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A MARKOV CHAIN MODEL FOR TRAFFIC EQUILIBRIUM PROBLEMS

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Abstract. We consider a stochastic approach in order to define an equilibrium model for a traffic-network problem. In particular, we assume a Markovian behaviour of the users in their movements throughout the zones of the traffic area. This assumption turns out to be effective at least in the context of urban traffic, where, in general, the users tend to travel by choosing the path they find more convenient and not necessarily depending on the already travelled part. The developed model is a homogeneous Markov chain, whose stationary distributions (if any) characterize the equilibrium.

Keywords. Traffic assignment problems, Markov chains, network flows.

1. Introduction

The complexity of the analysis of vehicular and pedestrian mobility in an urban area has the natural consequence that the related problems cannot be treated by means of a single mathematical model. In fact, traffic assignment problems are characterized by several aspects among which we mention the knowledge of the vehicular (pedestrian, etc.) demand, the management of the road network (streets to be made one-way only, numbers of lanes needed, semaphorical times, etc.) and the definition of the equilibrium flows.

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In the literature, there exists a very wide variety of models employed for the description of the traffic assignment problem, often related to the same aspect of the analysis: from the classic minimum-cost flow model up to variational inequality and stochastic queueing models (see *e.g.* [1, 4, 7, 12, 13, 15] and references therein).

In this paper we propose a discrete probabilistic approach to the traffic assignment problem for at least an urban area, based on the theory of Markov chains.

The model that we present can be particularly useful for a more accurate estimate of the traffic demand between two arbitrary (not necessarily adjacent) zones in the urban area that we suppose to be divided in n zones

$$Z_1, \dots, Z_n,$$

which are assumed to be homogeneous in the sense that the subdivision is made in such a way that the traffic inside each zone is less relevant than that between distinct zones.

The traffic demand is in general characterized by two distinct components: the trend and the fluctuations in the short period. The trend can be obtained by means of a statistical census, while the fluctuations are mainly evaluated by means of real time observations. Unfortunately, the data of a statistical census are not always very close to the actual conditions of the observed phenomenon, owing to the long time needed for the elaboration of the informations collected by means of the statistical sampling. On the contrary, the input data of the proposed model can be obtained, in real time, by detecting the passages of the users between adjacent zones. These data allow us to construct the transition matrix of a homogeneous Markov chain whose entries can be regarded as the conditional probabilities of passing from the zone Z_i to the zone Z_j, given the event of being in Z_i.

Fixed a suitable interval (of time) $[T_0, T_1]$, we suppose to know the numbers

$$s_{ij}, \quad i, j = 1, \dots, n; \tag{1}$$

the generic of them gives the number of movements between Z_i and Z_j during $[T_0, T_1]$. Put $s_i := \sum_{j=1}^{n} s_{ij}$, the total number of movements originating from Z_i, if $T := T_1 - T_0$ is sufficiently small we can suppose that s_{ij} is proportional to s_i, so that

$$s_{ij} = a_{ij} s_i, \quad i, j = 1, \dots, n, \tag{2}$$

where a_{ij} is a constant which represents the proportion of traffic from Z_i to Z_j, for $i, j = 1, \dots, n$.

The quantities s_{ij} can be determined by means of real observations of the traffic: each road may be provided with sensors that detect the number of passages of vehicles between the various zones. Equations (2) allow us to determine the constants a_{ij}. We will show that the matrix $A := [a_{ij}]$ is a powerful tool in order to develop the analysis of the behaviour of the traffic. By definition it is immediate to observe that

$$0 \leq a_{ij} \leq 1 \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} = 1, \quad i, j = 1, \ldots, n. \tag{3}$$

These properties suggest to adopt a homogeneous Markov chain as the model that describes the traffic in the urban area: actually, the constants a_{ij} can be considered as the conditional probabilities of passing from the zone Z_i to the zone Z_j, given the event of being in Z_i. As it is evident from the definition, such probabilities depend only on Z_i and Z_j; this is equivalent to assume that a user tends to travel a path which is mainly determined by the amount of traffic that he finds in each zone he has to cross in order to get to his destination: this is very likely in the short travels through an urban area. Furthermore, from the theory of the Markov chains, it follows that the equilibrium distribution of the model can be obtained (if any exists) computing a suitable eigenvector of the matrix A.

