GENERALIZATION OF THE TOTAL OUTER-CONNECTED DOMINATION IN GRAPHS

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Abstract. Let $G=(V, E)$ be a graph without an isolated vertex. A set $S \subseteq V$ is a total dominating set if $S$ is a dominating set, and the induced subgraph $G[S]$ does not contain an isolated vertex. The total domination number of $G$ is the minimum cardinality of a total dominating set of $G$. A set $D \subseteq V$ is a total outer-connected dominating set if $D$ is a total dominating set, and the induced subgraph $G[V \setminus D]$ is connected. The total outer-connected domination number of $G$ is the minimum cardinality of a total outer-connected dominating set of $G$. In this paper we generalize the total outer-connected domination number in graphs. Let $k \geq 1$ be an integer. A set $D \subseteq V$ is a total outer-$k$-connected component dominating set if $D$ is a total dominating and the induced subgraph $G[V \setminus D]$ has exactly $k$ connected component(s). The total outer-$k$-connected component domination number of $G$, denoted by $\gamma_{tc}^k(G)$, is the minimum cardinality of a total outer-$k$-connected component dominating set of $G$. We obtain several general results and bounds for $\gamma_{tc}^k(G)$, and we determine exact values of $\gamma_{tc}^k(G)$ for some special classes of graphs $G$.

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1. Introduction

For notation and terminology in general we follow [4]. Let $G=(V, E)$ be a simple graph of order $n=|V(G)|=|V|$ and size $e=|E(G)|=|E|$. We denote the open neighborhood of a vertex $v$ of $G$ by $N_G(v)$ or just $N(v)$, and its closed neighborhood by $N_G[v]=N[v]$. For a vertex set $S \subseteq V$, $N(S)=\bigcup_{v \in S}N(v)$ and $N[S]=\cup_{v \in S}N[v]$. The degree $\deg(x)$ of a vertex $x$ denotes the number of neighbors of $x$ in $G$. The maximum degree and minimum degree of $G$ are denoted by $\Delta(G)$ and $\delta(G)$, respectively. The distance between two vertices of a graph is the number of edges in a shortest path connecting them. The eccentricity of a vertex is the greatest distance between it and any other vertex. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the maximum eccentricity among all vertices of $G$. A set of vertices $S$ in $G$ is a dominating set, if $N[S]=V$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. If $S$ is a subset of $V$ then we denote by $G[S]$ the subgraph of $G$ induced by $S$. A dominating set $S$ of $G$ is a total dominating set if $G[S]$ has no isolated

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vertex. The total domination number of \( G \), denoted by \( \gamma_t(G) \), is the minimum cardinality of a total dominating set of \( G \).

**Total outer-connected domination** in graphs was introduced by Cyman in [1]. If \( G \) is without an isolated vertex, then a set \( D \subseteq V \) is a total outer-connected dominating set (TOCDS) of \( G \) if \( D \) is a total dominating set of \( G \) and the subgraph induced by \( V \setminus D \) is connected. The minimum cardinality of a total outer-connected dominating set in \( G \) is the total outer-connected domination number denoted \( \gamma_{tc}(G) \). A minimum TOCDS of a graph \( G \) is called a \( \gamma_{tc}(G) \)-set. Cyman in [1], Hattinig and Joubert in [3] obtained a lower bound for the total outer-connected domination number of a tree in terms of the order of the tree, and characterized trees achieving equality. Cyman and Raczek in [2] characterized trees with equal total domination and total outer-connected domination numbers. They also gave a lower bound for the total outer-connected domination number of a tree in terms of the order and the number of leaves of the tree, and characterized extremal trees. Jiang and Kang in [5] studied Nordhaus–Gaddum Typebounds for the total outer-connected domination number of a graph.

