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MODELLING AND ANALYSIS OF A BULK SERVICE QUEUEING MODEL WITH MULTIPLE WORKING VACATIONS AND SERVER BREAKDOWN

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Abstract. In this paper, a single server queue with variable batch size service, Poisson bulk arrival with multiple working vacations and server breakdown is considered. In working vacation, the server works with different rates rather than completely stoping the service during the vacation period. In this model, during the working vacation the server starts the service if it finds at least one customer in the queue with a maximum of 'b' customers, otherwise the server serves with variable batch size. Service time in working vacation and in regular period follows general distribution. The probability generating function of a queue size at an arbitrary time epoch as well as other completion epochs is derived. Expected queue length in a steady state is obtained. Also a numerical illustration is presented.

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1. Introduction

Bulk queueing models have been analyzed in the past by several authors. In these models, the customers arrive in bulk and provide the service in batches. Server vacation models are useful for the systems in which the server wants to utilize the idle time for different purposes. Application of vacation models can be found in production line systems, designing local area networks and data communication systems. Queueing systems with general bulk service and vacations have been studied extensively by many authors. Doshi [3] and Takagi [13] have made comprehensive surveys of queueing systems with vacations.

Lee [8] has developed a procedure to calculate the system size probabilities for a bulk queueing model. Krishna Reddy *et al.* [7], Arumuganathan and Jeyakumar [1] have studied Bulk queueing models with different parameters. Jeyakumar and Senthilnathan [5] have discussed a study on the behavior of the server breakdown without interruption in a $\mathrm{M}^x/\mathrm{G}(a,b)/1$ queueing system with multiple vacations and closedown time.

Working vacation policies of queueing systems have gained much attention in the literature recently. In many real life situations, the server can be used in various forms under different rates (service in low rate) during the vacation period. Such a queueing system is called a queue with working vacation and was introduced by Servi and Finn [12]. Zhang and Xu [19] have investigated an M/M/1 queue with multiple working vacations and N-policy using quasi birth and death processes through matrix-geometric solution method. Liu *et al.* [11] have discussed stochastic decompositions in the M/M/1 queue with working vacations. A batch arrival $\mathrm{M}^X/\mathrm{M}/1$

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queue with single working vacation was studied by Xu *et al.* [16]. They derived PGF of the stationary system length distribution by using matrix analytic method. Tian *et al.* [14] have discussed the M/M/1 Queue with Single Working Vacation.

Wu and Takagi [15] extended Servi and Finn [12] M/M/1/WV model to an M/G/1/WV model, where the service times during service period, the service times during working vacation as well as vacation times are generally distributed. Queue-length distribution of the M/G/1 queue with working vacations (M/G/1/WV) was presented by Kim *et al.* [6]. A GI/M/1 queue with multiple working vacations has been investigated by Baba [2]. Li and Tian [9] have provided a detailed discussion of a GI/M/1 queue with single working vacation. A bulk input $\mathrm{Geom}^{[X]}$/Geom/1 queue with single working vacation has been analyzed by Xu *et al.* [17]. They derived PGF of the stationary queue size using the matrix analysis method, from which they got the stochastic decomposition result for the PGF of the stationary queue size. Li *et al.* [10] have considered an M/G/1 queue with exponentially working vacations using matrix analytic approach and they derived the expected number of customers in the queue at an arbitrary time epoch and obtained other measures too. Yi *et al.* [18] have presented the steady-state queue-length distribution of the Geo/G/1 queue with disasters.

In the literature of queueing models with working vacation, it is observed that, working vacation queueing models with server breakdown is not available in the literature except Jain and Jain [4] also it is noted that papers on bulk service with working vacation do not exist in the literature. It is needed to steady the problem existing in a real time situation, which is discussed in the following section, which is the motivation for the development of this paper.

Our paper differs from the existing ones in the following way: for the first time to our knowledge the working vacation and server breakdown concept is newly considered for a variable batch size service queuing model which is of most practical importance. Probability generating function of queue length distribution at an arbitrary time epoch in steady state is obtained by using supplementary variable technique by Lee's method. Also the PGF of queue size at various completion epochs such as idle in working vacation, service in working vacation and renovation in working vacation are obtained.

This paper is organized as follows: In Section 2, the description of the queueing model and the steady state equations has been developed. Queue size distribution is discussed in Section 3. The probability generating function (PGF) of the queue length distribution in steady state at an arbitrary time completion epoch as well as the probability generating function of idle in working vacation, service completion in working vacation and service completion in regular period epochs are derived in Section 4. Expected queue length of the queueing system is obtained in Section 5. Numerical results are presented in Section 6. Finally the findings concluded in Section 7.

2. Description of the model and system equations

A practical situation for the proposed model is observed in flare nuts production industries wherein our attention is focused on turret lathe machine. By using this machine they produce the flare nuts by machining operations like turning, drilling, threading and cutting the brass rod. Turret lathe is a semi automatic machine and it has the provision to vary the speed of the machining operations by using the lever. The operator can provide the batch service for producing minimum '1' flare nut to maximum 'b' flare nuts in the working vacation period with slow service rate. After completing a working vacation period, if the requirement of the flare nuts is more than the threshold value (a), the operator provides the batch service for producing minimum 'a' flare nut to maximum 'b' flare nuts at a faster rate of speed by changing over the lever to complete the process early. This is a practical and existing situation in day-to-day affairs of a machine shop. In case of server break down, the server can manage the service to complete the batch by manual operation after which, the renovation process will take place.

The above process can be modeled as bulk service queueing model with multiple working vacations and server breakdown. In this queueing system, during the working vacation period, the server provides the batch service for minimum '1' customer to maximum of 'b' customers in slow service rate. Also if the system becomes empty, it remains idle. After completing a batch of service in working vacation, if there is any breakdown in the server with probability (π), the renovation of service station will be considered with slow renovation rates γ_1. After completing the renovation of the service station or if there is no breakdown of the server with probability $(1-\pi)$, again it starts the service (slow rate) with maximum of 'b' customers till the working vacation is complete. At the end of the working vacation, if it finds 'a' customers in queue, it starts the service at a fast rate (regular period) with maximum of 'b' customers. After completing a batch of service in regular period, if there is any breakdown in the server with probability (π), the renovation of service station will be carried out with different renovation rates γ_2. After completing the renovation of service station in regular period or if there is no breakdown of the server with probability $(1-\pi)$, if the queue length is greater than 'a', again it starts the regular service with maximum of 'b' customers. Otherwise, it continues with another working vacation and so on. Service time and renovation time in working vacation and in regular period are follows the general distribution.

2.1. Notations and assumptions

The following notations are used in this paper

Arrival rate is λ, let μ_v and μ_b be the service rate in working vacation and service rate in regular period respectively, γ_1 and γ_2 are the renovation rates in working vacation and renovation rate in regular period respectively. Working vacation duration follows an exponential distribution with parameterη. X is the group size random variable, g_k is the probability of $X = k$, $X(z)$ is the Probability generating function (PGF) of X. Let $S_v(.)$, $S_b(.)$, $R^{(W)}(.)$ and $R^{(B)}(.)$ represent the cumulative distribution function (CDF) of service time in working vacation, services time in regular period, renovation in working vacation, renovation in regular period respectively and their corresponding probability density functions be $s_v(x)$, $s_b(x)$, $r^{(W)}(x)$ and $r^{(B)}(x)$. Define $S_v^0(t)$, $S_b^0(t)$, $R_{(W)}^0(t)$ and $R_{(B)}^0(t)$ as the remaining service time in working vacation of a batch, remaining service time in regular period of a batch, remaining renovation time in working vacation of a batch, remaining renovation time in regular period of a batch at time 't' respectively and denote $\tilde{S}_v(\theta)$, $\tilde{S}_b(\theta)$, $R^{(W)}(\theta)$ and $R^{(B)}(\theta)$ the Laplace–Stieltje's transform (LST) of S_v, S_b, $R^{(W)}$ and $R^{(B)}$respectively.

We define,

$Y(t)$ = (0), [1], {2}, [3] and {4} if the server is on (idle in working vacation)
[Busy in working vacation], {Busy in regular period},
[renovation in working vacation] and {renovation in regular period}

$Z(t)$ = j, if the server is on jth working vacation.

$N_s(t)$ = Number of customers in the service at time 't'.

$N_q(t)$ = Number of customers in the queue at time 't'.

The supplementary variables $S_v^0(t)$, $S_b^0(t)$, $R_{(W)}^0(t)$ and $R_{(B)}^0(t)$ are introduced in order to obtain bivariate Markov process $\{N(t), Y(t)\}$, where $N(t) = \{Nq(t), Ns(t)\}$.

Let us define the probabilities as,

$$q_0(t) = P\left\{N_q(t) = 0,\ x \leqslant Q^0(t) \leqslant x + \mathrm{d}t,\ Y(t) = 0\right\}.$$

This denotes that there is no customer in the queue, if the system is idle.

$$W_{ij}(x,t)\mathrm{d}t = P\left\{N_s(t) = i,\ N_q(t) = j,\ x \leqslant S_v^0(t) \leqslant x + \mathrm{d}t,\ Y(t) = 1\right\},\ 1 \leqslant i \leqslant b,\ j \geqslant 0.$$
$$P_{ij}(x,t)\mathrm{d}t = P\left\{N_s(t) = i,\ N_q(t) = j,\ x \leqslant S_b^0(t) \leqslant x + \mathrm{d}t,\ Y(t) = 2\right\},\ a \leqslant i \leqslant b,\ j \geqslant 0.$$

That means there are 'i' customers under service, 'j' customers in the queue, the server is busy with remaining service time x in working vacation and regular period respectively.

In a similar manner, we define,

$$R_n^{(W)}(x,t)\mathrm{d}t = P\left\{N_q(t) = n,\ x \leqslant R_{(W)}^0(t) \leqslant x + \mathrm{d}t,\ Y(t) = 3\right\},\ n \geqslant 0.$$
$$R_n^{(B)}(x,t)\mathrm{d}t = P\left\{N_q(t) = n,\ x \leqslant R_{(B)}^0(t) \leqslant x + \mathrm{d}t,\ Y(t) = 4\right\},\ n \geqslant 0.$$

This means that there are 'n' customers waiting in the queue under renovation in working vacation and regular period.

We develop the system size equations. These equations provide the basis for the analysis given in sequel. These equations are obtained at time $t+\Delta$t considering all possibilities. One can note that when time t is increased by Δt, the remaining service time and renovation time in working vacation period and regular period will be reduced by $x-\Delta t$.

$$\begin{aligned} q_0(t+\Delta t) &= q_0(t)(1-\lambda\Delta t)+(1-\pi)\sum_{m=a}^{b} W_{m0}(0,t)\,\Delta t\,+R_0^{(W)}(0,t)\Delta t \\ &\quad +(1-\pi)\sum_{m=a}^{b} p_{m0}(0,t)\,\Delta t\,+R_0^{(B)}(0,t)\Delta t \end{aligned} \tag{2.1}$$

$$\begin{aligned} W_{i0}(x-\Delta t,\ t+\Delta t) &= W_{i0}(x,t)(1-\lambda\Delta t-\eta\Delta t)+(1-\pi)\sum_{m=1}^{b} W_{mi}(0,t)s_v(x)\Delta t+R_i^{(W)}(0,t)\,s_v(x)\Delta t \\ &\quad +\lambda q_0(t)\,g_i s_v(x)\,+(1-\pi)\sum_{m=a}^{b} P_{mi}(0,t)s_v(x)\Delta t \\ &\quad +R_i^{(B)}(0,t)\,s_v(x)\Delta t+\,\eta\,s_v(x)\int_0^\infty W_{i0}(y)\mathrm{d}y\Delta t,\quad 1\leqslant i\leqslant,a-1,\quad j\geqslant 1 \end{aligned} \tag{2.2}$$

$$\begin{aligned} W_{i0}(x-\Delta t,\ t+\Delta t) &= W_{i0}(x,t)(1-\lambda\Delta t-\eta\Delta t)+(1-\pi)\sum_{m=1}^{b} W_{mi}(0,t)\,s_v(x)\Delta t \\ &\quad +R_i^{(W)}(0,t)\,s_v(x)\Delta t+\lambda q_0(t)\,g_i s_v(x),\ a\leqslant i\leqslant b, j\geqslant 1 \end{aligned} \tag{2.3}$$

$$W_{ij}(x-\Delta t,\ t+\Delta t) = W_{ij}(x,t)(1-\lambda\Delta t-\eta\Delta t)+\sum_{k=1}^{j} W_{ij-k}(x,t)\lambda g_k\Delta t\ \ j\geqslant 1,\ 1\leqslant i\leqslant b-1 \tag{2.4}$$

$$\begin{aligned} W_{bj}(x-\Delta t,\ t+\Delta t) &= W_{bj}(x,t)(1-\lambda\Delta t-\eta\Delta t)+\sum_{k=1}^{j} W_{bj-k}(x,t)\lambda g_k\Delta t \\ &\quad +(1-\pi)\sum_{m=1}^{b} W_{mb+j}(0,t)s_v(x)\Delta t+R_{b+j}^{(W)}(0,t)\,s_v(x)\Delta t\,+\lambda q_0(t)\,g_{b+j}s_v(x), j\geqslant 1 \end{aligned} \tag{2.5}$$

