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THEORY AND APPLICATION OF RECIPROCAL TRANSFORMATION OF "PATH PROBLEM" AND "TIME FLOAT PROBLEM"

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Abstract. The concept of analytic geometry, *i.e.*, the reciprocal transformation of geometry and algebra, hints a prospect for the reciprocal transformation of the "path problem" and the "time float problem". A reciprocal transformation can be used to solve a complex problem in one field by translating it into a simpler one in another field. In this case, owing to the generalized concept of length, various types of non-path problems such as the optimum allocation problem and equipment replacement problem can be represented as "path problems". A "length network", which is generalized in nature, is translated into a "time network" by changing the meanings of arcs and lengths. Furthermore, "path problems" can be represented as "time float problems" by discovering the relationships of paths in the length network and time floats in the time network. Base on the relationships, "time float problems" also can be represented as "path problems". The relationships are keys to updating the mutual correspondence of path problems and time float problems. The relationships mirror the uniform qualities of networks in various disciplines and fields. We apply such relationships to solve optimum allocation problems, equipment replacement problems, and path problems with required lengths and to analyze anomalies in projects under generalized precedence relations. These applications test the effectiveness of our proposed approach to theoretical and applied researches in various disciplines and fields.

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1. Introduction

Scientific progress has often been realized by linking various disciplines and fields and translating a complex problem in one field into a simple one in another field. Descartes created the Cartesian coordinate system to translate between geometric and algebraic problems to found analytic geometry, an excellent example of this concept. This broke a long-standing bottleneck in geometrical studies and enabled, for example, conics to be translated into quadratic equations. As in analytic geometry, we try to establish a bridge between the "path problem" and the "time float problem". The "time float problem" mainly involves the calculations of time floats of activities in projects [4, 11, 12, 14, 21, 36] (see Sect. 2).

Keywords. Operations research, path, time float, abnormal critical activity, optimum allocation problem, equipment replacement problem.

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The concept of length is generalized to represent various concepts through various measurements. For instance, length in terms of distance, monetary, flux, and time units represents distance, costs and benefits, flow or capacity, and duration, respectively. Therefore, a path problem is a generalized one that could represent a non-path problem. Many studies have solved non-path problems by translating them into path problems. Zhang *et al.* [45] used dual theory to translate a minimum cut and maximum flow in a directed planar network into a shortest-path problem. Ranjbar [32] developed a path-relinking algorithm to tackle a project scheduling problem with a resource leveling objective function. Zeng *et al.* [44] used a similar algorithm to solve a bi-objective flow shop problem. The resource-constrained shortest-path problem (RCSP) is often used as a branch-and-price subproblem [48]. The solution of a path following problem contributes to the general motion planning problem [40] and the optimal power flow problem [34]. Some cost optimization problems can also correspond to shortest-path problems [2].

Path problems and relevant algorithms provide approaches for various non-path problems. Some such as the shortest- and longest-path problems, which can be solved using the classic Dijkstra [9] and Bellman [5] algorithms, are easy, and others are very complex. Classic path problems include shortest-path problems such as the traveling salesman problem [3, 15, 27, 37], shortest-path problem with cycles [1, 19], fuzzy shortest-path problem [46], stochastic shortest path problem [38], multi-objective shortest-path problem [30], and maximum capacity shortest path problem [6]; longest-path problems such as the longest Hamiltonian cycle problem [7], longest-path problem with cycles [16], probabilistic longest-path problem [28], fuzzy longest-path problem [17], and orderly colored longest path problem [35]; and other path problems such as the multi-constrained path problem [43], optimal path problem in a network [20], k-best path problem [29], minimum k-path connected vertex cover problem [23], and median path problem with bounded length [22]. Problems such as the traveling salesman problem and longest Hamiltonian cycle problem are NP-hard [7, 37].

In the time field, the time float concept is specific and includes total float, free float, safety float, and node float [4,11,12,14,36]. Comparing with path problems, time float problems such as computing the time float in the critical path method (CPM) [21] are generally simple. However, they may have abnormal characteristics under generalized precedence relations (GPRs) [8, 10, 13, 31, 33, 39, 41, 42], and classic algorithms may be inapplicable to linear scheduling [18, 25, 47]. Some authors have studied relationships between time floats problems and some other problems. Maculan *et al.* [26] provided insight on other considerations such as the similarities and differences between float time and flow problems. Lima *et al.* [24] provided a method for expressing problems as satisfiability whereby the set of so-called "primitives" can be expressed as a problem specification language and problems specification as path to-float time problems as sentences in that language.

Although the time float and the path containing its generalized objects exist in different fields, we discover that a time float is equivalent to the difference between the lengths of two types of paths in a time network; therefore,

$$\text{a time float} \Leftrightarrow \text{two paths}$$

Analogously with Descartes's work in analytic geometry, *i.e.*, linking a point in geometry and two numbers in algebra through Cartesian coordinates,

$$\text{a point} \Leftrightarrow \text{an ordered pair of numbers}$$

Therefore, the focus of this study is comparable to analytic geometry, and a complex path problem can be reciprocally transformed into a simple time float problem. Owing to the generalized characteristics of path problems, many unconventional ones also can be translated into time float problems and solved in the time field (see Sects 4.1 and 4.2), thus affording new approaches for complex problems. Commensurately, time float problems can be analyzed in a perspective of path length that contributes to the optimal solution (see Sect. 4.4).

This reciprocal transformation should also clarify uniformities in fields such as transportation, communications, computers, and finance. Developments in the economy and technology have afforded people increasing action spaces and forms of sociation. Accordingly, people have been structuring large transportation, communications, computers, and finance systems. These systems can mostly be represented as networks with length factors, such as transportation networks, communications networks, computer networks, finance networks, co-operation networks, social networks, and supply chain networks. Their structures should support travel, goods flow, cash flow, and information statistics, increase productivity, and improve quality of life. Therefore, it is important to study these systems. Currently, most relevant literatures focus on these networks in their own fields and on the uniform qualities of these networks' structures and operations in various fields.

This study applied the reciprocal transformation of the path problem and the time float problem to reveal the uniform qualities of various networks in terms of time float. It shows that time float parameters can be used to analyze and describe networks in various fields.

2. Time floats in time network

In a time network such as CPM or GPRs network, classic time floats mainly include total float TF_{ij}, free float FF_{ij}, safety float SF_{ij} of an activity (i,j), and note float TF_i of a node (i). Let ET_i and LT_i respectively indicate the earliest and latest time of a node (i) (see Appendix A) and d_{ij}, the duration of an activity. Then, the above time floats can be defined as follows:

Total Float. The total float TF_{ij} of an activity (i,j) is defined as

$$TF_{ij} = LT_j - ET_i - d_{ij} \tag{2.1}$$

which gives the permissible delay in an activity that does not delay a project.

Free Float. The free float FF_{ij} of an activity (i,j) is defined as

$$FF_{ij} = ET_j - ET_i - d_{ij} \tag{2.2}$$

which gives the permissible delay in an activity finish-time that does not affect the earliest start-time of its immediate succeeding activities.

Safety Float. The safety float SF_{ij} of an activity (i,j) is defined as

$$SF_{ij} = LT_j - LT_i - d_{ij} \tag{2.3}$$

which gives the number of periods by which an activity can be prolonged when its immediate preceding activities are completed at the latest completion time without delaying the project.

Node Float. The node float TF_i of node (i) is defined as

$$TF_i = LT_i - ET_i \tag{2.4}$$

which indicates the permissible delay in an activity that affects the earliest start-time of its immediate succeeding activities without delaying the project.

3. Relation laws of paths in length network and time floats in time network

In a length network, in which arc lengths that may be positive or negative and that cycles may exist, the path length is the sum of lengths of all arcs on the path. The length network can be considered as a time network by associating arcs to activities and arc lengths to activity durations (activity durations may be positive/negative and cycles may exist as in GPRs networks). In this case, it is advisable to let the length network have a single start node (1) and end node (n). If there exist more start or end nodes, dummy arcs and nodes can be added to the unique start and end nodes.

The relation laws of paths in a length network and time floats in the corresponding time network (when equating arcs and lengths to activities and durations) are given by the following theorems and corollaries. We name the path from the start node (1) to the end node (n) as "path" and other paths as "path sections" for convenience.

Lemma 3.1. *The length of the longest path section $\mu^{\nabla}_{1\to i}$ from the start node (1) to a node (i) in a length network, when equating the "length" to "time", is equal to the earliest time ET_i of the node (i) in the corresponding time network, that is,*

$$L(\mu^{\nabla}_{1\to i}) = ET_i \tag{3.1}$$

Proof. See Appendix B. □

Lemma 3.2. *The difference between the length of the longest path μ^{∇} and that of the longest path section $\mu^{\nabla}_{i\to n}$ from a node (i) to the end node (n) in a length network, when equating the "length" to "time", is equal to the latest time LT_i of the node (i) in the corresponding time network, that is,*

$$L(\mu^{\nabla}) - L(\mu^{\nabla}_{i\to n}) = LT_i \tag{3.2}$$

Proof. See Appendix C. □

Theorem 3.3. *The difference between the length of the longest path μ^{∇} and that of the longest path μ^{∇}_{ij} containing an arc (i, j) in a length network, when equating the "length" to "time", is equal to the total float TF_{ij} of the activity (i, j) in the corresponding time network, that is,*

$$L(\mu^{\nabla}) - L(\mu^{\nabla}_{ij}) = TF_{ij} \tag{3.3}$$

Proof. See Appendix D. □

Theorem 3.4. *The difference between the length of the longest path μ^{∇}_{j} containing a node (j) and that of the longest path μ^{∇}_{ij} containing an in-arc (i, j) of the node (j) in a length network, when equating the "length" to "time", is equal to the free float FF_{ij} of the activity (i, j) in the corresponding time network, that is,*

$$L(\mu^{\nabla}_{j}) - L(\mu^{\nabla}_{ij}) = FF_{ij} \tag{3.4}$$

Proof. See Appendix E. □

Theorem 3.5. *The difference between the length of the longest path μ^{∇}_{i} passing a node (i) and that of the longest path μ^{∇}_{ij} passing an out-arc (i, j) of the node (i) in a length network, when equating the "length" to "time", is equal to the safety float SF_{ij} of the activity (i, j) in the corresponding time network, that is,*

$$L(\mu^{\nabla}_{i}) - L(\mu^{\nabla}_{ij}) = SF_{ij} \tag{3.5}$$

Proof. It is similar to the proof of Theorem 3.4. □

Theorem 3.6. *The difference between the length of the longest path μ^{∇} and that of the longest path μ^{∇}_{i} passing a node (i) in a length network, when equating the "length" to "time", is equal to the time float TF_i of the node (i) in the corresponding time network, that is,*

$$L(\mu^{\nabla}) - L(\mu^{\nabla}_{i}) = TF_i \tag{3.6}$$

Proof. It is similar to the proof of Theorem 3.3. □

Theorem 3.7. *The difference between the length of the longest path μ^{∇} and that of a path μ in a length network, when equating the "length" to "time", is equal to the sum of free floats or safety floats of all activities on the path μ in the corresponding time network, that is,*

$$L(\mu^{\nabla}) - L(\mu) = \sum_{(u,v)\in\mu} FF_{uv} = \sum_{(u,v)\in\mu} SF_{uv} \tag{3.7}$$

