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A DYNAMIC ADVERTISING MODEL WITH REFERENCE PRICE EFFECT

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Abstract. This paper develops an advertising model in which goodwill affected by advertising effort has a positive effect on reference price and market demand. In a finite planning horizon, the optimal advertising strategy is provided by solving the optimization problem on the basis of Pontryagin's maximum principle, then the optimal sales price is obtained through one time pricing strategy. Furthermore, we extend this problem to an infinite planning horizon and present the corresponding optimal strategies. In addition, the relationships between system parameters and optimal solutions are analyzed. Numerical examples are employed to illustrate the effectiveness of the theoretical results, and to assess the sensitivity analysis of system parameters on the optimal strategies.

Keywords. Advertising, reference price, Goodwill, maximum principle.

Mathematics Subject Classification. 90B60, 49J15.

1. Introduction

As a significant factor affecting purchase decisions of consumers, reference price has received a great deal of attention recently. In fact, the research of reference price is derived from psychology. Helson [15] pointed out that the response of consumers to the sales price of a product came from comparing it with the reference price, the standard in their minds which relied on past price levels. Furthermore, Mayhew and Winer [27] divided reference price into internal reference price and external

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reference price. On this basis, Johnson and Cui [17] conducted a further research on external reference price strategies in pay-what-you-want pricing.

Numerous studies have indicated that reference price plays a critical role on consumer behavior. According to Fibich *et al.* [9], consumers will have a sense of gain when the sales price is less than the reference price in their minds. Such situation will lead to demand promotion. Conversely, consumers are likely to sense a loss when the sales price is greater than the reference price. In this case, there will be a decrease in demand. Putler [31] revealed that the effect of sales price on demand was different in accordance to whether it is higher or lower than the reference price. In the situation of asymmetric reference price effects, some researches, such as [3, 23, 32], stated that the effect of losses on demand was greater than that of gains, which means that consumers are more sensitive to a loss than to an equal gain. This result is in line with prospect theory proposed by Kahneman and Tversky [20].

With the development of theoretical studies on reference price, a growing number of researchers have focused on its formation and application in marketing. Most studies in this field assumed that the reference price was a weighted average of previous prices, so the exponential smoothing process was the most common formation of reference price. For instance, both Fibich *et al.* [9] and Zhang *et al.* [36] adopted this formation. As mentioned by Mazumdar *et al.* [28], reference price was also affected by many other factors such as advertising effort, quantity and so on. Nasiry and Ioana [29] formulated a peak-end model in which the reference price was a weighted average of the lowest and most recent prices. Mazumdar *et al.* [28] and Arslan and Kachani [2] provided a review of models with reference price. In addition, several literature associated reference price with pricing to study the optimal pricing strategies in an asymmetric framework. For example, Fibich *et al.* [9] suggested that a constant price was optimal when the effect of loss on demand was greater than the corresponding gains, otherwise cyclical pricing was a better choice, which was in line with the findings of Popescu and Wu [30]. Moreover, Urban [35] analyzed a single period joint inventory and pricing model with both symmetric and asymmetric reference price effects. Fibich *et al.* [10] studied the retailer's single promotion strategy with asymmetric reference price effects.

However, for tractability, some researchers explored pricing strategy under the symmetric reference price effect. For instance, Benchekroun *et al.* [4] explored the effect of myopia and farsighted behaviors on the profitability of the two channel structures. They first examined the problem in a bilateral monopoly where the manufacturer and the retailer controlled transfer price and retail price, respectively, and found that myopia behavior improved total channel profit when the reference price effect was relatively small. Martín-Herrán *et al.* [25] considered a two-echelon supply chain where the manufacturer set the wholesale price and the quality investment and the retailer set the retail price. They assumed that the retail price affects both the demand and the perceived quality of the brand and that its variations contribute to the building of an internal reference price. Through the research on the effect of retailer's myopia on the equilibrium solutions, they concluded that a myopic retailer set a low price, and the manufacturer invested less in quality as a response. Meanwhile, the channel were better when the retailer was nonmyopic. Zhang *et al.* [37] investigated the pricing strategies for the manufacturer and the retailer in a competitive supply chain with reference effects, and analyzed the solutions sensitivity with respect to various factors. Martín-Herrán and Taboubi [26] considered a differential game in a two-player supply chain with reference price effect, then presented the comparisons of the strategies and profits between integrated and decentralized channels at steady state and along the optimal time-path. The main findings indicated that, for some values of the initial reference price, there exists a time interval in which the decentralization is better than the integration.

Advertising is a common communication form to send messages of a product to consumers and induce them to purchase. A piece of successful advertisement not only builds a good brand image for a firm, but also improves its competitiveness, both of which contribute to promote the profits. Since advertising plays a significant role on consumers' purchasing behaviors, more and more firms prefer to spend much more on it. Meanwhile, studies on advertising also attract a great many researchers. Among them, Nerlove and Arrow first established a dynamic advertising model, which is widely used in this area by lots of researchers, such as Jøgensen et al.[18], Huang *et al.* [16] and so on. We also adopt this advertising model in this paper. Bykadorov and Ellero considered a firm who sold seasonal goods and sought to reach a fixed level of goodwill at the end of the selling period with the minimum total advertising and promotion expenditure. Furthermore, most researchers combined advertising with other market behaviors to investigate advertising strategies. For example, Fruchter [11] determined the optimal price and advertising effort *via* the modified Nerlove-Arrow model. Gozzi *et al.* [12] discussed a class of dynamic advertising problems with uncertainty in the presence of carryover effects. Grosset and Viscolani [14] considered the advertising issues in the competitive environment. Favaretto and Viscolani [8] provided an optimal advertising strategy and optimal production quantity for a seasonal good in a heterogeneous market.