The model analysed in the paper has been successfully experimented on the urban area of Pisa, within a co-operation among Italian National Research Council, the County of Pisa, and the University of Pisa.

The paper is organized as follows. In Section 2 we will state the general features of the model and point out the assumptions that allow us to prove that the classic relations, which characterize finite Markov chains, are fulfilled (Sect. 3). Section 4 will be devoted to the analysis of the dispersion of the traffic among the zones.

2. Basic setting

Let $[T_0, L]$ be an interval of time in which the behaviour of the traffic is homogeneous (for example, related to the early, the middle, or the final part of the day) and consider and a set of subintervals $[T_0, T_1]$, $[T_1, T_2]$, …, each of lenght $T \ll L$.

Each interval $[T_{k-1}, T_k]$ will be called the k-th period of the process, for $k = 1, \ldots,$ and we will suppose that, due to the homogeneity of the traffic condition during $[T_0, L]$, it can be characterized by the relations introduced in the previous section considering the interval $[T_0, T_1]$.

Let us summarize the assumptions (denoted by A1, A2, ...) and definitions (D1, D2, ...) that will be used in what follows:

D1. $S := \{1, \ldots, n\}$ is the set of indexes related to the zones $Z_1, \ldots, Z_n$, in which we suppose that the urban area is divided.

The subdivision of the urban area into zones can be made at several levels, each related to the accuracy of the analysis that we aim to reach. First of all it can be considered a macro subdivision of the whole area in order to have a general description of the traffic. Subsequently a refinement of the subdivision of the central zones, where we suppose that the major part of the traffic is concentrated, could be performed. Moreover, the output related to the macro or to the micro subdivisions may suggest different ways of splitting the area into zones: for example, if it results that the movements from the zone Z_1 are mainly directed to the zone Z_2 and *vice versa*, then these two zones could be grouped together in a new subdivision.

A1. We suppose that the total number of users present in the urban area is a constant M.

This assumption is not restrictive, since our analysis is related to a precise interval of time $[T_0, L]$, and we have assumed to consider only the traffic related to the urban area. Therefore, the external zones must be chosen in such a way that they are only origin of movements.

D2. $m_i^{(k)}$ is the number of users present in the zone Z_i in the period k, for $i \in S$ and $k = 1, 2, \dots$

D3. $x^k \in [0,1]^n$ is the vector of the traffic distribution in the set of the zones $Z_1, \dots, Z_n$ in the k-th period: we put

$$x_i^k = \frac{m_i^{(k)}}{M}, \quad i \in S. \tag{4}$$

By definition, x^0 is the initial distribution at time T_0.
The probability distribution x^k is a simple uniform distribution on the set S of the zones: by A1, it follows that

$$\sum_{i=1}^{n} m_i^{(k)} = M, \qquad k = 1, 2, \dots$$

which implies

$$\sum_{i=1}^{n} x_i^k = 1, \qquad k = 1, 2, \dots$$

D4. $s_{ij}^{(k)}$ is the number of the movements between Z_i and Z_j during the period k.

A2. We assume that the total number of movements originating from Z_i in the period k coincides with $m_i^{(k)}$. From the mathematical point of view, it means that the following relation holds:

$$\sum_{i=1}^{n} s_{ij}^{(k)} = m_i^{(k)}, \qquad k = 1, 2, \dots \tag{5}$$

A consequence of the assumption A2 is that all the users are supposed to be moving during a period: the stationary users in a given zone, are actually moving inside the same zone.
This might seem a drawback of the model because of the difficulty of the evaluation of the number of movements $s_{ii}^{(k)}$ inside the zone Z_i: in fact, these numbers must be obtained by means of statistical data, since they cannot be detected by means of the sensors as it happens for the

movements between different zones. Anyway, in order to overcome this problem, we have supposed that the traffic inside each zone is much less relevant than the one between different zones, so that the error in this particular evaluation can be neglected.

A3. We assume that

$$s_{ij}^{(k)} = a_{ij} m_i^{(k)}, \quad i, j \in S, \quad k = 1, 2, \dots \tag{6}$$

where a_{ij} are constants independent on k.