$$\begin{aligned} P_{i0}(x-\Delta t,\ t+\Delta t) &= P_{i0}(x,t)(1-\lambda\Delta t)+(1-\pi)\sum_{m=a}^{b} P_{mi}(0,t)\,s_b(x)\Delta t+R_i^{(B)}(0,t)\ s_b(x)\Delta t \\ &\quad +\ \eta s_b(x)\int_0^\infty W_{i0}(y)\mathrm{d}y\Delta t,\ a\leqslant i\leqslant b \end{aligned} \tag{2.6}$$

$$P_{ij}(x-\Delta t,\ t+\Delta t) = P_{ij}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{j} P_{ij-k}(x,t)\lambda g_k\Delta t,\ \ j\geqslant 1,\ a\leqslant i\leqslant b-1 \tag{2.7}$$

$$\begin{aligned} P_{bj}(x-\Delta t,\ t+\Delta t) &= P_{bj}(x,t)(1-\lambda\Delta t)+\sum_{k=1}^{j} P_{bj-k}(x,t)\lambda g_k\Delta t+(1-\pi)\sum_{m=a}^{b} P_{mb+j}(0,t)s(x)\Delta t \\ &\quad +R_{b+j}^{(B)}(0,t)\ s_b(x)\Delta t+\eta s_b(x)\int_0^\infty W_{bj}(y)\mathrm{d}y\Delta t,\ j\geqslant 1 \end{aligned} \tag{2.8}$$

$$R_0^{(W)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t - \eta\Delta t)R_0^{(W)}(x,t) + \pi\sum_{m=a}^{b} W_{m0}(0,t)\, r^{(W)}(x) \tag{2.9}$$

$$R_n^{(W)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t - \eta\Delta t)R_n^{(W)}(x,t) + \pi\sum_{m=a}^{b} W_{mn}(0,t)\, r^{(W)}(x) + \lambda\sum_{k=1}^{n} R_{n-k}^{(W)}(x) g_k,\ n \geqslant 1 \tag{2.10}$$

$$R_0^{(B)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)R_0^{(B)}(x,t) + \pi\sum_{m=a}^{b} P_{m0}(0,t)\, r^{(B)}(x) + +\eta r^B(x)\int_0^\infty R_0^{(W)}(y)\mathrm{d}y\Delta t \tag{2.11}$$

$$R_n^{(B)}(x-\Delta t, t+\Delta t) = (1-\lambda\Delta t)R_n^{(B)}(x,t) + \pi\sum_{m=a}^{b} P_{mn}(0,t)\, r^{(B)}(x) + \lambda\sum_{k=1}^{n} R_{n-k}^{(B)}(x) g_k + \eta r^B(x)\int_0^\infty R_n^{(W)}(y)\mathrm{d}y\Delta t,\ n \geqslant 1 \tag{2.12}$$

In steady state, let us define for $x>0$, $P_{i,j}(x) = \lim_{t\to\infty} P_{i,j}(x,t)$ $a \leqslant i \leqslant b$, $W_{i,j}(x) = \lim_{t\to\infty} W_{i,j}(x,t)$, $1 \leqslant i \leqslant b$ for and $j \geqslant 0$, $R_n^W(x) = \lim_{t\to\infty} R_n^W(x,t)$ and $R_n^B(x) = \lim_{t\to\infty} R_n^B(x,t)$ for $n \geqslant 0$.

2.2. Steady state system equations

Dividing equations (2.1)–(2.12) by Δt and letting the limit $\Delta t \to 0$, the steady state queue size equations are obtained as

$$\lambda q_0 = (1-\pi)\sum_{m=a}^{b} W_{m0}(0) + R_0^{(W)}(0) + (1-\pi)\sum_{m=a}^{b} p_{m0}(0) + R_0^{(B)}(0) \tag{2.13}$$

$$-W_{i0}'(x) = -(\lambda+\eta)W_{i0}(x) + (1-\pi)\sum_{m=1}^{b} W_{mi}(0)\, s_v(x) + R_i^{(W)}(0)\, s_v(x) + \lambda q_0\, g_i s_v(x) + (1-\pi)\sum_{m=a}^{b} P_{mi}(0)s_v(x) + R_i^{(B)}(0)\, s_v(x) + \eta\, s_v(x)\int_0^\infty W_{i0}(y)\mathrm{d}y, \quad 1 \leqslant i \leqslant a-1 \tag{2.14}$$

$$-W_{i0}'(x) = -(\lambda+\eta)W_{i0}(x) + (1-\pi)\sum_{m=1}^{b} W_{mi}(0,t)\, s_v(x) + R_i^{(W)}(0)\, s_v(x) + \lambda q_0(t)\, g_i s_v(x), \quad a \leqslant i \leqslant b \tag{2.15}$$

$$-W_{ij}'(x) = -(\lambda+\eta)W_{ij}(x) + \sum_{k=1}^{j} W_{ij-k}(x,t)\lambda g_k, \quad 1 \leqslant i \leqslant b-1,\ j \geqslant 1 \tag{2.16}$$

$$-W_{bj}'(x) = -(\lambda+\eta)W_{bj}(x) + \sum_{k=1}^{j} W_{bj-k}(x,t)\lambda g_k + (1-\pi)\sum_{m=1}^{b} W_{mb+j}(0,t)s_v(x) + R_{b+j}^{(W)}(0,t)\, s_v(x) + \lambda q_0(t)\, g_{b+j} s_v(x), \quad j \geqslant 1 \tag{2.17}$$

$$-P_{i0}^{'}(x) = -\lambda P_{i0}(x) + (1-\pi)\sum_{m=a}^{b} P_{mi}(0,t)\, s_b(x) + R_i^{(B)}(0,t)\ s_b(x)\ +\ \eta s_b(x)\int\limits_0^\infty W_{i0}(y)\mathrm{d}y, \quad a \leqslant i \leqslant b \tag{2.18}$$

$$-P_{ij}^{'}(x) = -\lambda P_{ij}(x) + \sum_{k=1}^{j} P_{ij-k}(x,t)\lambda g_k, \quad a \leqslant i \leqslant b-1,\ j \geqslant 1 \tag{2.19}$$

$$-P_{bj}^{'}(x) = -\lambda P_{bj}(x) + \sum_{k=1}^{j} P_{bj-k}(x,t)\lambda g_k + (1-\pi)\sum_{m=a}^{b} P_{mb+j}(0,t)s(x)$$
$$+\ R_{b+j}^{(B)}(0,t)\ s_b(x) + \eta s_b(x)\int\limits_0^\infty W_{bj}(y)\mathrm{d}y, \quad j \geqslant 1 \tag{2.20}$$

$$-R_0^{(W)'}(x) = -(\lambda+\eta)\, R_0^{(W)}(x) + \pi\sum_{m=a}^{b} W_{m0}(0,t)\, r^{(W)}(x) \tag{2.21}$$

$$-R_n^{(W)'}(x) = -(\lambda+\eta)R_n^{(W)}(x) + \pi\sum_{m=a}^{b} W_{mn}(0,t)\, r^{(W)}(x)\ +\lambda\sum_{k=1}^{n} R_{n-k}^{(W)}(x)g_k, \quad n \geqslant 1 \tag{2.22}$$

$$-R_0^{(B)'}(x) = -\lambda R_0^{(B)}(x) + \pi\sum_{m=a}^{b} P_{m0}(0,t)\, r^{(B)}(x) + \eta r^{(B)}(x)\int\limits_0^\infty R_0^{(W)}(y)\mathrm{d}y \tag{2.23}$$

$$-R_n^{(B)'}(x) = -\lambda R_n^{(B)}(x) + \pi\sum_{m=a}^{b} P_{mn}(0,t)\, r^{(B)}(x)\ +\lambda\sum_{k=1}^{n} R_{n-k}^{(B)}(x)g_k\ +\eta\, r^{(B)}(x)\int\limits_0^\infty R_n^{(W)}(y)\mathrm{d}y, \quad n \geqslant 1 \tag{2.24}$$

3. Queue size distribution

The Laplace–Stieltjes transforms of$P_{ij}(x)$,$W_{ij}(x)$, $R_n^{(W)}(x)$ and $R_n^{(B)}(x)$ are defined as follows:

$$\tilde{P}_{i\,j}(\theta) = \int\limits_0^\infty \mathrm{e}^{-\theta\, x} P_{ij}(x)\mathrm{d}x,\ \ \tilde{W}_{i\,j}(\theta) = \int\limits_0^\infty \mathrm{e}^{-\theta\, x} W_{ij}(x)\mathrm{d}x\,, \tilde{R}^{(B)}(\theta) = \int\limits_0^\infty \mathrm{e}^{-\theta\, x} R_n^{(B)}(x)\mathrm{d}x,$$

and

$$\tilde{R}_n^{(W)}(\theta) = \int\limits_0^\infty \mathrm{e}^{-\theta\, x} R_n^{(W)}(x)\mathrm{d}x.$$

Taking Laplace–Stieltje's transforms on both sides of the equations (2.13)–(2.24) we get,

$$\theta\tilde{W}_{i0}(\theta) - W_{i0}(0) = (\lambda+\eta)\tilde{W}_{i0}(\theta) - \tilde{s}_v(\theta)\Bigg\{(1-\pi)\sum_{m=1}^{b} W_{mi}(0)\ + R_i^{(W)}(0) + \lambda q_0\, g_i$$
$$+(1-\pi)\sum_{m=a}^{b} P_{mi}(0) + R_i^{(B)}(0) + \eta\int\limits_0^\infty W_{i0}(y)\mathrm{d}y\Bigg\}, \quad 1 \leqslant i \leqslant a-1 \tag{3.1}$$

$$\theta\tilde{W}_{i0}(\theta) - W_{i0}(0) = (\lambda+\eta)\tilde{W}_{i0}(\theta) - \tilde{s}_v(\theta)\left[(1-\pi)\sum_{m=1}^{b} W_{mi}(0) + R_i^{(W)}(0) + \lambda q_0\, g_i\right], \quad a \leqslant i \leqslant b \tag{3.2}$$

$$\theta\tilde{W}_{ij}(\theta) - W_{ij}(0) = (\lambda+\eta)\tilde{W}_{ij}(\theta) - \lambda\sum_{k=1}^{j} W_{ij-k}(\theta)\, g_k, \quad 1 \leqslant i \leqslant b-1, j \geqslant 1 \tag{3.3}$$

$$\theta\tilde{W}_{bj}(\theta) - W_{bj}(0) = (\lambda+\eta)\tilde{W}_{bj}(\theta) - \tilde{s}_v(\theta)\left[(1-\pi)\sum_{m=1}^{b} W_{mb+j}(0) \; + R^{(W)}_{b+j}(0) \; + \lambda q_0\, g_{b+j}\right] - \lambda\sum_{k=1}^{j} W_{bj-k}(\theta)\, g_k, \quad j \geqslant 1 \tag{3.4}$$

$$\theta\tilde{P}_{i0}(\theta) - P_{i0}(0) = \lambda\tilde{P}_{i0}(\theta) - \tilde{s}_b(\theta)\left[(1-\pi)\sum_{m=a}^{b} P_{mi}(0) + R^{(B)}_{i}(0) + \eta\int_0^\infty W_{i0}(y)\mathrm{d}y\right], \quad a \leqslant i \leqslant b \tag{3.5}$$

$$\theta\tilde{P}_{ij}(\theta) - P_{ij}(0) = \lambda\tilde{P}_{ij}(\theta) - \sum_{k=1}^{j} P_{ij-k}(\theta)\lambda g_k, \quad a \leqslant i \leqslant b-1,\, j \geqslant 1 \tag{3.6}$$

$$\theta\tilde{P}_{bj}(\theta) - P_{bj}(0) = \lambda\tilde{P}_{bj}(\theta) - \tilde{s}_b(\theta)\left[(1-\pi)\sum_{m=a}^{b} P_{m\,b+j}(0) \; + R^{(B)}_{b+j}(0) + \eta\int_0^\infty W_{bj}(y)\mathrm{d}y\right] - \sum_{k=1}^{j} P_{bj-k}(\theta)\lambda g_k, \quad j \geqslant 1 \tag{3.7}$$

$$\theta\tilde{R}^{(W)}_0(\theta) - R^{(W)}_0(0) = (\lambda+\eta)\tilde{R}^{(W)}_0(\theta) - \pi\sum_{m=1}^{b} W_{m0}(0)\,\tilde{R}^{(W)}(\theta) \tag{3.8}$$

$$\theta\tilde{R}^{(W)}_n(\theta) - R^{(W)}_n(0) = (\lambda+\eta)\tilde{R}^{(W)}_n(\theta) - \pi\sum_{m=1}^{b} W_{mn}(0)\,\tilde{R}^{(W)}(\theta) - \lambda\sum_{k=1}^{n} R^{(W)}_{n-k}(\theta) g_k, \quad n \geqslant 1 \tag{3.9}$$

$$\theta\tilde{R}^{(B)}_0(\theta) - R^{(B)}_0(0) = \lambda\tilde{R}^{(B)}_0(\theta) - \pi\sum_{m=a}^{b} P_{m0}(0)\,\tilde{R}^{(B)}(\theta) - \eta\tilde{R}^{(B)}(\theta)\int_0^\infty R^{(W)}_0(y)\mathrm{d}y \tag{3.10}$$

$$\theta\tilde{R}^{(B)}_n(\theta) - R^{(B)}_n(0) = \lambda\tilde{R}^{(B)}_n(\theta) - \pi\sum_{m=a}^{b} P_{mn}(0)\tilde{R}^{(B)}(\theta) - \lambda\sum_{k=1}^{n} R^{(B)}_{n-k}(\theta) g_k - \eta\tilde{R}^{(B)}(\theta)\int_0^\infty R^{(W)}_n(y)\mathrm{d}y, \quad n \geqslant 1 \tag{3.11}$$

