided by δ;
- subtract $(2^{n-1}-2)$ times the sum of the rows indexed by the points of $H_2 \setminus S$ divided by δ to the row a;
- subtract $(2^{n-2}-1)$ times the sum of the rows indexed by the points of $H_2 \setminus S$ divided by δ to the row b.

There results the matrix

$$
\begin{array}{c}
\begin{array}{c|c|c||c}
H_1 \setminus \{a,b\} & a & b & H_2 \setminus S
\end{array} \\
\left(\begin{array}{c|c|c||ccc}
A & k' & C' & & 0 & \\
\hline
k & \xi & \omega & & 0 & \\
\hline
C'^t & \sigma & \phi & & 0 & \\
\hline\hline
 & & & \mu & & \nu \\
0 & \tau & \rho & & \ddots & \\
 & & & \nu & & \mu
\end{array}\right)
\end{array}
\tag{16}
$$

where

$$\begin{aligned}
C' &= \left(\begin{array}{c} C'_2 \\ C'_1 \\ \hline \vdots \\ \hline C'_j \\ C'_i \end{array}\right), \\
k' &= k - \frac{2^{n-2}\tau}{\delta}, \\
C'_s &= C_s - \frac{2^{n-2}\rho}{\delta}, \\
\xi &= \epsilon' - (2^{n-1}-2)\frac{2^{n-2}\tau}{\delta}, \\
\omega &= \eta' - (2^{n-1}-2)\frac{2^{n-2}\rho}{\delta}, \\
\sigma &= \eta' - (2^{n-2}-1)\frac{2^{n-2}\tau}{\delta}, \\
\phi &= \beta' - (2^{n-2}-1)\frac{2^{n-2}\rho}{\delta}.
\end{aligned}$$

As μ differs from ν and from $-(2^{n-2}-1)\nu$ (see [8] for details), the lower right matrix has rank 2^{n-2} (*cf.* Lem. 1.2).

In the matrix from Equation (16), delete the row and the column indexed by the point a. In view of the rank values obtained for the lower right part in Equation (14), resp. Equation (16), it remains to prove that the following matrix has rank $2^{n-1}-1$:

$$\left(\begin{array}{cc|c|cc|c}
\alpha & 1 & & 0 & & C'_2 \\
1 & \alpha & & & & C'_1 \\
\hline
 & & \ddots & & & \vdots \\
\hline
 & 0 & & \alpha & 1 & C'_j \\
 & & & 1 & \alpha & C'_i \\
\hline
C_2 & C_1 & \cdots & C_j & C_i & \phi
\end{array}\right). \tag{17}$$

It is easy to see that the matrix in Equation (17) has rank at least $2^{n-1}-2$ (in view of the upper left submatrix on the points of $H_1 \setminus \{a,b\}$; notice $\alpha \neq 1,-1$). It remains to check that the last column of the matrix is not a linear combination of the $2^{n-1}-2$ first columns. Suppose to the contrary that for real values λ_1, λ_2,

$\cdots$, $\lambda_{2^{n-1}-2}$, there holds

$$\begin{array}{ccccccccccc}
\lambda_1 \cdot \alpha & + & \lambda_2 & & & & & & & = & C'_2 \\
\lambda_1 & + & \lambda_2 \cdot \alpha & & & & & & & = & C'_1 \\
 & & & \ddots & & & & & & & \vdots \\
 & & & & & & \lambda_k \cdot \alpha & + & \lambda_{k+1} & = & C'_{k+1} \\
 & & & & & & \lambda_k & + & \lambda_{k+1} \cdot \alpha & = & C'_k \\
\lambda_1 \cdot C_2 & + & \lambda_2 \cdot C_1 & + & \cdots & + & \lambda_k \cdot C_{k+1} & + & \lambda_{k+1} \cdot C_k & = & \phi,
\end{array}$$

where $k = 2^{n-1}-3$. The $2^{n-1}-2$ first equations imply for $i \in \{1, 3, 5, \ldots, 2^{n-1}-3\}$

$$\lambda_i = \frac{C'_{i+1} \cdot \alpha - C'_i}{\alpha^2 - 1} \qquad \text{and} \qquad \lambda_{i+1} = \frac{C'_i \cdot \alpha - C'_{i+1}}{\alpha^2 - 1}.$$

By carrying these values in the last equation, we obtain:

$$\begin{aligned}
\sum_{i \in \{1,2,\ldots,2^{n-1}-2\}} \alpha \cdot {C_i}^2 - \sum_{j \in \{1,3,\ldots,2^{n-1}-3\}} 2 \cdot C_j \cdot C_{j+1} & \\
+ \sum_{t \in \{1,2,\ldots,2^{n-1}-2\}} C_t \cdot \left(\frac{2^{n-2}\rho}{\delta} \right) \cdot (1-\alpha) \quad &= \quad (\alpha^2 - 1) \cdot \phi. \qquad (18)
\end{aligned}$$

Validity of Equation (18) is necessary and sufficient for the singularity of the matrix in equation (17). Let us show that equality (18) does not hold. By replacing α, ρ, ϕ and δ with their values (see Christophe [8] for the computations), equation (18) becomes:

$$\begin{aligned}
2^{n-2} \left(2^{n-4}+1\right) \left(\left(2^{n-3}+1\right) \left(2^{n-2}-1\right)\right) & \\
+2 \sum_{i \in \{1,3,\ldots,2^{n-1}-3\}} C_i \cdot C_{i+1} - \left(2^{n-3}+1\right) \sum_{i \in \{1 \ldots 2^{n_1}-2\}} C_i^2 \quad &= \quad 0. \qquad (19)
\end{aligned}$$