We generalize the total outer-connected domination number of a graph. Let \( G \) be a graph with no isolated vertex. For an integer \( k \geq 1 \), a subset \( S \) of the vertices of \( G \) is a total outer-\( k \)-connected component dominating set, or just TO\( k \)CDS, if \( S \) is a total dominating set of \( G \) and \( G[V \setminus S] \) has \( k \) connected components. The total outer-\( k \)-connected component domination number of \( G \), denoted by \( \gamma^k_{tc}(G) \), is the minimum cardinality of a TO\( k \)CDS of \( G \). In the case that there is no TO\( k \)CDS of \( G \), we define \( \gamma^k_{tc}(G) = 0 \). We also refer a \( \gamma^k_{tc}(G) \)-set in a graph \( G \) as a TO\( k \)CDS of cardinality \( \gamma^k_{tc}(G) \). Note that a TOCDS \( S \) is a TO\( 1 \)CDS if \( |S| < |V| \), and thus the concept of total outer-\( k \)-connected component domination is a generalization of the concept of total outer-connected domination.

In Section 2, we present some general results and bounds for the total outer-\( k \)-connected component domination number of graphs. In Section 3, we determine exact values of the total outer-\( k \)-connected component domination number for some special classes of graphs.

All graphs we consider in this paper are without isolated vertices and have at least three vertices. We recall that a leaf in a graph is a vertex of degree one, and a support vertex is one that is adjacent to a leaf. A pendant edge is an edge which at least one of its end-points is a leaf. We denote by \( L(G) \) and \( S(G) \) the set of all leaves and all support vertices of \( G \), respectively.

With \( K_n \) we denote the complete graph on \( n \) vertices, with \( P_n \) the path on \( n \) vertices, with \( C_n \) the cycle of length \( n \), and with \( W_n \) the wheel with \( n + 1 \) vertices. A bipartite graph is a graph whose vertex set can be partitioned into two sets of pair-wise non-adjacent vertices. We denote by \( K_{m,n} \) the complete bipartite graph which one partite set has cardinality \( m \) and the other partite set has cardinality \( n \). The corona \( cor(G) \) of a graph \( G \) is the graph obtained from \( G \) by adding a pendant edge to any vertex of \( G \). By \( \alpha(G) \) we denote the independence number of a graph \( G \).

## 2. General results and bounds

We begin with the following observation.

**Observation 2.1.** Let \( k \geq 1 \) be an integer, and let \( G \) be a graph without isolated vertices. If \( 0 < \gamma^k_{tc}(G) < n \), then \( \alpha(G) \geq k \), and \( \delta(G) \leq n - k \).

**Proof.** Assume that \( 0 < \gamma^k_{tc}(G) < n \) for some integer \( k \). Let \( S \) be a \( \gamma^k_{tc}(G) \)-set, and \( G_1, G_2, \ldots, G_k \) be the components of \( G[V \setminus S] \). Let \( x_i \) be a vertex in \( V(G_i) \) for \( i = 1, 2, \ldots, k \). Then clearly \( \{x_1, x_2, \ldots, x_k\} \) is an independent set, implying that \( \alpha(G) \geq k \). To complete the proof, note that, since \( x_1 \) is not adjacent to any \( x_i \), \( i = 2, 3, \ldots, k \), then \( \delta(G) \leq \deg(x_1) \leq (n - 1) - (k - 1) = n - k \). \( \square \)

**Lemma 2.2.** If \( \gamma^k_{tc}(G) = 0 \) for some integer \( k \), then for every \( m > k \), \( \gamma^m_{tc}(G) = 0 \).

**Proof.** Let \( \gamma^k_{tc}(G) = 0 \) for some integer \( k \) and \( m > k \) be an integer. Suppose to the contrary that \( \gamma^m_{tc}(G) \neq 0 \). Let \( S \) be a \( \gamma^k_{tc}(G) \)-set, and let \( G_1, G_2, \ldots, G_m \) be \( m \) connected components of \( G[V \setminus S] \). It is obvious that
Lemma 2.3. Let $k$ be the maximum integer such that $\gamma_{tc}^k(G) > 0$. If $S$ is a $TOK$-CDS, then every connected component of $G[V - S]$ is a complete graph.