Although there exists massive literature related to reference price and advertising, very little has been done to integrate them. In fact, as two main elements affecting market demand, reference price and advertising have a interaction relationship. In operation management, a firm is able to make larger profits considering the two elements together. Zhang *et al.* [36] considered them simultaneously and proposed a strategy to coordinate supply chain through cooperative advertising under reference price effect. They assumed that the advertising effort had a direct effect on the reference price and demand. Unlike their models, our paper relies on the assumption that reference price and demand are affected by goodwill which is controlled by advertising effort. Such model has not been studied in previous literature. Actually, consumers' evaluation of a product depends on its brand image, namely, goodwill. The higher the goodwill the product has, the higher evaluation it gets from consumers, which brings about a higher reference price. On the other hand, advertising effort of a product is unknown for consumers in most cases, hence it actually will not pose a direct effect on the reference price. The indirect effect takes place *via* the goodwill, which is mainly improved by advertising effort and will consequently raise the value of the product in the minds of consumers. This leads to a growth in the reference price. In addition, though a mass of researches showed that consumers are more sensitive to a loss than an equal gain, some empirical studies found the opposite result, such as those conducted by Greenleaf [13] and Briesch *et al.* [5]. It can be seen from their results that the response to the reference price seems to be very mixed. Hence, our paper just investigates the symmetric case for simplicity.

The purpose of the paper is to design the optimal advertising strategy with reference price effect. The strategy involves the consumer behavior, which is more close to the real market situation. Consequently, it will help the firm make more informed decisions, and bring about more profits. The main work and contributions are summarized in the following aspects. Firstly, a new model is established to show the relationships among advertising effort, goodwill and reference price. Secondly, optimal advertising effort in analytical form is obtained by solving the corresponding dynamic optimization problem on the basis of the maximum principle. Its characteristics and sensitivity to the system parameters are analyzed in details. This study provides a significant guidance for a firm to adjust its advertising effort to gain more profits when facing with different market environments, namely, different system parameters. Additionally, we provide the optimal sales price through one time pricing strategy. Meanwhile, the optimization problem is extended to an infinite planning horizon to analyze the firm's long-term advertising and pricing strategies.

The reminder of this paper is organized as follows. In Section 2, we develop the model of dynamic advertising with reference price effect. Section 3 provides the optimal advertising and pricing strategies in a finite planning horizon. Section 4 presents the corresponding optimal strategies in the infinite case. Numerical examples are given in Section 5 to illustrate the theoretical results. Finally, Section 6 concludes this study and presents possible extension of future research.

2. Model framework

To optimize its total profits, a firm will invest in advertising, a marketing tool, to improve sales. The advertising effort $u(t)$ contributes to the accumulation of the goodwill $G(t)$, which evolves according to the Nerlove-Arrow framework. This is the most widely used specification for the goodwill in marketing channels, *i.e.*,

$$\dot{G}(t) = u(t) - \delta G(t),\ G(0) = G_0, \tag{2.1}$$

where $G_0 > 0$ is the initial goodwill, and $\delta > 0$ is the decay coefficient of goodwill.

A reference price is constructed by consumers through personal shopping experience and exposure to the sales price. According to Kopalle and Winer [22], the reference price $r(t)$ is modeled as a continuous weighted average of past prices with an exponentially decaying weighting function, namely,

$$r(t) = \beta \mathrm{e}^{-\beta t} \int_{-\infty}^{t} \mathrm{e}^{\beta v} p(v) \mathrm{d}v,$$

where $\beta > 0$ called "memory parameter". An immediate consequence of the equation above is the differential equation

$$\dot{r}(t) = \beta(p - r(t)).$$

Based on the above formation, we assume that goodwill has a positive effect on the reference price, in other words, we make the assumption that a higher goodwill leads to an increasing reference price in consumer's mind. Taking it into account, the dynamic equation of the reference price can be written as

$$\dot{r}(t) = \beta(p - r(t)) + \alpha G(t),\ r(0) = r_0, \tag{2.2}$$

where r_0 is the initial reference price, and α represents the marginal contribution of goodwill to reference price.

We assume sales price p as a constant to be determined, which results from the following reasons. First, we focus on the firm's optimal advertising effort with consideration of consumer behavior, aiming to explore the impact of reference price on advertising strategy. Thus, similar to the studies in Jørgensen *et al.* [19], Sigué and Chintagunta [34] and Zhang *et al.* [36]), we also do not consider the dynamic pricing decision to concentrate on the study of advertising. Second, one may obtain the optimal sales price $p(t)$ under the framework of our work. However, the dynamic $p(t)$ means that the firm has to adjust the sales price over time, which is often restricted in most retail practice. Excessive promotion and price changing may adversely influence consumer behavior [24]. On one hand, when consumers face sustained price promotions, they may be accustomed to purchase the product on a promotion, which will decrease the reference prices in their minds. On the other hand, consumers may come to believe that frequent promotions are used as a "cover up" for insufficient quality. Consequently, the firm is unwilling to adjust sales price frequently. For instance, Hailanhome, a fashion apparel company in China, keeps the sales price of its products fixed all the time. Meanwhile, we assume that the sales price should satisfy $p \geq s$, of which s is the selling cost.

Considering the sales of the product over a finite planning horizon $[0, T]$, and the fact that both the goodwill and reference price have positive effects on the demand, we assume that the demand function takes the following form

$$D(t) = d + a(r(t) - p) + bG(t), \tag{2.3}$$

where d, a and b are positive constants, d represents the market potential, a implies the sensitivity of consumers to the gap between the reference price and the sales price, b denotes the marginal contribution of goodwill to the demand. The demand function which is linear in the price gap and goodwill, has been extensively applied in numerous literature by researchers, such as Greenleaf [13], Kopalle and Joan [23], Amrouche *et al.* [1], Karray and Martín-Herrán [21], Zhang *et al.* [36], Zhang *et al.* [37], and so on.

