RAIRO-Oper. Res. 50 (2016) 131–143
DOI: 10.1051/ro/2015020

RAIRO Operations Research
www.rairo-ro.org

COMPLEXITY ANALYSIS OF A WEIGHTED-FULL-NEWTON STEP INTERIOR-POINT ALGORITHM FOR $P_*(\kappa)$-LCP

Mohamed Achache[1]

Abstract. In this paper, a weighted-path-following interior point algorithm for $P_*(\kappa)$-linear complementarity problems ($P_*(\kappa)$-LCP) is presented. The algorithm uses at each weighted interior point iteration only feasible full-Newton steps and the strategy of the central-path for getting a solution for $P_*(\kappa)$-LCP. We prove that the proposed algorithm has quadratically convergent with polynomial time. The complexity bound, namely, $O((1+\kappa)\sqrt{n}\log\frac{n}{\epsilon})$ of the algorithm is obtained. Few numerical tests are reported to show the efficiency of the algorithm.

Mathematics Subject Classification. 90C33, 90C51, 11Y16.

Received December 11, 2014. Accepted June 10, 2015.

1. Introduction

In this paper, we consider the following linear complementarity problem (LCP): find a pair of vectors $(x, y) \in \mathbf{R}^n \times \mathbf{R}^n$ such that

$$y = Mx + q,\ x^T y = 0,\ (x,\ y) \geq 0, \tag{1.1}$$

where $M \in \mathbf{R}^{n\times n}$ is a $P_*(\kappa)$-matrix, and $q \in \mathbf{R}^n$.

The LCP contains several standard problems (*e.g.*, linear and quadratic optimization) and finds many applications in engineering and economic [4].

Feasible path-following algorithms are the most attractive interior point methods (IPMs) for solving a large wide of optimization problems. These algorithms achieved beautiful results such as polynomial complexity and numerical efficiency [11, 16]. They start with a strictly feasible centered starting point and maintain feasibility during the solution process. However, in practice these algorithms do not have always a strictly feasible centered starting point. So it is worth while to attention to other cases when the starting points are not centered. Thus leads to the concept of target-following IPMs introduced early by Jansen *et al.* [8]. These methods are based on the observation that with every algorithm which follows the central-path, we associate a target sequence on the central-path. Weighted path-following methods can be viewed as a particular case of target-following IPMs. These methods were studied by Ding *et al.* [6] for monotone LCP and by Roos *et al.* [11] for LO. By using a new technique for getting search directions Darvay [5] introduced a new weighted-path-following algorithm

Keywords. Linear complementarity problems, $P_*(\kappa)$-matrix, weighted-path-following, interior-point methods, polynomial complexity.

[1] Laboratoire de Mathématiques Fondamentales et Numériques. Université Ferhat Abbas, Sétif 1, 19000 Sétif, Algérie.
achache_m@yahoo.fr

for LO. Achache [1] and Wang [13] extend it for monotone standard and horizontal LCP. As a result, the best iteration bound is obtained by all of them for short-step algorithms. Later on, Wang *et al.* [13], presented new complexity analysis with a full-newton step feasible IPMs for $P_*(\kappa)$-LCP. The best known iteration bound for their algorithm is established. Also Wang *et al.* [12, 15], proposed polynomial interior point algorithms for $P_*(\kappa)$-LCP and $P_*(\kappa)$-horizontal LCP. They also established their polynomial complexity. Mansouri *et al.* [9] proposed a new short-step primal-dual path-following method for horizontal $P_*(\kappa)$-LCP. The complexity and some numerical results are stated. However, Pólik [10], treated also this problem in his Msc thesis where a class of primal-dual IPMs based on some self-regular functions is proposed. The complexity of this algorithm is also established. Cho [3], based on a kernel function, he proposed a large-update interior point algorithm for P_*-linear complementarity problem. The complexity of its algorithm is obtained.

Recently, Achache and Khebchache [2], proposed a new weighted short-step path-following method for monotone LCP, *i.e.*, $P_*(0)$-LCP, where the matrix M is assumed to be positive semidefinite. They proved that the corresponding short-step algorithm has the best well-known iteration bound, namely $O(\sqrt{n}\log\frac{n}{\epsilon})$.

The purpose of the paper is to generalize their results to $P_*(\kappa)$-LCP where the matrix M belongs to the class of $P_*(\kappa)$-matrix. At each iteration, the algorithm uses only full-Newton steps which have the advantage that no line searches are needed. We establish the currently best known iteration bound for $P_*(\kappa)$-LCP, namely, $O((1+\kappa)\sqrt{n}\log\frac{n}{\epsilon})$, which coincides with the bound derived for monotone LCP except that the iteration bound in $P_*(\kappa)$-LCP case are multiplied with the factor $(1+\kappa)$. Finally, few numerical results are reported to show the efficiency of the proposed algorithm.

The rest of the paper is built as follows. In Section 2, the basic ideas such as the weighted path, the Newton search directions and the proximity of a weighted-full-Newton-step interior-point algorithm for $P_*(\kappa)$-LCP are described. In Section 3, detailed proofs of complexity results are given. In Section 4, we give some numerical tests for our algorithm. Finally, some conclusions and remarks follow in Section 5.