For all $i \in \{1, 3, \ldots, 2^{n-1}-3\}$, we have $C_i + C_{i+1} = 2^{n-3}+2$. Thus Equation (19) becomes after some computation:

$$2^{2n-5} + 2^{n-3} - 1 \quad = \quad \sum_{i \in \{1,3,\ldots,2^{n-1}-3\}} C_i \cdot C_{i+1}. \qquad (20)$$

As the values for the coefficients C_i vary between 2 and 2^{n-3} and also for i odd $C_i + C_{i+1} = 2^{n-3}+2$, the product $C_i \cdot C_{i+1}$ takes at least the value 2^{n-2}. So the right-hand side of Equation (20) is at least $\left(2^{n-2}-1\right) 2^{n-2}$. For $n \geq 4$, the left-hand side is lesser. Hence Equation (20) cannot hold, and neither Equation (19) nor Equation(18). $\square$

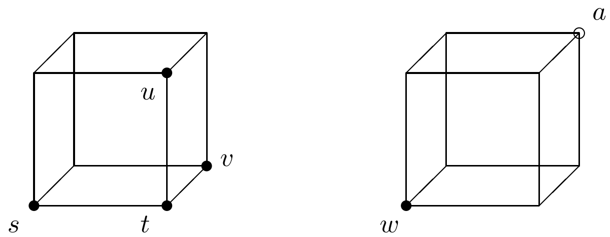

FIGURE 2. Illustration for the 4D-Frame Inequality with $m = 2$, $n = 4$ and $q = 2$.

2.6. 3D-Frame Inequality

Choose a 3-subspace in the affine space $AG = AG(n, GF(2))$ and, in this subspace, take an affine frame. We put the points i, j, k, l of the affine frame in the left-hand side and the other points of the 3-subspace, say s, t, u and v, in the right-hand side of the inequality. The 3D-Frame Inequality is as follows:

$$x_i + x_j + x_k + x_l - 2 \quad \leq \quad x_s + x_t + x_u + x_v. \tag{21}$$

Proposition 2.6. *Inequality* (21) *is facet-defining for* $P(m, n, 2)$ *if and only if* $2 \leq m < n - 1$.

There are $56 \cdot \frac{2^n}{2^3} \begin{bmatrix} n \\ 3 \end{bmatrix}_2$ facet-defining inequalities of this type and $6 \begin{bmatrix} n \\ m \end{bmatrix}_2^1 - 14 \begin{bmatrix} n \\ m \end{bmatrix}_2^2 + 8 \begin{bmatrix} n \\ m \end{bmatrix}_2^3$ vertices on each facet defined by such an FDI.

2.7. 4D-Frame Inequality

Choose a 4-subspace T in the affine space $AG = AG(n, GF(2))$ and an affine frame R in this subspace. Let S be the set of points of $T \setminus R$ which together with some 3 points from R form a plane of AG. As $\dim(T) = 4$, the set $T \setminus (R \cup S)$ contains only one point, say a (see Fig. 2 for an illustration in case $n = 4$). The inequality has the following form:

$$x_a - 2 + \sum_{i \in R} x_i \quad \leq \quad \sum_{s \in S} x_s. \tag{22}$$

The number of FDIs of this type equals $448 \cdot 2^{n-4} \begin{bmatrix} n \\ 4 \end{bmatrix}_2$. Each facet defined by one of these FDIs has a number of vertices equal to $15 \begin{bmatrix} n \\ m \end{bmatrix}_2^1 - 85 \begin{bmatrix} n \\ m \end{bmatrix}_2^2 + 150 \begin{bmatrix} n \\ m \end{bmatrix}_2^3 - 80 \begin{bmatrix} n \\ m \end{bmatrix}_2^4$.

Proposition 2.7. *Inequality* (22) *is facet defining for* $P(m, n, 2)$ *if and only if* $2 \leq m < n - 1$.

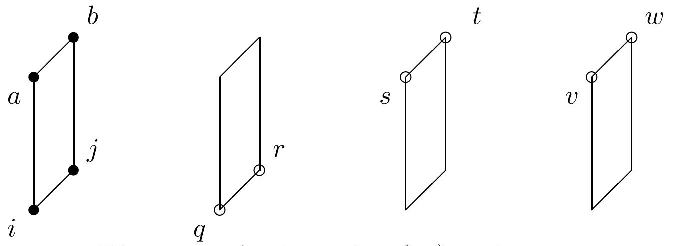

FIGURE 3. Illustration for Inequality (23) with $m = 2$, $n = 4$ and $q = 2$.

2.8. THE EIGHTH INEQUALITY

Our generalization works only for $m = 2$. Choose a plane $\{i, j, a, b\}$ in the affine space $AG = AG(n, GF(2))$, where $n \geq 3$, and partition AG into planes parallel to the plane $\{i, j, a, b\}$ (see Fig. 3 for an illustration with $n = 4$). In each plane of the partition, choose two points which form a line parallel to $\{i, j\}$. There results a set T of $2^{n-1} - 2$ points. Let $U = AG \setminus (\{i, j, a, b\} \cup T)$. The selection of points forming T is also subject to the following condition (which can be proved to be necessary for the next inequality to be facet defining):

There exists at least one plane containing both a and three points of U.

The inequality has the form:

$$2\,x_a + x_i + x_j - 2 \quad \leq \quad 2\,x_b + \sum_{t \in T} x_t. \tag{23}$$

Proposition 2.8. *Inequality (23), where T is the set of the $2^{n-1}-2$ points defined above, is facet defining if and only if $m = 2$ and $n \geq 4$.*

Proof. The proof of sufficiency is left to the reader. Assume now $m = 2$ and $n \geq 4$. Inequality (23) is valid and thus defines a face of $P(2, n, 2)$. Let us check this by evaluating the two sides at a vertex of the polytope $P(2, n, 2)$:

- The left-hand side takes a value less than or equal to 0. Then the inequality is trivially valid.
- The left-hand side takes value 1. Any plane in AG which gives this value to the left-hand side contains the point a and either i or j. By the construction of the set T, the plane contains also some point of T, and thus it gives a value 1 to the right-hand side of the inequality.
- The left-hand side takes value 2. This happens for a plane containing a, i and j. This plane then also contains b and gives a value of 2 to the right-hand side of the inequality.