To find the probability generating function (PGF), we define the PGF's as follows:

$$\tilde{P}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{P}_{ij}(\theta) z^j, \quad P_i(z,0) = \sum_{j=0}^{\infty} P_{ij}(0) z^j, \quad a \leqslant i \leqslant b$$

$$\tilde{W}_i(z,\theta) = \sum_{j=0}^{\infty} \tilde{W}_{ij}(\theta) z^j, \quad W_i(z,0) = \sum_{j=0}^{\infty} W_{ij}(0) z^j, \quad a \leqslant i \leqslant b$$

$$\tilde{R}^{(W)}(z,\theta) = \sum_{n=0}^{\infty} \tilde{R}^{W}_n(\theta) z^n, \quad R^{(W)}(z,0) = \sum_{n=0}^{\infty} R^{W}_n(0) z^n$$

$$\tilde{R}^{(B)}(z,\theta) = \sum_{n=0}^{\infty} \tilde{R}^{B}_n(\theta) z^n, \quad R^{(B)}(z,0) = \sum_{n=0}^{\infty} R^{B}_n(0) z^n \tag{3.12}$$

By multiplying the equations (3.1)–(3.11) with suitable power of z^n and summing over n, ($n = 0$ to ∞) and using (3.12)

$$\lambda q_0 = \left[(1-\pi)\sum_{m=1}^{b} W_{m0}(0) + R_0^{(W)}(0) + (1-\pi)\sum_{m=a}^{b} p_{m0}(0) + R_0^{(B)}(0)\right], \quad j \geqslant 1 \tag{3.13}$$

$$(\theta-\lambda-\eta+\lambda X(z))\tilde{W}_i(z,\theta) = W_i(z,0) - \tilde{S}_v(\theta)\left\{(1-\pi)\sum_{m=1}^{b} W_{mi}(0) + R_i^{(W)}(0) + \lambda q_0\, g_i \right.$$
$$\left. +(1-\pi)\sum_{m=a}^{b} P_{mi}(0) + R_i^{(B)}(0) + \eta\tilde{W}_i(z,0)\right\}, \quad 1 \leqslant i \leqslant a-1 \tag{3.14}$$

$$(\theta-\lambda-\eta+\lambda X(z))\tilde{W}_i(z,\theta) = W_i(z,0) - \tilde{S}_v(\theta)\left[(1-\pi)\sum_{m=1}^{b} W_{mi}(0) + R_i^{(W)}(0) + \lambda q_0 g_i\right] \quad a \leqslant i \leqslant b-1 \tag{3.15}$$

$$z^b(\theta-\lambda-\eta+\lambda X(z))\tilde{W}_b(z,\theta) = z^b W_b(z,0) - \tilde{S}_v(\theta)\left\{\sum_{m=1}^{b}(1-\pi)\left[W_m(z,0) - \sum_{j=0}^{b-1} W_{mj}(0) z^j\right]\right.$$
$$\left. + \left[R^{(W)}(z,0) - \sum_{j=0}^{b-1} R_j^{(W)}(0)\, z^j\right] + \lambda\left[q_0\left[X(z) - \sum_{k=1}^{b-1} g_k z^k\right]\right]\right\} \tag{3.16}$$

$$(\theta-\lambda+\lambda X(z))\tilde{P}_i(z,\theta) = P_i(z,0) - \tilde{S}_b(\theta)\left[(1-\pi)\sum_{m=a}^{b} P_{mi}(0) + R_i^{(B)}(0) + \eta\tilde{W}_i(z,0)\right], \quad a \leqslant i \leqslant b-1 \tag{3.17}$$

$$z^b(\theta-\lambda+\lambda X(z))\tilde{P}_b(z,\theta) = z^b P_b(z,0) - \tilde{S}_b(\theta)\left\{\sum_{m=a}^{b}(1-\pi)\left[P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0) z^j\right]\right.$$
$$\left. + \left[R^{(B)}(z,0) - \sum_{j=0}^{b-1} R_j^{(B)}(0)\, z^j\right] + \eta\tilde{W}_b(z,0)\right\} \tag{3.18}$$

$$(\theta-\lambda-\eta+\lambda X(z))\tilde{R}^{(W)}(z,\theta) = R^{(W)}(z,0) - \pi\,\tilde{R}^{(W)}(\theta)\sum_{m=a}^{b} W_m(z,0) \tag{3.19}$$

$$(\theta-\lambda+\lambda X(z))\tilde{R}^{(B)}(z,\theta) = R^{(B)}(z,0) - \pi\tilde{R}^{(B)}(\theta)\left[\sum_{m=a}^{b} P_m(z,0) + \eta\tilde{R}^{(W)}(z,0)\right]. \tag{3.20}$$

By substituting $\theta = \lambda + \eta - \lambda X(z)$ in the equations (3.14)–(3.16) and (3.19), $\theta = \lambda - \lambda X(z)$ in the equations (3.17), (3.18) and (3.20), we get,

$$W_i(z,0) = \tilde{S}_v(\lambda+\eta-\lambda X(z))\left\{(1-\pi)\sum_{m=1}^{b} W_{mi}(0) + R_i^{(W)'}(0) + \lambda q_0\, g_i\right.$$
$$\left. +(1-\pi)\sum_{m=a}^{b} P_{mi}(0) + R_i^{(B)'}(0) + \eta\tilde{W}_i(z,0)\right\}, \ 1 \leqslant i \leqslant a-1 \tag{3.21}$$

$$W_i(z,0) = \tilde{S}_v(\lambda+\eta-\lambda X(z)) \left[(1-\pi) \sum_{m=1}^{b} W_{mi}(0) + R_i^{(W)}(0) + \lambda q_0 g_i \right], \quad a \leqslant i \leqslant b-1 \tag{3.22}$$

$$z^b W_b(z,0) = \tilde{S}_v(\lambda+\eta-\lambda X(z)) \left\{ \sum_{m=1}^{b} (1-\pi) \left[W_m(z,0) - \sum_{j=0}^{b-1} W_{mj}(0) z^j \right] + \left[R^{(W)}(z,0) - \sum_{j=0}^{b-1} R_j^{(W)}(0)\, z^j \right] + \lambda \left[q_0 \left[X(z) - \sum_{k=1}^{b-1} g_k z^k \right] \right] \right\}, \quad j \geqslant 1$$

$$W_b(z,0) = \frac{\tilde{S}_V(\lambda-\lambda X(z)) f(z)}{z^b - (1-\pi)\,\tilde{S}_V(\lambda+\eta-\lambda X(z)) - \pi\,\tilde{S}_V(\lambda+\eta-\lambda X(z)) \tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))}$$

where

$$f_1(z) = \left[(1-\pi) + \pi\,\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z)) \right] \sum_{m=1}^{b-1} W_m(z,0) - (1-\pi) \sum_{j=0}^{b-1} \sum_{m=1}^{b} W_{mj}(0) z^j - \sum_{j=0}^{b-1} R_j^{(W)}(0)\, z^j + \lambda \left[q_0 \left[X(z) - \sum_{k=1}^{b-1} g_k z^k \right] \right] \tag{3.23}$$

$$P_i(z,0) = \tilde{S}_b(\lambda-\lambda X(z)) \left[(1-\pi) \sum_{m=a}^{b} P_{mi}(0) + R_i^{(B)}(0) + \eta \tilde{W}_i(z,0) \right], a \leqslant i \leqslant b-1 \tag{3.24}$$

$$z^b P_b(z,0) = \tilde{S}_b(\lambda-\lambda X(z)) \left\{ \sum_{m=a}^{b} (1-\pi) \left[P_m(z,0) - \sum_{j=0}^{b-1} P_{mj}(0) z^j \right] + \left[R^{(B)}(z,0) - \sum_{j=0}^{b-1} R_j^{(B)}(0)\, z^j \right] + \eta \tilde{W}_b(z,0) \right\}, \quad j \geqslant 1$$

$$P_b(z,0) = \frac{\tilde{S}_b(\lambda-\lambda X(z)) f(z)}{z^b - (1-\pi)\,\tilde{S}_b(\lambda-\lambda X(z)) - \pi\,\tilde{S}_b(\lambda-\lambda X(z)) \tilde{R}^{(B)}(\lambda-\lambda X(z))}$$

where,

$$f_2(z) = \left[(1-\pi) + \pi\,\tilde{R}^{(B)}(\lambda-\lambda X(z)) \right] \sum_{m=a}^{b-1} P_m(z,0) - \sum_{j=0}^{b-1} \sum_{m=a}^{b} P_{mj}(0) z^j - \sum_{j=0}^{b-1} R_j^{(B)}(0)\, z^j + \eta\pi\,\tilde{R}^{(B)}(\lambda-\lambda X(z)) \tilde{R}^{(W)}(z,0) + \eta \tilde{W}_b(z,0) \tag{3.25}$$

$$R^{(W)}(z,0) = \pi\,\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z)) \left[\sum_{m=1}^{b} W_m(z,0) \right] \tag{3.26}$$

$$R^{(B)}(z,0) = \pi\,\tilde{R}^{(B)}(\lambda-\lambda X(z)) \left[\sum_{m=1}^{b} P_m(z,0) + \eta \tilde{R}^{(W)}(z,0) \right]. \tag{3.27}$$

Using the equations (3.21) to (3.27) in (3.14) to (3.20) we get,

$$\tilde{W}_i(z,\theta)=\frac{\left[\tilde{S}_v(\lambda+\eta-\lambda X(z))-\tilde{S}_v(\theta)\right]\left\{(1-\pi)\sum\limits_{m=1}^{b}W_{mi}(0)+R_i^{(W)'}(0)+\lambda q_0\,g_i+(1-\pi)\sum\limits_{m=a}^{b}P_{mi}(0)+R_i^{(B)'}(0)+\eta\tilde{W}_i(z,0)\right\}}{(\theta-\lambda-\eta+\lambda X(z))},\quad 1\leqslant i\leqslant a-1 \tag{3.28}$$

$$\tilde{W}_i(z,\theta)=\frac{\left[\tilde{S}_v(\lambda+\eta-\lambda X(z))-\tilde{S}_v(\theta)\right]\left[(1-\pi)\sum\limits_{m=1}^{b}W_{mi}(0)+R_i^{(W)}(0)+\lambda q_0 g_i\right]}{(\theta-\lambda-\eta+\lambda X(z))},\quad a\leqslant i\leqslant b-1 \tag{3.29}$$

$$\tilde{W}_b(z,\theta)=\frac{\left[\tilde{S}_v(\lambda+\eta-\lambda X(z))-\tilde{S}_v(\theta)\right]f_1(z)}{\left[z^b-(1-\pi)\,\tilde{S}_V(\lambda+\eta-\lambda X(z))-\pi\,\tilde{S}_V(\lambda+\eta-\lambda X(z))\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))\right](\theta-\lambda-\eta+\lambda X(z))} \tag{3.30}$$

where

$$f_1(z)=\left[(1-\pi)+\pi\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))\right]\sum_{m=1}^{b-1}W_m(z,0)-(1-\pi)\sum_{j=0}^{b-1}\sum_{m=1}^{b}W_{mj}(0)z^j$$
$$-\sum_{j=0}^{b-1}R_j^{(W)}(0)\,z^j+\lambda\left[q_0\left[X(z)-\sum_{k=1}^{b-1}g_k z^k\right]\right]$$

$$\tilde{P}_i(z,\theta)=\frac{\left[\tilde{S}_b(\lambda-\lambda X(z))-\tilde{S}_b(\theta)\right]\left[(1-\pi)\sum\limits_{m=a}^{b}P_{mi}(0)+R_i^{(B)}(0)+\eta\,\tilde{W}_i(z,0)\right]}{(\theta-\lambda+\lambda X(z))},\quad a\leqslant i\leqslant b-1 \tag{3.31}$$

$$\tilde{P}_b(z,\theta)=\frac{\left[\tilde{S}_b(\lambda-\lambda X(z))-\tilde{S}_b(\theta)\right]f_2(z)}{\left[z^b-(1-\pi)\,\tilde{S}_b(\lambda-\lambda X(z))-\pi\,\tilde{S}_b(\lambda-\lambda X(z))\tilde{R}^{(B)}(\lambda-\lambda X(z))\right](\theta-\lambda+\lambda X(z))} \tag{3.32}$$

where,

$$f_2(z)=\left[(1-\pi)+\pi\,\tilde{R}^{(B)}(\lambda-\lambda X(z))\right]\sum_{m=a}^{b-1}P_m(z,0)-\sum_{j=0}^{b-1}\sum_{m=a}^{b}P_{mj}(0)z^j-\sum_{j=0}^{b-1}R_j^{(B)}(0)\,z^j$$
$$+\eta\pi\,\tilde{R}^{(B)}(\lambda-\lambda X(z))\tilde{R}^{(W)}(z,0)+\eta\tilde{W}_b(z,0)$$

$$\tilde{R}^{(W)}(z,\theta)=\frac{\pi\left[\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))-\tilde{R}^{(W)}(\theta)\right]\left[\sum\limits_{m=a}^{b}W_m(z,0)\right]}{(\theta-\lambda+\lambda X(z))} \tag{3.33}$$

$$\tilde{R}^{(B)}(z,\theta)=\frac{\pi\left[\tilde{R}^{(B)}(\lambda-\lambda X(z))-\tilde{R}^{(B)}(\theta)\right]\left[\sum\limits_{m=1}^{b}P_m(z,0)+\eta\tilde{R}^{(W)}(z,0)\right]}{(\theta-\lambda+\lambda X(z))}. \tag{3.34}$$