The notations used in this paper are as follows. $\mathbf{R}^n$, $\mathbf{R}^n_+$ and $\mathbf{R}^n_{++}$ denote the set of vectors with n components vectors, the set of nonnegative vectors, and the set of positive vectors, respectively. Given $x, y \in \mathbf{R}^n_{++}$, their Hadamard product is $xy = (x_1y_1, \ldots, x_ny_n)^T$. The expressions $\|u\| = \sqrt{u^Tu}$ and $\|u\|_\infty = \max_i |u_i|$ denote the Euclidean and the maximum norm for a vector u, respectively. Let $x, y \in \mathbf{R}^n_{++}$, $\sqrt{x} = (\sqrt{x_1}, \ldots, \sqrt{x_n})^T$, $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})^T$ and $\frac{x}{y} = (\frac{x_1}{y_1}, \ldots, \frac{x_n}{y_n})^T$. If $g(x) \geq 0$ is a real valued function of a real nonnegative variable, the notation $g(x) = O(x)$ means that $g(x) \leq cx$ for some positive constant c. For any $w \in \mathbf{R}^n$, $\min(w)$ (or $\max(w)$) denotes the smallest (or largest) component of w. Finally, $D := \text{Diag}\ (d)$ is the diagonal matrix of a vector d with $D_{ii} = d_i$ and the vector of all ones is denoted by e.

2. Weighted-full-Newton step interior-point algorithm for $P_*(\kappa)$-LCP

Throughout the paper, we make the following assumption on $P_*(\kappa)$-LCP.

Assumption 2.1. Without loss of generality we may assume that $P_*(\kappa)$-LCP satisfies the interior point condition, *i.e.*, there exists a pair of vectors (x^0, y^0) such that

$$y^0 = Mx^0 + q,\ x^0 > 0,\ y^0 > 0,$$

which implies that the solution set of $P_*(\kappa)$-LCP is not empty.

We recall that a matrix M is a $P_*(\kappa)$-matrix, if there exists a constant $\kappa \geq 0$ such that

$$(1+4\kappa)\sum_{i\in\mathcal{I}^+} x_i(Mx)_i + \sum_{i\in\mathcal{I}^-} x_i(Mx)_i \geq 0 \text{ for all } x\ \in \mathbf{R}^n$$

where

$$\mathcal{I}^+ = \{i : x_i(Mx)_i > 0\} \quad \text{ and } \mathcal{I}^- = \{i : x_i(Mx)_i < 0\}$$

are two index sets.

It is worth pointing out that for $\kappa = 0$ the $P_*(\kappa)$-matrix is reduced to the positive semidefinite matrix and so $P_*(0)$-LCP becomes the monotone LCP. The class of all $P_*(\kappa)$-matrices is denoted by $P_*(\kappa)$, and the class P_* is defined by

$$P_* = \bigcup_{\kappa \geq 0} P_*(\kappa),$$

i.e., M is a P_*-matrix if M belongs to $P_*(\kappa)$ for some $\kappa \geq 0$.

2.1. Weighted-path for $P_*(\kappa)$-LCP

Finding a solution of (1.1) is equivalent to solving the following system

$$\begin{array}{ll} Mx + q = y, \; x \geq 0, \\ xy \qquad\;\; = 0, \; y \geq 0. \end{array} \tag{2.1}$$

The basic idea of the weighted-path-following IPMs is to replace the second equation in (2.1), the so-called *complementarity condition* for $P_*(\kappa)$-LCP, by the parameterized equation $xy = w$ with $w > 0$ is a positive vector. Thus, one may consider

$$\begin{array}{ll} Mx + q = y, \; x \geq 0, \\ xy \qquad\;\; = w, \; y \geq 0. \end{array} \tag{2.2}$$

For each $w > 0$, the system (2.2) has a unique solution $x(w), y(w))$ (under given assumption) [10], which called the weighted-path of $P_*(\kappa)$-LCP. If w goes to zero then the limit of the weighted-path exists and since the limit point satisfies the complementarity condition it yields a solution of $P_*(\kappa)$-LCP. Note that if $w = \mu e$ with $\mu > 0$, then the weighted-path coincides with the classical central-path (*e.g.*, see [11]).

2.2. Newton search direction and proximity

The natural way to define a search direction is to follow the Newton approach and to linearize the second equation in (2.1) for the search directions Δx and Δy. This leads to the following system

$$\begin{array}{ll} M\Delta x \qquad\quad = \Delta y \\ y\Delta x + x\Delta y = w - xy. \end{array} \tag{2.3}$$