Advertising cost function $C(u)$ is quadratic, *i.e.*,

$$C(u) = \frac{1}{2}cu^2, \tag{2.4}$$

where $c > 0$ is the cost coefficient of the advertising investment.

In the following sections, we present the optimal joint pricing and dynamic advertising strategies for finite and infinite planning horizons, respectively.

3. Optimal strategies for finite planning horizon

In this section, we address the firm's optimization problem within the finite planning horizon $[0, T]$. The objective is to find the optimal joint pricing and dynamic advertising strategies while maximizing the total profit, specified as

$$\begin{aligned}
&\max_{u(\cdot),p} J = \int_0^T \left((p-s)\left(d + a(r(t)-p) + bG(t)\right) - \frac{1}{2}cu^2(t) \right) \mathrm{d}t \\
&\text{s.t.} \quad \dot{G}(t) = u(t) - \delta G(t),\, G(0) = G_0, \\
&\qquad\; \dot{r}(t) = \beta(p - r(t)) + \alpha G(t),\, r(0) = r_0, \\
&\qquad\; u(t) \geq 0, p \geq s.
\end{aligned} \tag{3.1}$$

For the optimization problem (3.1), we start with applying the maximum principle [33] to solve the problem of dynamic advertisement for a given sales price p, then take one time pricing to obtain the optimal pricing strategy.

Hence, we first form the following Hamiltonian function.

$$\begin{aligned}
H(G, r, u, \lambda_1, \lambda_2) = {} & (p-s)\big(a(r-p) + bG\big) \\
& - \frac{1}{2}cu^2 + \lambda_1(u - \delta G) + \lambda_2\big(\beta(p-r) + \alpha G\big),
\end{aligned} \tag{3.2}$$

where λ_1 and λ_2 are adjoint variables of the goodwill and reference price, respectively, and satisfy $\dot{\lambda}_1(t) = -\frac{\partial H}{\partial G}, \dot{\lambda}_2(t) = -\frac{\partial H}{\partial r}$, *i.e.*,

$$\dot{\lambda}_1(t) = -b(p-s) + \delta\lambda_1 - \alpha\lambda_2, \;\; \lambda_1(T) = 0, \tag{3.3}$$

$$\dot{\lambda}_2(t) = -a(p-s) + \beta\lambda_2, \qquad\quad \lambda_2(T) = 0. \tag{3.4}$$

By virtue of the maximum principle, the optimal advertising effort which maximizes the Hamiltonian function (3.2) is calculated as

$$u^*(t) = \begin{cases} 0, & \text{if } \lambda_1 < 0 \\ \dfrac{\lambda_1}{c}, & \text{otherwise.} \end{cases} \tag{3.5}$$

The optimal advertising strategy is shown in the following propositions. For the smoothness of the paper, all the proofs of main results are presented in Appendices.

Proposition 3.1. *When $\beta \neq \delta$, for a given sales price p, the optimal advertising strategy is*

$$u^*(t) = (p-s)\left(\frac{b\beta+a\alpha}{c\delta\beta} + e^{\beta(t-T)}\frac{a\alpha}{c\beta(\beta-\delta)} - e^{\delta(t-T)}\left(\frac{b\beta+a\alpha}{c\delta\beta} + \frac{a\alpha}{c\beta(\beta-\delta)}\right)\right). \tag{3.6}$$

It is easy to verify that $\dot{u}(t) < 0$ and $u(T) = 0$, equally means that the optimal advertising effort is positive initially and decreasing, then it vanishes by the end of the planning horizon.

For the optimal advertising effort $u^*(t)$, we have the following corollary.

Corollary 3.2. *For a given sales price p, the optimal advertising effort $u^*(t)$ described in (3.6) satisfies*

(1) *$u^*(t)$ is decreasing in t, and concave if $a\alpha < b\delta$;*
(2) *for all t, $u^*(t)$ is increasing in b, but decreasing in c;*
(3) *for all t, $u^*(t)$ is increasing in a and α while decreasing in β and δ if $\beta > \delta$.*

The corollary provides the sensitivities of the optimal advertising effort with respect to system parameters when it satisfies some certain sufficient conditions except for the second claim. For the third claim in the situation $\beta \leq \delta$, although we cannot give a strict mathematical proof, numerical simulations show that the optimal advertising effort shares the same properties with that of $\beta > \delta$.

The first statement in Corollary 3.2 indicates that the firm should keep a high advertising level initially owing to the well-known carryover effect of advertising. That is to say, advertising effort cannot play an active role on demand immediately, but a high advertising investment can shorten the delay time and lead to a high goodwill, which could finally bring about a high profit. On the other hand, at the end of the planning horizon, the firm should reduce its advertising effort to save cost. In addition, for the case $a\alpha < b\delta$ which implies a relatively high b or δ, the concave property of $u^*(t)$ shows that the firm should maintain a high advertising investment in the initial stage of the planning horizon.

In the proposed model, positive impact will increase with an increasing b which reflects the contribution of goodwill to demand. Therefore, a high advertising effort will push the demand to create more profits. On the contrary, advertising cost coefficient c negatively affects the advertising effort: a high c will cause a high advertising cost. As a result, it's wise to increase the advertising effort with a large b, but decrease the advertising effort when c is relatively large, as shown in the second claim in Corollary 3.2.