Proof. See Appendix F. □

Theorem 3.8. *For an arc (u, v) on the longest path section $\mu^{\nabla}_{1\to i}$ in a length network, when equating the "length" to "time", the free float FF_{uv} of the activity (u, v) is 0 in the corresponding time network, that is, if $(u, v) \in \mu^{\nabla}_{1\to i}$, then*

$$FF_{uv} = 0 \tag{3.8}$$

Proof. See Appendix G. □

Corollary 3.9. *A path section $\mu_{1\to i}$ is the longest one $\mu^{\nabla}_{1\to i}$ in a length network in case the free floats of all activities on $\mu_{1\to i}$ are 0 when equating the "length" to "time", that is, if $FF_{uv} = 0$ for $\forall(u, v) \in \mu^{\nabla}_{1\to i}$, then*

$$\mu^{\nabla}_{1\to i} = \mu_{1\to i} \tag{3.9}$$

Proof. See Appendix H. □

Theorem 3.10. *For an arc (u, v) on the longest path section $\mu^{\nabla}_{i\to n}$ in a length network, when equating the "length" to "time", the safety float SF_{uv} of an activity (u, v) is 0 in the corresponding time network, that is, if $(u, v) \in \mu^{\nabla}_{i\to n}$, then*

$$SF_{uv} = 0 \tag{3.10}$$

Proof. It is similar to the proof of Theorem 3.8. □

Corollary 3.11. *A path section $\mu_{i\to n}$ is the longest one $\mu^{\nabla}_{i\to n}$ in a length network in case the safety floats of all activities on $\mu_{i\to n}$ are 0 when equating the "length" to "time", that is, if $SF_{uv} = 0$ for $\forall(u, v) \in \mu^{\nabla}_{i\to n}$, then*

$$\mu^{\nabla}_{i\to n} = \mu_{i\to n} \tag{3.11}$$

Proof. It is similar to the proof of Corollary 3.9. □

Based on the generalized concept of a length network, the above relation laws between paths and time floats show uniform qualities of networks in various disciplines and fields and apply to various types of networks. In the next section, we illustrate applications of the above conclusions.

4. Illustration

The following illustrations validate the practicability of the reciprocal transformation of paths and time floats for theoretical and practical problems.

4.1. Optimum allocation problem

A power generation enterprise plans to build 7 power plants in 4 districts, with the corresponding profits as listed in Table 1. An optimum plan needs to be formulated subject to the following constraints: (1) at least one power plant is allocated in each district; (2) total profit is at most 50 less than the maximum profit (practical constraints on other objectives may result in unreality for the maximum profit); and (3) more power plants should be allocated in under-populated and undeveloped district C and less, in populous and developed district A.

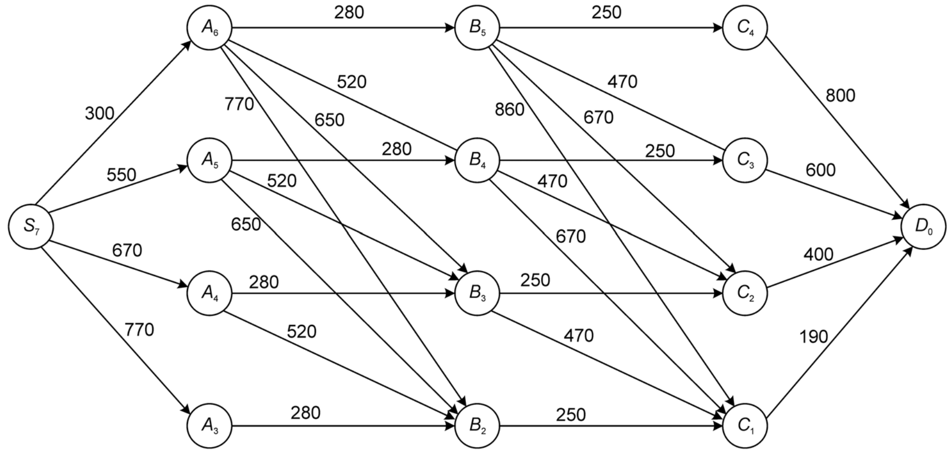

FIGURE 1. Length network to represent the power plant allocation problem.

TABLE 1. Profit of power plant allocation and number.

Area	1	2	3	4	5	6	7
A	300	550	670	770	850	930	1000
B	280	520	650	770	870	960	1050
C	250	470	670	860	990	1020	1100
D	190	400	600	800	1000	1200	1400

Process

The problem is to make a multiple objective decision. The primary objective is to allocate at least one power plant in each district and make a total profit at most 50 less than the maximum profit. The secondary objective is to ensure a suitable number of power plants in districts A and C.

(1) For convenience, we use the network shown in Figure 1 to represent the problem. Node S_7 indicates the 7 power plants at the initial time; node A_i, the remaining i power plants after allocation in district A; length of arc (S_7, A_i), the profit of the $7 - i$ power plants in district A; node B_j, the remaining j power plants after allocation in districts A and B; length of arc (A_i, B_j), the profit of the $i - j$ power plants in district B; node C_k, the remaining k power plants after allocation in districts A, B, and C; length of arc (B_j, C_k), the profit of the $j - k$ power plants in district C; node D_0, there were no power plants left after allocation in district D; and length of arc (C_k, D_0), the profit of the k power plants in district D.
In Figure 1, a path indicates a plan and its length, the profit. Thus, making a plan with profit at most 50 less than the maximum profit is translated into finding a path at most 50 shorter than the longest path.

(2) Simplify the path problem.
Simplification involves removing all paths more than 50 shorter than the longest path. However, this incurs a high computational load, making it difficult to directly remove paths shown in Figure 1. We test and verify that simplification is much simpler after translating the path problem into a time float problem.

The arc lengths in Figure 1 are considered as "time" *i.e.* activity durations to translate the length network into a time network. The time parameters of nodes in the time network are computed as shown in Figure 2.

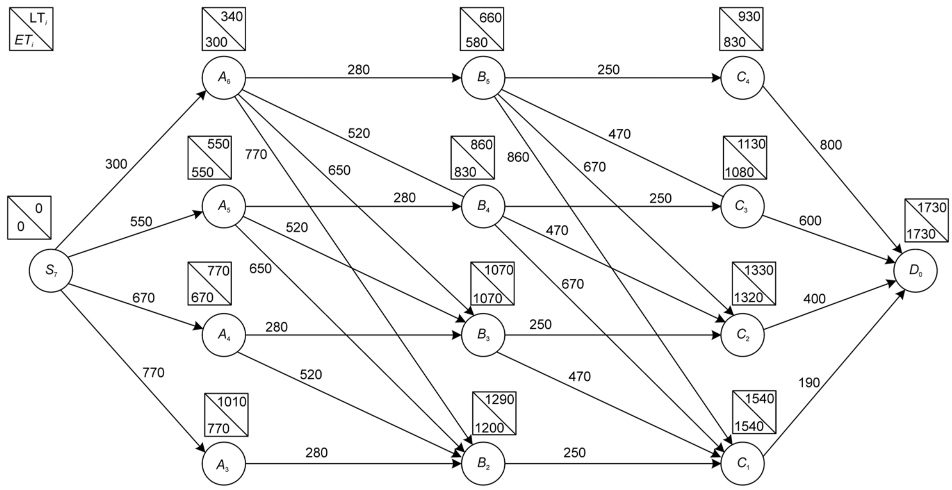

FIGURE 2. Length network shown in Figure 1 represented as a time network.

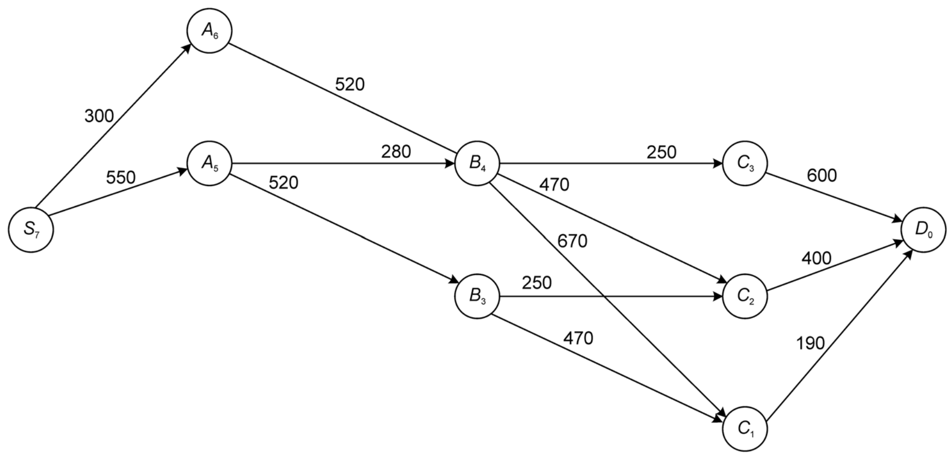

FIGURE 3. Simplification of network shown in Figure 1.

(1) According to equation (3.4), $TF_i = LT_i - ET_i$. If $TF_i > 50$, then in the original length network, $LT_i - ET_i > 50$; each path μ_i passing the node (i) is more than 50 shorter than the longest path μ^∇ and can be removed by removing the node (i) and its adjacent arcs.

(2) Similarly, according to equation (3.1), if $TF_{ij} > 50$, then in the original length network, each path μ_{ij} passing the arc (i, j) can be removed by removing the arc (i, j).

Therefore, in Figure 2, we only need to compute the node floats and total floats of activities to remove paths more than 50 shorter than the longest path μ^∇ from the original length network, to give the simplified network shown in Figure 3.

(3) Choose the optimum plan.

In Figure 3, the length of the longest path is 1730, indicating that the greatest profit is 1730. Paths with lengths $\geq 1730 - 50 = 1680$ indicate other feasible plans meeting the primary objective. Then, for the secondary objective, two plans give the most $(4 - 1 = 3)$ power plants in district C:

Plan 1: $(S_7) \to (A_6) \to (B_4) \to (C_1) \to (D_0)$

Plan 2: $(S_7) \to (A_5) \to (B_4) \to (C_1) \to (D_0)$

Table 2. Equipment cost.

Year	1	2	3	4	5	6	7
Price p_i	21	22	23	24	25	26	7

Table 3. Maintenance cost and benefit of equipment.