Note that under our assumptions, *i.e.*, since M is a $P_*(\kappa)$-matrix and (1) is strictly feasible, the system (2.3) has a unique solution $(\Delta x, \Delta y)$. Hence, the new iterate is obtained by taking a full-Newton step according to

$$x_+ = x + \Delta x \quad \text{and} \quad y_+ = y + \Delta y.$$

For the analysis of the algorithm, we define a norm-based proximity measure as follows

$$\delta := \delta(x, y; w) = \frac{1}{2\sqrt{\min(w)}} \left\| \frac{w - xy}{\sqrt{xy}} \right\|$$

which vanishes if $(x, y) = (x(w), y(w))$ and positive otherwise. Hence, the value of δ can be considered as a measure for the distance between a given pair (x, y) and $(x(w), y(w))$.
Now to simplify the matters, we define the vectors

$$v = \sqrt{xy} \quad \text{and} \quad d = \sqrt{xy^{-1}}$$

where all the operations are understood to be componentwise. Note that

$$v^2 = w \Leftrightarrow xy = w.$$

The vector d is used to scale x and y to the same vector v as

$$d^{-1}x = \mathrm{d}y = v \tag{2.4}$$

as well as for the original directions to the scaling directions

$$d_x = d^{-1}\varDelta x \quad \text{and} \quad d_y = d\varDelta y.$$

It follows that

$$x\varDelta y + y\varDelta x = v(d_x + d_y)$$

and

$$d_x d_y = \varDelta x \varDelta y = \varDelta y M \varDelta x.$$

By using the above relations, the system (2.3) becomes

$$\begin{array}{ll} \bar{M} d_x & = d_y \\ d_x + d_y & = p_v, \end{array} \tag{2.5}$$

where

$$p_v = v^{-1}(w - v^2)$$

and $\bar{M} = DMD$ with D :=Diag (d). Also our proximity becomes

$$\delta(v; w) := \frac{\|p_v\|}{2\sqrt{\min(w)}} = \frac{\left\|v^{-1}(w - v^2)\right\|}{2\sqrt{\min(w)}}. \tag{2.6}$$

Let denote another measure as follows

$$\sigma_C(w) = \frac{\max(w)}{\min(w)}. \tag{2.7}$$

Note that $\sigma_C(w) \geq 1$ and $\sigma_C(w) = 1$ if w is on the central-path. This measure is an indicator of the closeness of w to the central-path μe in the sense that if the $\sigma_C(w)$ is close to one then w is near the central-path.

2.3. Generic weighted-full-Newton-step interior-point algorithm for $P_*(k)$-LCP

The generic weighted-full-Newton step interior-point algorithm for $P_*(k)$-LCP is presented in Figure 1.

3. Complexity analysis of the interior-point algorithm for $P_*(k)$-LCP

In this Section, we first give sufficient conditions for the strict feasibility of the full-Newton step. Then, we prove the local quadratically convergence of the iterates. Finally, the complexity bound of the algorithm is obtained.

3.1. Feasibility of the full-Newton step

It is well-known that the scaled search directions d_x and d_y are orthogonal in LO case since $d_x^T d_y = 0$ (see [11, 12]). Meanwhile, in $P_*(k)$-LCP case, this propriety failed, *i.e.*, $d_x^T d_y \neq 0$. This yields difficulties in the analysis of the algorithm. To overcome these difficulties, we develop some new results on the scaled search directions.

Input:
An accuracy parameter ϵ;
an update parameter $\theta, 0 < \theta < 1$;
a proximity parameter $\tau, (\ 0 < \tau < 1)$;
a strictly feasible pair (x^0, y^0) and $w^0 > 0$ such that $\delta(x^0, y^0; w^0) \leq \tau$.
begin
$x := x^0;\ y := y^0;\ w := w^0;$
while $x^T y > \epsilon$ **do**
begin
compute $(\Delta x, \Delta y)$ from (2.3);
update $x := x + \Delta x;\ y := y + \Delta y;$
$w := (1 - \theta) w;$
end
end.

FIGURE 1.

Lemma 3.1. *Let* (d_x, d_y) *be a solution of* (2.5), *and suppose* $w > 0$. *If* $\delta := \delta(v; w)$, *then one has*

$$-4\kappa \min(w)\delta^2 \leq d_x^T d_y \leq \delta^2 \min(w) \tag{3.1}$$

and

$$\|d_x d_y\| \leq \left(\sqrt{2} + 4\kappa\right) \min(w)\delta^2. \tag{3.2}$$

Proof. Let (d_x, d_y) be a solution of (2.5), and consider the index sets

$$\mathcal{I}^+ = \{i : (d_x)_i (d_y)_i > 0\}, \ \ \mathcal{I}^- = \{i : (d_x)_i (d_y)_i < 0\}.$$

By the following inequality

$$0 < 4(d_x)_i (d_y)_i \leq ((d_x)_i + (d_y)_i)^2 = (p_v)_i^2 \ \forall i \in \mathcal{I}^+,$$

it follows on one hand that

$$\sum_{i \in \mathcal{I}^+} |(d_x)_i (d_y)_i| \leq \frac{1}{4} \|p_v\|^2 = \min(w)\, \delta^2$$

and on the other hand

$$\begin{aligned} d_x^T d_y &= \sum_{i \in \mathcal{I}^+} (d_x)_i (d_y)_i + \sum_{i \in \mathcal{I}^-} (d_x)_i (d_y)_i \\ &\leq \sum_{i \in \mathcal{I}^+} (d_x)_i (d_y)_i \leq \frac{1}{4} \|p_v\|^2 = \min(w)\, \delta^2. \end{aligned}$$