The vertices of the face defined by Inequality (23) are the following planes:

1. the plane $\{i, j, a, b\}$;
2. the $2^{n-1} - 2$ planes containing a, i, one point of T and one point of U;
3. the $2^{n-1} - 2$ planes formed with a, j, one point of T and one point of U;
4. the $2^{n-2} - 1$ planes containing i, j and two points from U;

5. the planes containing a and three points of U (their number depends on the choice of T).

To prove that Inequality (23) defines a facet of $P(2,n,2)$, we proceed similarly as in the proofs of Propositions 2.1 and 2.5. Considering the vertices of Types 1, 2, 3, and 4 together with only one vertex of type 5, we form the matrix M holding the columns of coordinates in $\mathbb{R}^{2^n}$ of all these vertices. It is sufficient to show that $M \cdot M^t$ has rank $2^n - 1$. Take the three lines parallel to $\{i,j\}$ which contain each a point of the selected plane of type 5, and call R their union.

To form M, and thus also $M \cdot M^t$, we list the points of AG in the following ordering: first i, j, then the points in R, next the other points in U, all these points being grouped two by two according to the lines parallel to $\{i,j\}$ they form; next, a, b and then the points of T grouped two by two according to the lines parallel to $\{i,j\}$ they form, and in an ordering which corresponds to the one followed for the points of U (meaning that the first pair of points from T forms with the first pair of points from U a plane of the partition, similarly for the second pairs of points, and so on).

With this ordering of the points of AG, the matrix $M \cdot M^t$ takes the form

$$\begin{array}{c} \begin{array}{cc|c|c|c|c} i & j & U & a & b & T \end{array} \\ \left(\begin{array}{c|c|c|c|c} \begin{matrix} 3\cdot 2^{n-2}-2 & 2^{n-2} \\ 2^{n-2} & 3\cdot 2^{n-2}-2 \end{matrix} & 2 & 2^{n-1}-1 & 1 & 1 \\ \hline 2 & A & B & 0 & C \\ \hline 2^{n-1}-1 & B^t & 2^n-2 & 1 & 2 \\ \hline 1 & 0 & 1 & 1 & 0 \\ \hline 1 & C^t & 2 & 0 & \begin{matrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{matrix} \end{array}\right) \end{array} \tag{24}$$

where A, B and C are the following matrices:

$$A = \begin{array}{c} \begin{array}{c|c|c|c} \quad R \quad & \quad & \ldots & \quad \end{array} \\ \left(\begin{array}{c|c} \begin{matrix} 4 & 1 & 1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 4 & 1 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 \\ 1 & 0 & 1 & 0 & 4 & 1 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{matrix} & 0 \\ \hline 0 & \begin{array}{c|c|c} \begin{matrix} 3 & 1 \\ 1 & 3 \end{matrix} & & 0 \\ \hline & \ddots & \\ \hline 0 & & \begin{matrix} 3 & 1 \\ 1 & 3 \end{matrix} \end{array} \end{array}\right) \end{array}, \tag{25}$$

$$B = \left(\begin{array}{c} 3 \\ 2 \\ 3 \\ 2 \\ 3 \\ 2 \\ \hline 2 \\ \vdots \\ 2 \end{array} \right), \qquad C = \left(\begin{array}{cc|c|cc} 1 & 1 & & & \\ 1 & 1 & & 0 & \\ \hline & & \ddots & & \\ \hline & 0 & & 1 & 1 \\ & & & 1 & 1 \end{array} \right). \tag{26}$$

Perform the following operations on the matrix from Equation (24) (the goal is to set to zero the entries with column indices in T and row indices outside T):

- subtract from the first two rows (indexed by i and j) $1/2$ times each row indexed by a point of T;
- from each row indexed by a point of U, subtract $1/2$ times two selected rows indexed by points of T;
- from the row indexed by a, subtract each row indexed by a point of T.

We then obtain the following matrix:

$$\begin{array}{c} \begin{array}{cc|c|c|c||c} i & j & U & a & b & T \end{array} \\ \left(\begin{array}{cc|c|c|c||c} 3 \cdot 2^{n-1} - 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 2^{n-1} - 1 & & & & \\ \hline \multicolumn{2}{c|}{1} & A' & B' & 0 & 0 \\ \hline \multicolumn{2}{c|}{1} & B'^t & 2 & 1 & 0 \\ \hline \multicolumn{2}{c|}{1} & 0 & 1 & 1 & 0 \\ \hline\hline \multicolumn{2}{c|}{1} & C^t & 2 & 0 & \begin{matrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{matrix} \end{array} \right) \end{array}, \tag{27}$$

where A' and B' are the following matrices:

$$A' = \left(\begin{array}{cccccc|ccc} 3 & 0 & 1 & 0 & 1 & 0 & & & \\ 0 & 2 & 0 & 0 & 0 & 0 & & & \\ 1 & 0 & 3 & 0 & 1 & 0 & & 0 & \\ 0 & 0 & 0 & 2 & 0 & 0 & & & \\ 1 & 0 & 1 & 0 & 3 & 0 & & & \\ 0 & 0 & 0 & 0 & 0 & 2 & & & \\ \hline & & & & & & 2 & & 0 \\ & & 0 & & & & & \ddots & \\ & & & & & & 0 & & 2 \end{array} \right), \qquad B' = \left(\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ \hline 0 \\ \vdots \\ 0 \end{array} \right). \tag{28}$$