We now turn to the third claim. Generally speaking, a higher advertising effort induces higher goodwill and reference price. An increase in a also enlarges the positive effect on the demand from the reference price r, so it is reasonable to invest more in advertising to increase the profit. In addition, α reflects the active effect of goodwill on reference price, which means the driving force from goodwill on reference price will become larger with the increase of α, so it is better to spend more on advertising to improve reference price so as to increase the demand. On the other hand, the increase in advertising cost can be compensated by the increased profit. For the effect of β on the advertising effort, as a large β means the consumer has shorter term memory and less loyalty to the products, the firm cannot achieve a higher profit through a high level advertisement. Hence, it is reasonable for the firm to reduce the advertising effort when it faces a large β. Moreover, the increase in δ will accelerate the decline of goodwill, which implies that a high advertising effort can only lead to a low return. Consequently, it's not worth investing more in advertising with a large δ.

As Proposition 3.1 gives the optimal advertising strategy under the situation where $\beta \neq \delta$, from the perspective of continuity, the following proposition provides the optimal advertising strategy when $\beta = \delta$.

Proposition 3.3. *When $\beta = \delta$, for a given sales price p, the optimal advertising strategy is*

$$u^*(t) = (p-s)\left(\frac{b\beta + a\alpha + a\alpha\beta \mathrm{e}^{\beta(t-T)}t - (b\beta + a\alpha + a\alpha\beta T)\mathrm{e}^{\beta(t-T)}}{c\beta^2}\right)\cdot \quad (3.7)$$

It should be noted that the optimal advertising effort in Proposition 3.3, as a continuity result of Proposition 3.1, shares the same characteristics with Proposition 3.1. In other words, when β approaches δ, the results of Proposition 3.1 will converge to those in Proposition 3.3. Similarly, the optimal advertising effort $u^*(t)$ described in (3.7) increases in a, b and α, but decreases in c and β.

Substituting (3.6) into equation (2.1), yields the following time path for goodwill.

$$G(t) = (p-s)\left(\frac{\xi_1}{\delta} + \frac{\xi_2 \mathrm{e}^{\beta(t-T)}}{\beta+\delta} - \frac{(\xi_1+\xi_2)\mathrm{e}^{\delta(t-T)}}{2\delta}\right) + \xi_3 \mathrm{e}^{-\delta t}, \quad (3.8)$$

where

$$\xi_1 = \frac{b\beta + a\alpha}{c\beta\delta},\ \xi_2 = \frac{a\alpha}{c\beta(\beta-\delta)},\ \xi_3 = G_0 - (p-s)\left(\frac{\xi_1}{\delta} + \frac{\xi_2}{\beta+\delta}\mathrm{e}^{-\beta T} - \frac{\xi_1+\xi_2}{2\delta}\mathrm{e}^{-\delta T}\right)\cdot$$

According to (2.2) and (3.8), the time path for reference price is calculated as

$$r(t) = p + \alpha(p-s)\left(\frac{\xi_1}{\beta\delta} + \frac{\xi_2 \mathrm{e}^{\beta(t-T)}}{2\beta(\beta+\delta)} - \frac{(\xi_1+\xi_2)\mathrm{e}^{\delta(t-T)}}{2\delta(\beta+\delta)}\right) + \frac{\alpha\xi_3 \mathrm{e}^{-\delta t}}{\beta-\delta} + \xi_4 \mathrm{e}^{-\beta t}, \quad (3.9)$$

where $\xi_4 = r_0 - p - \alpha(p-s)\left(\frac{\xi_1}{\beta\delta} + \frac{\xi_2 \mathrm{e}^{-\beta T}}{2\beta(\beta+\delta)} - \frac{(\xi_1+\xi_2)\mathrm{e}^{-\delta T}}{2\delta(\beta+\delta)}\right) - \frac{\alpha\xi_3}{\beta-\delta}$.

Substituting equations (3.8) and (3.9) into the firm's profit function in (3.1), yields

$$J = (H_1 + H_6)p^2 + (H_2 + H_3 - 2sH_6)p + H4 + s^2 H_6, \quad (3.10)$$

where

$$
\begin{aligned}
\eta_1 &= \frac{\xi_1 T}{\beta\delta} + \frac{\xi_2(1-\mathrm{e}^{-\beta T})}{2\beta^2(\beta\delta)} - \frac{(\xi_1+\xi_2)(1-\mathrm{e}^{-\delta T})}{2\delta^2(\beta+\delta)},\\
\eta_2 &= \frac{\xi_1}{\delta} + \frac{\xi_2\mathrm{e}^{-\beta T}}{\beta+\delta} - \frac{(\xi_1+\xi_2)\mathrm{e}^{-\delta T}}{2\delta},\\
\eta_3 &= \frac{\xi_1}{\beta\delta} + \frac{\xi_2\mathrm{e}^{-\beta T}}{2\beta(\beta+\delta)} - \frac{(\xi_1+\xi_2)\mathrm{e}^{-\delta T}}{2\delta(\beta+\delta)},\\
\eta_4 &= \frac{\xi_1 T}{\delta} + \frac{\xi_2(1-\mathrm{e}^{-\beta T})}{\beta(\beta+\delta)} - \frac{(\xi_1+\xi_2)(1-\mathrm{e}^{-\delta T})}{2\delta^2},\\
H_1 &= a\alpha\eta_1 - \frac{\eta_2}{\delta}\Big(\frac{a\alpha}{\beta-\delta}+b\Big)(1-\mathrm{e}^{-\delta T}) + \frac{a}{\beta}\Big(\frac{\alpha\eta_2}{\beta-\delta} - \alpha\eta_3 - 1\Big)(1-\mathrm{e}^{-\beta T}) + b\eta_4,\\
H_2 &= -2a\alpha s\eta_1 + \frac{a\alpha}{\delta(\beta-\delta)}(G_0+2s\eta_2)(1-\mathrm{e}^{-\delta T})\\
&\quad + \frac{a}{\beta}\Big(r_0 + s + 2\alpha s\eta_3 - \frac{\alpha}{\beta-\delta}(G_0+2s\eta_2)\Big)(1-\mathrm{e}^{-\beta T}),\\
H_3 &= -2bs\eta_4 + \frac{b}{\delta}(G_0+2s\eta_2)(1-\mathrm{e}^{-\delta T}) + \mathrm{d}T,\\
H_4 &= a\alpha s^2\eta_1 + bs^2\eta_4 - \frac{s}{\delta}(G_0+s\eta_2)(b+\frac{a\alpha}{\beta-\delta})(1-\mathrm{e}^{-\delta T})\\
&\quad - \frac{as}{\beta}\Big(r_0 + \alpha s\eta_3 - \frac{\alpha}{\beta-\delta}(G_0+s\eta_2)\Big) - s\mathrm{d}T,\\
H_5 &= -\frac{c}{2}\Bigg(\xi_1^2 T + \frac{\xi_2^2}{2\beta}(1-\mathrm{e}^{-2\beta T}) + \frac{(\xi_1+\xi_2)^2}{2\delta}(1-\mathrm{e}^{-2\delta T}) + \frac{2\xi_1\xi_2}{\beta}\left(1-\mathrm{e}^{-\beta T}\right)\\
&\quad - \frac{2\xi_1}{\delta}(\xi_1+\xi_2)\left(1-\mathrm{e}^{-\delta T}\right) - \frac{2\xi_2(\xi_1+\xi_2)}{\beta+\delta}\left(1-\mathrm{e}^{-(\beta+\delta)T}\right)\Bigg).
\end{aligned}
$$