The relations (6) are obtained assuming that the considerations made in Section 1 for the interval $[T_0, T_1]$ are extended to a generic period. Note that, by A2, the total number of movements originating from the zone Z_i coincides with the numbers of users present in Z_i, in the period k. Therefore (6) follows from (2) replacing s_i with $m_i^{(k)}$.
The assumption (6) is of crucial importance in the development of the analysis since it is the key tool in order to prove that the proposed model can be represented by a Markov chain.

D5. $A := [a_{ij}] \in \mathbb{R}^{n \times n}$ will be called the (one period) transition matrix.

D6. $A^{(k)} := [a_{ij}^{(k)}] \in \mathbb{R}^{n \times n}$ is the (k-periods) transition matrix.

The meaning of the matrix A has been widely described in the previous section: each component a_{ij} represents the conditional probability that a user, that in the period k finds himself in the zone Z_i, is in the zone Z_j at the period $k + 1$. The matrix $A^{(k)}$ has an analogous meaning, with the only difference that the transition time from one zone to another is given by exactly k periods. Therefore $a_{ij}^{(k)}$ is the conditional probability of passing from the zone Z_i to the zone Z_j in exactly k periods, given the event of being in Z_i at the beginning of the first of the k periods: this will be proved in the Proposition 3.1.

Since A is independent on k, the model is completely determined by the matrix A and the vector x^0 (see Prop. 3.1), so that it will be denoted by the couple (x^0, A).

Finally, we mention a minor drawback of the model, that can be easily overcome. It lies on the fact that there is not the possibility to distinguish the behaviour of different users. In order to clarify this aspect, consider the following situations:

i) at the period k the user 1 goes from the zone Z_i to Z_j and at the period $k + 1$, from Z_j to Z_k;
ii) at the period k the user 2 goes from the zone Z_i to Z_j and at the period $k + 1$, the user 3 goes from Z_j to Z_k.

We observe that the model cannot detect the two different occurences since only the transitions between the zones are considered without taking into account the users that perform them.

3. Analysis of the model

The first step in our analysis consists in showing that the model (x^0, A) coincides with a homogeneous Markov chain having A as transition matrix and x^0 as initial distribution: next result shows that the classic relations which characterize finite Markov chains actually hold.

Proposition 3.1. *The following relations hold for $k = 1, 2, \ldots$:*

$$x^{k+1} = x^k A, \tag{7}$$

$$A^{(k)} = A^k. \tag{8}$$

Proof. Fixed any $k = 1, 2, \ldots$ and $j \in S$, it results

$$m_j^{(k+1)} = m_j^{(k)} + \sum_{i=1}^{n} s_{ij}^{(k)} - \sum_{i=1}^{n} s_{ji}^{(k)} = m_j^{(k)} + \sum_{i=1}^{n} a_{ij} m_i^{(k)} - \sum_{i=1}^{n} a_{ji} m_j^{(k)}.$$

Therefore, it follows that

$$m_j^{(k+1)} = \sum_{i=1}^{n} a_{ij} m_i^{(k)}, \quad j \in S.$$

Dividing both members by M and recalling the definition (4), we obtain (7).

As regards (8), we observe that from (7) it follows that for each $q = 0, 1, \ldots$

$$x^{k+q} = x^q A^k, \quad k = 1, 2, \ldots \tag{9}$$

Therefore A^k represents the transition matrix related to k successive periods. □

The relation (9) is a direct generalization of (7) and states the Markov property of lack of memory of the process. This roughly means that once that the process has reached the period q, the distributions related to the successive periods do not depend on the conditions of the process before the period q. This assumption fits particularly well in the context of a traffic problem: actually, if it is known the distribution x^q of the traffic at the period q, we expect that that the distributions x^{q+k} do not depend on the distributions related to the periods which precede q, but only on x^q.

By (7), we can easily compute the probability distributions of the traffic related to each period: the question that arises is if these distributions converge to some probability vector, in order that the traffic process tends to a precise configuration. The following definition clarifies the concept of equilibrium distribution for the traffic model.

Definition 3.1. *The traffic model (x^0, A) is said consistent iff there exists*

$$\lim_{k\to\infty} x^k = \bar{x}. \tag{10}$$

$\bar{x}$ is said the equilibrium distribution of the model.

Next result provides a simple necessary condition in order to compute the eventual equilibrium distribution.