In Equation (27), the rank of the submatrix in the bottom right part is equal to $2^{n-1} - 2$. Now subtract the row indexed by b from the first two rows and from the row indexed by a. Next, subtract the row indexed by a from rows numbers 3, 5 and 7. We obtain the matrix:

$$
\begin{array}{c}
\begin{array}{cc|c||c|c} \quad i \quad & \quad j \quad & \quad U \quad & a & b \end{array} \\
\left(\begin{array}{cc|c||c|c}
3\cdot 2^{n-1}-2 & 0 & \multirow{2}{*}{1} & 0 & 0 \\
0 & 2^{n-1}-2 & & & \\
\hline
\multicolumn{2}{c|}{1} & \begin{matrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{matrix} & 0 & 0 \\
\hline\hline
\multicolumn{2}{c|}{1} & B'^{t} & 1 & 0 \\
\hline
\multicolumn{2}{c|}{1} & 0 & 1 & 1
\end{array}\right)
\end{array}. \tag{29}
$$

The rank of the right bottom matrix is clearly 2. In the top left matrix, subtract to the two first rows $1/2$ times each of the rows indexed by the points of U. This gives the matrix

$$
\begin{array}{c}
\begin{array}{cc||c} \quad i \quad & \quad j \quad & \quad U \quad \end{array} \\
\left(\begin{array}{cc||c}
3\cdot 2^{n-2}-1 & 1-2^{n-2} & \multirow{2}{*}{0} \\
1-2^{n-2} & 2^{n-2}-1 & \\
\hline\hline
\multicolumn{2}{c||}{1} & \begin{matrix} 2 & & 0 \\ & \ddots & \\ 0 & & 2 \end{matrix}
\end{array}\right)
\end{array}. \tag{30}
$$

The rank of the bottom right matrix is equal to $2^{n-1}-2$ and the rank of the top left matrix is 1. In view of all computations done, the rank of the matrix $M \cdot M^t$ is then equal to $2^{n-1}-2+2+2^{n-1}-2+1 = 2^n - 1$. □

3. Tangles

As it was indicated in the Introduction, Anglada and Maurras [2] obtain a complete linear description for the polytope $P(n-1, n, q)$ of hyperplanes of $AG = AG(n, GF(q))$. We summarize their result. A *tangle* in AG is a set of $q^{n-1} + q^{n-2} + \cdots + 1$ hyperplanes that contains exactly one hyperplane per class of parallel hyperplanes. There are $q^{q^{n-1}+\cdots+1}$ tangles. Select one tangle, and denote it by $\mathcal{T}$. For a point i of AG, let t_i be the number of hyperplanes in $\mathcal{T}$ which contain the point i. Then define t and x to be the vectors of components t_i and x_i respectively, for $i \in AG$. Let $\beta = (q^{n-1} + \cdots + q) \cdot q^{n-2}$. To each tangle $\mathcal{T}$, there corresponds the affine inequality

$$t \cdot x \quad \geq \quad \beta. \tag{31}$$

Anglada and Maurras [2] prove that Inequality (31) defines a facet of the polytope $P(n-1, n, q)$ and, conversely, that each facet is defined by Inequality (31) for some

TABLE 3. For each of the 14 orbits of tangles for $n = 4$, $q = 2$, the coefficients of a corresponding inequality are given, as well as the number of tangles in the orbit.

	Coefficients of the tangle inequality	Number of tangles
1	15 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7	16
2	8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 0	16
3	14 8 6 8 6 8 8 8 6 6 8 6 6 8 6 8	240
4	9 7 7 9 7 9 9 7 9 7 7 9 9 7 7 1	240
5	13 9 7 7 7 7 9 9 7 5 7 5 7 7 5 9	1680
6	10 8 8 10 8 10 8 8 8 6 6 2 8 6 6 8	1680
7	12 8 8 8 8 8 8 8 8 4 8 4 8 8 4 8	560
8	11 7 7 11 7 11 7 7 7 7 7 3 7 7 7 7	560
9	12 8 8 8 8 8 10 8 6 6 6 4 6 6 6 10	6720
10	11 7 9 9 9 9 9 7 7 5 7 3 7 7 5 9	6720
11	11 9 9 9 7 7 11 7 5 5 7 5 7 5 7 9	6720
12	10 8 10 10 8 8 10 6 6 4 8 4 8 6 6 8	6720
13	10 6 10 10 6 10 10 6 6 6 10 6 6 6 6 6	448
14	9 9 9 9 5 9 9 9 9 5 9 5 9 5 5 5	448

tangle $\mathcal{T}$. Moreover, the vertices of this facet are all the hyperplanes not taken in the tangle $\mathcal{T}$.

It is tempting to try to generalize Anglada and Maurras' characterization of facets of $P(n-1, n, q)$ to $P(m, n, q)$, for a given m satisfying $1 \leq m \leq n-2$. When adequately rewritten, some of the FDIs described in Section 2 for $P(m, n, q)$ have a left-hand side which is obtained from some tangle as in the case of the hyperplane polytope (see previous paragraph), but with a different right-hand side. In the opposite direction, only some of the tangle inequalities provide the left-hand side of a FDI of $P(m, n, q)$.

Let us be more explicit for the example of $P(2, 4, 2)$. In $AG(4, GF(2))$, the 2^{15} tangles are sorted into 14 orbits under the action of the affine group. Table 3 lists the coefficients of a corresponding inequality for each orbit, together with the number of tangles in the orbit.