Note that profit function (3.10) is quadratic about the sales price p. Thus we obtain the optimal pricing strategy in the following results.

Proposition 3.4. *When $\beta \neq \delta$, the optimal pricing strategy is*

$$p^* = \frac{2sH_6 - H_2 - H_3}{2(H_1 + H_6)}, \tag{3.11}$$

if $H_1 + H_6 < 0$, and $p^* > s$. Otherwise, the optimal sales price $p^* = s$.

Similarly, in the case $\beta = \delta$, we can also calculate the goodwill, reference price and the total profits on the basis of the given advertising strategy. The optimal sales price can be easily obtained by solving the corresponding optimization problem. Here, we will not discuss in more details about this.

4. Optimal strategies for infinite planning horizon

In this section, we explore the firm's long-term strategies. The optimization problem for the firm can be expressed as

$$\begin{aligned} \max_{u(\cdot),p} J &= \int_0^\infty \mathrm{e}^{-\rho t}\left((p-s)\left(d+a(r(t)-p)+bG(t)\right)-\frac{1}{2}cu^2(t)\right)\mathrm{d}t \\ \text{s.t.}\quad & \dot{G}(t)=u(t)-\delta G(t),\ G(0)=G_0, \\ & \dot{r}(t)=\beta(p-r(t))+\alpha G(t),\ r(0)=r_0, \\ & u(t)\geq 0, p\geq s, \end{aligned} \tag{4.1}$$

where ρ is a positive discount rate.

With a similar method presented in the previous section, we obtain the optimal advertising strategy in the following proposition.

Proposition 4.1. *For a given sales price p, the optimal advertising strategy in the infinite planning horizon is*

$$u^* = \frac{p-s}{c(\delta+\rho)}\left(b+\frac{a\alpha}{\beta+\rho}\right)\cdot \tag{4.2}$$

Note that the optimal advertising effort is a constant, making the firm easy to follow. Specifically, the advertising effort is composed of two parts: $\frac{b(p-s)}{c(\delta+\rho)}$ and $\frac{a\alpha(p-s)}{c(\delta+\rho)(\beta+\rho)}$. The first part captures the long-term effect of advertising on demand through goodwill, while the second part represents the impact of reference price effect on advertising. When $\alpha=0$, the effect of advertising on reference price disappears, resulting in the absence of the second part. Also, the situation where $a=0$ also leads to the same result. Moreover, we find that, when $\rho=0$, the optimal advertising effort in (4.2) is equivalent to that in (3.6) when $T\to\infty$, which implies that the optimal advertising strategy in the finite planning horizon is consistent with that in the infinite case when T approaches infinity.

Similarly, the optimal advertising strategy in the infinite planning horizon shares the same parameter sensitivity with that in the finite case, that is, it increases with a, b, α, but decreases with c, β, δ. Consistent with the findings of [26,36], the optimal advertising effort decreases with discount rate ρ, which indicates that a patient firm is likely to invest more in advertising, while a myopic firm is not willing to pay more investment on advertising.

Substituting (4.2) into equation (2.1), yields the time path of goodwill as follows.

$$G(t) = (G_0 - G_{ss})\mathrm{e}^{-\delta t} + G_{ss}, \tag{4.3}$$

where G_{ss} is the steady-state goodwill, calculated as

$$G_{ss} = \frac{p-s}{c\delta(\delta+\rho)}\left(b+\frac{a\alpha}{\beta+\rho}\right)\cdot \tag{4.4}$$

Then, substituting (4.3) into equation (2.2), we obtain the time path of reference price below.

$$r(t) = \frac{\alpha(G_0 - G_{ss})}{\beta - \delta}(\mathrm{e}^{-\delta t} - \mathrm{e}^{-\beta t}) + (r_0 - r_{ss})\mathrm{e}^{-\beta t} + r_{ss}, \tag{4.5}$$

where $\beta \neq \delta$, and r_{ss} is the steady-state reference price, given by

$$r_{ss} = p + \frac{\alpha G_{ss}}{\beta} \cdot \tag{4.6}$$

Note from (4.4) and (4.6) that both the steady-state goodwill and reference price increase with a, b, α, but decrease with c, β, δ, ρ, which is in line with the parameter sensitivity of the optimal advertising effort. This result is intuitive due to the fact that higher advertising effort leads to higher goodwill and consequently to higher reference price at the steady state.