Year	0–1	1–2	2–3	3–4	4–5	5–6	6–7
Maintenance m_i	5	6	7	9	11	14	18
Benefit b_i	49	39	31	27	25	24	22
Net benefit $b_i - m_i$	44	33	24	18	14	10	4

Hence, Plan 1 with less $(7-6=1)$ power plants in district A is the optimum one. This plan gives $6-4=2$ power plants in district B and 1 power plant in district D. The total profit is $300+520+670+190=1680$.

4.2. Equipment replacement problem

A factory purchases equipment at the beginning of the 1st year, and the manager plans for its maintenance or replacement (early in one year) over the following seven years. The price, maintenance cost, and benefit of this equipment are shown in Tables 2 and 3. Consider the following questions:

(1) How should a plan with the greatest net benefit be formulated?
(2) More expensive equipment has higher quality, lower energy consumption, and reduced emissions. Thus, how should the optimum plan with (1) net benefits at most 3% less than the greatest one and (2) the higher equipment cost be formulated?

Process

Subproblem (2) is a multi-object decision-making problem in which the primary objective is a net benefit of not more than 3% less than the greatest one and a secondary objective of higher equipment cost.

(1) For convenience, we use the network shown in Figure 4 to represent the problem. Nodes (1)–(7) indicate early times in the 1st–7th years and node (8) indicates a late time in the 7th year; and arc (i,j) indicates running the new equipment from early in the ith year to late in the $j-1$th year and its length l_{ij} indicates the net benefit in the $j-i$ years, that is,

$$l_{ij} = \sum_{k=i}^{j-1}(b_i - m_i) - p_i$$

Thus, each path $\mu_{1\to 8}$ in Figure 4 indicates a plan for equipment replacement in seven years, and its length indicates the net benefit. Therefore, the plan with the greatest net benefit is the longest path in Figure 4, and the plan with net benefit not more than 3% less than the greatest one is the path that is not more than 3% shorter than the longest one.

(2) Solve subproblem (1) and simplify subproblem (2).

We consider the arc lengths in Figure 4 as "time" *i.e.* activity durations, and we compute the time parameters of nodes as shown in Figure 5. The project duration in Figure 5 is 186 (the bold path – (1) → (3) → (5) → (8) – is the critical (longest) path). This indicates that the plan with the greatest net benefit (186) in the original problem involves replacing the equipment early in the 3rd and 5th years.

As in Section 4.1, the simplification of subproblem (2) involves removing paths that are more than $186\times 3\% \approx 6$ shorter than the longest path shown in Figure 4. We compute the time floats of the nodes shown in Figure 5,

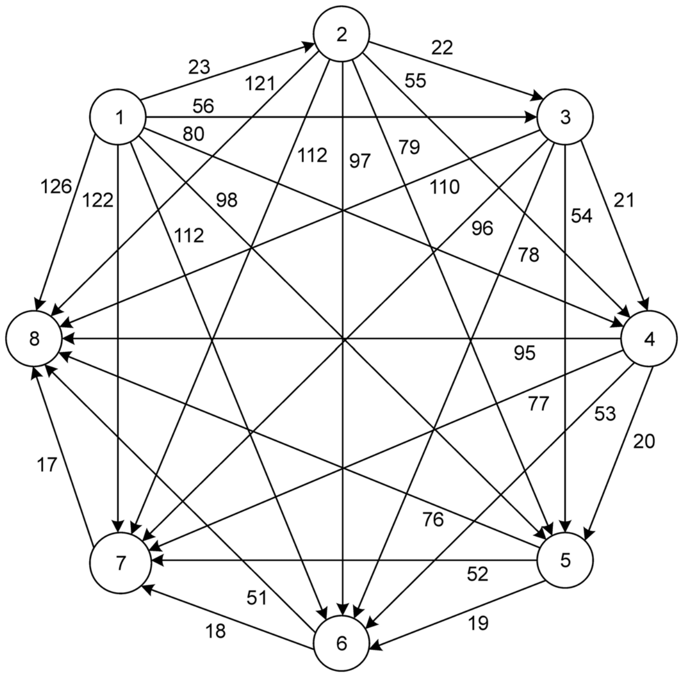

FIGURE 4. Length network representing the equipment replacement problem.

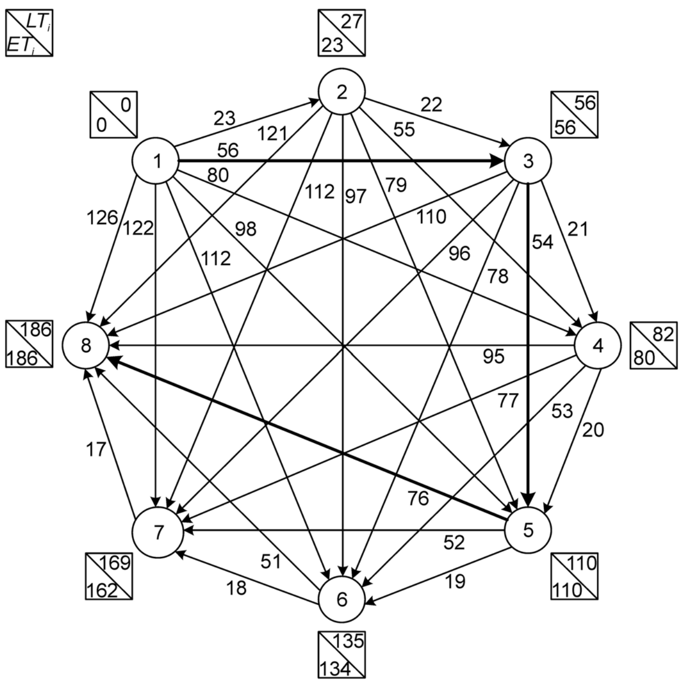

FIGURE 5. Time network translated from length network shown in Figure 4.

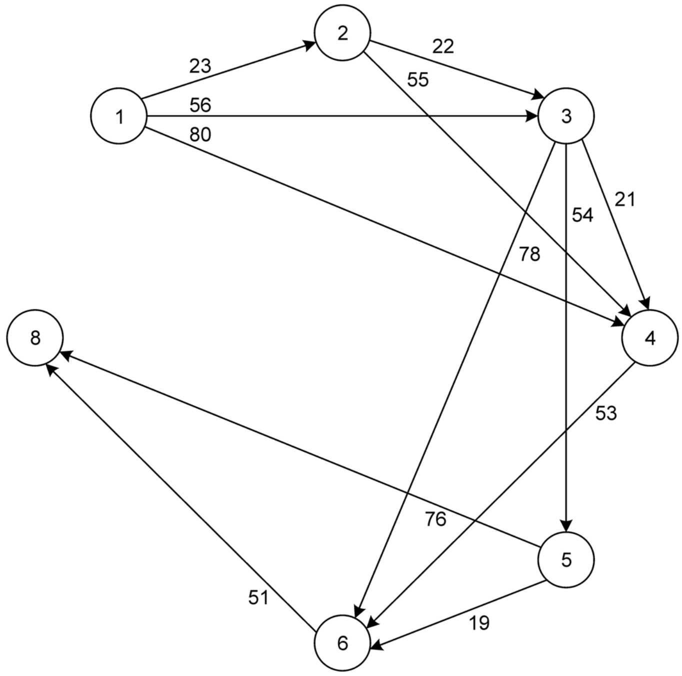

FIGURE 6. Length network after simplification.

remove nodes with $TF_i > 186 - 180 = 6$ and their adjacent arcs, compute the total floats of the remaining activities, and remove activities with $TF_{ij} > 6$, as shown in Figure 6.

(3) Choose optimum plan

We easily find the following paths that are not shorter than $186 - 6 = 180$ in Figure 6:

$$\mu_1 = (1) \to (2) \to (4) \to (6) \to (8)$$
$$\mu_2 = (1) \to (3) \to (5) \to (8)$$
$$\mu_3 = (1) \to (3) \to (5) \to (6) \to (8)$$
$$\mu_4 = (1) \to (4) \to (6) \to (8)$$

Their lengths are 182, 186, 180, and 184, respectively. The corresponding equipment replacement costs are 83, 70, 96, and 71, as listed in Table 2. Therefore, the plan with the highest equipment cost (96) corresponding to μ_3 is the optimum one (the net benefit is 180), that is, replacing the equipment early in the 3rd, 5th, and 6th years.

4.3. Path problem with required length

The longest- and shortest-path problems basically involve finding the required paths. The optimal solutions of practical problems are often the paths with the required lengths instead of the longest or shortest ones. For instance, the path for a city marathon is one with the required length. Current approaches for these path problems are based upon lengths. In this study, we translate path problems into time float problems to realize simpler approaches.

4.3.1. Method

We can use one of the following methods to find paths with length λ in a length network. We consider arc lengths in the network as "time" *i.e.* activity durations and compute the time parameters of nodes.

Method 1

Find a node (i) with $TF_i = ET_n - \lambda$. In the original length network, the longest path μ_i^∇ passing the node (i) is the path with length λ. Next, we obtain μ_i^∇ as follows:

(1) In the time network, find an immediate preceding activity (h, i) with $FF_{hi} = 0$ of the node (i); find an immediate preceding activity (g, h) with $FF_{gh} = 0$ of the activity (h, i); $\ldots$; and so on until we reach the start node (1) and obtain a path section $\mu_{1\to i} = (1) \to \cdots \to (g) \to (h) \to (i)$. In the original length network, $\mu_{1\to i}$ is the longest path section $\mu_{1\to i}^\nabla$ from the start node (1) to the node (i).
(2) Find an immediate succeeding activity (i, j) with $SF_{ij} = 0$ of the node (i); find an immediate succeeding activity (j, k) with $SF_{jk} = 0$ of the activity (i, j); $\ldots$; and so on until we reach the end node (n) and obtain a path section $\mu_{i\to n} = (i) \to (j) \to (k) \to \cdots \to (n)$. In the original length network, $\mu_{i\to n}$ is the longest path section $\mu_{i\to n}^\nabla$ from the node (i) to the end node (n). Then,

$$\mu_i^\nabla = \mu_{1\to i}^\nabla + \mu_{i\to n}^\nabla$$

Method 2

Find an activity (i, j) with $TF_{ij} = ET_n - \lambda$. In the original length network, the longest path μ_{ij}^∇ passing the arc (i, j) is the path with length λ. Next, we obtain μ_{ij}^∇ by the same process as μ_i^∇.