Remember that each tangle produces an affine inequality having in its right-hand side the value β as in Equation (31). Moreover, each such inequality defines a facet of $P(3, 4, 2)$. On the other hand, the eight families of FDIs of $P(2, 4, 2)$ are provided in Table 1. Because the polytope $P(2, 4, 2)$ is not full-dimensional, each facet is described by several affine inequalities. Taking this into account, we checked which facets are defined by an inequality coming from some tangle. Our results (independently obtained by Anglada [1]) are summarized in Table 4. There is no tangle inequality producing a Point-Hyperplane Facet, nor the 3D-Frame Facet.

TABLE 4. For $P(2,4,2)$, correspondence between facets and tangle inequalities written in the form $\sum_{i \in AG} a_i x_i \geq c$.

Facets of $P(2,4,2)$	Associated tangle ineq.	Right-hand side
Trivial of Type I	15 7 7 7 7 7 7 7 7 7 7 7 7 7 7 7	28
Trivial of Type II	8 8 8 8 8 8 8 8 8 8 8 8 8 8 8 0	24
Point-Hyperplane	-	-
Plane-Frame	12 8 8 8 8 8 8 8 8 4 8 4 8 8 4 8	24
Fifth	10 8 10 10 8 8 10 6 6 4 8 4 8 6 6 8	24
3D-Frame	-	-
4D-Frame	9 9 9 9 5 9 9 9 9 5 9 5 9 5 5 5	24
Eighth	11 7 9 9 9 9 9 7 7 5 7 3 7 7 5 9	24

We also tried to generalize tangles to "tangles of m-subspaces" but were not successful in obtaining FDIs. We refer the reader to Christophe [8] for the full story.

4. The automorphism group

For some polytopes, their automorphism groups helped produce new facets from known ones (see for instance [5],[9] or [10]). Automorphisms also help sort facets into meaningful families. With these remarks in mind, we now aim at determining all the combinatorial automorphisms of the polytope we are studying, that is the polytope $P(m,n,q)$ of m-subspaces of $AG(n,GF(q))$.

Let us first recall some notions about finite affine spaces (*cf.* [4]). A *semiaffinity* of $AG = AG(n,GF(q))$, where $n > 1$, is any permutation $\alpha : AG \to AG$ which maps any line to a line; in case $q = 2$, we also require than any plane be mapped to a plane (for $q > 2$, the latter condition follows from the first). It can be shown that semiaffinities admit an easy description when a frame is selected in AG (so that we can use coordinates). Then, a semiaffinity is exactly a mapping $\alpha : AG \to AG : x \to \alpha(x)$ where the i-th coordinate of the image is given by

$$(\alpha(x))_i \quad = \quad \sum_{j=1}^{n} m_{ij}\, \sigma(x_j) \; + \; t_i \tag{32}$$

for some regular matrix (m_{ij}) in $(GF(q))^{n \times n}$, some automorphism σ of the Galois field $GF(q)$ and some translation vector (t_i) having n coordinates in $GF(q)$. The semiaffinities of AG form the *semiaffine group* $A\Gamma L(n,GF(q)) = A\Gamma L(AG)$. When we impose that σ be the identity on $GF(q)$, we get the *affinities* of AG, which form the affine group $AGL(n,GF(q)) = AGL(AG)$.

Any semiaffinity α of $AG = AG(n,GF(q))$ permutes the points of AG among themselves and also m-subspaces of AG among themselves, while preserving the incidence relation. Hence, α induces on $\mathbb{R}^{q^n}$ a linear permutation α' which accordingly permutes the coordinates x_i indexed by points i in AG, and moreover this linear permutation α' sends any vertex of the polytope $P(m,n,q)$ onto a vertex of the same polytope. Hence, the semiaffinity α induces a combinatorial automorphism of the polytope $P(m,n,q)$ which is the restriction of α' to the set of vertices of $P(m,n,q)$. Remember that a *combinatorial automorphism* of a polytope can be defined as a permutation of the vertices which maps, for each facet F, the set of vertices of F onto the set of vertices of some facet of the polytope. Among combinatorial automorphisms, we find those induced by linear permutations (or more generally, affinities) of the ambient real space that map the polytope onto itself. For instance, in our case, we just saw that semiaffinities of AG produce linear permutations preserving the polytope $P(m,n,q)$, which in turn induce combinatorial automorphisms. When $0 < m < n-1$, it happens that all combinatorial automorphisms occur in this way.

Proposition 4.1. *Any semiaffinity of $AG = AG(n, GF(q))$ induces as just indicated a combinatorial automorphism of the polytope $P(m,n,q)$. For $1 \leq m \leq n-2$, each automorphism of the polytope is induced in this way by some semiaffinity of AG.*

As a consequence, the automorphism group of the polytope $P(m,n,q)$ is canonically isomorphic to the semiaffine group $A\Gamma L(AG(n, GF(q)))$. We start the proof with two lemmas.

Lemma 4.2. *Let $1 \leq m \leq n-2$. In the n-dimensional affine space $AG(n,K)$ over a skew-field K, select one m-subspace per m-direction in such a way that any two of the selected m-subspaces meet. Then the collection $\mathcal{F}$ of all these m-subspaces has a nonempty intersection.*

Proof. We proceed by recurrence on m. The case $m = 1$ is trivial (because $n \geq 3$): Take two of the lines in $\mathcal{F}$, say D and E, and let a be their intersection point. For each line direction not parallel to the plane generated by D and E, the line in that direction which lies in $\mathcal{F}$ must contain a. It is then clear that a belongs to all lines from $\mathcal{F}$.

Assuming $m > 1$, let us suppose the statement is true for the case of $(m-1)$-subspaces, and prove it for the case of m-subspaces. We first show that $\mathcal{F}$ has the following property:

(*) Consider some $(m-1)$-direction D. Then the m-subspaces of $\mathcal{F}$ whose direction contains D have as their intersection some $(m-1)$-subspace in direction D.