Proof. Let $k$ be the maximum integer such that $\gamma_{tc}^k(G) > 0$, and let $S$ be a $TOK$-CDS. Suppose to the contrary that there is a connected component $G_1$ of $G[V - S]$ such that $G_1$ is not complete. Let $x, y$ be two non-adjacent vertices in $G_1$. Then $S \cup (V(G_1) - \{x, y\})$ is a $TO(k + 1)$-CDS for $G$, a contradiction.

Lemma 2.4. If a graph $G$ has a $TOK$-CDS, then it has a $TOt$-CDS for any integer $t < k$.

Proof. Let $S$ be a $TOK$-CDS for a graph $G$, where $k > 1$, and let $G_1, G_2, \ldots, G_k$ be the components of $G[V - S]$. Let $t < k$. Then $S \cup (V(G_1) \cup V(G_2) \cup \ldots \cup V(G_{k-t}))$ is a $TOt$-CDS for $G$.

Lemma 2.5. Let $G$ be a connected graph. If $k$ is the maximum integer such that $\gamma_{tc}^k(G) > 0$, then $\text{diam}(G) \leq 3k - 1$.

Proof. If $k$ is the maximum integer such that $\gamma_{tc}^k(G) > 0$, then $\gamma_{tc}^r(G) = 0$ for each $r \geq k + 1$. Suppose to the contrary that $\text{diam}(G) \geq 3k$. Let $x_0, x_1, x_2, \ldots, x_{2k}$ be a diametrical path in $G$ such that $d = 3p + t$ with an integer $0 \leq t \leq 2$, and let $L_t$ be the set of leaves of $G$ adjacent to $x_i$ for $1 \leq i \leq d - 1$. Let $B$ be the subset of vertices $x_{3i}$ such that $|L_{3i}| = 0$ for $i = 1, 2, \ldots, p - 1$, and define the set $A$ by

$$A = \{x_0, x_3, \ldots, x_{3(p-1)}, x_d\} \cup L_{3i} \setminus B.$$ 

Then $S = V \setminus A$ is a $TO(p + 1)$-CDS for $G$. Since $p + 1 \geq k + 1$, we obtain a contradiction to the hypothesis, and the proof is complete.

Theorem 2.6. Let $G$ be a connected graph $G$ of order $n \geq 3$. Then $\gamma_{tc}^2(G) = 0$ if and only if $G \in \{P_3, C_4, C_5, K_n\}$.

Proof. First notice that $\gamma_{tc}^1(K_n) = \gamma_{tc}^1(P_3) = \gamma_{tc}^1(C_4) = 2$, $\gamma_{tc}^1(C_5) = 3$, and $\gamma_{tc}^k(K_n) = \gamma_{tc}^k(P_3) = \gamma_{tc}^k(C_4) = \gamma_{tc}^k(C_5) = 0$ for any $k \geq 2$. Let $G$ be a graph of order at least three and $\gamma_{tc}^2(G) = 0$. Since $G$ is connected, we have $\gamma_{tc}^k(G) > 0$. By Lemma 2.5, $\text{diam}(G) \leq 2$. If $\text{diam}(G) = 1$, then clearly $G$ is a complete graph. Thus assume that $\text{diam}(G) = 2$. Let $x, y$ be two diametrical vertices with $d(x, y) = \text{diam}(G) = 2$.