Method 3

Find an immediate preceding activity (i, j) with $0 < FF_{ij} < ET_n - \lambda$ of a critical node (with node float 0) and its immediate preceding activity (h, i) with $FF_{hi} = ET_n - \lambda - FF_{ij}$. In the original length network, the longest path passing the two arcs (h, i) and (i, j) is the path with length λ, that is,

$$\mu_{hi\to ij}^\nabla = \mu_{1\to h}^\nabla + (h, i) + (i, j) + \mu_{j\to n}^\nabla$$

Method 4

Find an immediate succeeding activity (i, j) with $0 < SF_{ij} < ET_n - \lambda$ of a critical node and its immediate succeeding activity (j, k) with $SF_{jk} = ET_n - \lambda - SF_{ij}$. In the original length network, the longest path passing the two arcs (i, j) and (j, k) is the path with length λ, that is,

$$\mu_{ij\to jk}^\nabla = \mu_{1\to i}^\nabla + (i, j) + (j, k) + \mu_{k\to n}^\nabla$$

Method 5

Find an immediate preceding activity (i, j) with $0 < FF_{ij} < ET_n - \lambda$ of a critical node; activities (h, i), (g, h), (f, g), $\ldots$, (b, c) with free floats 0; and an immediate preceding activity (a, b) with $FF_{ab} = ET_n - \lambda - FF_{ij}$ of the activity (b, c). In the original length network, the longest path passing these arcs is the path with length λ, that is,

$$\mu_{ab\to ij}^\nabla = \mu_{1\to a}^\nabla + (a, b) + (b, c) + \cdots + (f, g) + (g, h) + (h, i) + (i, j) + \mu_{j\to n}^\nabla$$

Method 6

Find an immediate succeeding activity (i, j) with $0 < SF_{ij} < ET_n - \lambda$ of a critical node; activities (j, k), (k, l), (l, m), $\ldots$, (u, v) with safety floats 0; and an immediate succeeding activity (v, r) with $SF_{vr} = ET_n - \lambda - SF_{ij}$ of the activity (r, s). In the original length network, the longest path passing these arcs is the path with length λ, that is,

$$\mu_{ij\to uv}^\nabla = \mu_{1\to i}^\nabla + (i, j) + (j, k) + (k, l) + (l, m) + \cdots + (v, r) + (r, s) + \mu_{s\to n}^\nabla$$

The processes of finding $\mu_{1\to i}^\nabla$ and $\mu_{j\to n}^\nabla$ in Methods 2–6 are same to that in Method 1. The proofs of the above methods are presented in Appendix I.

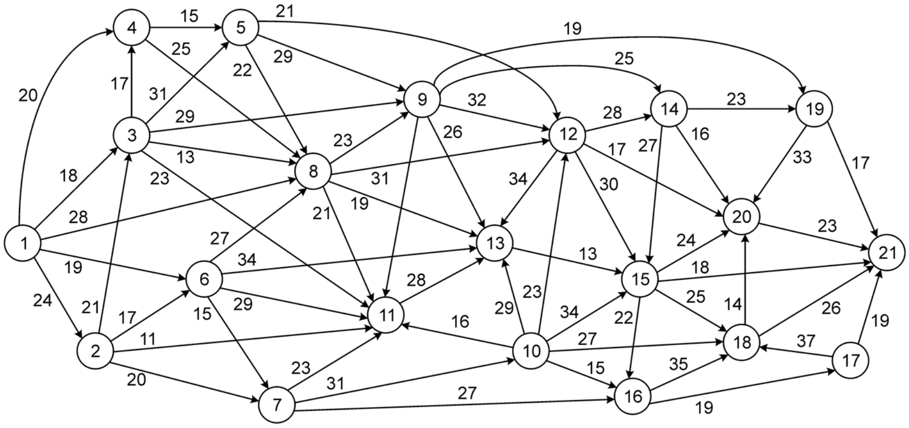

FIGURE 7. Example of a length network.

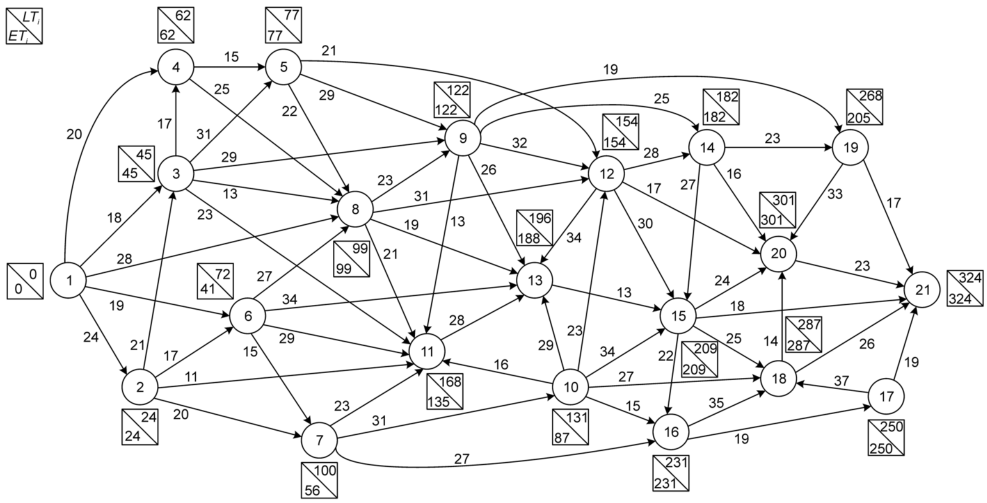

FIGURE 8. Time network translated from length network shown in Figure 7.

4.3.2. Illustration

We show an example of finding paths with lengths 280 and 244 in Figure 7.

We consider the arc lengths in Figure 7 as activity durations and compute the time parameters of nodes, as shown in Figure 8.

(1) Find the path with length 280.

The node float of node (7) is $TF_7 = 324 - 280 = 44$; hence, in Figure 7, the longest path μ_7^{∇} passing the node is the required path with length 280. Next, we obtain μ_7^{∇} as follows:

(a) In Figure 8, find immediate preceding activity $(6, 7)$ with $FF_{6,7} = 0$ of node (7); find immediate preceding activity $(2, 6)$ with $FF_{2,6} = 0$ of activity $(6, 7)$; $\ldots$; and so on, until we reach start node (1). The longest path section from start node (1) to node (7) in Figure 7 is

$$\mu_{1\to 7}^{\nabla} = (1) \to (2) \to (6) \to (7)$$

(b) Find immediate succeeding activity $(7,10)$ with $SF_{7,10}=0$ of node (7); find immediate succeeding activity $(10,12)$ with $SF_{10,12}=0$ of activity $(7,10)$; ...; and so on, until we reach end node (21). The longest path from node (7) to end node (21) in Figure 7 is

$$\mu^{\nabla}_{7\to 21}=(7)\to(10)\to(12)\to(14)\to(15)\to(16)\to(17)\to(18)\to(20)\to(21).$$

Therefore,

$$\begin{aligned}\mu^{\nabla}_{7}&=\mu^{\nabla}_{1\to 7}+\mu^{\nabla}_{7\to 21}\\&=(1)\to(2)\to(6)\to(7)\to(10)\to(12)\to(14)\to\\&\quad(15)\to(16)\to(17)\to(18)\to(20)\to(21).\end{aligned}$$

(2) Find the path with length 244.

There are no nodes (i) with $TF_i=324-244=80$. The total float of activity $(10,13)$ is $TF_{10,13}=80$; hence, in Figure 7, the longest path $\mu^{\nabla}_{10,13}$ passing arc $(10,13)$ is the required path with length 244. Next, we use a method similar to that in step (1) to obtain

$$\begin{aligned}\mu^{\nabla}_{10,13}=&(1)\to(2)\to(6)\to(7)\to(10)\to(13)\to(15)\to\\&(16)\to(17)\to(18)\to(20)\to(21).\end{aligned}$$

These methods are mainly suited for special cases. For other generalized cases of finding paths with required lengths, we can improve and develop these methods, *e.g.* develop methods for the kth shortest or longest paths problems based on the principles of Methods 5 and 6.

4.4. Analysis of laws of anomalies in project under GPRs

Elmaghraby and Kamburowski [13] discovered an anomaly in GPRs networks: if durations of some critical activities are prolonged, the length of the critical path (*i.e.* project duration) decreases. Now, a critical activity's duration could be prolonged infinitely, but the project duration obviously cannot be shortened infinitely; therefore, the above anomaly can only appear in a certain duration range. Therefore, the following questions need to be solved urgently: (1) How to identify the range? (2) What phenomenon will appear beyond this range? (3) What laws apply to the anomaly?

For simplifying these time problems, we use the relation laws of paths and time floats to translate them into path problems and derive some conclusions. Figure 9 shows an abnormal critical activity A in a GPRs network. Next, we explain the anomaly and research laws of the project duration when prolonging the duration of activity A.

The critical path (bold path) in Figure 9 is $\mu^{\nabla}=(1)\to\cdots\to(k)\to(j)\to(i)\to(r)\to\cdots\to(n)$. Durations $d_{ij}=d_A$ and $d_{ji}=-d_A$ cause $d'_{ij}=d_{ij}+\Delta d$ and $d'_{ji}=d_{ji}-\Delta d$ when $d'_A=d_A+\Delta d$, $\Delta d>0$. Furthermore, $(j,i)\in\mu^{\nabla}_{ji}=\mu^{\nabla}$ and $d'_{ji}=d_{ji}-\Delta d$ cause $L(\mu^{\nabla\prime})=L(\mu^{\nabla})-\Delta d$, that is, the project duration will be shortened when prolonging the duration d_A of activity A.

In Figure 9, if assuming $FF_{pi}<SF_{jq}$ and that activity A will keep criticality while its duration d_A is prolonged, we can derive the following conclusions:

(1) If $0<\Delta d\le FF_{pi}$, the project duration will be shortened when increasing d_A. That is, duration $d'_A=d_A+\Delta d$ for $0<\Delta d\le FF_{pi}$ causes $L(\mu^{\nabla\prime})=L(\mu^{\nabla})-\Delta d$.
(2) If $FF_{pi}<\Delta d\le SF_{jq}$, the project duration will keep $L(\mu^{\nabla\prime})=L(\mu^{\nabla})-FF_{pi}$ when increasing d_A. That is, duration $d'_A=d_A+\Delta d$ for $FF_{pi}<\Delta d\le SF_{jq}$ causes $L(\mu^{\nabla\prime})=L(\mu^{\nabla})-FF_{pi}$.
(3) If $SF_{jq}<\Delta d\le FF_{pi}+SF_{jq}$, the project duration $\left(L(\mu^{\nabla\prime})=L(\mu^{\nabla})-FF_{pi}\right)$ will be prolonged to $L(\mu^{\nabla})$ at most when increasing d_A. That is, duration $d'_A=d_A+\Delta d$ for $SF_{jq}<\Delta d\le FF_{pi}+SF_{jq}$ causes $L(\mu^{\nabla\prime})=L(\mu^{\nabla})+\Delta d-(FF_{pi}+SF_{jq})\le L(\mu^{\nabla})$.
(4) If $FF_{pi}+SF_{jq}<\Delta d<+\infty$, the project duration will be prolonged beyond $L(\mu^{\nabla})$. That is, duration $d'_A=d_A+\Delta d$ for $FF_{pi}+SF_{jq}<\Delta d<+\infty$ causes $L(\mu^{\nabla\prime})=L(\mu^{\nabla})+\Delta d-(FF_{pi}+SF_{jq})>L(\mu^{\nabla})$.