Indeed, let A, B be two m-subspaces from $\mathcal{F}$ which are parallel to the $(m-1)$-direction D. As by assumption they share a common point, their intersection is an $(m-1)$-subspace with direction D. Thus A and B generate an $(m+1)$-subspace, let it be Z. Now take any m-direction E containing the $(m-1)$-direction D but not parallel to Z. There exists such a direction because $m+1 < n$. Take the m-subspace C of $\mathcal{F}$ in direction E. As C meets A, then $A \cap C$ is an $(m-1)$-subspace with $(m-1)$-direction D, that is, parallel to $A \cap B$. Moreover, $C \cap Z$ is also an $(m-1)$-subspace, so $A \cap C = Z \cap C$. The only way C can meet B is then to have $A \cap B = A \cap C = B \cap C$. This is true for all C's of $\mathcal{F}$ whose direction contains the

$(m-1)$-direction D but which are not parallel to Z. Take then an element F of $\mathcal{F}$ whose direction contains the $(m-1)$-direction D and which is parallel to Z. As F must meet C in a $(m-1)$-subspace, we derive $D \cap C = A \cap B$, and Statement (*) is proved.

Let us come back to the induction proof of Lemma 4.2. By applying Statement (*) to each $(m-1)$-direction, we obtain a family $\mathcal{H}$ of $(m-1)$-subspaces. Let us prove that if U, V are in $\mathcal{H}$, then U and V meet. Suppose to the contrary $U \cap V = \varnothing$. Then there exists some m-subspace A in $\mathcal{F}$ such that U is contained in A and also $A \cap V = \varnothing$. Next, there exists some subspace B in $\mathcal{F}$ such that V is contained in B and $A \cap B = \varnothing$. We have then a contradiction because by assumption we must have $A \cap B \neq \varnothing$. So we can apply the recurrence to $\mathcal{H}$. Thus, there exists some point belonging to each $(m-1)$-subspace of $\mathcal{H}$. This point also belongs to all the elements of $\mathcal{F}$. □

Lemma 4.3. *For $1 \leq m \leq n-2$, any facet F of $P(m,n,q)$ has at most $(q^{n-m}-1) \cdot \begin{bmatrix} n \\ m \end{bmatrix}_q$ vertices. Moreover, if F has exactly this number of vertices, then F is a Trivial Facet of Type I.*

Proof. As before, we identify any m-subspace with its characteristic vector. For each m-direction of the affine space $AG = AG(n, GF(q))$, the q^{n-m} parallel m-subspaces in this direction form a partition of AG. The point $(q^{m-n}, q^{m-n}, \dots, q^{m-n})$ in $\mathbb{R}^{q^n}$ is the center of gravity for all the (characteristic vectors of) m-subspaces in a given direction. Then at most $q^{n-m}-1$ m-subspaces in such a direction give a vertex of the given facet F. Indeed, if F contained all the m-subspaces of some m-direction, then F would contain the corresponding center of gravity. But this center is also the center of gravity of all of the m-subspaces in any other m-direction, hence F would contain all the vertices of the polytope. We conclude that F has at most $(q^{n-m}-1) \cdot \begin{bmatrix} n \\ m \end{bmatrix}_q$ vertices.

Let us prove the 2nd assertion. Suppose the facet F contains exactly $(q^{n-m}-1) \cdot \begin{bmatrix} n \\ m \end{bmatrix}_q$ vertices. Thus, from previous paragraph, F contains for each m-direction $q^{n-m}-1$ of the q^{n-m} m-subspaces in that direction. To prove that F is a Trivial Facet of type I, we consider the family $\mathcal{A}$ of all vertices, or m-subspaces, not in F and exhibit some point i of AG which belongs to each member of $\mathcal{A}$.

A first step is to prove that any two members of $\mathcal{A}$ intersect. Suppose that two m-subspaces A and A' of AG are disjoint and also $A \in \mathcal{A}$. Let us then prove $A' \notin \mathcal{A}$.

The union of all m-subspaces parallel to A that meet A' forms a subspace T of some dimension t. Clearly, T is partitioned into m-subspaces parallel to A; call their collection $\mathcal{U}$ and notice $A \notin \mathcal{U}$. Similarly, T is partitioned into m-subspaces parallel to A'; call their collection $\mathcal{U}'$ and notice $A' \in \mathcal{U}'$. For U in $\mathcal{U}$ or in $\mathcal{U}'$, we denote by x^U the characteristic vector of U. All vectors x^U belong to $\mathbb{R}^{q^n}$ and they obviously satisfy

$$\frac{1}{q^{m-t}} \sum_{U \in \mathcal{U}} x^U \quad = \quad \frac{1}{q^{m-t}} \left(x^{A'} + \sum_{V \in \mathcal{U}' \setminus \{A'\}} x^V \right).$$

The left-hand side gives the center of gravity of some vertices of the facet F (because all m-spaces in the direction of A but A belong to F). Hence, the right-hand side is a point in F and so A' is a vertex of F, that is $A' \notin \mathcal{A}$.

The second and final step of the proof is to show that all members of $\mathcal{A}$ have a common point. This follows from Lemma 4.2. □

Proof of Proposition 4.1. For any semiaffinity α of AG, denote by $f(\alpha)$ the combinatorial automorphism of the polytope $P(m, n, q)$ induced by the linear permutation α' (see paragraph before the statement of Prop. 4.1). Clearly, the mapping $f : A\Gamma L(n, GF(q)) \to \mathrm{Aut}(P)$ is an injective homomorphism of groups. We proceed to prove that f is surjective.