4. Probability generating function of queue size

4.1. PGF of queue size at various epochs

The PGF of queue size at various completion epochs are obtained as follows:

(a) Service in working vacation completion epoch

Using equations (3.28) to (3.30) and substituting $\theta = 0$ and after some algebra, we get,

$$P_{SW}(z) = \frac{\begin{array}{c}(\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)\left\{\sum\limits_{i=1}^{b-1}(z^b-z^i)\,d_i^{(W)} + \left\{\sum\limits_{i=1}^{b-1}(z^b-z^i)\,g_i + X(z)\right\} d_0^{(B)} \right.\\ \left. + \left\{\left[\sum\limits_{i=1}^{b-1}(z^b-z^i)g_i + X(z)\right]-1\right\} d_0^{(W)} + z^b\sum\limits_{i=1}^{a-1} d_i^{(B)} + z^b\eta\right\}\end{array}}{\left[z^b-(1-\pi)\tilde{S}_v(\lambda+\eta-\lambda X(z))-\pi\tilde{S}_v(\lambda+\eta-\lambda X(z))\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))\right](-\lambda-\eta+\lambda X(z))}. \tag{4.1}$$

(b) Renovation in working vacation completion epoch

Using equations (3.33) and substituting $\theta = 0$ and after some computation, we get,

$$R_{RW}(z) = \frac{\begin{array}{c}I_{10}\left\{\sum\limits_{i=1}^{b-1}(z^b-z^i)\,d_i^{(W)} + \left\{\sum\limits_{i=1}^{b-1}(z^b-z^i)g_i + X(z)\right\} d_0^{(B)} \right.\\ \left. + \left\{\left[\sum\limits_{i=1}^{b-1}(z^b-z^i)g_i + X(z)\right]-1\right\} d_0^{(W)} + z^b\sum\limits_{i=1}^{a-1} d_i^{(B)} + z^b\eta\right\}\end{array}}{\left[z^b-(1-\pi)\tilde{S}_v(\lambda+\eta-\lambda X(z))-\pi\tilde{S}_v(\lambda+\eta-\lambda X(z))\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z))\right](-\lambda-\eta+\lambda X(z))}$$

$$I_{10} = \left[\pi\tilde{S}_v(\lambda+\eta-\lambda X(z))\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z)) - \pi\tilde{S}_v(\lambda+\eta-\lambda X(z))\right]. \tag{4.2}$$

(c) Idle period completion epoch

From equation (3.13),

$$P_{WI}(z) = \frac{\left\{d_0^{(W)} + d_0^{(B)}\right\}}{\lambda}, \tag{4.3}$$

$$O_3 = \left[(1-\pi)\tilde{S}_v(\lambda+\eta-\lambda X(z)) + \pi\tilde{S}_v(\lambda+\eta-\lambda X(z))\tilde{R}^{(W)}(\lambda+\eta-\lambda X(z)) - 1\right],$$

$$O_4 = \left[(1-\pi)\tilde{S}_b(\lambda+\eta-\lambda X(z)) + \pi\tilde{S}_b(\lambda+\eta-\lambda X(z))\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z)) - 1\right],$$

$$O_5 = O_3(-\lambda+\lambda X(z))\left[z^b - \tilde{S}_b(\lambda+\eta-\lambda X(z)) - O_2\right].$$

4.2. PGF of queue size at an arbitrary time epoch

The PGF of the queue size at an arbitrary time epoch is obtained as

$$P(z) = \sum_{m=1}^{b-1}\tilde{W}_m(z,0)+\tilde{W}_b(z,0) + \sum_{m=a}^{b-1}\tilde{P}_m(z,0)+\tilde{P}_b(z,0) + q_0 + \tilde{R}^{(w)}(z,0) + \tilde{R}^{(B)}(z,0). \tag{4.4}$$

Substitute (3.28) to (3.34) in (4.4) with $\theta = 0$ we get,

$$P(z) = \frac{\begin{array}{l}\lambda(-\lambda-\eta+\lambda X(z))\left[z^b-\tilde{S}_v(\lambda+\eta-\lambda X(z))-O_1\right]O_4\left[\sum\limits_{i=a}^{b-1}(z^b-z^i)d_i^{(B)}-\sum\limits_{i=1}^{a-1}d_i^{(B)}z^i-d_0^{(B)}\right]\\ +\left\{O_5+O_6+\left\{\eta\lambda\left\{z^b(\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)O_4\left\{(\tilde{S}_v(\lambda+\eta-\lambda X(z))+O_1)\sum\limits_{k=1}^{b-1}g_k+\left[X(z)-\sum\limits_{k=1}^{b-1}g_kz^k\right]\right\}\right.\right.\right.\\ \left.\left.\left.+\left\{\pi\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z))O_1O_4+\pi\left(z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right)O_1O_2\right\}\left\{\left[\sum\limits_{i=1}^{b-1}(z^b-z^i)g_i+X(z)\right]-1\right\}\right\}\right\}\right\}d_0^{(B)}\\ +\left\{O_5+\left\{z^b\eta\lambda\left\{(\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)(\tilde{S}_v(\lambda+\eta-\lambda X(z))+O_1)O_4\right.\right.\right.\\ \left.\left.\left.+\left\{\pi\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z))O_1O_4+\pi\left(z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right)O_1O_2\right\}\right\}\right\}\right\}\left[\sum\limits_{i=1}^{a-1}d_i^{(B)}+\sum\limits_{i=1}^{b-1}d_i^{(W)}\right]\\ -\left\{O_5+\eta\lambda\left\{z^b\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)O_4+\left\{\pi\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z))O_1O_4\right.\right.\right.\\ \left.\left.\left.+\pi\left(z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right)O_1O_2\right\}\right\}\right\}\sum\limits_{i=1}^{b-1}d_i^{(W)}z^i\\ +\left\{O_5+O_6+\left\{\eta\lambda\left\{z^b(\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)O_4\left\{\left\{(\tilde{S}_v(\lambda+\eta-\lambda X(z))+O_1)\sum\limits_{k=1}^{b-1}g_k+\left[X(z)-\sum\limits_{k=1}^{b-1}g_kz^k\right]\right\}-1\right\}\right.\right.\right.\\ \left.\left.\left.+\left\{\pi\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z))O_1O_4+\pi\left(z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right)O_1O_2\right\}\left\{\left[\sum\limits_{i=1}^{b-1}(z^b-z^i)g_i+X(z)\right]-1\right\}\right\}\right\}\right\}d_0^{(W)}\\ +z^b\eta\left\{\left\{\lambda O_4\left\{(-\lambda-\eta+\lambda X(z))\left[z^b-\tilde{S}_v(\lambda+\eta-\lambda X(z))-O_1\right]+\eta(\tilde{S}_v(\lambda+\eta-\lambda X(z))-1)\left[(\tilde{S}_v(\lambda+\eta-\lambda X(z))+O_1\right]\right.\right.\right.\\ \left.\left.\left.+\eta\left\{\pi\tilde{R}^{(B)}(\lambda+\eta-\lambda X(z))O_1O_4+\pi\left(z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right)O_1O_2\right\}\right\}\right\}+O_5\right\}\end{array}}{(-\lambda-\eta+\lambda X(z))(-\lambda+\lambda X(z))\left[z^b-\tilde{S}_v(\lambda+\eta-\lambda X(z))-O_1\right]\left[z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right]}. \tag{4.5}$$

This represents the PGF of number of customers in queue at an arbitrary time epoch.

4.3. Steady state condition

The probability generating function defined in (4.5) has to satisfy $P(1) = 1$. In order to satisfy this condition, applying L'Hospital rules and equating the expression to 1, we have,

$$\begin{array}{l}\left\{2\eta\lambda(N_2+N_5)\sum\limits_{i=a}^{b-1}(b-i)d_i^{(B)}+\eta\lambda(N_1+N_6)\sum\limits_{i=1}^{a-1}id_i^{(B)}\right.\\ \quad+\left\{O_5''+O_6''+\left\{\eta\lambda(N_1+bN_2+(SV1)\,O_4')\left\{\left\{((D+D^{(1)})-1)+\ (D+D^{(1)})(N_3+N_4)-1\right\}\sum\limits_{k=1}^{b-1}g_k+1\right\}\right\}\right.\\ \qquad\qquad\left.\left.-\ \eta\lambda N_2\sum\limits_{k=1}^{b-1}kg_k+\eta\lambda N_2(X1)\right\}\right\}\\ \qquad\qquad\left.+2N_5\sum\limits_{i=1}^{b-1}(b-i)g_i+(N_6+2(Rb1)N_5+2(SV1)^{(1)}O_4')(2N_5(X1)-1)\right\}d_0^{(B)}\\ \quad+\left\{O_5''+\lambda\eta\left\{(D+D^{(1)})(N_1+bN_2+(SV1)\,O_4'+N_3+N_4)+(N_6+2(Rb1)N_5+2(SV1)^{(1)}O_4'+2bN_5)\right.\right.\\ \qquad\qquad\left.\left.+2\pi D^{(1)}(b-(Sb1)-(Sb1)^{(2)})O_2'\right\}\left[\sum\limits_{i=1}^{a-1}d_i^{(B)}+\sum\limits_{i=1}^{b-1}d_i^{(W)}\right]\right.\\ \quad-\left\{O_5''+\lambda\eta\left\{(N_1+bN_2+(SV1)\,O_4'((D+D^{(1)})-1))+(D+D^{(1)})(N_3+N_4)\right.\right.\\ \quad\left.\left.+(N_6+2(Rb1)N_5+2(SV1)^{(1)}O_4'+2bN_5)+2\pi D^{(1)}(b-(Sb1)-(Sb1)^{(2)})O_2\right\}\sum\limits_{i=1}^{b-1}d_i^{(W)}-(\eta\lambda N_1+2N_5)\sum\limits_{i=1}^{b-1}id_i^{(W)}\right.\end{array}$$

$$
\begin{aligned}
&+\left\{O_5''+O_6''+\left\{\eta\lambda(N_1+bN_2+(SV1)\,O_4')\left\{\left\{((D+D^{(1)})-1)+\ (D+D^{(1)})(N_3+N_4)-1\right\}\sum_{k=1}^{b-1}g_k+1\right\}\right\}\right.\\
&\qquad\left.+\eta^2\lambda(D+D^{(1)})(\,N_1+(SV1)\,O_4'\ +N_3+N_4)\ -\ \eta\lambda N_2\sum_{k=1}^{b-1}kg_k+\eta\lambda N_2(X1)\right\}\Bigg\}\\
&\qquad+2N_5\sum_{i=1}^{b-1}\,(b-i)g_i+(N_6+2(Rb1)N_5+2(SV1)^{(1)}O_4')(2N_5(X1)-1)\Bigg\}\,d_0^{(B)}\\
&+\Big\{\eta\lambda(N_1+N_6)+(\lambda X1)(N_2+N_5)+\lambda\eta(bO_4'-(N_3+N_4)+\eta^2\lambda(D+D^{(1)})(\,N_1+(SV1)\,O_4'\ +N_3+N_4)\\
&\qquad+(N_6+2(Rb1)N_5+2(SV1)^{(1)}O_4')+2\pi D^{(1)}(b-(Sb1)-(Sb1)^{(2)})O_2\Big\}\Big\}\\
&\qquad=2\eta\lambda(\tilde{S}_v(\eta)-1)(\lambda X_1)\,(b-\lambda E(X)E(S_b)-\pi\lambda E(X)E(R_b)).
\end{aligned}
\tag{4.6}
$$

Since $p_i^{(B)}$, $p_i^{(W)}$ are the probabilities of 'i' customers being in the queue, it follows that left hand side of the above expression must be positive. Thus $P(1)=1$ is satisfied only iff

$$
b-\lambda E(X)E(S_b)-\pi\lambda E(X)E(R_b)>0,\ \text{ if } \rho=\lambda E(X)E(S_b)-\pi\lambda E(X)E(R_b)/b \tag{4.7}
$$

then $\rho<1$ is the condition to be satisfied for the existence of steady state for the model under consideration.