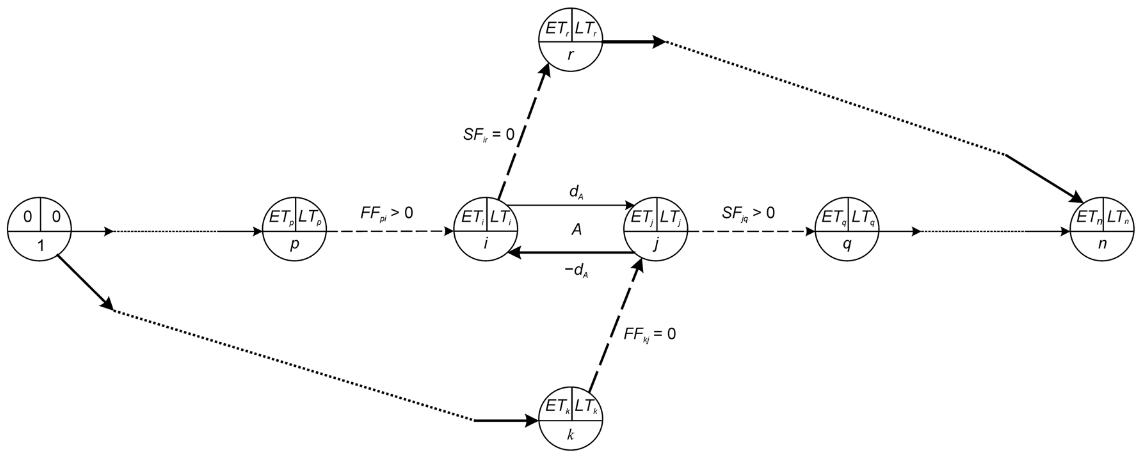

FIGURE 9. Example of a GPRs network with abnormal critical activity A.

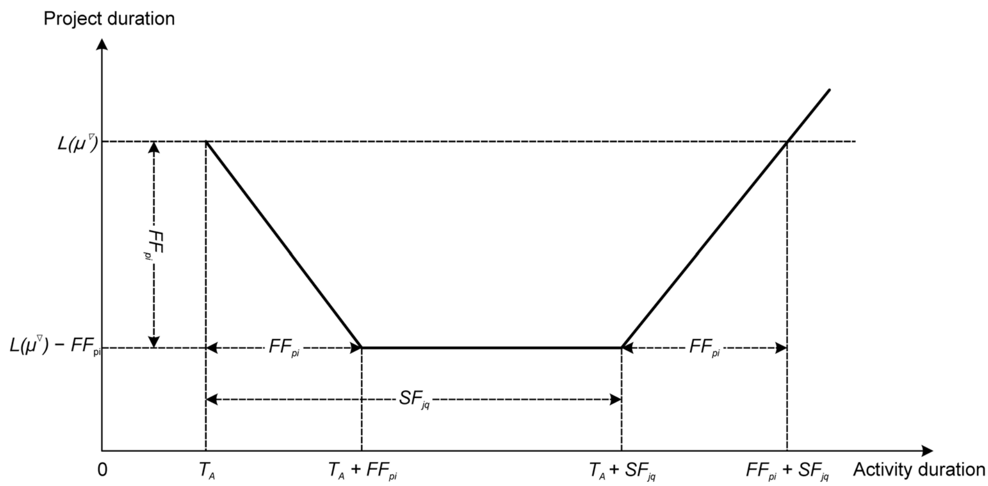

FIGURE 10. Diagram representing the laws of the project duration when prolonging the duration of abnormal critical activity A.

The following corollary summarizes the above conclusions:

Corollary 4.1. *In Figure 9, if $\Delta d > FF_{pi} + SF_{jq}$, then $L(\mu^{\nabla\prime}) > L(\mu^{\nabla})$ and the project duration will be prolonged; if $\Delta d \le FF_{pi} + SF_{jq}$, then $L(\mu^{\nabla\prime}) \le L(\mu^{\nabla})$ and the project duration will be unchanged or shortened.*

Figure 10 shows the above conclusions, and the proof is presented in Appendix J.

The above examples clearly show that some difficult "path problems" become much simpler after being translated into "time float problems" (Sects. 4.1–4.3) and that some difficult "time float problems" become much simpler after being translated into "path problems" (Sect. 4.4). The reciprocal transformation provides a new approach for studies and applications of these problems.

5. Conclusion

Descartes created the Cartesian coordinate system to translate difficult geometric problems into simpler algebraic problems, thus breaking bottlenecks in geometry and founding analytic geometry. Difficult algebraic problems can also be translated into simpler geometric problems by using Cartesian coordinates. We used this example to study the laws of relations of paths in length networks and time floats in time network and then explore approaches for the reciprocal transformation of path problems and time float problems. By using the generalized concept of length, many problems in other fields can also be translated into time float problems. Furthermore, the reciprocal transformation of length and time networks embodies uniform qualities among networks corresponding to various systems. In this study, we discussed illustrative problems in which reciprocal transformation of path and time float problems were used to (1) simplify the optimum allocation problem and equipment replacement problem, (2) easily solve the path problem with required lengths in the length network, and (3) analyze anomalies and their laws in the GPRs network.

In accordance with the conclusions in Section 3 and the illustrations in Section 4, the conversion of the path problems tackled to the corresponding time float problems is mainly to compute time parameters in time network. In the computational process, we made a finite-time consideration to each arc so that the complexity of the conversion is $O(m)$ and m indicates the number of arcs. The conversion maintains the essences of original problems, however, it provide an approach to simplify the original problems and decrease in the number of variable (see Sects. 4.1–4.3). These applications in Section 4 validate the effectiveness of our proposed approach to theoretical and applied researches in various disciplines and fields.

Appendix A. Algorithms of the earliest and latest times of nodes in time network

Assume nodes (1) and (n) respectively indicate start and end nodes of a time network, d_{ij} indicates duration of an activity (i, j), and ET_i and LT_i respectively indicate the earliest and latest times of a node (i).

(1) For the CPM network, algorithms for ET_i and LT_i are as follows:

Step 1. Let $j = 2, 3, \ldots, n$,

$$\begin{cases} ET_1 = 0 \\ ET_j = \max\limits_{i}\{ET_i + d_{ij}\} \end{cases} \tag{A.1}$$

Step 2. Let $j = n-1, n-2, \ldots, 1$,

$$\begin{cases} LT_n = ET_n \\ LT_j = \min\limits_{k}\{LT_k - d_{jk}\} \end{cases} \tag{A.2}$$

(2) For the GPRs network, algorithms for ET_i and LT_i are as follows:

Step 1. Let

$$ET_1 = 0 \tag{A.3}$$

Let

$$w^{(1)}(1, i) = d_{1i},\ i = 2, 3, \ldots, n$$

For $t = 2, 3, \ldots$,

$$w^{(t)}(1, i) = \max_{h}\{w^{(t-1)}(1, h) + d_{hi}\},\ i = 2, 3, \ldots, n$$

If $t := k$ and for $i = 2, 3, \ldots, n$,

$$w^{(k)}(1, i) = w^{(k-1)}(1, i)$$

then,

$$ET_j = w^{(k)}(1,i) \tag{A.4}$$

Step 1. Let

$$LT_n = ET_n \tag{A.5}$$

Let

$$w^{(1)}(i,n) = LT_n - d_{in},\ \ i = 1,2,\ldots,n-1$$

For $t = 2,3,\ldots$,

$$w^{(t)}(i,n) = \min_j\{w^{(t-1)}(j,n) - d_{ij}\},\ \ i = 1,2,\ldots,n-1$$

If $t := k$ and for $i = 1,2,\ldots,n-1$,

$$w^{(k)}(i,n) = w^{(k-1)}(i,n)$$

then,

$$LT_j = w^{(k)}(j,n) \tag{A.6}$$

Appendix B. Proof of Lemma 3.1

In a length network, the length of the longest path section $\mu^{\nabla}_{1\to i}$ from start node (1) to a node (i) is

$$L(\mu^{\nabla}_{1\to i}) = \sum_{(u,v)\in\mu^{\nabla}_{1\to i}} l_{uv} = \max \sum_{(u,v)\in\mu_{1\to i}} l_{uv}$$

l_{uv} indicates the length of the arc (u,v) in the length network, and when equating "length" to "time", it indicates the duration d_{uv} of the activity (u,v) in the corresponding time network, that is, $d_{uv} = l_{uv}$.

According to Appendix A, in a time network such as a CPM network or GPRs network, the earliest times of a node (i) and its immediate preceding nodes satisfy

$$ET_i = \max_h\{ET_h + d_{hi}\}$$

then,

$$\begin{aligned}
ET_i &= \max_h\{ET_h + d_{hi}\}\\
&= \max_h\{\max_g\{ET_g + d_{gh}\} + d_{hi}\}\\
&= \max_h\{\max_g\{\max_f\{ET_f + d_{fg}\} + d_{gh}\} + d_{hi}\}\\
&= \max_h\{\max_g\{\max_f\{\cdots\{\max_1\{ET_1 + d_{1a}\} + d_{ab}\} + \cdots + d_{fg}\} + d_{gh}\} + d_{hi}\}\\
&= ET_1 + \max\{d_{1a} + d_{ab} + \cdots + d_{fg} + d_{gh} + d_{hi}\}
\end{aligned}$$

According to equations (A.1) and (A.3), $ET_1 = 0$, then

$$ET_i = \max\{d_{1a} + d_{ab} + \cdots + d_{fg} + d_{gh} + d_{hi}\}$$

Corresponding to the original length network, $d_{uv} = l_{uv}$ and

$$\begin{aligned}
\max\{d_{1a} + d_{ab} + \cdots + d_{fg} + d_{gh} + d_{hi}\} &= \max\{l_{1a} + l_{ab} + \cdots + l_{fg} + l_{gh} + l_{hi}\}\\
&= \max \sum_{(u,v)\in\mu_{1\to i}} l_{uv} = L(\mu^{\nabla}_{1\to i})
\end{aligned}$$

therefore,

$$ET_i = L(\mu^{\nabla}_{1\to i})$$

Equation (3.1) and Lemma 3.1 are correct. This completes the proof.

Appendix C. Proof of Lemma 3.2

In a length network, the length of the longest path section $\mu^{\nabla}_{i\to n}$ from a node (i) to end node (n) is

$$L(\mu^{\nabla}_{i\to n}) = \sum_{(u,v)\in\mu^{\nabla}_{i\to n}} l_{uv} = \max \sum_{(u,v)\in\mu_{i\to n}} l_{uv}$$

The arc (u,v) with length l_{uv} in the length network, when equating "length" to "time", indicates the activity (u,v) with duration d_{uv} in the corresponding time network, that is, $d_{uv} = l_{uv}$.