Now since M is $P_*(k)$-matrix, we deduce that

$$\begin{aligned} d_x^T d_y &= \sum_{i \in \mathcal{I}^+} |(d_x)_i (d_y)_i| + \sum_{i \in \mathcal{I}^-} (d_x)_i (d_y)_i \\ &= (1 + 4\kappa) \sum_{i \in \mathcal{I}^+} |(d_x)_i (d_y)_i| + \sum_{i \in \mathcal{I}^-} (d_x)_i (d_y)_i - 4\kappa \sum_{i \in \mathcal{I}^+} |(d_x)_i (d_y)_i| \\ &\geq -4\kappa \sum_{i \in \mathcal{I}^+} |(d_x)_i (d_y)_i| \geq -\kappa \|p_v\|^2 = -4\kappa \min(w)\delta^2. \end{aligned}$$

This gives the proof of the first part of the lemma. For the second part, we have

$$\begin{aligned}\|d_x d_y\|^2 &= \sum_{i\in\mathcal{I}^+}(d_x)_i^2(d_y)_i^2 + \sum_{i\in\mathcal{I}^-}(d_x)_i^2(d_y)_i^2 \\ &\le \frac{1}{16}\sum_{i\in\mathcal{I}^+}(p_v)_i^4 + \left(\sum_{i\in\mathcal{I}^-}(d_x)_i(d_y)_i\right)^2 \\ &\le \frac{1}{16}\|p_v\|^4 + \left(\sum_{i\in\mathcal{I}^-}(d_x)_i(d_y)_i\right)^2.\end{aligned}$$

Also because M is a $P_*(k)$-matrix, it is easy to see that

$$\begin{aligned}\sum_{i\in\mathcal{I}^-}|(d_x)_i(d_y)_i| &\le (1+4\kappa)\sum_{i\in\mathcal{I}^+}|(d_x)_i(d_y)_i| \\ &\le \left(\frac{1}{4}+\kappa\right)\|p_v\|^2.\end{aligned}$$

Hence, we get

$$\begin{aligned}\|d_x d_y\|^2 &\le \left(\left(\frac{1}{4}+\kappa\right)^2 + \frac{1}{16}\right)\|p_v\|^4 \\ &\le \left(\frac{1}{\sqrt{8}}+\kappa\right)^2\|p_v\|^4.\end{aligned}$$

Using (2.6), the result follows. This completes the proof. □

Lemma 3.2. *The full-Newton step is strictly feasible if and only if* $w + d_x d_y > 0$.

Proof. Assume that the full-Newton step is positive, we have

$$\begin{aligned}x_+y_+ &= (x+\Delta x)(y+\Delta y) \\ &= xy + x\Delta y + y\Delta x + \Delta x\Delta y \\ &= xy + (w - xy) + d_x d_y = w + d_x d_y.\end{aligned}$$

The above equality makes clear that $w + d_x d_y > 0$. Now, proving "the only" part of the statement in the lemma, we introduce a steplength α with $\alpha \in [0,1]$, and we define

$$x^\alpha = x + \alpha\Delta x, \quad y^\alpha = y + \alpha\Delta y.$$

We then have $x^0 = x,\ x^1 = x_+$ and similar relations for y, hence $x^0y^0 = xy > 0$. We may write

$$\begin{aligned}x^\alpha y^\alpha &= (x+\alpha\Delta x)(y+\alpha\Delta y) \\ &= xy + \alpha(x\Delta y + y\Delta x) + \alpha^2\Delta x\Delta y.\end{aligned}$$

But $w - xy = x\Delta y + y\Delta x$ substitution gives

$$\begin{aligned}x^\alpha y^\alpha &= xy + \alpha(w - xy) + \alpha^2\Delta x\Delta y \\ &= (1-\alpha)xy + \alpha(w + \alpha\Delta x\Delta y) \\ &= (1-\alpha)xy + \alpha(w + \alpha d_x d_y).\end{aligned}$$

We assume that $w + d_x d_y > 0$ which is equivalent to $d_x d_y > -w$. Substitution gives

$$x^\alpha y^\alpha > (1-\alpha)xy + \alpha(1-\alpha)w.$$

Since $(1-\alpha)xy + \alpha(1-\alpha)w \geq 0$, it follows that $x^\alpha y^\alpha > 0$ for any $0 \leq \alpha \leq 1$. Since x^α and y^α are linear functions of α and since $x^0 > 0$ and $y^0 > 0$, it follows that x^1 and y^1 must be positive. This completes the proof. □

Lemma 3.3. *If* $\delta(v;w) < \frac{1}{\sqrt{\sqrt{2}+4\kappa}}$. *Then the full-Newton step is strictly feasible.*

Proof. Lemma 3.2 implies that x_+ and y_+ are strictly feasible if and only if $w + d_x d_y > 0$. So $w + d_x d_y > 0$ holds if $w_i + (d_x)_i(d_y)_i > 0$, for all i. Now since $w_i + (d_x)_i(d_y)_i \geq w_i - |(d_x)_i(d_y)_i| \geq \min(w) - \|d_x d_y\|$ for all i. According to Lemma 3.1 (3.2), it follows that

$$\min(w) - \|d_x d_y\| \geq \min(w)(1 - (\sqrt{2} + 4\kappa)\delta^2).$$

Thus $w + d_x d_y > 0$ holds if $\delta(v;w) < \frac{1}{\sqrt{\sqrt{2}+4\kappa}}$. □