Assume first that $\text{deg}(x) \geq 3$. We show that $G[N(x)]$ is complete. Assume there are two non-adjacent vertices $a, b$ in $N(x)$. Since $V - \{a, b\}$ is not a $TO2$-CDS, we obtain that there is a vertex $z$ such that $N(z) \subseteq \{a, b\}$. If $z \neq y$, then $V - \{y, z\}$ is a $TO2$-CDS for $G$, a contradiction. So $z = y$. Let $c \in N(x) - \{a, b\}$. Then $V - \{y, c\}$ is a $TOk$-CDS for some $k \geq 2$, and by Lemma 2.4, $G$ has a $TO2$-CDS, a contradiction. We deduce that $G[N(x)]$ is complete. Now $N(x)$ is a $TO2$-CDS for $G$, a contradiction. Thus $\text{deg}(x) \leq 2$. We also have $\text{deg}(y) \leq 2$. First assume that $\text{deg}(x) = 1$. Let $w \in N(x)$. If $\text{deg}(w) \geq 3$, then $V - \{x, y\}$ is a $TO2$-CDS for $G$, a contradiction. Thus $\text{deg}(w) = 1$, and so $G = P_3$. Assume thus that $\text{deg}(x) = 2$ and $\text{deg}(y) = 2$. Let $N(x) = \{a, w\}$, where $w \in N(y)$. If $a \in N(w)$, then $V - \{x, y\}$ is a $TO2$-CDS for $G$, a contradiction. So $a \not\in N(w)$. If there is a vertex $z \in N(a) - \{x, y\}$ such that $z \not\in N(y)$, then $V - \{y, z\}$ is a $TO2$-CDS for $G$, a contradiction. Thus each vertex of $N(a) - \{x, y\}$ is adjacent to $y$. Similarly, each vertex of $N(w) - \{x, y\}$ is adjacent to $y$. If $|N(a) - \{x, y\}| \geq 2$ or $|N(w) - \{x, y\}| \geq 2$, then $V - \{x, z\}$ is a $TO2$-CDS for $G$, where $z \in N(a) - \{x, y\}$ or $z \in N(w) - \{x, y\}$, a contradiction. Thus $|N(a) - \{x, y\}| \leq 1$ and $|N(w) - \{x, y\}| \leq 1$. Let $N(a) - \{x, y\} = \{z\}$. If $a \in N(y)$, then $V - \{x, z\}$ is a $TO2$-CDS for $G$, a contradiction. So assume that $a \not\in N(y)$. If $w \in N(z)$, then $V - \{x, y\}$ is a $TO2$-CDS for $G$, a contradiction. Thus assume now that $w \not\in N(z)$. Then $G = C_5$ or $N(w) - \{x, y\} = \{z_1\}$ with $z_1 \neq z$. However, then $V - \{x, y\}$ is a $TO2$-CDS for $G$, a contradiction. Since $\text{diam}(G) = 2$ we deduce that $a \in N(y)$. If $N(w) - \{x, y\} = \{z\}$, then we observe that then $V - \{x, z\}$ is a $TO2$-CDS for $G$, a contradiction. Thus $|N(w) - \{x, y\}| = 0$. Hence $G = C_4$. □
In the following we obtain the total outer-$k$-connected component domination number of a disconnected graph $G$ in terms of the total outer-$k$-connected component domination numbers of its components. For this purpose we define $\gamma_{tc}^0(G) = |V|$.

**Theorem 2.7.** Let $G$ be a disconnected graph with $m$ connected components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Then

$$\gamma_{tc}^k(G) = \min \left( \sum_{i=1}^{m} \gamma_{tc}^{l_i}(G_i) \right)$$

where $l_i \in \{0, 1, 2, \ldots, k\}$.

**Proof.** Let $G$ be a disconnected graph with $m$ connected components $G_1, G_2, \ldots, G_m$, and let $k \geq m$. Let $S_i$ be a $\gamma_{tc}^{l_i}(G_i)$-set for $i = 1, 2, \ldots, m$ if $G_i$ has a TO$k$CDS, where $0 \leq l_i \leq k - m + 1$ and $\sum_{i=1}^{m} l_i = k$. It is obvious that $\bigcup_{i=1}^{m} S_i$ is a TO$k$CDS for $G$. This implies that

$$\gamma_{tc}^k(G) \leq \min \left( \sum_{i=1}^{m} \gamma_{tc}^{l_i}(G_i) \right).$$

On the other hand let $S$ be a TO$k$CDS for $G$. Let $S_i = S \cap V(G_i)$ for $i = 1, 2, \ldots, m$. If $l_i$ is the number of components of $G_i - S_i$, then $S_i$ is a TO$l_i$CDS for $G_i$. This completes the proof. \[\square\]

We next obtain lower bounds for the total outer-$k$-connected component domination number of a graph $G$.