Recall form Section 2.1 that for each point i of AG, there is a Trivial Facet of type I with equation $x_i \geq 0$. Denote by TFI the collection of all these facets. The resulting correspondence between points of AG and facets in TFI is clearly one-to-one. Lemma 4.3 entails that TFI exactly consists of those facets having the maximum number of vertices. Hence, any combinatorial automorphism β of $P(m, n, q)$ maps any element of TFI to some element of TFI. Consequently, β, which actually is a permutation of the vertices of $P(m, n, q)$, in other words of the m-subspaces of AG, maps all the m-subspaces avoiding a given point i on all the m-subspaces avoiding some point. Denoting as $\alpha(i)$ the latter point, we see that β induces a permutation α of the points of AG. Moreover, for any point p and m-subspace M of AG, we have $p \in M$ implies $\alpha(p) \in \beta(M)$. It follows that α maps any m-subspace to some m-subspace, and then also any line to some line. Finally, α is a semiaffinity which induces the given combinatorial automorphism β. □

This completes the proof of Proposition 4.1, which describes the automorphism group of the polytope $P(m, n, q)$ in case $1 \leq m \leq n - 2$. In the case of $P(2, 4, 2)$, it is then easily seen that the eight families of FDIs in Table 1 define orbits of the action of the automorphism group on the set of all facets. The same assertion is not true for the generalized families considered in Section 2: for instance, the choice in Section 2.5 of various subsets S (not affinely equivalent subsets S, to be precise) will deliver facets not lying in a same orbit.

For the hyperplane polytope $P(n - 1, n, q)$, all facets have the same number of vertices: this follows from the characterization of the facets due to Anglada and Maurras [2] and recalled in Section 3. So, the last assertion of Lemma 4.3 does not extend to this polytope. Even more, the automorphism group of the polytope $P(n - 1, n, q)$ has a completely different structure than the one of $P(m, n, q)$ for $1 \leq m \leq n - 2$.

Proposition 4.4. *The automorphism group of the polytope $P(n-1,n,q)$ consists of all permutations of the collection of hyperplanes of $AG(n,GF(q))$ that stabilize the partition into classes of parallel hyperplanes.*

Proof. The proof is immediate in view of the characterization of the vertex set of a facet: by Anglada and Maurras [2] (*cf.* Sect. 3 here), each facet of $P(n-1,n,q)$ arises from some tangle, and its vertices are all hyperplanes outside this tangle. Thus the set of vertices of a facet consists in a collection of hyperplanes which misses exactly one hyperplane per direction. □

The order of the automorphism group of the hyperplane polytope $P(n-1,n,q)$ equals $(\frac{q^n-1}{q-1})!\ (q!)^{\frac{q^n-1}{q-1}}$. Notice that the polytope $P(n-1,n,q)$, with $n \geq 2$, admits combinatorial automorphisms that are not induced by any semiaffinity of AG.

5. Adjacency of $P(m,n,q)$

Knowing the adjacency relationship of vertices could help in future search for facets of $P(m,n,q)$. Recall that two vertices of a polytope are *adjacent* if they are the vertices of a 1-dimensional face. Equivalently, the intersection of all facets containing both vertices does not contain any other vertex of the polytope; this is the criterion used in the proofs of this section. Adjacency of vertices $P(m,n,q)$ is characterized first for $q=2$, then for $q \geq 3$. To formulate some conditions about directions of lines in $AG = AG(n,GF(q))$, we make use of the projective space at the infinity of the affine space AG (*cf.* [3,4]).

Proposition 5.1. *Two vertices of $P(m,n,2)$ are adjacent if and only if, as m-subspaces of the affine space $AG = AG(n,GF(2))$, they have a nonempty intersection or their common line directions form in the hyperplane at infinity a projective subspace of dimension strictly less than $m-2$.*

Proof. First, we prove that if two m-subspaces A and B of AG have a nonempty intersection, they deliver two adjacent vertices of the polytope $P(m,n,2)$. To show this, we prove that there are no other vertices of the polytope in the intersection of all the facets which contain the vertices (associated to) A and B. Remark that A and B belong in particular to the facets with equations:

- $x_i = 0$ (Trivial Facet of Type I) for all $i \in AG \setminus \{A \cup B\}$;
- $x_i = 1$ (Trivial Facet of Type II) for all $i \in A \cap B$.

There does not exist any m-subspace different from A and B belonging to the intersection of the facets just listed. Suppose on the contrary there exists one, say C. Then C must be included in $A \cup B$ (because of the Trivial Facets of Type I) and also have at least one point in $A \setminus B$ and at least one point in $B \setminus A$. The m-subspace C must also contain $A \cap B$ (because of the Trivial Facets of Type II). Then, the fourth point of the plane generated by a point of C in $A \setminus B$, a point of C in $B \setminus A$ and a point of C in $A \cap B$ lies in C and in $AG \setminus \{A \cup B\}$. This gives a contradiction. So A and B are adjacent vertices of the polytope.

Let us now prove that two m-subspaces A and B with an empty intersection and a shared subspace at infinity of dimension less than $m-3$ are adjacent. Assume some m-subspace C belongs to all facets which contain both A and B. We show that $C \notin \{A, B\}$ leads to a contradiction. Assuming $C \notin \{A, B\}$, we notice first $C \subseteq A \cup B$, because for $i \in AG \setminus (A \cup B)$, the Trivial Facet of Type I with equation $x_i \geq 0$ contains the vertices A and B, thus also vertex C. Necessarily, C contains at least one point in A and one point in B. So C can be partitioned into two parallel subspaces of dimension $m-1$, one included in A, the other in B. It results that the subspaces A and B have in common at infinity a projective subspace of dimension $m-2$. This contradicts our assumption.