Substituting (4.2), (4.3) and (4.5) into (4.1), the firms' optimal profit can be calculated as

$$J = (K_4 - K_3)p^2 + (K_2 + K_5 + sK_3)p - sK_2 + K_6, \tag{4.7}$$

where

$$K_1 = \frac{b(\beta+\rho) + a\alpha}{(\beta+\rho)(\delta+\rho)},\ K_2 = \frac{ar_0}{\beta+\rho} + K_1 G_0 + \frac{d}{\rho} + \frac{a\alpha s K_1}{\beta c\delta(\beta+\rho)} + \frac{sK_1}{c\delta}\Big(K_1 - \frac{b}{\rho}\Big),$$
$$K_3 = \frac{a}{\beta+\rho}\Big(\frac{\alpha K_1}{\beta c\delta} + 1\Big) + \frac{K_1}{c\delta}\Big(K_1 - \frac{b}{\rho}\Big),\ K_4 = \frac{a\alpha K_1}{\beta c\delta\rho} - \frac{K_1^2}{2c\rho},$$
$$K_5 = \frac{sK_1^2}{c\rho} - \frac{2a\alpha s K_1}{\beta c\delta\rho},\ K_6 = \frac{sK_1}{c\rho}\Big(\frac{a\alpha s}{\beta\delta} - \frac{sK_1}{2}\Big) \cdot$$

Obviously, the profit function (4.7) in the infinite planning horizon is quadratic with respect to sales price p, then we characterize the optimal pricing strategy in the following proposition.

Proposition 4.2. *When $\beta \neq \delta$, the optimal pricing strategy for the infinite planning horizon is*

$$p^* = \frac{K_2 + K_5 + sK_3}{2(K_3 - K_4)}, \tag{4.8}$$

if $K_4 - K_3 < 0$, and $p^* > s$. Otherwise, the optimal sales price $p^* = s$.

Likewise, when $\beta = \delta$, we can also work out the goodwill, reference price as well as profit corresponding to the optimal advertising effort presented in (4.2). As a consequence, the optimal advertising strategy can be similarly obtained by solving the optimization problem.

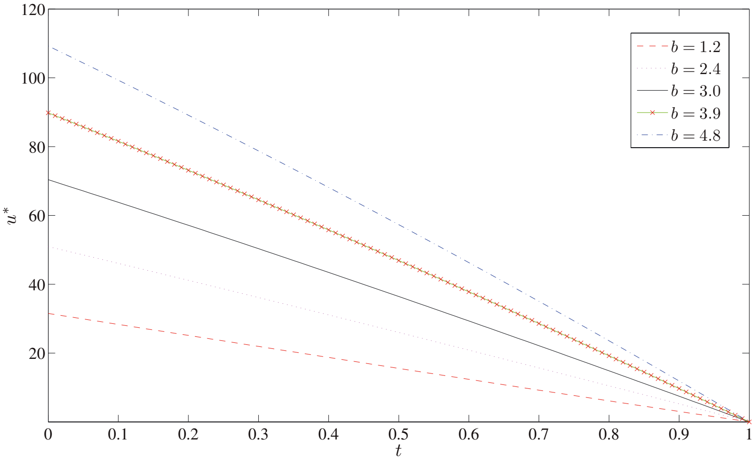

FIGURE 1. The optimal advertising effort paths with different b.

5. NUMERICAL EXAMPLES

In this section, we present three numerical examples to illustrate the theoretical results, in which Example 5.1 provides the optimal advertising strategy with the fixed sales price, and Example 5.2 explores the optimal advertising and pricing strategies for the finite planning horizon, while Example 5.3 presents the optimal advertising and pricing strategies for the infinite planning horizon.

Example 5.1. Consider the following parameters: $a = 6,\ b = 3,\ c = 0.4,\ \alpha = 0.1,\ \beta = 0.6,\ \delta = 0.3,\ T = 1,\ G_0 = 10,\ r_0 = 25,\ p = 20,\ s = 10,\ d = 100$,. We assume the sales price p is exogenous and given. Note that $\beta \neq \delta$. So the optimal advertising effort can be obtained from (3.6) according to Proposition 3.1, *i.e.*,

$$u^*(t) = 333 + 83\mathrm{e}^{0.6t-0.6} - 416\mathrm{e}^{0.3t-0.3},$$

which is displayed as the solid line in Figure 1, and the total profits $J_1^* = 1849.5$.

Next, the impact on the advertising effort from the changes of parameters are investigated. The parameters are varied by -60%, -30%, 0, $+30\%$ and $+60\%$, and when a parameter is varied, others remain unchanged. In Figures 1 and 2, the optimal advertising paths under the variations of b and δ are depicted, respectively.

As shown in Figure 1, with the increase of parameter b, the optimal advertising effort enjoys a great growth. In marketing, advertising effort has been regarded as a significant tool to build a good brand-image and attract more consumers so as to maximize the profit. In particular, goodwill is a vital contributor to drive demand in the luxury industry where b is relatively higher. As a result, it is not surprising that firms in this field are willing to invest more in advertising effort.