**Theorem 2.8.** Let $G$ be a graph of order $n$ and size $e$, and let $k \geq 2$. If $\gamma_{tc}^k(G) > 0$, then

$$\gamma_{tc}^k(G) \geq \frac{2e - (n - k + 1)(n - k)}{2(k - 1)}$$

**Proof.** Let $S$ be a $\gamma_{tc}^k(G)$-set of cardinality $s$. If $G_1, G_2, \ldots, G_k$ are the components of $G[V - S]$ such that $|V(G_i)| = n_i$ for $i = 1, 2, \ldots, k$, then

$$e \leq \sum_{i=1}^{k} \frac{n_i(n_i - 1)}{2} + \frac{s(s - 1)}{2} + \sum_{i=1}^{k} sn_i.$$ 

The right hand side of this inequality becomes maximum when $n_1 = n_2 = \ldots = n_{k-1} = 1$ and $n_k = n - s - (k-1)$. Therefore we obtain

$$e \leq \frac{(n - s - k + 1)(n - s - k)}{2} + \frac{s(s - 1)}{2} + s(n - s)$$

$$= \frac{(n - k + 1)(n - k)}{2} + s(k - 1),$$

and this leads to the desired bound immediately. \[\square\]

Let $k, p, s$ be integers such that $p \geq 1$ and $k, s \geq 2$. Now let the graph $H$ consist of the disjoint union of $K_s$, $K_p$ and $k - 1$ isolated vertices $v_1, v_2, \ldots, v_{k-1}$ such that all vertices of $K_s$ are adjacent to all vertices of $K_p$ and $v_1, v_2, \ldots, v_{k-1}$ are adjacent to all vertices of $K_s$. Then it is straightforward to verify that

$$\gamma_{st}^k(H) = s = \frac{2e(H) - (n(H) - k + 1)(n(H) - k)}{2(k - 1)}.$$ 

This family of examples show that the bound of Theorem 2.8 is sharp. Since $\alpha(H) = k$, we see that the bound $\alpha(G) \geq k$ in Observation 2.1 is sharp too.
Theorem 2.9. For a graph $G$ of order $n$, size $e$ and $\gamma_{tc}^k(G) > 0$, $$\gamma_{tc}^k(G) \geq \left\lceil \frac{4n - 2k - 2e}{3} \right\rceil.$$ 

Proof. Let $S$ be a $\gamma_{tc}^k(G)$-set of cardinality $s$, and $G_1, G_2, \ldots, G_k$ be the connected components of $G[V - S]$. Suppose that $|V(G_i)| = n_i$ for $1 \leq i \leq k$. Since $S$ is a dominating set of $G$, any vertex in $G_i$ has at least one neighbor in $S$ for $1 \leq i \leq k$. On the other hand $G_i$ is connected and so has at least $n_i - 1$ edges for $i = 1, 2, \ldots, k$. Also $G[S]$ has no isolated vertex. Thus, we obtain

$$e \geq \sum (n_i - 1) + \sum n_i + \frac{s}{2}.$$ 

Since $\sum n_i = n - s$ we have $e \geq 2n - \frac{3}{2} - k$. This implies that $s \geq \frac{4n - 2k - 2e}{3}$, and the proof is complete. \hfill $\Box$

An immediate consequence of Theorem 2.9 with $k = 1$ is the following corollary for trees which is a main result of [1].

Corollary 2.10 [1]. For a tree $T$ of order $n$, $\gamma_{tc}(T) \geq \frac{2n}{3}$.