3.2. Quadratic convergence of the iterates

For convenience, we may write

$$v_+ := \sqrt{x_+ y_+}.$$

Lemma 3.4. *If* $\delta < \frac{1}{\sqrt{\sqrt{2}+4\kappa}}$. *Then*

$$\|v_+^{-1}\| \leq \frac{1}{\sqrt{\min(w)\left(1 - (\sqrt{2} + 4\kappa)\delta^2\right)}}.$$

Proof. Since $v_+^2 = w + d_x d_y$, it follows that

$$v_+^{-1} = \frac{e}{\sqrt{w + d_x d_y}}.$$

Now, since $\|d_x d_y\|_\infty \leq \|d_x d_y\|$, we deduce that

$$\begin{aligned}\|v_+^{-1}\| &= \left\|\frac{e}{\sqrt{w + d_x d_y}}\right\| = \frac{1}{\|\sqrt{w + d_x d_y}\|} \\ &\leq \frac{1}{\sqrt{\min(w) - \|d_x d_y\|_\infty}} \leq \frac{1}{\sqrt{\min(w) - \|d_x d_y\|}}.\end{aligned}$$

Finally, the result follows from Lemma 3.1 (3.2). This completes the proof. □

The next lemma shows the influence of a full-Newton step on the proximity measure.

Lemma 3.5. *If* $\delta < \frac{1}{\sqrt{\sqrt{2}+4\kappa}}$. *Then*

$$\delta_+ := \delta(v_+;w) \leq \frac{(\sqrt{2} + 4\kappa)\delta^2}{2\sqrt{1 - (\sqrt{2} + 4\kappa)\delta^2}}.$$

Proof. We have

$$\delta_+ = \frac{1}{2\sqrt{\min(w)}} \left\| v_+^{-1}(w - v_+^2) \right\|,$$

but $w - v_+^2 = -d_x d_y$ then

$$\delta_+ \leq \frac{1}{2\sqrt{\min(w)}} \|v_+^{-1}\| \, \|d_x d_y\|.$$

By Lemma 3.4 and (3.2), we get the required result. □

Corollary 3.6. *If* $\delta \leq \left(2(\sqrt{2}+4\kappa)\right)^{-1}$, *then* $\delta_+ \leq \left(\sqrt{\sqrt{2}+4\kappa}\delta\right)^2$, i.e., *local quadratically convergence of the full-Newton step is obtained.*

In the next lemma, we discuss the influence on the proximity measure of the update parameter $w_+ = (1-\theta)w$ on the Newton process along the w-path.

Lemma 3.7. *Suppose that* $\delta \leq \frac{1}{2(\sqrt{2}+4\kappa)}$ *and* $w_+ = (1-\theta)w$ *where* $0 < \theta < 1$. *Then*

$$\delta(v_+; w_+) \leq \frac{\theta\sqrt{n}\sigma_c(w)}{2\sqrt{1-\theta}\sqrt{1-\left(\sqrt{2}+4\kappa\right)\delta^2}} + \frac{\left(\sqrt{2}+4\kappa\right)\delta^2}{2\sqrt{1-\theta}\sqrt{1-\left(\sqrt{2}+4\kappa\right)\delta^2}}.$$

In addition, if $\theta = \frac{1}{2(\sqrt{2}+4\kappa)\sqrt{n}\sigma_c(w)}$ *and* $n \geq 4$, *then we have*

$$\delta(v_+; w_+) \leq \left(2(\sqrt{2}+4\kappa)\right)^{-1}.$$

Proof. We have

$$\begin{aligned}
\delta(v_+; w_+) &= \frac{1}{2\sqrt{\min(w_+)}} \left\| v_+^{-1}(w_+ - v_+^2) \right\| \\
&= \frac{1}{2\sqrt{(1-\theta)\min(w)}} \left\| v_+^{-1}(w_+ - w + w - v_+^2) \right\|.
\end{aligned}$$

By triangle inequality, it follows that

$$\delta(v_+; w_+) \leq \frac{1}{2\sqrt{(1-\theta)\min(w)}} \left\| v_+^{-1} \right\| \left(\|w - w_+\| + \left\| w - v_+^2 \right\| \right).$$

Substitution $w - v_+^2 = -d_x d_y$ and $\|w - w_+\| = \theta\|w\|$ and with the fact that $\|w\| \leq \sqrt{n}\, \|w\|_\infty$,we get

$$\delta(v_+; w_+) \leq \frac{\|v_+^{-1}\|(\theta\sqrt{n}\max(w) + \|d_x d_y\|)}{2\sqrt{(1-\theta)\min(w)}}.$$

Finally, by (2.7) and (3.2) and Lemma 3.4, it follows that

$$\delta(v_+; w_+) \leq \frac{1}{2\sqrt{1-\theta}\sqrt{1-(\sqrt{2}+4\kappa)\delta^2}} \left(\theta\sqrt{n}\sigma_C(w) + \left(\sqrt{2}+4\kappa\right)\delta^2 \right).$$

This gives the first part of the lemma.