According to Appendix A, in a time network, the latest times of a node (i) and its immediate succeeding nodes satisfy

$$LT_i = \min_j\{LT_j - d_{ij}\}$$

then,

$$\begin{aligned}
LT_i &= \min_j\{LT_j - d_{ij}\} \\
&= \min_j\{\min_k\{LT_k - d_{jk}\} - d_{ij}\} \\
&= \min_j\{\min_k\{\min_l\{LT_l - d_{kl}\} - d_{jk}\} - d_{ij}\} \\
&= \min_j\{\min_k\{\min_l\{\cdots\{\min_n\{LT_n - d_{yn}\} - d_{xy}\} - \cdots - d_{kl}\} - d_{jk}\} - d_{ij}\} \\
&= LT_n + \min\{-d_{ij} - d_{jk} - d_{kl} - \cdots - d_{xy} - d_{yn}\} \\
&= LT_n - \max\{d_{ij} + d_{jk} + d_{kl} + \cdots + d_{xy} + d_{yn}\}
\end{aligned}$$

Corresponding to the original length network, $d_{uv} = l_{uv}$ and

$$\begin{aligned}
\max\{d_{ij} + d_{jk} + d_{kl} + \cdots + d_{xy} + d_{yn}\} &= \max\{l_{ij} + l_{jk} + l_{kl} + \cdots + l_{xy} + l_{yn}\} \\
&= \max \sum_{(u,v)\in\mu_{i\to n}} l_{uv} = L(\mu^{\nabla}_{i\to n})
\end{aligned}$$

Furthermore, according to equation (3.1), let $i = n$; then,

$$ET_n = L(\mu^{\nabla}_{1\to n}) = L(\mu^{\nabla})$$

According to equations (A.2) and (A.5),

$$LT_n = ET_n = L(\mu^{\nabla}) \tag{C.1}$$

therefore,

$$\begin{aligned}
LT_i &= LT_n - \max\{d_{ij} + d_{jk} + d_{kl} + \cdots + d_{xy} + d_{yn}\} \\
&= L(\mu^{\nabla}) - L(\mu^{\nabla}_{i\to n})
\end{aligned}$$

Equation (3.2) and Lemma 3.2 are correct. This completes the proof.

Appendix D. Proof of Theorem 3.3

In a length network, the longest path μ^{∇}_{ij} passing an arc (i,j) is composed of $\mu^{\nabla}_{1\to i}$, (i,j), and $\mu^{\nabla}_{j\to n}$. The arc (i,j) with length l_{ij} in the length network, when equating "length" to "time", indicates the activity (i,j)

with duration d_{ij} in the corresponding time network, that is, $d_{ij} = l_{ij}$. According to equations (2.1), (3.1), (3.2), and (C.1),

$$\begin{aligned} L(\mu^{\nabla}) - L(\mu^{\nabla}_{ij}) &= L(\mu^{\nabla}) - \big(L(\mu^{\nabla}_{1\to i}) + l_{ij} + L(\mu^{\nabla}_{j\to n})\big) \\ &= LT_n - (ET_i + d_{ij} + LT_n - LT_j) \\ &= LT_j - ET_i - d_{ij} \\ &= TF_{ij} \end{aligned}$$

Equation (3.3) and Theorem 3.3 are correct. This completes the proof.

Appendix E. Proof of Theorem 3.4

In a length network, the longest path $\mu^{\nabla}_j = \mu^{\nabla}_{1\to j} + \mu^{\nabla}_{j\to n}$ and $\mu^{\nabla}_{ij} = \mu^{\nabla}_{1\to i} + (i,j) + \mu^{\nabla}_{j\to n}$. The arc (i,j) with length l_{ij} in the length network, when equating "length" to "time", indicates the activity (i,j) with duration d_{ij} in the corresponding time network. According to equations (2.2), (3.1) and (3.2),

$$\begin{aligned} L(\mu^{\nabla}_j) - L(\mu^{\nabla}_{ij}) &= \big(L(\mu^{\nabla}_{1\to j}) + L(\mu^{\nabla}_{j\to n})\big) - \big(L(\mu^{\nabla}_{1\to i}) + l_{ij} + L(\mu^{\nabla}_{j\to n})\big) \\ &= (ET_j + LT_n - LT_j) - (ET_i + d_{ij} + LT_n - LT_j) \\ &= ET_j - ET_i - d_{ij} \\ &= FF_{ij} \end{aligned}$$

Equation (3.4) and Theorem 3.4 are correct. This completes the proof.

Appendix F. Proof of Theorem 3.7

In a length network, assume a path

$$\mu = (1) \to (p) \to (q) \to (r) \to \cdots \to (u) \to (v) \to (n)$$

whose length is $L(\mu) = l_{1p} + l_{pq} + l_{qr} + \cdots + l_{uv} + l_{vn} = \sum\limits_{(i,j)\in\mu} l_{ij}$. The arc (i,j) with length l_{ij} in the length network, when equating "length" to "time", indicates the activity (i,j) with duration d_{ij} in the corresponding time network. And here μ is composed of the activities $(1,p),(p,q),(q,r),\cdots,(u,v),(v,n)$. According to equation (2.2),

$$\begin{aligned} \sum_{(i,j)\in\mu} FF_{ij} &= FF_{1p} + FF_{pq} + FF_{qr} + \cdots + FF_{uv} + FF_{vn} \\ &= (ET_p - ET_1 - d_{1p}) + (ET_q - ET_p - d_{pq}) + (ET_r - ET_q - d_{qr}) + \cdots \\ &\quad + (ET_v - ET_u - d_{uv}) + (ET_n - ET_v - d_{vn}) \\ &= ET_n - ET_1 - (d_{1p} + d_{pq} + d_{qr} + \cdots + d_{uv} + d_{vn}). \end{aligned}$$

According to equations (A.1), (A.3), and (C.1), $ET_1 = 0$ and $ET_n = L(\mu^{\nabla})$; then,

$$\sum_{(i,j)\in\mu} FF_{ij} = L(\mu^{\nabla}) - (d_{1p} + d_{pq} + d_{qr} + \cdots + d_{uv} + d_{vn}).$$

Corresponding to the original length network, $d_{ij} = l_{ij}$ and

$$d_{1p} + d_{pq} + d_{qr} + \cdots + d_{uv} + d_{vn} = l_{1p} + l_{pq} + l_{qr} + \cdots + l_{uv} + l_{vn} = L(\mu^{\nabla})$$

therefore,

$$L(\mu^{\nabla}) - L(\mu) = \sum_{(i,j)\in\mu} FF_{ij}.$$

Similarly, it can be proved that

$$L(\mu^{\nabla}) - L(\mu) = \sum_{(i,j)\in\mu} SF_{ij}.$$

Equation (3.7) and Theorem 3.7 are correct. This completes the proof.

Appendix G. Proof of Theorem 3.8

In a length network, for an arc $(i,j) \in \mu^{\nabla}_{1\to i}$, $(i), (j) \in \mu^{\nabla}_{1\to i}$ causes $\mu^{\nabla}_{1\to u}$ and $\mu^{\nabla}_{1\to v}$ satisfy $\mu^{\nabla}_{1\to u}, \mu^{\nabla}_{1\to v} \subset \mu^{\nabla}_{1\to i}$ and $\mu^{\nabla}_{1\to v} = \mu^{\nabla}_{1\to u} + (u,v)$. Therefore,

$$L(\mu^{\nabla}_{1\to v}) - L(\mu^{\nabla}_{1\to u}) - l_{ij} = 0 \tag{G.1}$$

The arc (i,j) with length l_{ij} in the length network, when equating "length" to "time", indicates the activity (i,j) with duration d_{ij} in the corresponding time network. According to equation (3.1), $L(\mu^{\nabla}_{1\to v}) = ET_v$, $L(\mu^{\nabla}_{1\to u}) = ET_u$; we then substitute them in equation (G.1) such that

$$ET_v - ET_u - d_{uv} = 0.$$

According to equation (2.2),

$$FF_{uv} = ET_v - ET_u - d_{uv} = 0.$$

Equation (3.8) and Theorem 3.8 are correct. This completes the proof.

Appendix H. Proof of Corollary 3.9

In a length network, assume a path section

$$\mu_{1\to i} = (1) \to (e) \to (f) \to (r) \to \cdots \to (p) \to (q) \to (i).$$

When equating "length" to "time", as in Appendix F,

$$\begin{aligned}
\sum_{(u,v)\in\mu_{1\to i}} FF_{uv} &= FF_{1e} + FF_{ef} + FF_{fr} + \cdots + FF_{pq} + FF_{qi} \\
&= (ET_e - ET_1 - d_{1e}) + (ET_f - ET_e - d_{ef}) + (ET_r - ET_f - d_{fr}) + \cdots \\
&\quad + (ET_q - ET_p - d_{pq}) + (ET_i - ET_q - d_{qi}) \\
&= ET_i - ET_1 - (d_{1e} + d_{ef} + d_{fr} + \cdots + d_{pq} + d_{qi}) \\
&= L(\mu_{1\to i}) - (d_{1e} + d_{ef} + d_{fr} + \cdots + d_{pq} + d_{qi}).
\end{aligned}$$

Corresponding to the original length network, $d_{ij} = l_{ij}$ and

$$d_{1e} + d_{ef} + d_{fr} + \cdots + d_{pq} + d_{qi} = l_{1e} + l_{ef} + l_{fr} + \cdots + l_{pq} + l_{qi} = L(\mu_{1\to i})$$

therefore,

$$L(\mu^{\nabla}_{1\to i}) - L(\mu_{1\to i}) = \sum_{(i,j)\in\mu_{1\to i}} FF_{uv}$$

and obviously $\mu^{\nabla}_{1\to i} = \mu_{1\to i}$ for $\sum\limits_{(i,j)\in\mu_{1\to i}} FF_{uv} = 0$. According to Appendix A, $ET_v = \max\{ET_u + d_{uv}\}$; then $ET_v \geq ET_u + d_{uv}$ and $FF_{uv} \geq 0$ based on equation (2.2). Hence, $\sum\limits_{(i,j)\in\mu_{1\to i}} FF_{uv} = 0$ means $FF_{uv} = 0$ for $\forall (i,j) \in \mu_{1\to i}$, and equation (3.9) and Corollary 3.9 are correct. This completes the proof.