It is obvious that $\gamma_{tc}^k(G) \leq n - k$. To characterize graphs achieving equality for the upper bound of the above inequality, we need to introduce a family of graphs. For $k > 1$, let $\mathcal{G}_k$ be the class of all graphs $G$ such that $G \in \mathcal{G}_k$ if and only if $V = A \cup B$ such that $|A| = n - k$, $G[A]$ has no isolated vertex, $G[B] = \overline{K}_k$, and no subset $S \subseteq A \cup B$ with $|S| < n - k$ is a total outer-$k$-connected component dominating set for $G$. The following is a characterization for graphs $G$ with $\gamma_{tc}^k(G) = n - k$. The proof is straightforward and is omitted.

Theorem 2.11. For a connected graph $G$ of order $n$, $\gamma_{tc}^k(G) = n - k$ if and only if $G \in \mathcal{G}_k$.

3. Exact values

In this section we determine the total outer-$k$-connected component domination number for some special classes of graphs.

Proposition 3.1. For $n \geq 3$, $\gamma_{tc}^k(K_n) = \begin{cases} 2 & \text{if } k = 1 \\ 0 & \text{if } k \geq 2. \end{cases}$

Proof. Let $n \geq 3$. If $S$ is a TO$k$CDS in $K_n$, then $k = 1$, since $K_n[V - S]$ contains exactly one connected component. Thus $\gamma_{tc}^1(K_n) = 0$ if $k \geq 2$. Now it is obvious that $\gamma_{tc}^1(K_n) = \gamma_t(K_n) = 2$. \hfill $\Box$

Proposition 3.2. For $2 \leq m \leq n$, $\gamma_{tc}^k(K_{m,n}) = \begin{cases} 0 & \text{if } n < k \\ 2 & \text{if } n = k \\ m + n - k & \text{if } n \geq k, k \geq 2. \end{cases}$

Proof. Let $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be the two partite sets of $K_{m,n}$. Assume that $\gamma_{tc}^k(K_{m,n}) > 0$. So $k \leq n$. If $k = 1$, then $\gamma_{tc}^1(K_{m,n}) = \gamma_t(K_{m,n}) = 2$. So we assume that $k \geq 2$. Let $S$ be a $\gamma_{tc}^k(K_{m,n})$-set. Since $K_{m,n}[X \cup Y - S]$ is disconnected, it follows that either $X \subseteq S$ or $Y \subseteq S$. Therefore $K_{m,n}[X \cup Y - S]$ consists of isolated vertices. As $K_{m,n}[X \cup Y - S]$ has exactly $k$ connected components, we deduce that $|S| \geq m + n - k$. On the other hand $X \cup \{y_1, y_2, \ldots, y_{n-k}\}$ is a TO$k$CDS for $K_{m,n}$ of cardinality $m + n - k$. This completes the proof. \hfill $\Box$

For $n \geq 3$, we have the following.
Theorem 3.3. \( \gamma_{tc}^k(P_n) = \begin{cases} 0 & \text{if } n < 3k - 2 \\ 2k - 2 & \text{if } 3k - 2 \leq n \leq 4k - 4 \\ 2k - 1 & \text{if } n = 4k - 3 \\ 2k & \text{if } 4k - 2 \leq n \leq 4k - 1 \\ 2k + 1 & \text{if } n = 4k \\ 2k + 2 & \text{if } n = 4k + 1 \\ n - 2k & \text{if } n \geq 4k + 2. \end{cases} \)

Proof. Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \), where \( v_i \) is adjacent to \( v_{i+1} \) for \( i = 1, 2, \ldots, n - 1 \). Assume that \( \gamma_{tc}^k(P_n) > 0 \). Let \( S \) be a \( \gamma_{tc}^k(P_n) \)-set, and let \( G_1, G_2, \ldots, G_k \) be the connected components of \( G - S \). Then \( G[S] \) has at least \( k - 1 \) components. Since any component of \( G[S] \) has at least two vertices, we obtain \( n \geq k + 2(k - 1) = 3k - 2 \).

We deduce, in particular, that \( \gamma_{tc}^k(P_n) = 0 \) if \( n \leq 3k - 2 \).