Proposition 3.2. *If $\bar{x}$ is an equilibrium distribution for (x^0, A) then $\bar{x}$ is a left (probability) eigenvector of the matrix A corresponding to the eigenvalue 1.*

Proof. In order to prove our statement, it is enough to pass to the limit for $k \to \infty$ in (7), taking into account (10). □

Remark 3.1. We observe that the matrix A always admits the eigenvalue 1. In fact, since A is stochastic (*i.e.* the relations (3) hold) we have that $\det(I - A) = 0$ so that the solution set, say $\mathcal{M}$, of the system

$$xA = x, \quad \sum_{i=1}^{n} x_i = 1, \quad x \in [0,1]^n \tag{11}$$

is nonempty.

$\mathcal{M}$ can be considered as the set of absorbing distributions for the model. Actually from (7) it follows that if there exists $k \geq 0$ such that $x^k \in \mathcal{M}$, then x^k is an equilibrium distribution.

The consistency of the model is a necessary condition in order to have some practical utility from the output solution: anyway this is not enough to be able to affirm that such a solution really reflects the real situation. The equilibrium distribution in general depends on the value of the initial distribution x^0 as can be easily seen taking, for example $A = I$, the identity matrix. Therefore, if there is any mistake in the evaluation of x^0, then, even though the model is consistent, we will obtain a not satisfactory solution.

We will say that the model is stable when this circumstance does not occur.

Definition 3.2. *The traffic model (x^0, A) is said to be stable iff it is consistent and the equilibrium point $\bar{x}$ does not depend on the choice of x^0.*

The concept of stability is of great importance in our particular context. Actually in the analysis of the urban traffic, it is very difficult to estimate the initial distribution x_0, except for particular periods of the day as early morning or late evening. On the contrary, it is likely to have a good approximation of the transition matrix A in real time.

We remark that a stable model can be a powerful tool in order to estimate the fluctuations of the demand in the short period.

A sufficient condition for the stability of the model is given by the following theorem:

Theorem 3.1. *Suppose that the following condition holds:*

$$\text{there exists } k \in N \text{ such that } A^k > 0. \tag{12}$$

Then the model (x^0, A) *is stable.*

Proof. It is a consequence of a well-known result concerning positive matrices obtained by Perron and Frobenius (see *e.g.* [6], Th. 4.2). □

Remark 3.2. Condition (12) characterizes regular positive matrices and, in the theory of Markov processes, regular Markov chains.

In the next section we will deepen the analysis of the meaning of the single components of the matrix A^k in the context of our traffic model.

4. Dispersion of the traffic among the zones

In this section we analyse more in details the traffic between two specific zones of the urban area. This will allow us to have a better understanding of the behaviour of the users and of the nature of the equilibrium distribution. To this aim, it will be useful to consider a further representation of our model given by the directed graph $\mathcal{G} := (\mathcal{N}, \mathcal{A})$, where

$$\mathcal{N} := \{1, \ldots, n\} \quad \text{and} \quad \mathcal{A} := \{(i,j) \in \mathcal{N} \times \mathcal{N} : a_{ij} > 0,\ i,j \in \mathcal{N}\}\cdot$$

The graph $\mathcal{G}$ is the classic auxiliary graph associated to the Markov chain having A as transition matrix.

Let Z_i and Z_j be fixed distinct zones.

Definition 4.1. *We say that* Z_j *is connected to* Z_i *iff*

$$\text{there exists } k \in N \text{ such that } a_{ij}^{(k)} > 0. \tag{13}$$

Using the graph interpretation of the model, it is possible to show that the connection between the zones Z_i and Z_j is equivalent to the existence of a path in $\mathcal{G}$ between the node i and the node j (see *e.g.* [5], Th. 1.12).

Let us consider the asymptotic behaviour of the traffic between Z_i and Z_j; to this end suppose that there exists

$$\lim_{k\to\infty} a_{ij}^{(k)} = \pi_{ij}, \quad \forall i,j \in S. \tag{14}$$

Let $\Pi := [\pi_{ij}]_{i,j\in S}$ be the limit matrix of the sequence $\{A^{(k)}\}_{k\in N}$.

Remark 4.1. It is possible to show that Π is a stochastic matrix and that all the rows of Π belong to the set $\mathcal{M}$ of the absorbing distributions for the model (x^0, A) (see *e.g.* [6] and Rem. 3.1).