For the second part, if $\theta = \frac{1}{2\sqrt{n}\sigma_C(w)(\sqrt{2}+4\kappa)}$ observe that $\sigma_C(w) \geq 1$ and for $n \geq 4$, then $\theta \leq \frac{1}{4(\sqrt{2}+4\kappa)}$. It follows that

$$\delta(v_+; w_+) \leq \frac{1}{2\sqrt{1-\theta}\sqrt{1-(\sqrt{2}+4\kappa)\delta^2}}\left(\frac{1}{2(\sqrt{2}+4\kappa)} + (\sqrt{2}+4\kappa)\delta^2\right).$$

But since $\delta \leq \frac{1}{2(\sqrt{2}+4\kappa)}$, and the two expressions $\frac{1}{\sqrt{1-(\sqrt{2}+4\kappa)\delta^2}}$ and δ^2 are monotonic increasing functions with respect to $\delta \geq 0$, so we get

$$\delta(v_+; w_+) \leq f(\theta)$$

where

$$f(\theta) = \frac{1}{2\sqrt{1-\theta}\sqrt{1-\frac{1}{4(\sqrt{2}+4\kappa)}}}\left(\frac{3}{4(\sqrt{2}+4\kappa)}\right).$$

The function $f(\theta)$ is continuous and monotonic increasing function on the interval $\left[0, \frac{1}{4(\sqrt{2}+4\kappa)}\right]$, we have

$$f(\theta) \leq f\left(\frac{1}{4\sqrt{2}+4\kappa}\right) = \frac{g(\kappa)}{\sqrt{2}+4\kappa},$$

where

$$g(\kappa) = \frac{3}{8\left(1-\frac{1}{4(\sqrt{2}+4\kappa)}\right)}.$$

Since $g(\kappa)$ is a monotonic decreasing function with respect to $\kappa \geq 0$, it follows that $g(\kappa) \leq g(0)$ and so

$$f(\theta) \leq \frac{g(0)}{\sqrt{2}+4\kappa} \leq \frac{1}{2(\sqrt{2}+4\kappa)}$$

since $g(0) = \frac{3}{8-\sqrt{2}} = 0.45553$. Hence $\delta(v_+; w_+) \leq \frac{1}{2(\sqrt{2}+4\kappa)}$. This completes the proof. □

Note that, in all the iterates produced by the algorithm, we have $\sigma_C(w) = \sigma_C(w_0)$. It follows from Lemma 3.6, that for defaults $\theta = \frac{1}{2\sqrt{n}\sigma_C(w)(\sqrt{2}+4\kappa)}$ and $\tau = \frac{1}{2(\sqrt{2}+4\kappa)}$, the conditions $x, y > 0$ and $\delta(v_+; w_+) \leq \tau$ are maintained during the algorithm. Thus confirms that the algorithm is well defined.

In the next lemma, we study the effect of a full-Newton step on the duality gap.

Lemma 3.8. *Let $\delta \leq \frac{1}{2(\sqrt{2}+4\kappa)}$. Then after a full-Newton step the duality gap satisfies*

$$x_+^T y_+ \leq 2n \max(w).$$

Proof. By Lemma 3.1, (3.1) and as $v_+^2 = w + d_x d_y$ and $e^T w \leq n \max(w)$, one has

$$x_+^T y_+ = e^T v_+^2 = e^T w + d_x^T d_y \leq e^T w + \min(w)\delta^2 \leq (n+1)\max w \leq 2n \max w,$$

since $\delta < 1$ for all $\kappa \geq 0$, and $(n+1) \leq 2n$ for all $n \geq 1$. This proves the lemma. □

3.3. Complexity bound

Lemma 3.9. *Let x^k and y^k be the vectors obtained after k iterations by the algorithm with $w := w_k$. Then the inequality $(x^k)^T y^k \leq \epsilon$ is satisfied for*

$$k \geq \frac{1}{\theta} \log \frac{2n \max(w_0)}{\epsilon}.$$

Proof. Lemma 3.8 implies that

$$(x^k)^T y^k \leq 2n \max(w_k) = 2n(1-\theta)^k \max w_0.$$

Then, the inequality $(x^k)^T y^k \leq \epsilon$ holds if

$$(1-\theta)^k 2n \max(w_0) \leq \epsilon.$$

Taking logarithms, we get $k\log(1-\theta) \leq \log\epsilon - \log 2n\max(w_0)$. Since $-\log(1-\theta) \geq \theta$, then the above inequality holds if

$$k\theta \geq \frac{\log 2n \max(w_0)}{\epsilon}.$$

This implies the lemma. □

Theorem 3.10. *Let* $\theta = \frac{1}{2\sqrt{n}\sigma_C(w_0)(\sqrt{2}+4\kappa)}$ *and suppose that* $w_0 = \frac{x^0y^0}{2\max(x^0y^0)}$ *with* x^0 *and* y^0 *are strictly feasible starting points for* (1.) *with* $\delta(x^0, y^0; w_0) \leq \tau$. *Then the algorithm requires at most*

$$O\left((1+\kappa)\sigma_C(w_0)\sqrt{n}\log\frac{n}{\epsilon}\right)$$

iterations to obtain an ϵ*-approximate solution of* $P_*(\kappa)$*-LCP.*

In particular, if $w_0 = \frac{\epsilon}{2}$ *then the algorithm requires at most*

$$O\left((1+\kappa)\sqrt{n}\log\frac{n}{\epsilon}\right)$$

iterations which is the currently best known complexity for such short-step methods.