Assume that \( n \geq 4k + 2 \). Any component of \( G[V - S] \) has at most two vertices, so \( |V - S| \leq 2k \). This implies that \( |S| \geq n - 2k \). On the other hand \( \{v_{4i+1}, v_{4i+2} : 0 \leq i \leq k - 1\} \cup \{v_j : j \geq 4k + 1\} \) is a \( \text{TO}(n - 2k) \)-CDS for \( P_n \), and thus \( \gamma_{tc}^k(P_n) = n - 2k \).

Next we assume that \( 3k - 2 \leq n \leq 4k - 4 \). It is obvious that \( G[S] \) has at least \( k - 1 \) components, and each component of \( G[S] \) has at least two vertices. Thus \( |S| \geq 2(k - 1) = 2k - 2 \). Let \( D = \{v_{3i+2}, v_{3i+3} : 0 \leq i \leq k - 2\} \). Then \( D \) is a \( \text{TO}k \)-CDS for \( P_{3k - 2} \) of cardinality \( 2k - 2 \). If \( t = n - 3k + 2 \), then we subdivide the edges \( v_{3i+3}v_{3i+4} \) for \( i = 1, 2, \ldots, t \) to obtain a path \( P_n \) from \( P_{3k - 2} \). Then \( D \) is still a \( \text{TO}k \)-CDS for \( P_n \). Thus \( \gamma_{tc}^k(P_n) \leq 2k - 2 \) and the result follows.

Assume next that \( n = 4k - 3 \). Suppose that \( |S| \leq 2k - 2 \). For \( S \) to dominate maximum number of vertices, without loss of generality, we may assume that each connected component of \( G[S] \) is \( K_2 \), and each component of \( G[S] \) dominates two vertices of \( G[V - S] \). Then \( |N[S]| \leq 2(\frac{2k - 2}{2}) + |S| < n \), a contradiction. Thus \( |S| \geq 2k - 1 \).

On the other hand \( \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k - 2\} \cup \{v_{n-1}\} \) is a \( \text{TO} \)-CDS for \( P_n \) of cardinality \( 2k - 1 \). Thus \( \gamma_{tc}^k(P_{4k - 3}) = 2k - 1 \).

Next assume that \( 4k - 2 \leq n \leq 4k - 1 \). Suppose that \( |S| \leq 2k - 1 \). If each component of \( G[S] \) is a \( K_2 \), then \( |S| \leq 2k - 2 \) and \( S \) dominates at most \( 4\frac{|S|}{2} \leq 4k - 4 < n \) vertices of \( P_n \), a contradiction. Thus \( |S| \geq 2k - 1 \).

On the other hand \( \{v_{4i+2}, v_{4i+3} : 0 \leq i \leq k - 2\} \cup \{v_n, v_{n-1}\} \) is a \( \text{TO} \)-CDS for \( P_n \) of cardinality \( 2k \). Thus \( \gamma_{tc}^k(P_n) = 2k \).

The proof for \( n \in \{4k, 4k + 1\} \) is similar.

The following theorem can be proved in a similar manner as in the proof of Theorem 3.3, and so we omit the proof.

Theorem 3.4. \( \gamma_{tc}^k(C_n) = \begin{cases} 0 & \text{if } n < 3k \\ 2k & \text{if } 3k \leq n \leq 4k \\ n - 2k & \text{if } n \geq 4k + 1. \end{cases} \)

For a wheel and an integer \( k > 1 \) the center of the wheel is in any \( \text{TO}k \)-CDS. So the following is easily verified.

Theorem 3.5. Let \( k \geq 2 \) be a positive integer, and let \( W_n \) be a wheel with \( n \geq 3 \). Then

\[ \gamma_{tc}^k(W_n) = \begin{cases} 0 & \text{if } n < 2k \\ k + 1 & \text{if } n \geq 2k. \end{cases} \]

We close with the following problems.

Problem 1. Find sharp upper and lower bounds for the total outer-\( k \)-connected component domination number of a graph.
Problem 2. Determine the complexity issue of the total outer-$k$-connected component domination number.

Problem 3. Determine the total outer-$k$-connected component domination number in grid graphs.

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