If (14) holds then, passing to the limit (for $k \to \infty$) in (9), we obtain that

$$\bar{x} := x^0 \Pi \tag{15}$$

is the equilibrium distribution of the model (x^0, A).

Definition 4.2. *We will say that Z_j is a transit zone iff*

$$\pi_{ij} = 0, \quad \forall i \in S. \tag{16}$$

We observe that a transit zone is a zone where no user is expected to be after a sufficiently large number of periods. By Definition 4.2 and taking into account (15), we infer that

$$\bar{x}_j = 0, \quad \text{for every transit zone } j.$$

We have shown (see Th. 3.1 and Rem. 3.2) that the regularity of the matrix A is a sufficient condition for the stability of the model. It also guarantees that no transit zone exists.

Theorem 4.1. *Suppose that the matrix A is regular. Then (14) holds and*

$$\pi_{ij} > 0, \quad \forall i, j \in S. \tag{17}$$

Proof. See *e.g.* [6], Theorem 4.2. □

The regularity of the matrix A can be interpreted saying that, in a sufficiently large number of periods, the traffic has a complete diffusion in the urban area; actually (17) ensures that there exists a certain percentage of movements from any zone Z_i to any Z_j, whatever i and j may be chosen.

Let us turn our attention on the traffic related to a suitable subset $S' \subset S$ of the zones of the urban area. It may occur that there exists a subset of zones S' that is not connected with the remaining zones: in this case we say that the set S' is closed. An index of the presence of closed subsets of zones is

$$\det(I - T),$$

where T is a principal submatrix of A.

Proposition 4.1. *Let T be the principal submatrix of A obtained by selecting the rows (and columns) of indexes in the subset $S' \subset S$. If there exists a closed subset of zones $S'' \subseteq S'$, then*

$$\det(I - T) = 0. \tag{18}$$

Proof. Since S'' is a closed set of zones then the submatrix $B \subset A$ obtained by selecting the rows and the columns of indexes in S'' is stochastic. Therefore T

contains a principal stochastic submatrix which implies (18). In fact $|\det(I-T)| = |\det(I-T')|$ where

$$T' = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$$

has been obtained by a permutation of the rows and the columns of T in order to let B coincide with the first $|S''|$ rows and columns of T' and C is a suitable square matrix of order $|S'| - |S''|$.

Let I_1 and I_2 be the identity matrices of order $|S''|$ and $|S'| - |S''|$, respectively. Since B is stochastic then

$$\det(I - T') = \det(I_1 - B)\det(I_2 - C) = 0,$$

and, therefore $\det(I - T) = 0$. □

Remark 4.2. If $|S'| = 2$ then it is simple to prove that (18) is a sufficient condition for the existence of a closed subset of zones $S'' \subseteq S'$.

Example 4.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/4 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Obviously any principal submatrix T containing the first row and the first column of A fulfils (18). In fact the zone Z_1 is absorbing, since no movements towards any other zone are registered.

Going into details in our analysis, fixed a zone Z_i we may be interested in evaluating the dispersion of the traffic related to this zone.

Denote by D_{rs} the submatrix of $I - A$ obtained by deleting the row r and the column s.

Definition 4.3.

$$\alpha_{ij} := \frac{\det(D_{jj})}{\det(D_{ii})}$$

is called the index of dispersion of the traffic of the zone Z_i with respect to Z_j, $i, j \in S$ and $j \neq i$.

If $\det(D_{ii}) = 0$ we assume $\alpha_{ij} = +\infty$, $\forall j \neq i$.

Observe that, whenever there exists a closed subset of zones S'' then it results $\alpha_{ij} = +\infty$, for every couple (i, j) with $i \notin S''$ and $j \in S''$, since, by Proposition 4.1, we have that $\det(D_{ii}) = 0$, for every $i \notin S''$. This means that there is a possibility not to come back anymore to Z_i.

Of particular interest is the case where A is a regular matrix: actually under this hypothesis we have that

$$\det(D_{ii}) \neq 0, \quad \forall i \in S.$$

In this case α_{ij} is a finite number, for every i, $j \in S$ and the meaning of the indexes of dispersion is closely related to the concept of recurrence time in the zone Z_i:

$$\tau_i = \min(k \geq 1\text{: a user is in } Z_i).$$

Remark 4.3. The condition "$k \geq 1$" implies that the value τ_i does not take into account the initial distribution x^0, therefore, if $x_i^0 \neq 0$, then τ_i indicates the first period when the user comes back to the zone Z_i: that's why we talk about recurrence time.