Appendix I. Proof of algorithms in Section 4.3

(1) The proof of Method 1.
According to equations (3.6) and (C.1), $L(\mu^{\nabla}_i) = ET_n - TF_i$. If $TF_i = ET_n - \lambda$, then

$$\begin{aligned} L(\mu^{\nabla}_i) &= ET_n - TF_i \\ &= ET_n - (ET_n - \lambda) \\ &= \lambda \end{aligned}$$

In the length network, $\mu^{\nabla}_i = \mu^{\nabla}_{1\to i} + \mu^{\nabla}_{i\to n}$. According to Theorem 3.8, if seeing the network as a time network, $FF_{ef} = 0$ for $\forall (e,f) \in \mu^{\nabla}_{1\to i}$ and $SF_{uv} = 0$ for $\forall (u,v) \in \mu^{\nabla}_{i\to n}$. Therefore, $\mu^{\nabla}_{1\to i}$ could be found by finding immediate preceding activities with free floats 0 one-by-one from the node (i) to start node (1), and $\mu^{\nabla}_{i\to n}$ could be found by finding immediate succeeding activities with safety floats 0 one-by-one from the node (i) to end node (n). Then, we obtain the path $\mu^{\nabla}_{1\to i}$.

(2) The proof of Method 2 is similar to that in (1).

(3) The proof of Method 3.
The longest path passing two connected arcs (h,i) and (i,j) is $\mu^{\nabla}_{hi\to ij} = \mu^{\nabla}_{1\to h} + (h,i) + (i,j) + \mu^{\nabla}_{j\to n}$. If seeing the network as a time network, according to equations (3.7) and (3.8),

$$\begin{aligned} L(\mu^{\nabla}_{hi\to ij}) &= ET_n - \sum_{(u,v)\in\mu^{\nabla}_{hi\to ij}} FF_{uv} \\ &= ET_n - \left(\sum_{(e,f)\in\mu^{\nabla}_{1\to h}} FF_{ef} + FF_{hi} + FF_{ij} + \sum_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv} \right) \end{aligned}$$

According to Theorem 3.8, $\sum\limits_{(e,f)\in\mu^{\nabla}_{1\to h}} FF_{ef} = 0$. The activity (i,j) is an immediate preceding one of the critical node (j); hence, $\mu^{\nabla}_{j\to n} \subset \mu^{\nabla}$ and $\sum\limits_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv} = 0$. Therefore,

$$\begin{aligned} L(\mu^{\nabla}_{hi\to ij}) &= ET_n - \left(\sum_{(e,f)\in\mu^{\nabla}_{1\to h}} FF_{ef} + FF_{hi} + FF_{ij} + \sum_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv} \right) \\ &= ET_n - (FF_{hi} + FF_{ij}) \end{aligned}$$

If $FF_{hi} = ET_n - \lambda - FF_{ij}$, that is, $FF_{hi} + FF_{ij} = ET_n - \lambda$, then

$$\begin{aligned} L(\mu^{\nabla}_{hi\to ij}) &= ET_n - (FF_{hi} + FF_{ij}) \\ &= ET_n - (ET_n - \lambda) \\ &= \lambda \end{aligned}$$

(4) The proof of Method 4 is similar to that in (3).

(5) The proof of Method 5.
According to the known conditions, $FF_{bc} = \cdots = FF_{fg} = FF_{gh} = FF_{hi} = 0$, that is, $\sum_{(p,q)\in\mu_{b\to i}} FF_{pq} = 0$, $\mu_{b\to i} = (b) \to (c) \to \cdots \to (g) \to (h) \to (i)$. The longest path passing arc (a,b), path section $\mu_{b\to i}$, and arc (i,j) are $\mu^{\nabla}_{ab\to ij} = \mu^{\nabla}_{1\to a} + (a,b) + \mu_{b\to i} + (i,j) + \mu^{\nabla}_{j\to n}$. If seeing "length" as "time", according to equations (3.7) and (3.8),

$$\begin{aligned} L(\mu^{\nabla}_{ab\to ij}) &= ET_n - \sum_{(u,v)\in\mu^{\nabla}_{ab\to ij}} FF_{uv} \\ &= ET_n - \left(\sum_{(e,f)\in\mu^{\nabla}_{1\to a}} FF_{ef} + FF_{ab} + \sum_{(p,q)\in\mu^{\nabla}_{b\to i}} FF_{pq} + FF_{ij} + \sum_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv}\right) \end{aligned}$$

According to Theorem 3.8, $\sum_{(e,f)\in\mu^{\nabla}_{1\to a}} FF_{ef} = 0$. The activity (i,j) is an immediate preceding one of the critical node (j); hence, $\mu^{\nabla}_{j\to n} \subset \mu^{\nabla}$ and $\sum_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv} = 0$. Therefore,

$$\begin{aligned} L(\mu^{\nabla}_{ab\to ij}) &= ET_n - \left(\sum_{(e,f)\in\mu^{\nabla}_{1\to a}} FF_{ef} + FF_{ab} + \sum_{(p,q)\in\mu^{\nabla}_{b\to i}} FF_{pq} + FF_{ij} + \sum_{(u,v)\in\mu^{\nabla}_{j\to n}} FF_{uv}\right) \\ &= ET_n - (FF_{ab} + FF_{ij}) \end{aligned}$$

If $FF_{ab} = ET_n - \lambda - FF_{ij}$, that is, $FF_{ab} + FF_{ij} = ET_n - \lambda$, then

$$\begin{aligned} L(\mu^{\nabla}_{ab\to ij}) &= ET_n - (FF_{ab} + FF_{ij}) \\ &= ET_n - (ET_n - \lambda) \\ &= \lambda \end{aligned}$$

(6) The proof of Method 6 is similar to that in (5).

This completes the proof.

Appendix J. Proof of conclusions in Section 4.4

According to equation (2.1),

$$\begin{aligned} TF_{pi} &= LF_i - ET_p - d_{ij} \\ &= (LF_i - ET_i) + (ET_i - ET_p - d_{ij}) \\ &= TF_i - FF_{pi} \end{aligned}$$

In Figure 9, the activity A is a critical one and $(j,i) \in \mu^{\nabla}$; hence, $TF_i = 0$ and $TF_{pi} = FF_{pi}$. If $FF'_{pi} = 0$, a new critical path will appear. Similarly, if $SF'_{jq} = 0$, a new critical path will appear. Therefore, we first study the laws of FF_{pi} and SF_{jq} when prolonging d_A. Assume $FF_{pi} < SF_{jq}$.

(1) Study the law of FF_{pi} when prolonging d_A.
According to equation (3.4), $FF_{pi} = L(\mu^{\nabla}_i) - L(\mu^{\nabla}_{pi})$; FF_{pi} could be studied in terms of μ^{∇}_i and μ^{∇}_{pi}. For μ^{∇}_i,

$$\mu^{\nabla}_i = (1) \to \cdots \to (k) \to (j) \to (i) \to (r) \to \cdots \to (n) = \mu^{\nabla}$$

$(j,i) \in \mu_i^\nabla = \mu^\nabla$ and $d_{ji} = -d_A$ cause $d'_{ji} = d_{ji} - \Delta d$ and $L(\mu^{\nabla\prime}) = L(\mu_i^{\nabla\prime}) = L(\mu_i^\nabla) - \Delta d = L(\mu^\nabla) - \Delta d$ when $d'_A = d_A + \Delta d$. For μ_{pi}^∇,

$$\mu_{pi}^\nabla = (1) \to \cdots \to (p) \to (i) \to (r) \to \cdots \to (n)$$

$(i,j) \notin \mu_{pi}^\nabla$ and $(j,i) \in \mu_{pi}^\nabla$ cause $L(\mu_{pi}^{\nabla\prime}) = L(\mu_{pi}^\nabla)$ when $d'_A = d_A + \Delta d$. Hence, if $d'_A = d_A + \Delta d$, then

$$\begin{aligned} FF'_{pi} &= L(\mu_i^{\nabla\prime}) - L(\mu_{pi}^{\nabla\prime}) \\ &= \big(L(\mu_i^\nabla) - \Delta d\big) - L(\mu_{pi}^\nabla) \\ &= \big(L(\mu_i^\nabla) - L(\mu_{pi}^\nabla)\big) - \Delta d \\ &= FF_{pi} - \Delta d \end{aligned}$$

1. If $0 < \Delta d < FF_{pi}$, then $FF'_{pi} = FF_{pi} - \Delta d > 0$ and $L(\mu^{\nabla\prime}) = L(\mu^\nabla) - \Delta d$, that is,

$$L(\mu^{\nabla\prime}) = L(\mu^\nabla) - \Delta d$$

   for $d_A < d_A + \Delta d \le d_A + FF_{pi}$, as in Figure 10.
2. If $\Delta d = FF_{pi}$, then $FF'_{pi} = 0$ and μ_{pi}^∇ will become a new critical path, that is,

$$L(\mu_{pi}^\nabla) = L(\mu^{\nabla\prime}) = L(\mu^\nabla) - \Delta d = L(\mu^\nabla) - FF_{pi}$$

3. If $\Delta d > FF_{pi}$, because $A \notin \mu^\nabla = \mu_{pi}^\nabla$, the path length $L(\mu_{pi}^\nabla) = L(\mu^{\nabla\prime}) = L(\mu^\nabla) - FF_{pi}$ will be unchanged when prolonging d_A, that is, the critical path length $L(\mu^{\nabla\prime}) = L(\mu^\nabla) - FF_{pi}$ for $d'_A > d_A + FF_{pi}$. However, the prolongation Δd of d_A is restricted by SF_{jq}, and therefore, it cannot be indefinite, as in Figure 10.

(2) Study the law of SF_{jq} when prolonging d_A.
According to equation (3.5), $SF_{jq} = L(\mu_j^\nabla) - L(\mu_{jq}^\nabla)$; then, SF_{jq} could be studied in terms of two paths μ_j^∇ and μ_{jq}^∇. For μ_{jq}^∇,

$$\mu_{jq}^\nabla = (1) \to \cdots \to (k) \to (j) \to (q) \to \cdots \to (n)$$

$(i,j) \notin \mu_{jq}^\nabla$ and $(j,i) \notin \mu_{jq}^\nabla$ cause $L(\mu_{jq}^{\nabla\prime}) = L(\mu_{jq}^\nabla)$ and will be unchanged when $d'_A = d_A + \Delta d$. For μ_j^∇,

$$\mu_j^\nabla = (1) \to \cdots \to (k) \to (j) \to (i) \to (r) \to \cdots \to (n)$$

$d_{ji} = -d_A$ and $(j,i) \notin \mu_j^\nabla$ cause $d'_{ji} = d_{ji} - \Delta d$ and $L(\mu_j^{\nabla\prime}) = L(\mu_j^\nabla) - \Delta d$ when $d'_A = d_A + \Delta d$. According to equation (3.5),

$$\begin{aligned} SF'_{jq} &= L(\mu_j^{\nabla\prime}) - L(\mu_{jq}^{\nabla\prime}) \\ &= \big(L(\mu_j^\nabla) - \Delta d\big) - L(\mu_{jq}^\nabla) \\ &= \big(L(\mu_j^\nabla) - L(\mu_{jq}^\nabla)\big) - \Delta d \\ &= SF_{jq} - \Delta d \end{aligned}$$

1. If $0 < \Delta d < SF_{jq}$, then $SF'_{jq} = SF_{jq} - \Delta d > 0$. Based on Corollary 3.11, the path μ_{jq}^∇ passing the arc (j,q) will not be a new critical path and will not affect the project duration.
2. If $\Delta d = SF_{jq}$, then $SF'_{jq} = SF_{jq} - \Delta d = 0$, which means that the path μ_{jq}^∇ passing the arc (j,q) (also path μ_{ij}^∇) will be a new critical path, that is,

$$\mu^\nabla = \mu_{jq}^\nabla = \mu_{ij}^\nabla = (1) \to \cdots \to (p) \to (i) \to (j) \to (q) \to \cdots \to (n)$$

   Therefore, if $FF_{pi} \le \Delta d \le SF_{jq}$, then $L(\mu^{\nabla\prime}) = L(\mu^\nabla) - FF_{pi}$, that is,

$$L(\mu^{\nabla\prime}) = L(\mu^\nabla) - FF_{pi}$$

   for $d_A + FF_{pi} \le d_A + \Delta d \le d_A + SF_{jq}$, as in Figure 10.