Computational aspects: Equation (4.5) gives has "$2b$" unknowns $p_0^{(B)}$, $p_1^{(B)}$, $\dots p_{b-1}^{(B)}$ and $p_0^{(W)}$, $p_1^{(W)}$, $\dots p_{b-1}^{(W)}$. The denominator of (4.5) has the polynomial with degree '2b' therefore it has '$2b$' roots. By Rouche's theorem of complex variables, $\left[z^b-\tilde{S}_v(\lambda+\eta-\lambda X(z))-O_1\right]\left[z^b-\tilde{S}_b(\lambda-\lambda X(z))-O_2\right]$ has $2b-1$ zeros inside and one on the unit circle $|z|=1$. Since $P(z)$ is analytic within and on the unit circle, the numerator must vanish at these points, which gives '$2b$' equations in '$2b$' unknowns. These equations can be solved by using any suitable numerical technique, for instance MATLAB routines.

5. Expected queue length

5.1. Expected queue length at various completion epochs

Expected queue length at various completion epoch are obtained to know the effect of the idle period in working vacation, service in working vacation and service in regular period.

(a) Service in regular period completion epoch

The mean queue length $E_{SB}(Q)$ at a service in regular period completion epoch is got by differentiating the PGF and substitute $z=1$ which is obtained from (3.31) and (3.32) with$\theta=0$. We get

$$
E_{SB}(Q)=\frac{\begin{array}{c}U_2^{(1)}\sum\limits_{i=a}^{b-1}(b-i)d_i^{(B)}+U_3^{(1)}\sum\limits_{i=a}^{b-1}(b(b-1)-i(i-1))d_i^{(B)}\ +(L_1^{(1)}+\pi J_1^{(2)}+J_2^{(1)})\sum\limits_{i=1}^{a-1}d_i^{(B)}+L_2^{(1)}\sum\limits_{i=1}^{a-1}i\,d_i^{(B)}\\ +L_3^{(1)}\sum\limits_{i=1}^{a-1}i(i-1)d_i^{(B)}\ +\left(L_1^{(1)}\ +U_1^{(1)}\right)d_0^{(B)}+\left(J_1^{(1)}+U_1^{(1)}\right)d_0^{(W)}++(J_1^{(1)}+\pi J_1^{(2)}+J_2^{(1)}-\pi J_1')\sum\limits_{i=1}^{b-1}d_i^{(W)}\\ +(J_3^{(1)}+\pi J_3^{(2)})\sum\limits_{i=1}^{b-1}i\,d_i^{(W)}+(J_4^{(1)}+\pi J_4^{(2)})\sum\limits_{i=1}^{b-1}i(i-1)d_i^{(W)}\ +\eta L_1^{(1)}\ +b\eta L_2^{(1)}+\ b(b-1)\eta L_3^{(1)}+\eta\,(J_2^{(1)}+\pi J_1')\end{array}}{24\eta^2A^2\left[(\lambda X_1)(b-Sb1-\pi Rb1)\right]^2}
\tag{5.1}
$$

where the expression for $U_i^{(1)'}S$, $L_i^{(1)'}S$, and $J_i^{(1)'}S$ are defined in Appendix A.

(b) Renovation in regular period completion epoch

The mean queue length $E_{RB}(Q)$ at a renovation in regular period completion epoch is found by differentiating the PGF and substitute $z=1$ which is obtained from (3.34) with $\theta=0$. We get

$$E_{RB}(Q)=\frac{\begin{array}{l} U_2^{(2)}\sum\limits_{i=a}^{b-1}(b-i)d_i^{(B)}+U_3^{(2)}\sum\limits_{i=a}^{b-1}(b(b-1)-i(i-1))d_i^{(B)}+(L_1^{(2)}+\pi J_1^{(4)}+J_2^{(2)})\sum\limits_{i=1}^{a-1}d_i^{(B)}+L_2^{(2)}\sum\limits_{i=1}^{a-1}i\,d_i^{(B)} \\ +L_3^{(2)}\sum\limits_{i=1}^{a-1}i(i-1)d_i^{(B)}+\left(L_1^{(2)}+U_1^{(2)}\right)d_0^{(B)}+\left(J_1^{(3)}+U_1^{(2)}\right)d_0^{(W)}+(J_1^{(3)}+\pi J_1^{(4)}+J_2^{(2)}-\pi J_1'')\sum\limits_{i=1}^{b-1}d_i^{(W)} \\ +(J_3^{(3)}+\pi J_3^{(4)})\sum\limits_{i=1}^{b-1}i\,d_i^{(W)}+(J_4^{(3)}+\pi J_4^{(4)})\sum\limits_{i=1}^{b-1}i(i-1)d_i^{(W)}+\eta L_1^{(2)}+b\eta L_2^{(2)}+b(b-1)\eta L_3^{(2)}+\eta\,(J_2^{(2)}+\pi J_1'') \end{array}}{24\eta^2A^2\left[(\lambda X_1)(b-Sb1-\pi Rb1)\right]^2} \tag{5.2}$$

where the expression for $U_i^{(2)'}S$, $L_i^{(2)'}S$, $J_i^{(3)'}S$, $J_i^{(4)'}S$ and $J_i^{(2)'}S$ are defined in Appendix B.

(c) Service in working vacation completion epoch

By calculating $P'_{SW}(1)$, one can get $E_{SW}(Q)$ at service in working vacation completion epoch,

$$E_{SW}(Q)=\frac{I_4^{(1)}\sum\limits_{i=1}^{b-1}(b-i)W_i+Q_1^{(1)}\sum\limits_{i=1}^{a-1}P_i+Q_2^{(1)}P_0+(Q_2^{(1)}+Q_3^{(1)})W_0+Q_1^{(1)}}{\eta^2A^2} \tag{5.3}$$

where,

$$\begin{aligned} I_1^{(1)}&=\eta\ A\,(SV1),\ I_2^{(1)}=\ (\lambda X_1)A\,(D-1),\ I_3^{(1)}=\eta\ B(D-1),\ I_4^{(1)}=\eta\ A\,(D-1),\ Q_1^{(1)}\\ &=-\eta\left[bI_4^{(1)}+I_1^{(1)}+I_2^{(1)}-I_3^{(1)}\right]\\ Q_2^{(1)}&=\left[I_4^{(1)}\left[(b-i)\sum_{i=1}^{b-1}g_i+X1\right]+I_1^{(1)}+I_2^{(1)}-I_3^{(1)}\right],\ Q_3^{(1)}=\left[I_2^{(1)}-I_3^{(1)}\right] \end{aligned}$$

and

$$\begin{aligned} A&=\left[1-(1-\pi)\tilde{S}_V(\eta)-\pi\tilde{S}_V(\eta)\tilde{R}_V(\eta)\right],B=\left[b-(1-\pi)(SV1)-\pi(SV1)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV1)\right],\\ C&=\left[b(b-1)-(1-\pi)(SV2)-\pi(SV2)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV2)-2\pi(SV1)(RV1)\right]. \end{aligned}$$

(d) Renovation in working vacation completion epoch

$E_{RW}(Q)$ at service in working vacation completion epoch is obtained by substituting $z=1$ in $P'_{RW}(z)$.

$$E_{RW}(Q)=\frac{I_4^{(2)}\sum\limits_{i=1}^{b-1}(b-i)W_i+Q_1^{(2)}\sum\limits_{i=1}^{a-1}P_i+Q_2^{(2)}P_0+(Q_2^{(2)}+Q_3^{(2)})W_0+Q_1^{(2)}}{\eta^2A^2} \tag{5.4}$$

where,

$$\begin{aligned} I_1^{(2)}&=\eta\ A\,(SV1)^{(1)},\ I_2^{(2)}=\ (\lambda X_1)A\,(D^{(1)}-1),\ I_3^{(2)}=\ \eta\ B(D^{(1)}-1),\ I_4^{(2)}=\eta\ A\,(D^{(1)}-1),\ Q_1^{(2)}\\ &=-\eta\left[bI_4^{(2)}+I_1^{(2)}+I_2^{(2)}-I_3^{(2)}\right]\\ Q_2^{(2)}&=\left[I_4^{(2)}\left[(b-i)\sum_{i=1}^{b-1}g_i+X1\right]+I_1^{(2)}+I_2^{(2)}-I_3^{(2)}\right],\ Q_3^{(2)}=\ \left[I_2^{(2)}-I_3^{(2)}\right] \end{aligned}$$

and

$$A=\left[1-(1-\pi)\tilde{S}_V(\eta)-\pi\tilde{S}_V(\eta)\tilde{R}_V(\eta)\right], B=\left[b-(1-\pi)(SV1)-\pi(SV1)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV1)\right],$$
$$C=\left[b(b-1)-(1-\pi)(SV2)-\pi(SV2)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV2)-2\pi(SV1)(RV1)\right].$$

5.2. Expected queue length at an arbitrary time epoch

Differentiating (4.5) with respect to z, and $z=1$, then the mean queue length $E(Q)$ at an arbitrary time epoch is,

$$E(Q)=\frac{\begin{array}{l}(U_2^{(1)}+U_2^{(2)})\sum\limits_{i=a}^{b-1}(b-i)d_i^{(B)}+(U_3^{(1)}+U_3^{(2)})\sum\limits_{i=a}^{b-1}(b(b-1)-i(i-1))d_i^{(B)}\\
+\left[L_1^{(1)}+L_1^{(2)}+\pi(J_1^{(2)}+J_1^{(4)})+J_2^{(1)}+J_2^{(2)}+24T_1^2(Q_1^{(1)}+Q_1^{(2)})\right]\sum\limits_{i=1}^{a-1}d_i^{(B)}\\
+(L_2^{(1)}+L_2^{(2)})\sum\limits_{i=1}^{a-1}id_i^{(B)}+(L_3^{(1)}+L_3^{(2)})\sum\limits_{i=1}^{a-1}i(i-1)d_i^{(B)}+\left[L_1^{(1)}+L_1^{(2)}+U_1^{(1)}+U_1^{(2)}+24T_1^2(Q_2^{(1)}+Q_2^{(2)})\right]d_0^{(B)}\\
+\left[J_1^{(1)}+J_1^{(3)}+U_1^{(1)}+U_1^{(2)}+24T_1^2(Q_2^{(1)}+Q_2^{(2)}+Q_3^{(1)}+Q_3^{(2)})\right]d_0^{(W)}+24T_1^2(I_4^{(1)}+I_4^{(2)})\sum\limits_{i=1}^{b-1}(b-i)d_i^{(W)}\\
+\left[J_1^{(1)}+J_1^{(3)}+\pi(J_1^{(2)}+J_1^{(4)})+J_2^{(1)}+J_2^{(2)}-\pi(J_1'+J_1'')\right]\sum\limits_{i=1}^{b-1}d_i^{(W)}\\
+(J_3^{(1)}+J_3^{(3)}+\pi(J_3^{(2)}+J_3^{(4)}))\sum\limits_{i=1}^{b-1}id_i^{(W)}+(J_4^{(1)}+J_4^{(3)}+\pi(J_4^{(2)}+J_4^{(4)}))\sum\limits_{i=1}^{b-1}i(i-1)d_i^{(W)}\\
+\left\{\eta(L_1^{(1)}+L_1^{(2)})+b\eta(L_2^{(1)}+L_2^{(2)})+b(b-1)\eta(L_3^{(1)}+L_3^{(2)})+\eta(J_2^{(1)}+J_2^{(2)}+\pi(J_1'+J_1'')+24T_1^2(Q_1^{(1)}+Q_1^{(2)})\right\}\end{array}}{24\eta^2A^2\left[(\lambda X_1)(b-Sb1-\pi Rb1)\right]^2} \tag{5.5}$$

where the expression for $U_i^{(1)'}S$, $L_i^{(1)'}S$, $J_i^{(1)'}S$, $U_i^{(2)'}S$, $L_i^{(2)'}S$, $J_i^{(3)'}S$, $J_i^{(4)'}S$ and $J_i^{(2)'}S$ are defined in Appendices A and B.

6. Numerical illustration

To illustrate the impact of proposed model we analyze the model numerically. In the flare nuts production industries, flare nuts are ordered in bulk and it follows Poisson distribution. The nuts are produced in turret lathe machine (TLM), whose service follows an exponential distribution. At a time 10 flare nuts can be produced in regular period with fast service rate by TLM After the service completion, if the requirement of the flare nuts is less than the threshold values it performs the service in slow service rate in working vacation In case of server break down, the server can manage the service to complete the batch by manual operation after which, the renovation process will take place.

Computational analysis for various combinations of service time distributions is exhibited. The unknown probabilities of the PGF of the queue size at an arbitrary time epoch and other computations are carried out using MATLAB. Numerical results are shown in the tables that are self-explanatory (Tabs. 1 through 7). The notations used in these tables are same as those defined earlier in the text.

6.1. Effect of duration of working vacation and service rates on queue length

The expected queue length $E(Q)$ is computed for various values of η versus different service rates in working vacation are presented in Tables 1 and 2. Let us assume $a=3$, $b=10$, $\mu_b=9$, $\lambda=4$, $\gamma_1=2$, $\gamma_2=4$ and η follows exponential distribution.