Proof. By taking θ and w_0 in Lemma 3.9, the proof is straightforward. □

4. Numerical results

In this Section, we present numerical results for Algorithm 2.3 using $\epsilon = 10^{-6}$, $\theta = \frac{1}{2\sqrt{n}\sigma_C(w_0)\left(\sqrt{2}+4k\right)}$ and $\tau = \frac{1}{2\left(\sqrt{2}+4k\right)}$. The algorithm has been applied to two $P_*(0)$-linear complementarity problems and to one $P_*(\kappa)$-LCP, with $\kappa \neq 0$.

Problem 1. The LCP is given by:

$$M = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 \\ -1 & -1 & -2 & 0 \end{pmatrix}, \; q = (-8, -6, -4, 3)^T.$$

The initial starting point is:

$$x^0 = (1.5,\ 0.4,\ 0.2,\ 7)^T.$$

An exact solution is:

$$x^{\text{exact}} = (2.5,\ 0.5,\ 0,\ 2.5)^T.$$

The numerical results for Problem 1 with the theoretical choice $w_0 = 0.5e$, and with relaxed weights as $w_0^k = ke$ are summarized in Table 1. (in this case $\sigma_C(w^0) = 1$)). Now, with the weights $w_0^k = k\frac{x^0y^0}{\max(x^0y^0)}$ and with the same choice of θ, the numerical results are summarized in Table 2.

TABLE 1.

$w_0^k \backslash \theta$	$\theta = \dfrac{1}{2\sqrt{2n}}$	$\theta = \dfrac{1}{\sqrt{2n}}$
$w_0^{0.5}$	61	28
$w_0^{0.05}$	49	23
$w_0^{0.005}$	37	17
$w_0^{0.0005}$	25	12

TABLE 2.

$w_0^k \backslash \theta$	$\theta = \dfrac{1}{2\sigma_C(w^0)\sqrt{2n}}$	$\theta = \dfrac{1}{\sigma_C(w^0)\sqrt{2n}}$
$w_0^{0.5}$	230	113
$w_0^{0.05}$	197	97
$w_0^{0.005}$	150	74
$w_0^{0.0005}$	103	51

TABLE 3.

$w_0^k \backslash \theta$	$\theta = \dfrac{1}{2\sqrt{2n}}$	$\theta = \dfrac{1}{\sqrt{2n}}$
$w_0^{0.5}$	84	39
$w_0^{0.05}$	68	32
$w_0^{0.005}$	52	25
$w_0^{0.0005}$	36	17

Problem 2.

$$M = \begin{pmatrix} 4 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}, \; q = \big(-1, \ldots, -1\big)^T,$$

The initial starting point is:

$$x^0 = (0.65,\, 0.65,\, 0.65,\, 0.65,\, 0.65,\, 0.65,\, 0.65)^T.$$

An exact solution is:

$$x^{\text{exact}} = (0.3660, 0.4639, 0.4897, 0.4948, 0.4897, 0.4639, 0.3660)^T.$$

The numerical results for Problem 2 with the theoretical choice $w_0 = 0.5e$, and with relaxed weights as $w_0^k = ke$ are summarized in Table 3. (in this case $\sigma_C(w^0) = 1$)). Now, with the weights $w_0^k = k\frac{x^0y^0}{\max(x^0y^0)}$ and with the choice of θ, the numerical results are summarized in Table 4.

Problem 3. For $c \geq 0$, the $P_*(\kappa)$-LCP is given by:

$$M = \begin{pmatrix} 0 & 1+4\kappa & 0 \\ -1 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \; q = (0.01,\; 0.501,\; -0.49)^T. \tag{4.1}$$

TABLE 4.

$w_0^k \backslash \theta$	$\theta = \frac{1}{2\sigma_C(w^0)\sqrt{2n}}$	$\theta = \frac{1}{\sigma_C(w^0)\sqrt{2n}}$
$w_0^{0.5}$	263	129
$w_0^{0.05}$	210	103
$w_0^{0.005}$	156	77
$w_0^{0.0005}$	103	51

TABLE 5.

$w_0^k \backslash \theta$	$\theta = \frac{1}{5\sqrt{n}}$	$\theta = \frac{1}{\sqrt{n}}$
$w_0^{0.5}$	113	17
$w_0^{0.05}$	94	15
$w_0^{0.005}$	75	12
$w_0^{0.0005}$	57	9

TABLE 6.