3. If $\Delta d > SF_{jq}$, then $(j,i) \in \mu^{\nabla} = \mu_{ij}^{\nabla}$ causes

$$\begin{aligned} L(\mu^{\nabla\prime}) &= L(\mu^{\nabla}) - FF_{pi} + \Delta d - SF_{jq} \\ &= L(\mu^{\nabla}) - (FF_{pi} + SF_{jq}) + \Delta d \end{aligned}$$

Therefore, if $SF_{jq} \leq \Delta d \leq FF_{pi} + SF_{jq}$, then $L(\mu^{\nabla\prime}) \leq L(\mu^{\nabla})$ and the project duration will be unchanged or shortened; and if $\Delta d > FF_{pi} + SF_{jq}$, then $L(\mu^{\nabla\prime}) > L(\mu^{\nabla})$ and the project duration will be prolonged.

This completes the proof.

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References

[1] A. Aini and A. Salehipour, Speeding up the Floyd–Warshall algorithm for the cycled shortest path problem. *Appl. Math. Lett.* **25** (2012) 1–5.

[2] R. Alur, S. Kannan, K. Tian and Y. Yuan, On the complexity of shortest path problems on discounted cost graphs. In: *Language and Automata Theory and Applications.* Springer (2013) 44–55.

[3] M. Battarra, A.A. Pessoa, A. Subramanian and E. Uchoa, Exact algorithms for the traveling salesman problem with draft limits. *Eur. J. Oper. Res.* **235** (2014) 115–128.

[4] A. Battersby, *Network Analysis for Planning and Scheduling*, 3 edition. Wiley, New York (1970).

[5] R. Bellman, On a routing problem. *Quarterly Appl. Math.* **16** (1958) 87–90.

[6] T.B. Boffey, R.C. Williams, B. Pelegrin and P. Fernandez, The maximum capacity shortest path problem: generation of efficient solution sets. *RAIRO: OR* **36** (2002) 1–19.

[7] H. Broersma, F.V. Fomin, P.V. Hof and D. Paulusma, Exact algorithms for finding longest cycles in claw-free graphs. *Algorithmica* **65** (2013) 129–145.

[8] Y.L. Chen, D. Rinks and K. Tang, Critical path in an activity network with time constraints. *Eur. J. Oper. Res.* **100** (1997) 122–133.

[9] E.W. Dijkstra, A note on two problems in connexion with graphs. *Numer. Math.* **1** (1959) 269–271.

[10] S.E. Elmaghraby, An algebra for the analysis of generalized activity networks. *Manage. Sci.* **10** (1964) 494–514.

[11] S.E. Elmaghraby, *Activity Networks: Project Planning and Control by Network Models.* Wiley, New York (1977).

[12] S.E. Elmaghraby and J. Kamburowski, On project representation and activity floats. *Arab. J. Sci. Eng.* **15** (1990) 627–637.

[13] S.E. Elmaghraby and J. Kamburowski, The analysis of activity networks under generalized precedence relations (GPRs). *Manage. Sci.* **38** (1992) 1245–1263.

[14] S.E. Elmaghraby, Activity nets: A guided tour through some recent developments. *Eur. J. Oper. Res.* **82** (1995) 383–408.

[15] J.B. Escario, J.F. Jimenez and J.M. Giron-Sierra, Ant colony extended: Experiments on the travelling salesman problem. *Expert Syst. Appl.* **42** (2015) 390–410.

[16] T. Feder, R. Motwani and C. Subi, Approximating the longest cycle problem in sparse graphs. *SIAM J. Comput.* **31** (2002) 1596–1607.

[17] K. Ghoseiri and A. Moghadam, Continuous fuzzy longest path problem in project networks. *J. Appl. Sci.* **8** (2008) 4061–4069.

[18] D.J. Harmelink, Linear scheduling model: Float characteristics. *J. Constr. Eng. Manage.* **127** (2001) 255–260.

[19] M. Haouari, N. Maculan and M. Mrad, Enhanced compact models for the connected subgraph problem and for the shortest path problem in digraphs with negative cycles. *Comput. Oper. Res.* **40** (2013) 2485–2492.

[20] H. Huang and S. Gao, Optimal paths in dynamic networks with dependent random link travel times. *Transp. Res. B* **46** (2012) 579–598.

[21] J.E. Kelley Jr and M.R. Walker, Critical-path planning and scheduling. In: Papers presented at the December 1–3, 1959, eastern joint *IRE-AIEE-ACM computer conference, ACM* (1959) 160–173.

[22] I. Lari, F. Ricca and A. Scozzari, Comparing different metaheuristic approaches for the median path problem with bounded length. *Eur. J. Oper. Res.* **190** (2008) 587–597.

[23] X. Liu, H. Lu, W. Wang and W. Wu, PTAS for the minimum k-path connected vertex cover problem in unit disk graphs. *J. Global Optim.* **56** (2013) 449–458.

[24] P.M.V. Lima, G.C. Pereira, M.M. Morveli-Espinoza and F.M.G. Franca, Mapping and Combining Combinatorial Problems into Energy Landscapes via Pseudo-Boolean Constraints. Vol. 3704 of *Lect. Notes Comput. Sci.* Springer, Berlin (2005) 308–317.

[25] G. Lucko and A.A. Peña Orozco, Float types in linear schedule analysis with singularity functions. *J. Constr. Eng. Manage.* **135** (2009) 368–377.

[26] N. Maculan, G. Plateau and A. Lisser, Integer linear models with a polynomial number of variables and constraints for some classical combinatorial optimization problems. *Pesquisa Operacional* **23** (2003) 161–168.

[27] I. Muter, A new formulation and approach for the black and white traveling salesman problem. *Comput. Oper. Res.* **53** (2015) 96–106.

[28] C. Murat and V.T. Paschos, The probabilistic longest path problem. *Networks* **33** (1999) 207–219.

[29] M.M.B. Pascoal and A. Sedeno-Noda, Enumerating K best paths in length order in DAGs. *Eur. J. Oper. Res.* **221** (2012) 308–316.

[30] F.J. Pulido, L. Mandow and J.L. Prez de la Cruz, Multiobjective shortest path problems with lexicographic goal-based preferences. *Eur. J. Oper. Res.* **239** (2014) 89–101.

[31] J.X. Qi and Z.X. Su, Analysis of an anomaly: The increase in time float following consumption. *Sci. World J.* **2014** (2014) 415870.

[32] M. Ranjbar, A path-relinking metaheuristic for the resource levelling problem. *J. Oper. Res. Soc.* **64** (2012) 1071–1078.

[33] B. Roy, Graphes et ordonnancements. *Revue Francaise de Recherche Operationelle* **25** (1962) 323–326.

[34] M. Su, L. Liu, J. Wang and H. Cai, Solving the optimal power flow problem via a non-interior path following algorithm. *J. Comput Inf. Syst.* **9** (2013) 1191–1198.

[35] M. Szachniuk, M.C. De Cola, G.. Felici and J. Blazewicz, The orderly colored longest path problema survey of applications and new algorithms. *RAIRO: OR* **48** (2014) 25–51.

[36] W. Thomas, Four float measures for critical path scheduling. *J. Ind. Eng.* **10** (1969) 19–23.

[37] A. Toriello, W.B. Haskell and M. Poremba, A dynamic traveling salesman problem with stochastic arc costs. *Oper. Res.* **62** (2014) 1107–1125.

[38] F.W. Trevizan and M.M. Veloso, Depth-based short-sighted stochastic shortest path problems. *Artif. Intell.* **216** (2014) 179–205.

[39] V. Valls and P. Lino, Criticality analysis in activity-on-node networks with Minimal time lags. *Ann. Oper. Res.* **102** (2001) 17–37.

[40] W. Van Loock, S. Bellens, G. Pipeleers, J. De Schutter and J. Swevers, Time-optimal parking and flying: Solving path following problems efficiently. In: *2013 IEEE International Conference on. Mechatronics (ICM)*. IEEE (2013) 841–846.

[41] J.D. Wiest, Precedence diagramming method: Some unusual characteristics and their implications for project managers. *J. Oper. Manage.* **1** (1981) 121–130.

[42] S.H. Yakhchali and S.H. Ghodsypour, Computing latest starting times of activities in interval-valued networks with minimal time lags. *Eur. J. Oper. Res.* **200** (2010) 874–880.

[43] S. Yussof and H.S. Ong, A robust GA-based QoS routing algorithm for solving multi-constrained path problem. *J. Comput.* **5** (2010) 1322–1334.

[44] R.Q. Zeng, M. Basseur and J.K. Hao, Solving bi-objective flow shop problem with hybrid path relinking algorithm. *Appl. Soft Comput.* **13** (2013) 4118–4132.

[45] X.C. Zhang, H. Jiang, and G.L. Chen, Minimum cuts and maximum flows in directed planer networks with both node and edge capacities. *Chinese J. Comput.* **29** (2006) 544–551.

[46] X.G. Zhang, Q. Wang, A. Adamatzky, F.T.S. Chan, S. Mahadevan and Y. Deng, A biologically inspired optimization algorithm for solving fuzzy shortest path problems with mixed fuzzy arc lengths. *J. Optim. Theory Appl.* **163** (2014) 1049–1056.

[47] L.H. Zhang, X. Zou and Z.N. Kan, Improved strategy for resource allocation in repetitive projects considering the learning effect. *J. Constr. Eng. Manage.* **140** (2014) 04014053.

[48] X.Y. Zhu and W.E. Wilhelm, Implementation of a three-stage approach for the dynamic resource-constrained shortest-path sub-problem in branch-and-price. *Comput. Oper. Res.* **40** (2013) 385–394.