It is assume that, the service rate in working vacation and in regular period follow exponential distribution with parameters μ_v and μ_b respectively. The $E(Q)$ is presented for different values of η in Table 1.

It is assume that, the service rate in working vacation and in regular period follow 2-Erlangian distribution with parameters μ_v and μ_b respectively. The $E(Q)$ is presented for different values of η in Table 2.

TABLE 1. Expected queue length for M^X/ M(1,10); M(3,10)/1.

η/μ_V	Expected queue length							
	0.5	1.0	1.5	2	2.5	3	3.5	4
0.5	5.5666	5.0697	4.5254	4.0369	3.1569	2.9364	2.5587	2.1671
1.0	6.1524	5.7158	5.0299	4.8555	4.5986	4.0444	3.5222	3.0349
1.5	7.0999	6.3524	5.9754	5.5698	5.0854	4.5398	4.0497	3.4697
2.0	7.5268	7.0241	6.4698	6.0698	5.7222	5.2975	4.9697	4.0482
2.5	7.9531	7.4232	7.0254	6.3584	6.0697	5.5264	5.0666	4.5297
3.0	8.5112	8.0658	7.2231	6.8152	6.4842	6.0854	5.7987	5.1234

TABLE 2. Expected queue length for M^X/E_2(1,10); E_2(3,10)/1.

η/μ_V	Expected queue length							
	0.5	1.0	1.5	2	2.5	3	3.5	4
0.5	2.0928	1.9812	1.7129	1.5123	1.5621	1.3111	1.2817	1.0812
1.0	2.4567	2.1612	1.9812	1.8721	1.7812	1.5211	1.4128	1.2181
1.5	2.9118	2.5281	2.3212	2.1281	2.0821	1.8821	1.7181	1.6721
2.0	3.5219	3.1728	2.9181	2.7218	2.5232	2.3211	2.1221	1.9812
2.5	3.9128	3.7182	3.5627	3.1672	2.9123	2.7813	2.5672	2.3621
3.0	4.5522	3.9977	3.7672	3.5272	3.3321	3.0128	2.8123	2.5123

TABLE 3. Arrival rate and Service rate in working vacation (2-Erlangian) *vs*. Expected queue length (with server breakdown $\mu_b = 5$, $\eta = 3$).

λ/μ_v	Expected queue length							
	0.5	1.0	1.5	2	2.5	3	3.5	4
0.5	0.9521	0.7547	0.6214	0.5587	0.4224	0.2234	0.1872	0.1255
1.0	1.4101	1.3870	1.2116	1.0697	0.9784	0.847	0.7241	0.6241
1.5	3.0231	2.8036	2.6297	2.4541	2.2296	2.041	1.8222	1.5222
2.0	3.5204	3.0893	2.8234	2.6887	2.4269	2.2557	2.0014	1.8341
2.5	5.2267	4.5777	4.0248	3.8574	3.6224	3.4784	3.2241	3.1254
3.0	9.7208	8.5254	7.0555	6.8111	6.5201	6.0248	5.8221	5.5221

From the Tables 1 and 2, we observe that while the duration of the working vacation increases, the E(Q) increases and when service rate in working vacation increases then the E(Q) decreases. For instance, when service rate $\mu_v = 1.5$, we get $E(Q) = 3.7672$ for $\eta = 3$ whereas for service rate $\mu_v = 2.5$ we get $E(Q) = 3.3321$ for $\eta = 3$ (see Tab. 2).

TABLE 4. Arrival rate and Service rate in regular period (2-Erlangian) *vs*. Expected queue length (with server breakdown $\mu_v = 2$, $\eta = 3$).

λ/μ_b	Expected queue length							
	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.5	2.9458	2.7012	2.5369	2.4258	2.3256	2.2258	1.9222	1.7251
1.0	5.0254	4.8254	4.6685	4.4698	4.0333	3.7241	3.5585	3.0240
1.5	8.5324	7.0555	6.5254	6.3222	6.0265	5.8222	5.4541	3.3258
2.0	10.5412	9.1241	8.5222	8.0542	7.4254	7.0695	6.8264	5.0975
2.5	12.1211	11.5825	11.2872	10.8064	9.7241	8.7587	7.1255	6.0485
3.0	19.5254	17.1236	16.2268	15.1269	14.1221	13.2257	12.1298	10.8263

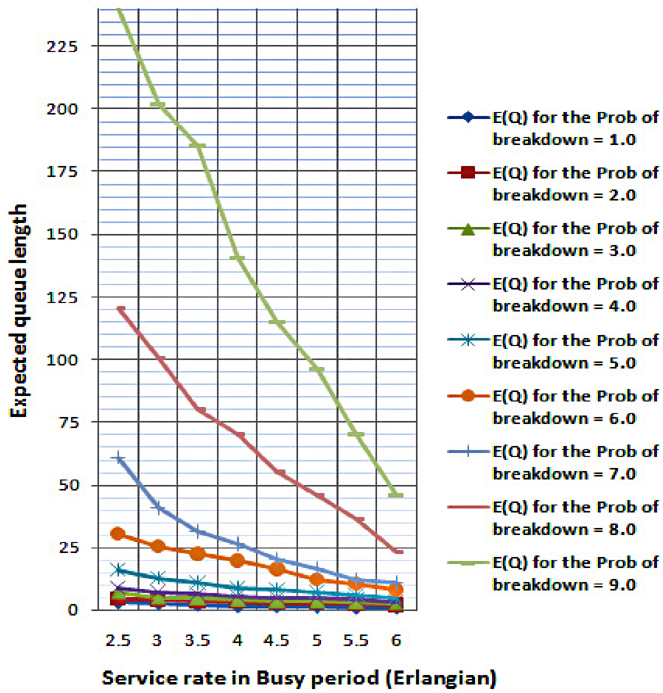

FIGURE 1. Prob. of breakdown *vs*. Queue length (for various service rates in MWV).

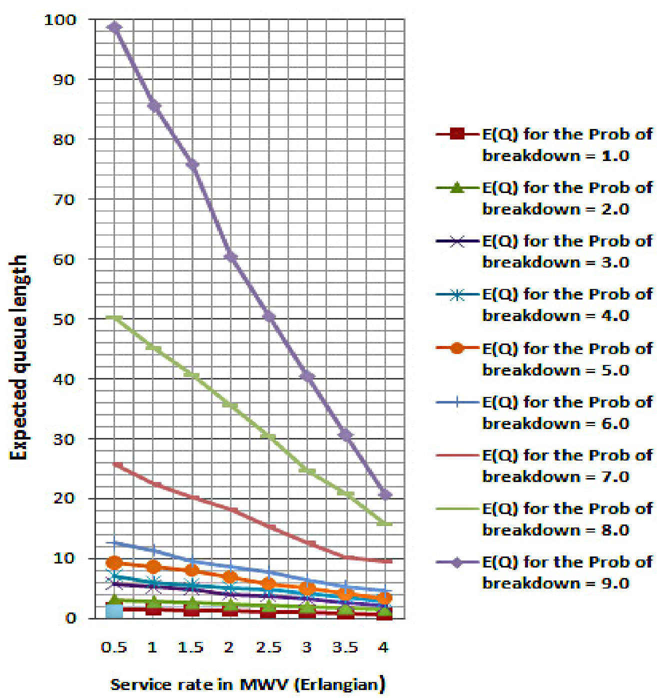

FIGURE 2. Prob. of breakdown *vs*. Queue length (for various service rates in regular period).

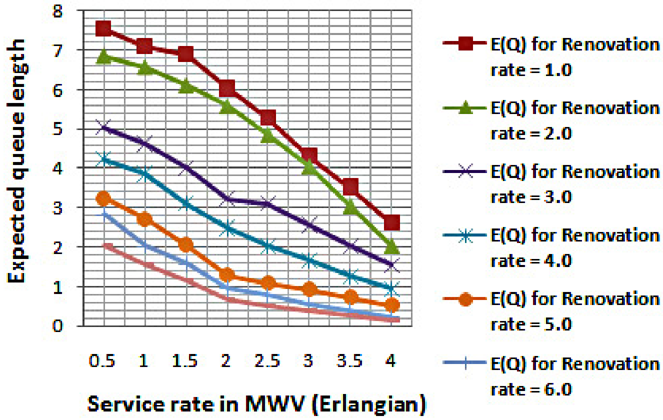

FIGURE 3. Renovation rate *vs*. Queue length (for various service rates in MWV).

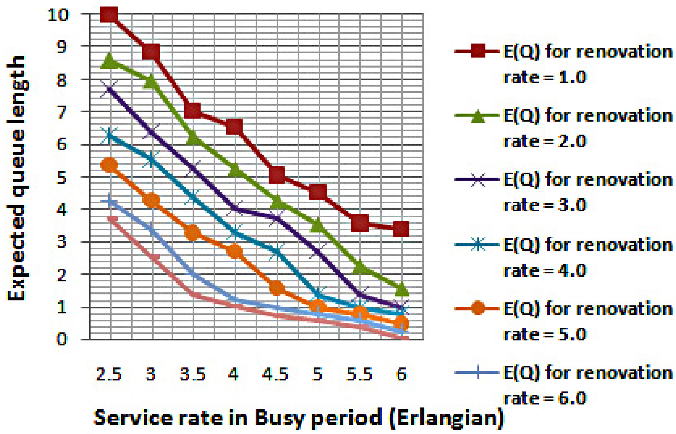

FIGURE 4. Renovation rate *vs*. Queue length (for various service rates in regular period).

It is observed from the above table that the expected queue length is higher if the service rates follow the exponential distribution than the service rate that follows the 2-Erlangian distribution for a fixed value of η. For example, when service rate $\mu_v = 1.5$, we get $E(Q) = 3.7672$ for $\eta = 3$ (see Tab. 2). Whereas for service rate $\mu_v = 1.5$ we get $E(Q) = 7.2231$ for $\eta = 3$ (see Tab. 1).

6.2. Effect of arrival rate on queue length

The expected queue length $E(Q)$ computed for various arrival rates are presented in the Tables 3 and 4. We show the effect of λ and μ_v on the queue length when μ_b and η are fixed in the Table 3 also the effect of λ and μ_b on the queue length when μ_v and η are fixed in Table 4.

From Tables 3 and 4, it could be seen that when the arrival rate of the customer increases then the $E(Q)$ increases and when the service rate in working vacation and in regular period increases then the $E(Q)$ decreases. For instance, when service rate $\mu_v = 2.0$, we get $E(Q) = 2.6887$ for $\lambda = 2.0$ whereas for service rate $\mu_v = 3.0$ we get $E(Q) = 2.2557$ for $\lambda = 2.0$ (see Tab. 3). Also, when service rate $\mu_b = 4.0$, we get $E(Q) = 8.0542$ for $\lambda = 2.0$ whereas for service rate $\mu_b = 5.0$ we get $E(Q) = 7.0695$ for $\lambda = 2.0$ (see Tab. 4). These trends, in fact, resemble the performance of the traditional queueing models.

TABLE 5. Arrival rate and Service rate in working vacation (2-Erlangian) *vs*. Expected queue length (without server breakdown $\mu_b = 5$, $\eta = 3$).

λ/μ_v	Expected queue length							
	0.5	1.0	1.5	2	2.5	3	3.5	4
0.5	0.4521	0.3542	0.2548	0.1452	0.1054	0.0975	0.0854	0.0754
1.0	0.9421	0.8546	0.7542	0.6542	0.5864	0.4625	0.3564	0.2685
1.5	1.5542	1.4724	1.3542	1.2642	1.1524	1.0245	0.8654	0.6521
2.0	2.9854	2.5342	2.0542	1.9654	1.7240	1.6524	1.4210	1.0524
2.5	4.5421	3.9450	3.0524	2.8540	2.3024	2.1354	1.9542	1.7243
3.0	6.5241	5.854	5.2542	4.0524	3.5243	3.0421	2.521	2.0213

TABLE 6. Arrival rate and Service rate in regular period (2-Erlangian) *vs*. Expected queue length (without server breakdown $\mu_v = 2$, $\eta = 3$).

λ/μ_b	Expected queue length							
	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
0.5	0.8754	0.7235	0.5042	0.3254	0.2452	0.1254	0.1054	0.0854
1.0	1.8634	1.5520	1.0654	0.7524	0.5521	0.3640	0.2854	0.1984
1.5	3.2584	2.8521	2.6540	1.9542	1.6653	1.0653	0.9875	0.7356
2.0	5.6265	4.4521	4.2640	3.8650	2.5654	2.0540	1.5640	1.1560
2.5	7.0254	6.8542	5.6552	4.7589	4.0587	3.5524	3.0421	2.7524
3.0	9.2215	8.1245	7.9654	5.2546	5.7541	4.0524	3.5524	2.8542

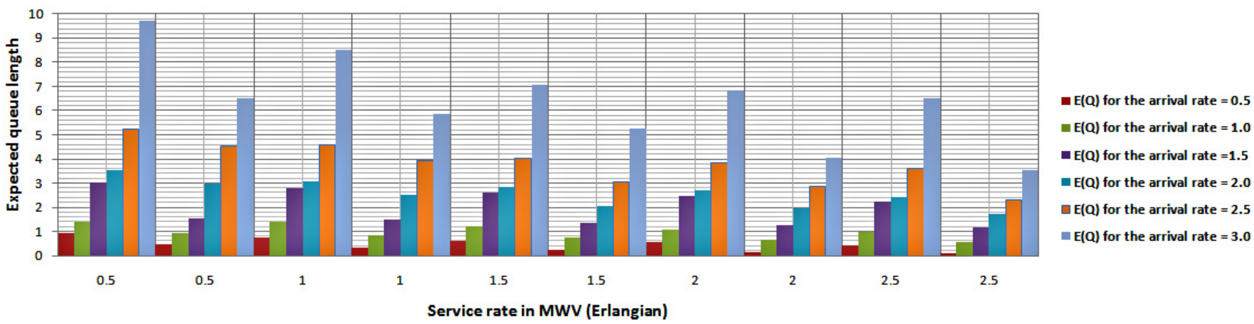

FIGURE 5. Comparison bar chart for $E(Q)$ with breakdown and without breakdown (various service rate in working vacation).