In Figure 2, it can be observed that the decline of parameter δ pushes an increase of advertising effort. Obviously, for the products with a high goodwill decay

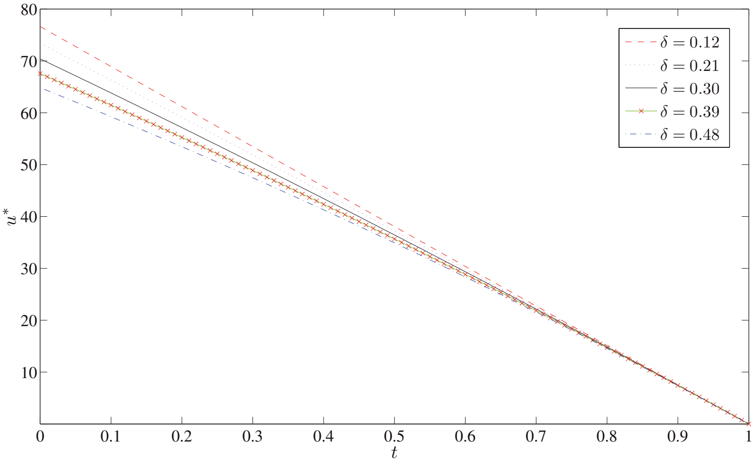

FIGURE 2. The optimal advertising effort paths with different δ.

rate, a high investment only brings about a low income, which means the advertising effort cannot significantly promote the profits. Therefore, it is not worth investing more in advertising. Conversely, products with low goodwill decay rate requires low investment but will produce a high yield. Hence, it is more profitable to have intensive advertising effort when the decay coefficient gets lower. For example, high advertising effort may generate an economical burden to a firm in the small commodities market where products with high goodwill decay rates have characteristics of homogeneity, low-cost and low margin. This is because that high costs of advertising cannot be compensated by the increased profits. Hence, there is no need to invest in advertising.

Now we take a look at the reference price. It is obvious that the reference price is mainly affected by parameters β and α. Here, Figure 3 presents the effect of the memory parameter β on the reference price. Note that the reference price in the benchmark case is plotted by solid line.

The solid line in Figure 3 displays that the reference price decreases at first, then increases for an extended period of time. That's because the sales price p is lower than the initial reference price r_0 and the initial goodwill G_0 is relatively low, which makes the negative effect of the discrepancy between p and r play a dominate role on reference price. Subsequently, with the growth of goodwill, the driving force to the increment of reference price gets much larger, so the reference price keeps sustained growth over a period of time. In our daily life, if the sales price of a product is lower than the reference price, its value is certainly degraded in our minds, but when the goodwill increases, its good brand-image would help to increase the reference price, which is consistent with our outcomes.

As shown in Figure 3, the reference price decreases with respect to β. Note that the sales price p is always lower than the reference price r, which has a negative effect on the reference price. Thus, the impact will become much greater when β

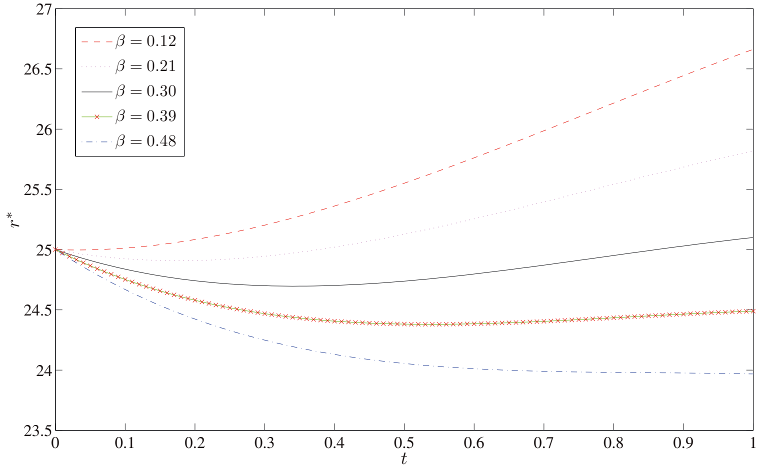

FIGURE 3. The optimal reference price paths with different β.

TABLE 1. Impacts of parameters on optimal solutions in finite case.

PV	p^*	J^*	PV	p^*	J^*
9	16.28	321.64	0.1	15.61	285.10
a 10	15.65	289.92	b 0.2	15.65	289.92
11	15.14	263.98	0.3	15.71	294.87
0.5	15.66	290.22	0.01	15.63	287.53
c 1.0	15.65	289.92	α 0.02	15.65	289.92
1.5	15.64	289.82	0.03	15.68	292.34
0.1	15.39	276.27	0.3	15.66	290.54
β 0.2	15.65	289.92	δ 0.4	15.65	289.92
0.3	15.93	304.01	0.5	15.64	289.34

Note: PV denotes parameter value.

is relatively large. Meanwhile, Corollary 3.2 has shown that a large β can cause a decreasing advertising effort. That will generate a relatively low goodwill, which also slows down the growth of reference price.

Example 5.2. Consider another parameter setting: $a = 10,\ b = 0.2,\ c = 1,\ \alpha = 0.02,\ \beta = 0.2,\ \delta = 0.4,\ T = 1,\ G_0 = 10,\ r_0 = 10,\ s = 10,\ d = 100$.

According to Propositions 3.3 and 3.4, the optimal sales price $p^* = 15.65$, and the optimal advertising effort $u^*(t) = 16.95 - 28.25\mathrm{e}^{0.2t-0.2} + 11.30\mathrm{e}^{0.4t-0.4}$. The corresponding profit $J^* = 289.92$. To investigate the effects of system parameters on optimal sales price and profit, we present the parameter sensitivities of optimal solutions in Table 1.