Assume that at time T_0 the traffic is concentrated in the zone Z_i, and denote by $E_i(\tau_i)$ the mean value of τ_i under the probability obtained supposing $x_i^0 = 1$. The following proposition relates the indexes of dispersion α_{ij} with the mean recurrence time in Z_i.

Proposition 4.2. *Suppose that the matrix A is regular. The mean recurrence time in the zone Z_i is given by*

$$E_i(\tau_i) = \sum_{j=1}^{n} \alpha_{ij}, \quad i \in S.$$

Proof. When the matrix A is regular it is possible to show that (see *e.g.* [10])

$$E_i(\tau_i) = \frac{1}{\bar{x}_i}, \quad i \in S$$

and

$$\bar{x}_i = \frac{\det(D_{ii})}{\sum_{j=1}^{n} \det(D_{jj})}, \quad i \in S.$$

Eliminating the dependence on $\bar{x}_i$ in the previous equalities we prove the statement. □

We observe that α_{ij} is an index of the contribution that the zone Z_j gives to the value of the mean recurrence time in the zone Z_i.

5. Concluding remarks

We have considered a Markov chain model for the traffic assignment problem in an urban area. The analysis has been carried out under the main assumption that the users adopt a Markovian behaviour in their movements, which amounts to say the paths they use is determined more by the traffic conditions they find while travelling, than by a previously determined rule. This assumption is particularly effective in the analysis of the urban traffic characterized by frequent and short travels.

The proposed model is particularly relevant in view of the applications to the estimate of the fluctuations in the traffic demand in a short period. As we observed, the input data s_{ij} can be obtained in real time by counting the movements from the zone Z_i to the zone Z_j, for every i, j. These data, collected electronically by means of the sensors collocated at street crossings, can be instantaneously elaborated by the computational program. It is known that a homogeneous Markov chain converges to an equilibrium distribution in a substantially short number of iterations, which allows us to have immediately a faithful description of the behaviour of the traffic at that moment.

A further nice feature of our approach is that it does not necessarily require the use of simulation techniques that are often needed in probabilistic models for road traffic problems.

6. Appendix A

We now outline the possible developments of the analysis of traffic equilibrium problems, based on the theory of Markov chains:

- models based on more general definitions of the states of the process;
- dynamic models formalized as continuous Markov processes;
- Markovian models with control;
- distinction of different types of traffic (*i.e.* car traffic, bus traffic, etc.) by means of appropriate Markov chains;
- connections with others equilibrium models as the input-output Leontief models [9] and the Wardrop equilibrium model.

Models based on more general definitions of the states of the process

In order to consider a greater accuracy in the analysis, the states of the process may be defined in the following way:

Definition 6.1. *We will say that a vehicle is in the state s_{ij} if in the period k is in the zone Z_j, and in the period $k-1$ is in the zone Z_i.*

Obviously, the only possible states will be those related to adjacent zones. Denote by $S(k)$ the state of the process at the period k and by $Z(k)$ the zone in which the process is at period k.

Let π_{ihj} the probability of transition from s_{ih} to s_{hj}. We have

$$\pi_{ihj} = P(Z_h \to Z_j | Z_i \to Z_h) = P(S(k) = s_{hj} | S(k-1) = s_{ih}).$$

If we suppose that the Markov property

$$P(S(k) = s_{hj}|S(k-1) = s_{ih}, S(k-2) = s_{ki}, ...) = P(S(k) = s_{hj}|S(k-1) = s_{ih}),$$

holds, then we have

$$P(Z(k) = Z_j|Z(k-1) = Z_h, Z(k-2) = Z_i, Z(k-3) = Z_k, ...)$$

$$= P(Z(k) = Z_j|Z(k-1) = Z_h, Z(k-2) = Z_i).$$

Therefore, the process defined by the sequence $\{Z(k)\}$ is characterized by the fact that the conditional probabilities of being in a certain zone depend on the two preceding states, instead of one (as in the model presented in the paper).