From Figures 1 and 2 it is observed that while the probability of breakdown increases the $E(Q)$ will also increase. When the service rate $\mu_v = 2.0$ we get $E(Q) = 5.0254$ for probability of breakdown $\pi = 0.4$, whereas for probability of breakdown $\pi = 0.7$, we get $E(Q) = 18.2542$. Also, the service rate $\mu_b = 4.0$ we get $E(Q) = 5.4875$ for probability of breakdown $\pi = 0.4$, whereas for probability of breakdown $\pi = 0.7$, we get $E(Q) = 26.1472$.

It is observed that from Figures 3 and 4; while the renovation rate increases the $E(Q)$ will also decrease. When the service rate $\mu_v = 2.0$, $E(Q)$ is 3.2219 for renovation rate $\gamma_1 = 3.0$, whereas for renovation rate $\gamma_1 = 6.0$, $E(Q)$ is 0.9631. Also for the service rate $\mu_b = 3.5$ we get $E(Q) = 7.0254$ for renovation rate $\gamma_2 = 3.0$, whereas for renovation rate $\gamma_2 = 6.0$, $E(Q)$ is 4.3690.

From Tables 3–6, Figures 5 and 6; it is observed that the expected queue length is higher when the breakdown occurs.

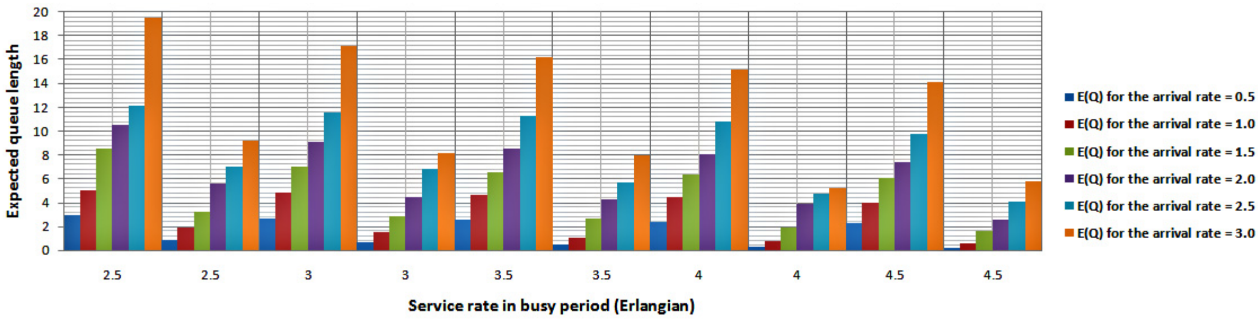

FIGURE 6. Comparison bar chart for $E(Q)$ with breakdown and without breakdown (various service rate in working vacation).

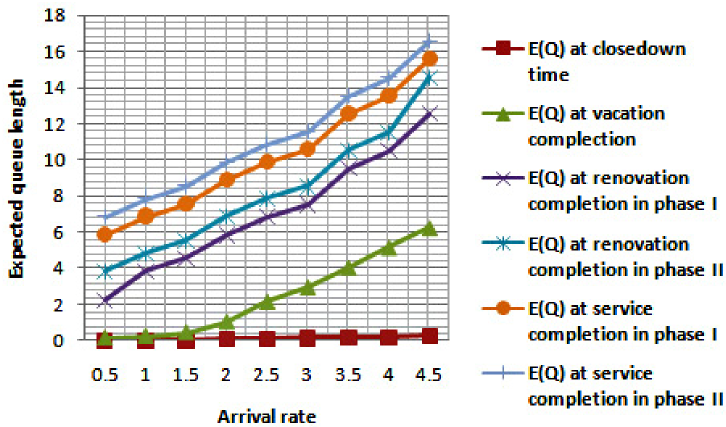

FIGURE 7. Queue length at various completion epochs.

6.3. Expected queue length at different completion epochs

The numerical results for expected queue length at each completion epoch for the values $\mu_v = 2$, $\mu_b = 3$, $\eta = 4$ is given below.

From Figure 7 it is observed that E(Q) is higher at service in regular period completion epoch and is lower at renovation in working vacation completion epoch than the others for the same arrival rate.

7. CONCLUSION

A variable bulk service queueing model with multiple working vacations and server breakdown is analyzed here. Probability generating function of queue size at an arbitrary time epoch and at different completion epochs are obtained. Queue length for varying arrival rate, service rate in working vacation, service rate in regular period and duration of working vacation are obtained. A numerical illustration is presented to show the various performance measures of the proposed model.

Appendix A

The $U_i^{(1)'}S$, $L_i^{(1)'}S$, and $J_i^{(1)'}S$ in (4.7) and (5.5) are defined as follows:

$$A = \left[1-(1-\pi)\tilde{S}_V(\eta)-\pi\tilde{S}_V(\eta)\tilde{R}_V(\eta)\right], B = \left[b-(1-\pi)(SV1)-\pi(SV1)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV1)\right],$$
$$C = \left[b(b-1)-(1-\pi)(SV2)-\pi(SV2)\tilde{R}_V(\eta)-\pi\tilde{S}_V(\eta)(RV2)-2\pi(SV1)(RV1)\right],\ D=\tilde{S}_V(\eta)$$

where,

$$Sb1 = \lambda.X1.E(Sb), Sb2 = \lambda.X2.E(Sb)+\lambda^2.X1^2.E(Sb)$$
$$Sb3 = \lambda.X3.E(Sb)+3.\lambda^2.X1.X2.E(Sb^2)+\lambda^3.X1^3.E(Sb^3)$$
$$Rb1 = \lambda.X1.E(Rb), Rb2 = \lambda.X2.E(Rb)+\lambda^2.X1^2.E(Rb)$$
$$Rb3 = \lambda.X3.E(Rb)+3.\lambda^2.X1.X2.E(Rb^2)+\lambda^3.X1^3.E(Rb^3)$$
$$SV1 = \lambda.X1.E(SV), SV2 = \lambda.X2.E(SV)+\lambda^2.X1^2.E(SV^2)$$
$$SV3 = \lambda.X3.E(SV)+3.\lambda^2.X1.X2.E(SV^2)+\lambda^3.X1^3.E(SV^3)$$
$$RV1 = \lambda.X1.E(RV), RV2 = \lambda.X2.E(RV)+\lambda^2.X1^2.E(RV^2)$$
$$RV3 = \lambda.X3.E(RV)+3.\lambda^2.X1.X2.E(RV^2)+\lambda^3.X1^3.E(RV^3), X1=E(X)=X'(1), X2=X''(1), 3=X'''(1)$$
$$T_1 = (\lambda X_1)(b-Sb1-\pi.Rb1), T_2 = (\lambda X_2)(b-Sb1-\pi.Rb1)+(\lambda X_1)(b(b-1)-Sb2-\pi.Rb2-2.\pi.(Sb1)(Rb1))$$
$$T_3 = (\lambda X_2)(b(b-1)-Sb2-\pi.Rb2-2.\pi.(Sb1)(Rb1)), T_5=(\lambda X_3)(b-Sb1-\pi.Rb1)$$
$$T_4 = (\lambda X_1)(b(b-1)(b-2)-Sb3-\pi.Rb3-3.\pi.(Sb2)(Rb1)-3.\pi.(Sb1)(Rb2))$$
$$J_1^{(1)} = \{(D-1)H_1-[(SV1)H_2-(SV2)H_3]\}$$
$$J_2^{(1)} = \Big\{[2D-1]\left[((SV1)^{(1)}+(SV1))H_2-((SV2)^{(1)}+(SV2))H_3\right]$$
$$-(D^{(1)}+D)(D-1)H_1-2(SV1)((SV1)^{(1)}+(SV1))H_3\Big\}$$
$$J_3^{(1)} = \{2(SV1)H_3-(D-1)H_2\}, J_4=\{(D-1)H_3\}, J_5=\{(D-1)H_4-[(SV1)H_5-(SV2)H_3]\}$$
$$H_1 = f_1R_1+f_2R_2+f_4R_3+f_9R_4-2\eta^2[2f_3-3f_5-9f_6-6f_7-6f_8-6f_{10}-18f_{11}-18f_{12}]$$
$$H_2 = 12\eta^2[2f_9+f_4-f_2]-2f_1R_2, H_3=12\eta^2 f_1, H_5=12\eta^2[2f_9+f_4-f_2]-2f_1R_6$$
$$H_4 = f_1R_5+f_2R_6+f_4R_7+f_9R_8+2\eta^2[2f_3-3f_5-9f_6-6f_7-6f_8-6f_{10}-18f_{11}-18f_{12}]$$
$$f_1 = AT_1(Sb1), f_2=AT_1(Sb2), f_3=AT_1(Sb3), f_4=AT_2(Sb1), f_5=AT_2(Sb2), f_6=AT_3(Sb1)$$
$$f_7 = AT_4(Sb1), f_8=AT_5(Sb1), f_9=BT_1(Sb1), f_{10}=BT_1(Sb2), f_{11}=BT_2(Sb1), f_{12}=CT_1(Sb1)$$
$$R_1 = 36\eta(\lambda X2)+24b\eta(\lambda X1)+12b(b-1)\eta^2, R_2=12b\eta^2+12\eta(\lambda X1), R_3=-12b\eta^2+36\eta(\lambda X1)$$
$$R_4 = -24b\eta^2+72\eta(\lambda X1), R_8=-24b\eta^2+72\eta(\lambda X1)-24X1\eta^2$$
$$R_5 = 36\eta(\lambda X2)+24b\eta(\lambda X1)+12b(b-1)\eta^2+12(X2)\eta^2+24(X1)b\eta^2+24(X1)\eta(\lambda X1)$$
$$R_6 = 12b\eta^2+12\eta(\lambda X1)+12(X1)\eta^2, R_7=-12b\eta^2+36\eta(\lambda X1)-48(X1)\eta^2$$
$$J_1^{(2)} = \left\{(D^{(1)}-1)H_1^{(1)}-\left[(SV1)^{(1)}H_2^{(1)}-(SV2)^{(1)}H_3^{(1)}\right]\right\},$$
$$J_1' = \left\{(D^{(1)}-1)H_1'-\left[(SV1)^{(1)}H_2'-(SV2)^{(1)}H_3^{(1)}\right]\right\}$$
$$J_3^{(2)} = \left\{2(SV1)^{(1)}H_3^{(1)}-(D^{(1)}-1)H_2^{(1)}\right\}, J_4^{(1)}=\left\{(D^{(1)}-1)H_3^{(1)}\right\},$$
$$J_5^{(2)} = \left\{(D^{(1)}-1)H_4^{(1)}-\left[(SV1)^{(1)}H_5^{(1)}-(SV2)^{(1)}H_3^{(1)}\right]\right\}$$