It remains to prove that two m-subspaces A and B which have an empty intersection and a shared subspace at infinity of dimension $m-1$ or $m-2$ are not adjacent. In both cases, $A \cup B$ can be partitioned into two other m-subspaces C and D. Let p be the middle point of (the characteristic vectors of) A and B. Then p is also the middle point of the vertices C and D, thus A and B are nonadjacent vertices of the polytope $P(m, n, 2)$. □

When $q \geq 3$, any two vertices of the polytope $P(m, n, q)$ are adjacent. More generally, we have the following result. Remember that a polytope P is *k-neighbourly* if $1 \leq k \leq \dim(P)$ and every k of its vertices are exactly the vertices of some proper face of the polytope.

Proposition 5.2. *For $0 \leq m \leq n-1$ and $q \geq 2$, the polytope $P(m, n, q)$ is $(q-1)$-neighbourly.*

For $m = n-1$, the same result appears in Anglada and Maurras [2].

Proof. Let S_1, S_2, $\ldots$, S_{q-1} be m-subspaces and let F be the intersection of all the facets containing the (characteristic vectors of) S_1, S_2, $\ldots$, S_{q-1}. We show that any m-subspace T which belongs to F is one of S_1, S_2, $\ldots$, S_{q-1}. Among the facets which contain F, we have the Trivial Facets of Type I with equation $x_i = 0$ for all points i of $AG \setminus (S_1 \cup S_2 \cup \cdots \cup S_{q-1})$. Consequently we have $T \subset S_1 \cup S_2 \cup \cdots \cup S_{q-1}$. Because the q^m points of T are distributed in the $q-1$ m-subspaces S_i, the average number of points in $T \cap S_i$ is equal to or greater than $q^m/(q-1)$. As $q^m/(q-1) > q^{m-1}$, for at least one value of i, the m-subspace S_i contains more than q^{m-1} points of T. Hence $T = S_i$. □

6. Conclusion

Several pieces of information have been given on the polytope $P(m, n, q)$ of m-subspaces of $AG(n, GF(q))$. Families of facets are established, the automorphism group is described and the adjacency of vertices is characterized. Future work on this polytope should aim at finding more facets, in particular for the case $q > 2$. In passing, we notice that we left aside the question of separation by the facet-defining inequalities we found. The question of separation becomes interesting when linear optimization on the polytope $P(m, n, q)$ is required. Here, it is left as an open problem.

After completion of the results presented here, we produced new families of facets by first generating the ridges of known facets, and then finding a second facet containing a given ridge. In another approach, we directly generalized FDIs from the case $m = n - 1$ to other values of m. The various outcomes (reported in Christophe [8]) make even more apparent that the structure of $P(m, n, q)$ is wild. We have at this time no hope to forge a full description of $P(m, n, q)$. This is in strong contrast with the case of the polytope $P(n - 1, n, q)$ of hyperplanes of $AG(n, GF(q))$, for which Anglada and Maurras [2] provide a complete characterization of the facets. In this case, the FDIs are built in a combinatorial manner just from the tangles of $AG(n, GF(q))$. It would be interesting to better understand what makes this result possible, and only possible in the case of hyperplanes.

References

[1] O. Anglada, *Quelques polyèdres combinatoires bien décrits.* Ph.D. thesis, Université de la Méditerranée Aix-Marseille II, Faculté des Sciences de Luminy (2004).

[2] O. Anglada and J.F. Maurras, Enveloppe convexe des hyperplans d'un espace affine fini. *RAIRO Oper. Res.* **37** (2003) 213–219.

[3] E. Artin, *Geometric Algebra.* Wiley Classics Library. John Wiley & Sons Inc., New York (1988).

[4] A. Beutelspacher and U. Rosenbaum, *Projective Geometry: from foundations to applications.* Cambridge University Press, Cambridge (1998).

[5] G. Bolotashvili, M. Kovalev, and E. Girlich, New facets of the linear ordering polytope. *SIAM J. Discrete Math.* **12** (1999) 326–336.

[6] A. Brøndsted, *An Introduction to Convex Polytopes.* Graduate Texts in Mathematics, vol. 90. Springer-Verlag, New York (1983).

[7] T. Christof, PORTA – a POlyhedron Representation Transformation Algorithm. version 1.3.2 (1999), written by T. Christof, maintained by A. Loebel and M. Stoer, available at `http://www.informatik.uni-heidelberg.de/groups/comopt/software/PORTA/`.

[8] J. Christophe, *Le polytope des sous-espaces d'un espace affin fini.* Ph.D. thesis, Université Libre de Bruxelles, 2006. Accessible on line at `http://theses.ulb.ac.be:8000/ETD-db/collection/available/ULBetd-01222007-165129/ro.html`

[9] J. Christophe, J.-P. Doignon and S. Fiorini, The biorder polytope. *Order* **21** (2004) 61–82.

[10] J.-P Doignon and M. Regenwetter, An approval-voting polytope for linear orders. *J. Math. Psych.* **41** (1997) 171–188.

[11] E. Gawrilow and M. Joswig, POLYMAKE: a software package for analyzing convex polytopes. Version 1.4 (Dec. 2000), available at url `http://www.math-tu-berlin.de/diskregeom/polymake/`.

[12] B. Grünbaum, *Convex Polytopes*, Graduate Texts in Mathematics, vol. 221. Springer-Verlag, New York, second edition (2003).

[13] J.F. Maurras, The line-polytope of a finite affine plane. *Discrete Math.* **115** (1993) 283–286.

[14] J.H. van Lint and R.M. Wilson, *A Course in Combinatorics.* Cambridge University Press, Cambridge, UK (1992).

[15] G.M. Ziegler, *Lectures on Polytopes*, Graduate Texts in Mathematics, vol. 152. Springer-Verlag (1995).