A drawback in this approach is certainly given by the estimate of the transition probabilities π_{ihj}. Actually, in this case an observer situated in the zone Z_h should distinguish the veichles passing in transit from the zone Z_i to Z_j.

Dynamic models

Consider the fundamental relation

$$x^{k+1} = x^k A. \tag{19}$$

Suppose that we want to replace the discrete parameter $k \in N$ with the continuous parameter $t \in \mathbb{R}_+$. In this case (19) becomes

$$x(t + \Delta t) = x(t)A(t, \Delta t), \tag{20}$$

where $A(t, \Delta t)$ is the transition matrix related to the interval of time $[t, t + \Delta t]$. Set

$$a_{ij} = b_{ij}(t)\Delta t, \quad i \neq j$$
$$a_{ii} = 1 + b_{ii}(t)\Delta t.$$

Substituting the previous relations in (20) and letting $\Delta t \to 0$, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(t)B(t), \quad x(0) = c, \tag{21}$$

where $B(t) := [b_{ij}(t)]$, $i, j = 1, \dots, n$ and $c \in [0, 1]^n$.

The differential system (21) is the continuous counterpart of (19).

Markovian models with control

A first important application of the continuous processes is given by dynamic models with control. It is assumed that the matrix A depends on the further control variable $q = q(t)$, so that

$$a_{ij} = b_{ij}(q, t)\Delta t, \quad i \neq j$$

$$a_{ii} = 1 + b_{ii}(q, t)\Delta t.$$

q is a parameter that, at each stage of the process, is chosen in order to maximize the probability of being in a given zone, say Z_1.

With these positions the differential system (21) can be rewritten in the following way:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \max_q \sum_{j=1}^{n} b_{1j}(q,t)x_j(t), \quad x_1(0) = c_1, \tag{22}$$

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sum_{j=1}^{n} b_{ij}(q^*,t)x_j(t), \quad x_i(0) = c_i,\ i = 2,\dots,n \tag{23}$$

where $q^* = q^*(t)$ is one of the control functions which maximizes (22).

An exhaustive analysis of the properties of the differential system (22, 23) can be found in [2].

Distinction of different types of traffic by means of appropriate Markov chains

An important aspect of the Markov chain model lies in the fact that it allows to study separately and (or) simultaneously the equilibrium of different ways of transport (for example by car or by bus). Actually the number s_{ij} of movements between the zone Z_i and the zone Z_j might be related only to a certain kind of means of transport. In this way, we could define a set of transition matrices

$$A_1, \dots, A_s$$

such that A_i represents the percentage of movements of the i-th mean of transport. For example, A_1 could be related to only cars, A_2 to buses, A_3 to two wheels vehicles up to a certain weight T, A_4 greater than T, and so on.

Therefore, we have s parallel Markov chains which can be separately analysed. The main question that arises adopting this setting is the relation between the single Markov chain (y^0, A_i) with the (global) chain (x^0, A). A comparison between the stationary distributions of the various chains with that of (x^0, A) leads us to understand the influence that the traffic due to a specific mean of transport has in the global equilibrium.

Connections with the Leontief input-output model

The open Leontief model [9] considers an economy in which there are r industries producing exactly one kind of goods each.

This model can be represented in terms of the following notations:

- q_{ij} is the amount of the output of the industry j needed by the industry i in order to produce a unit of output;
  Q is the $r \times r$ matrix with entries q_{ij}.

- x_i is the output of the i-th industry and $x := (x_1, \dots, x_r)$ is the row vector of outputs;
- $\gamma := (c_1, \dots, c_r)$ is the demand vector.

The fundamental relation that characterizes the open Leontief model is:

$$x(I - Q) = \gamma. \tag{24}$$

The system (24) has a solution if the matrix $I - Q$ has an inverse that will be denoted by N. The entries of the matrix N have a direct interpretation in the context of the economy: n_{ij} is the amount of output that the industry j must produce in order to fill a unit order for industry i (*i.e.* $\gamma = e_i$, the i-th unit vector).

Recalling the analysis developed in Section 4, it is of interest to interpret the results obtained by means of the Leontief model in terms of the Markov traffic equilibrium model. In Section 4, we turned our attention to the principal submatrices T, of the transition matrix A, such that $\det(I - T) \neq 0$. Suppose to replace the matrix Q with T in the relation (24).