$$D^{(1)} = \pi\tilde{S}_V(\eta)\tilde{R}_V(\eta) - \pi\tilde{S}_V(\eta).(SV1)^{(1)} = \pi(SV1)\tilde{R}_V(\eta) + \pi\tilde{S}_V(\eta)(RV1) - \pi(SV1)$$
$$(SV2)^{(1)} = \pi(SV2)\tilde{R}_V(\eta) + \pi\tilde{S}_V(\eta)(RV2) + 2\pi(SV1)(RV1) - \pi(SV2)$$
$$(Sb1)^{(1)} = \pi(Sb1), (Sb2)^{(1)} = \pi(Sb2) + 2\pi(Sb1)(Rb1), (Sb3)^{(1)} = \pi(Sb3) + 3\pi(Sb2)(Rb1) + 3\pi(Sb1)(Rb2)$$
$$H_1^{(1)} = f_1^{(1)}R_1' + f_2^{(1)}R_2' + f_4^{(1)}R_3' + f_9^{(1)}R_4' - 2\eta^2\left[2f_3^{(1)} - 3f_5^{(1)} - 9f_6^{(1)} - 6f_7^{(1)} - 6f_8^{(1)} - 6f_{10}^{(1)} - 18f_{11}^{(1)} - 18f_{12}^{(1)}\right]$$
$$H_2^{(1)} = 12\eta^2\left[2f_9^{(1)} + f_4^{(1)} - f_2^{(1)}\right] - 2f_1^{(1)}R_2', H_3^{(1)} = 12\eta^2 f_1^{(1)}, H_5^{(1)} = 12\eta^2\left[2f_9^{(1)} + f_4^{(1)} - f_2^{(1)}\right] - 2f_1^{(1)}R_6'$$
$$H_4^{(1)} = f_1^{(1)}R_5' + f_2^{(1)}R_6' + f_4^{(1)}R_7' + f_9^{(1)}R_8' + 2\eta^2\left[2f_3^{(1)} - 3f_5^{(1)} - 9f_6^{(1)} - 6f_7^{(1)} - 6f_8^{(1)} - 6f_{10}^{(1)} - 18f_{11}^{(1)} - 18f_{12}^{(1)}\right]$$
$$f_1^{(1)} = AT_1(Sb1)^{(1)}, f_2^{(1)} = AT_1(Sb2)^{(1)}, f_3^{(1)} = AT_1(Sb3)^{(1)}, f_4^{(1)} = AT_2(Sb1)^{(1)}, f_5^{(1)} = AT_2(Sb2)^{(1)},$$
$$f_6^{(1)} = AT_3(Sb1)^{(1)} f_7^{(1)} = AT_4(Sb1)^{(1)}, f_8^{(1)} = AT_5(Sb1)^{(1)}, f_9^{(1)} = BT_1(Sb1)^{(1)}, f_{10}^{(1)} = BT_1(Sb2)^{(1)},$$
$$f_{11}^{(1)} = BT_2(Sb1)^{(1)}, f_{12}^{(1)} = CT_1(Sb1)^{(1)}$$
$$R_1' = 36\eta(\lambda X2), R_2' = 12\eta(\lambda X1), R_3' = 36\eta(\lambda X1), R_4' = 72\eta(\lambda X1), R_8' = 72\eta(\lambda X1) - 24(X1)\eta^2$$
$$R_5' = 36\eta(\lambda X2) + 12(X2)\eta^2 + 24(X1)\eta(\lambda X1), R_6' = 12\eta(\lambda X1) + 12(X1)\eta^2, R_7' = 36\eta(\lambda X1) - 48(X1)\eta^2$$
$$H_1' = f_1^{(1)}R_1 + f_2^{(1)}R_2 + f_4^{(1)}R_3 + f_9^{(1)}R_4 - 2\eta^2\left[2f_3^{(1)} - 3f_5^{(1)} - 9f_6^{(1)} - 6f_7^{(1)} - 6f_8^{(1)} - 6f_{10}^{(1)} - 18f_{11}^{(1)} - 18f_{12}^{(1)}\right]$$
$$H_2' = 12\eta^2\left[2f_9^{(1)} + f_4^{(1)} - f_2^{(1)}\right] - 2f_1^{(1)}R_2$$
$$L_1 = 24f_1M_1 + 48f_4M_2 + M_1[2f_3 - 3f_5 - 9f_6 - 6f_7 - 6f_8] - 24f_9M_4, L_2 = 6[f_2 - f_4]M_3$$
$$L_3 = 6f_1M_3, M_1 = \eta A(\lambda X2) + 4\eta B(\lambda X1) - \eta^2 C - A(\lambda X1)^2, M_2 = \eta A(\lambda X1) - \eta^2 B,$$
$$M_3 = 2\eta^2 A, M_4 = \eta^2 B$$
$$U_1 = \left[(J_1 + \pi J_1' + \pi J_1^{(1)}) + J_2 + k(J_3 + \pi J_3^{(1)}) + k(k-1)(J_4 + \pi J_4^{(1)})\right]\sum_{k=1}^{b-1} g_k + (J_5 + \pi J_5^{(1)})$$
$$U_2 = 12\eta^2 A^2[T_1(Sb2) - T_2(Sb1)], U_3 = 12\eta^2 A^2 T_1(Sb1)$$
$$O_1 = \pi\tilde{S}_V(\lambda + \eta - \lambda X(z))\tilde{R}^{(W)}(\lambda + \eta - \lambda X(z)) - \pi\tilde{S}_V(\lambda + \eta - \lambda X(z))$$
$$O_2 = \pi\tilde{S}_b(\lambda + \eta - \lambda X(z))\tilde{R}^{(B)}(\lambda + \eta - \lambda X(z)) - \pi\tilde{S}_b(\lambda + \eta - \lambda X(z))$$

Appendix B

The $U_i^{(2)'}S$, $L_i^{(2)'}S$, $J_i^{(3)'}S$, $J_i^{(4)'}S$ and $J_i^{(2)'}S$ in (5.1) and (5.5) are defined as follows:

$$J_1^{(3)} = \left\{(D-1)H_1^{(2)} - \left[(SV1)H_2^{(2)} - (SV2)H_3^{(2)}\right]\right\}$$
$$J_2^{(2)} = \left\{[2D-1]\left[((SV1)^{(1)} + (SV1))H_2^{(2)} - ((SV2)^{(1)} + (SV2))H_3^{(2)}\right]\right.$$
$$\left. -(D^{(1)} + D)(D-1)H_1^{(2)} - 2(SV1)((SV1)^{(1)} + (SV1))H_3^{(2)}\right\}$$
$$J_3^{(3)} = \left\{2(SV1)H_3^{(2)} - (D-1)H_2^{(2)}\right\}, J_4 = \left\{(D-1)H_3^{(2)}\right\}, J_5 = \left\{(D-1)H_4^{(2)} - \left[(SV1)H_5^{(2)} - (SV2)H_3^{(2)}\right]\right\}$$
$$H_1^{(2)} = f_1^{(2)}R_1 + f_2^{(2)}R_2 + f_4^{(2)}R_3 + f_9^{(2)}R_4 - 2\eta^2\left[2f_3^{(2)} - 3f_5^{(2)} - 9f_6^{(2)} - 6f_7^{(2)} - 6f_8^{(2)} - 6f_{10}^{(2)} - 18f_{11}^{(2)} - 18f_{12}^{(2)}\right]$$
$$H_2^{(2)} = 12\eta^2\left[2f_9^{(2)} + f_4^{(2)} - f_2^{(2)}\right] - 2f_1^{(2)}R_2, H_3^{(2)} = 12\eta^2 f_1^{(2)}, H_5^{(2)} = 12\eta^2\left[2f_9^{(2)} + f_4^{(2)} - f_2^{(2)}\right] - 2f_1^{(2)}R_6$$

$$H_4^{(2)} = f_1^{(2)}R_5+f_2^{(1)}R_6+f_4^{(1)}R_7+f_9^{(1)}R_8+2\eta^2\left[2f_3^{(1)}-3f_5^{(1)}-9f_6^{(1)}-6f_7^{(1)}-6f_8^{(1)}-6f_{10}^{(1)}-18f_{11}^{(1)}-18f_{12}^{(1)}\right]$$

$$f_1^2 = AT_1(Sb1)^{(2)}, f_2^2 = AT_1(Sb2)^{(2)}, f_3^2 = AT_1(Sb3)^{(2)}, f_4^2 = AT_2(Sb1)^{(2)}, f_5^2 = AT_2(Sb2)^{(2)},$$

$$f_6^2 = AT_3(Sb1)^{(2)} f_7^2 = AT_4(Sb1)^{(2)}, f_8^2 = AT_5(Sb1)^{(2)}, f_9^2 = BT_1(Sb1)^{(2)}, f_{10}^2 = BT_1(Sb2)^{(2)},$$

$$f_{11}^2 = BT_2(Sb1)^{(2)}, f_{12}^2 = CT_1(Sb1)^{(2)}$$

$$R_1 = 36\eta(\lambda X2)+24b\eta(\lambda X1)+12b(b-1)\eta^2, R_2 = 12b\eta^2+12\eta(\lambda X1), R_3 = -12b\eta^2+36\eta(\lambda X1)$$

$$R_4 = -24b\eta^2+72\eta(\lambda X1), R_8 = -24b\eta^2+72\eta(\lambda X1)-24X1\eta^2$$

$$R_5 = 36\eta(\lambda X2)+24b\eta(\lambda X1)+12b(b-1)\eta^2+12(X2)\eta^2+24(X1)b\eta^2+24(X1)\eta(\lambda X1)$$

$$R_6 = 12b\eta^2+12\eta(\lambda X1)+12(X1)\eta^2, R_7 = -12b\eta^2+36\eta(\lambda X1)-48(X1)\eta^2$$

$$J_1^{(4)} = \left\{(D^{(2)}-1)H_1^{(3)}-\left[(SV1)^{(2)}H_2^{(3)}-(SV2)^{(2)}H_3^{(3)}\right]\right\},$$

$$J_1'' = \left\{(D^{(2)}-1)H_1''-\left[(SV1)^{(2)}H_2''-(SV2)^{(2)}H_3^{(2)}\right]\right\}$$

$$J_3^{(4)} = \left\{2(SV1)^{(2)}H_3^{(3)}-(D^{(2)}-1)H_2^{(3)}\right\}, J_4^{(2)} = \left\{(D^{(2)}-1)H_3^{(3)}\right\},$$

$$J_5^{(4)} = \left\{(D^{(2)}-1)H_4^{(3)}-\left[(SV1)^{(2)}H_5^{(3)}-(SV2)^{(2)}H_3^{(3)}\right]\right\}$$

$$D^{(2)} = D^{(1)}, (SV1)^{(2)} = (b-(Sb1)+(1+\pi)(SR1))D^{(1)}+(SV1)^{(1)}$$

$$(SV2)^{(2)} = (b(b-1)-(Sb2)-2\pi(Sb1)(SR1)+(1+\pi)(SR2))D^{(1)}+(b-(Sb1)(1+\pi)(SR1))\pi(SV1)^{(1)}+(SV2)^{(1)}$$

$$(Sb1)^{(2)} = \pi(Rb1), (Sb2)^{(2)} = \pi(Rb2)+2\pi(Sb1)(Rb1), (Sb3)^{(2)} = \pi(Rb3)+3\pi(Sb2)(Rb1)+3\pi(Sb1)(Rb2)$$

$$H_1^{(3)} = f_1^{(2)}R_1'+f_2^{(2)}R_2'+f_4^{(2)}R_3'+f_9^{(2)}R_4'-2\eta^2\left[2f_3^{(2)}-3f_5^{(2)}-9f_6^{(2)}-6f_7^{(2)}-6f_8^{(2)}-6f_{10}^{(2)}-18f_{11}^{(2)}-18f_{12}^{(2)}\right]$$

$$H_2^{(3)} = 12\eta^2\left[2f_9^{(2)}+f_4^{(2)}-f_2^{(2)}\right]-2f_1^{(2)}R_2', H_5^{(3)} = 12\eta^2\left[2f_9^{(2)}+f_4^{(2)}-f_2^{(2)}\right]-2f_1^{(2)}R_6'$$

$$H_4^{(3)} = f_1^{(2)}R_5'+f_2^{(2)}R_6'+f_4^{(2)}R_7'+f_9^{(2)}R_8'+2\eta^2\left[2f_3^{(2)}-3f_5^{(2)}-9f_6^{(2)}-6f_7^{(2)}-6f_8^{(2)}-6f_{10}^{(2)}-18f_{11}^{(2)}-18f_{12}^{(2)}\right]$$

$$H_1'' = f_1^{(2)}R_1+f_2^{(2)}R_2+f_4^{(2)}R_3+f_9^{(2)}R_4-2\eta^2\left[2f_3^{(2)}-3f_5^{(2)}-9f_6^{(2)}-6f_7^{(2)}-6f_8^{(2)}-6f_{10}^{(2)}-18f_{11}^{(2)}-18f_{12}^{(2)}\right]$$

$$H_2'' = 12\eta^2\left[2f_9^{(2)}+f_4^{(2)}-f_2^{(2)}\right]-2f_1^{(2)}R_2$$

$$L_1^{(2)} = 24f_1^{(2)}M_1+48f_4^{(2)}M_2+M_1\left[2f_3^{(2)}-3f_5^{(2)}-9f_6^{(2)}-6f_7^{(2)}-6f_8^{(2)}\right]-24f_9^{(2)}M_4,$$

$$L_2^{(2)} = 6\left[f_2^{(2)}-f_4^{(2)}\right]M_3, L_3^{(2)} = 6f_1M_3, M_1 = \eta A(\lambda X2)+4\eta B(\lambda X1)-\eta^2C-A(\lambda X1)^2,$$

$$M_2 = \eta A(\lambda X1)-\eta^2B, M_3 = 2\eta^2A, M_4 = \eta^2B$$

$$U_1^{(2)} = \left[(J_1^{(3)}+J_1''+\pi J_1^{(4)})+J_2^{(2)}+k(J_3^{(3)}+\pi J_3^{(4)})+k(k-1)(J_4^{(3)}+\pi J_4^{(4)})\right]\sum_{k=1}^{b-1}g_k+(J_5^{(3)}+\pi J_5^{(4)})$$

$$U_2^{(2)} = 12\eta^2A^2\left[T_1(Sb2)^{(2)}-T_2(Sb1)^{(2)}\right], U_3^{(2)} = 12\eta^2A^2T_1(Sb1)^{(2)}.$$

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