$w_0^k \backslash \theta$	$\theta = \frac{1}{5\sigma_C(w^0)\sqrt{n}}$	$\theta = \frac{1}{\sigma_C(w^0)\sqrt{n}}$
$w_0^{0.5}$	187	34
$w_0^{0.05}$	149	27
$w_0^{0.005}$	110	20
$w_0^{0.0005}$	71	14

The matrix M is a $P_*(\kappa)$-matrix for all $\kappa \geq 0$ (see [3]). For example if $\kappa = \frac{1}{4}$ and $c = 1$, then the $P_*(\frac{1}{4})$-LCP (a non monotone LCP) is given by:

$$M = \begin{pmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad q = (0.01,\ 0.501,\ -0.49)^T.$$

Also the defaults θ and τ in the algorithm become

$$\theta = \frac{1}{2\sqrt{n}\sigma_C(w^0)(\sqrt{2}+1)} \simeq \frac{1}{5\sqrt{n}\sigma_C(w^0)} \text{ and } \tau = \frac{1}{2(\sqrt{2}+1)} \simeq \frac{1}{5}.$$

The initial starting point is:

$$x^0 = (0.2, 0.02, 0.5)^T.$$

An exact solution is:

$$x^{\text{exact}} = (0,\ 0,\ 0.49)^T.$$

The numerical results for the Problem 3 with the theoretical choice $w_0 = 0.5e$ and with relaxed weights as $w_0^k = ke$ are summarized in Table 5 (in this case $\sigma_C(w^0) = 1$)). For the choice $w_0 = k\frac{x^0 y^0}{\max(x^0 y^0)}$ and with the choice of θ, the numerical results are summarized in Table 6.

5. Conclusions and remarks

In this paper, we have presented a feasible weighted full-Newton-step interior point algorithm for solving $P_*(\kappa)$-LCP.The currently best known iteration bound for $P_*(\kappa)$-LCP is derived, namely $O((1+\kappa)\sqrt{n}\log\frac{n}{\epsilon})$, which almost coincides with bound derived for monotone LCP except that the iteration bound in $P_*(\kappa)$-LCP case are multiplied with the factor $(1+\kappa)$. Preliminary numerical results obtained by the algorithm are encouraging. An interesting topic remains for further research is the generalization of the analysis of this algorithm for symmetric cone $P_*(\kappa)$-LCP.

References

[1] M. Achache, A weighted path-following method for the linear complementarity problem. *Studia Univ. Babes-Bolyai, Informatica* **XLIX** (2004) 61–73.

[2] M. Achache and R. Khebchache, A full-Newton step feasible weighted primal-dual interior point algorithm for monotone LCP. *Afrika Matematika* **26** (2015) 139–151.

[3] G.M. Cho, Large-update interior point algorithm for P_*-linear complementarity problem. *J. Inequalities Appl.* **363** (2014) 1–12.

[4] R.W. Cottle, J.S. Pang and R.E. Stone, The linear complementarity problem. Academic Press, San Deigo (1992).

[5] Zs. Darvay, A weighted-path-following method for linear optimization. *Studia Univ. Babes-Bolyai, Informatica* **XLVII** (2002).

[6] J. Ding and T.Y. Li, An algorithm based on weighted logarithmic barrier functions for linear complementarity problems. *Arab. J. Sci. Eng.* **15** (1990) 679–685.

[7] T. Illés and M. Nagy, The Mizuno-Todd-Ye predictor-corrector algorithm for sufficient matrix linear complementarity problem. *Oper. Res. report* (2004).

[8] B. Jansen, C. Roos, T. Terlaky and J.Ph. Vial, Primal-dual target-following algorithms for linear programming. *Ann. Oper. Res.* **62** (1997) 197–231.

[9] H. Mansouri and H. Asadi, A quadratically convergent $O((1+\kappa)\sqrt{n})$ interior point algorithm for the $P_*(\kappa)-$ matrix horizontal linear complementarity problem. *J. Sci. Islamic Republic of Iran* **23** (2012) 237–244.

[10] I. Pólik, *Novel analysis of interior point methods for linear optimization problems [in Hungarian]*. Msc thesis, Eotvos Lorand University of Sciences. Faculty of Natural Sciences, Budapest (2002).

[11] C. Roos, T. Terlaky and J. Ph. Vial, Theory and Algorithms for Linear optimization. An Interior Point Approach. John Wiley and Sons, Chichester (1997).

[12] G.Q. Wang and Y.Q. Bai, Polynomial interior-point algorithms for $P_*(\kappa)$-horizontal linear complementarity problem. *J. Comput. Appl. Math.* **233** (2009) 249–263.

[13] G.Q. Wang, Y.J. Yue and X.Z. Cai, A weighted path-following method for monotone horizontal linear complementarity problem. *Fuzzy Inform. Eng.* **54** (2009) 479–487.

[14] G.Q. Wang, X.J. Fan, D.T. Zhu and D.Z. Wang, New complexity analysis of a full-Newton step feasible interior point algorithm for $P_*(\kappa)$-LCP. *Optim. Lett.* **9** (2015) 1105–1119.

[15] G.Q. Wang, C.J. Yu and K.L. Teo, A full-Newton step feasible interior-point algorithm for $P_*(\kappa)$-linear complementarity problem. *J. Global Optim.* **59** (2014) 81–99.

[16] S.J. Wright, Primal-dual interior point methods. Copyright by SIAM (1997).