We conjecture that the entry n_{ij} of the matrix $(I - T)^{-1}$ represents the contribution of the zones Z_i, in front of a unit traffic demand related to Z_j. Moreover, it should be investigated if the indexes of dispersion, introduced in Definition 4.3, may have any interpretation in the Leontief model.

Further developments of the analysis can be obtained by embedding the Leontief model in a suitable Markov chain [8], which turns out to be closely related to the Markov chain associated to the traffic equilibrium model.

7. Appendix B

In this Appendix we report some of the peculiar properties of a Markov chain that can be of interest in the analysis developed in the present paper.

The next result provides a sufficient condition in order to obtain that the reverse implication holds in the Proposition 4.1.

Proposition 7.1. *Let T be a positive matrix of order m and H be a regular stochastic matrix of order m such that $T \leq H$. If $\det(I - T) = 0$, then $T = H$ so that T is stochastic.*

Proof. It is a consequence of the following result concerning regular matrices (see *e.g.* [14]).

Let C be a regular matrix of order m and B any matrix of order m such that $0 \leq B \leq C$; let λ_1 be the greatest real eigenvalue of C and β be an eigenvalue of B. If it results $|\beta| = \lambda_1$ then $B = C$.

Let us prove our proposition. Since H is stochastic it is well known that $\lambda_1 = 1$. The condition $\det(I - T) = 0$ implies that T admits the eigenvalue $\beta = 1$. By the above mentioned result we obtain that $T = H$. □

Corollary 7.1. *Let T be the principal submatrix of A obtained by selecting the rows (and columns) of indexes in the subset $S' \subset S$. If $T > 0$ and $\det(I - T) = 0$, then T is stochastic, so that S' is a closed subset of zones.*

The following theorem states the existence of the limit matrix Π defined by (14):

Theorem 7.1. *Condition (14) holds if and only if the unique eigenvalue λ of A, with modulus 1, is $\lambda = 1$ and in case $\lambda = 1$ has multiplicity k, there exist k linearly independent eigenvectors associated to this eigenvalue.*

The previous theorem states a necessary and sufficient condition such that there exist an equilibrium distribution for the model, which allows us to treat also the non regular cases. To this end it is of interest the following result:

Proposition 7.2. Π *is a stochastic matrix such that*

- *the rows are left (probability) eigenvectors of A corresponding to the eigenvalue 1;*
- *the columns are right eigenvectors of A corresponding to the eigenvalue 1.*

Proof. It is enough to notice that the matrix Π fulfils the relations

$$\Pi A = A\Pi = \Pi.$$

□

The following example is concerned with a non regular model.

Example 7.1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1/6 & 1/6 & 2/3 & 0 \\ 1/4 & 0 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easy to see that the characteristic equation associated to A is

$$(1 - \lambda)(1/6 - \lambda)(1/2 - \lambda)(1 - \lambda) = 0$$

so that the eigenvalue $\lambda = 1$ has multiplicity 2.

The (probability) eigenvectors associated to this eigenvalue are

$$(1, 0, 0, 0) \quad \text{and} \quad (0, 0, 0, 1)$$

so that, by Theorem 7.1, the limit matrix Π exists. In order to find the matrix Π, we can apply Proposition 7.2.

The matrix Π must be of the form

$$\Pi = \begin{pmatrix} a & 0 & 0 & b \\ c & 0 & 0 & d \\ e & 0 & 0 & f \\ g & 0 & 0 & h \end{pmatrix}.$$

Obviously, since the states 1 and 4 are absorbing, it will be

$$a = 1,\ b = 0,\ g = 0,\ h = 1.$$

By the second statement of Proposition 7.2, we have that the following system must be fulfilled:

$$\begin{cases} \frac{1}{6} + \frac{1}{6}c + \frac{2}{3}e = c \\ \frac{1}{4} + \frac{1}{2}e = e \\ \frac{1}{6}d + \frac{2}{3}f = d \\ \frac{1}{2}f + \frac{1}{4} = f \end{cases}$$

which solution is $c = \frac{3}{5}$, $d = \frac{2}{5}$, $e = f = \frac{1}{2}$.

The limit matrix is

$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 3/5 & 0 & 0 & 2/5 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

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