Note from Table 1 that the optimal sales price and profit increase with b, α, β, but decrease with a, c, δ. A large value of parameter b, meaning a high contribution

TABLE 2. Impacts of parameters on optimal solutions in infinite case.

	p^*	u^*	G_{ss}^*	r_{ss}^*	J^*
$a(9;\ 10;\ 11)$	$-$	$+$	$+$	$-$	$-$
$b(0.1;\ 0.2;\ 0.3)$	$+$	$+$	$+$	$+$	$+$
$c(0.5;\ 1;\ 1.5)$	$-$	$-$	$-$	$-$	$-$
$\alpha(0.01;\ 0.02;\ 0.03)$	$+$	$+$	$+$	$+$	$+$
$\beta(0.1;\ 0.2;\ 0.3)$	$+$	$+$	$+$	$+$	$+$
$\delta(0.3;\ 0.4;\ 0.5)$	$-$	$-$	$-$	$-$	$-$
$\rho(0.08;\ 0.1;\ 0.12)$	$-$	$-$	$-$	$-$	$-$

of goodwill to demand, allows the firm to raise price to increase marginal profit without worrying the comparatively low demand brought by the price, which consequently leads to an increase in profit. Since α captures the positive effect of goodwill on reference price, its increase will lead to the increase of reference price on consumers' mind, thus, a high price is still acceptable for consumers. The positive effect of memory parameter β on sales price indicates that the firm is supposed to increase marginal profit through a high sales price when it faces disloyal consumers to this brand. However, the negative effect of parameter a implies that it's beneficial for the firm to lower the price to boost market demand. Besides, parameter c plays negative effects on the optimal solutions. From the view of investment cost, when the cost parameter is relatively large, the firm is likely to reduce the advertising effort to lower cost, which produces the low goodwill and reference price. To improve market demand, the firm has to set a relatively low sales price. As a result, the low sales price and advertising effort create less profit for the firm. In addition, a high goodwill decay δ generates a relatively low goodwill, ultimately results in the decrease of demand. To reduce the drain on demand, the firm prefers to adopt markdown strategy.

Example 5.3. We consider T is infinity and discount rate $\rho = 0.1$, with other parameter values fixed as in Example 5.2. According to Propositions 4.1 and 4.2, we obtain the optimal sales price $p^* = 37.78$, the optimal advertising effort $u^* = 48.15$, and the corresponding profit $J^* = 14130$. Table 2 summaries the impacts of system parameters on the optimal solutions and the steady-state goodwill and reference price, in which "+" denotes a positive relationship between the parameter and the optimal solutions, whereas "$-$" stands for a negative relationship.

From Table 2, the following messages are drawn out: parameters b, α, β exert positive effects on the optimal sales price, advertising effort, profit, and the steady-state goodwill and reference price, while parameters c, δ, ρ exert negative ones. Regarding the effect of parameter a, we find that the optimal advertising effort and the steady-state goodwill increase with a, but the optimal sales price, the steady-state reference price and profit decrease with a. The reasons for this result derive from the following aspects. On one hand, a large a means a great contribution of reference to demand, which stimulates the firm to improve advertisement to increase goodwill and reference, consequently to raise market demand. On the other hand, an increase of a also results in a greater negative effect of price on demand, which prevents the firm from setting a high price. Meanwhile, the steady-state goodwill increases due to the high advertising effort. However, owing to the relatively low price, the steady-state reference price decreases. Ultimately, a low marginal profit and a high investment cost, resulting from the low price and the high advertising effort, lead to the decrease of profit.

Furthermore, the results above indicates negative impacts of discount rate ρ on the optimal solutions, suggesting that, a myopic firm, *i.e.*, a high ρ, is willing to choose the low sales price and the low advertising effort, ending up with the low steady-state goodwill and reference price, as well as the relatively low profit.

Likewise, the impacts of parameters $b, c, \alpha, \beta, \delta$ on the optimal solutions in the infinite planning horizon are the same with those in the finite case. We do not make detailed explanations any more.

6. Conclusions

Reference price, as a key factor when consumers decide whether to buy a product or not, has posed a vital effect on consumer behaviors. Taking it into account, this paper provides the optimal advertising effort for a firm to maximize its total profits. In the proposed model, we assume that the advertising effort has an indirect and positive effect on reference price through goodwill, which is not involved in previous literature. The joint pricing and dynamic advertising strategies are obtained in finite and infinite planning horizons, respectively. It is shown that, in the finite planning horizon, the optimal advertising effort has a decreasing trend until it reduces to zero at the end of the period. However, the optimal advertising effort is a constant in the infinite case, which makes the firm easy to follow. Further, some managerial insights are obtained for the firm's advertising and pricing strategies through parameter sensitivities analysis.

In this paper, the sales price is assumed time-invariant. Our main goal is to propose the optimal advertising strategy with reference price effect. However, with the development of e-commerce, as well as fiercer market competition, dynamic pricing is becoming a more and more popular strategy for a firm to pursue the maximum profit. Thus, it would be of interest to consider dynamic pricing and advertising simultaneously, but computing such optimal solution would be quite challenging. Additionally, we only consider a linear demand function. Another potential extension is to establish an dynamic advertising and pricing model with reference price effect by considering the demand rate as a general function, as proposed in Dixit and Stiglitz [7].

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Appendix

Proof of Proposition 3.1. When $\beta \neq \delta$, it can be calculated from (8) and (9) that

$$\lambda_1(t) = (p-s)\left(\frac{b\beta + a\alpha}{\delta\beta} + \mathrm{e}^{\beta(t-T)}\frac{a\alpha}{\beta(\beta-\delta)} - \mathrm{e}^{\delta(t-T)}\left(\frac{b\beta + a\alpha}{\delta\beta} + \frac{a\alpha}{\beta(\beta-\delta)}\right)\right),$$

$$\lambda_2(t) = (p-s)\frac{a}{\beta}\left(1 - \mathrm{e}^{\beta(t-T)}\right).$$

Differentiating $\lambda_1(t)$ with respect to t, yields,

$$\dot{\lambda}_1(t) = \frac{(p-s)\mathrm{e}^{\delta(t-T)}\left(a\alpha\beta\mathrm{e}^{(\beta-\delta)(t-T)} - (b\beta + a\alpha)(\beta-\delta) - a\alpha\delta\right)}{\beta(\beta-\delta)}.$$

Obviously, $\mathrm{e}^{(\beta-\delta)(t-T)} < 1$ when $\beta > \delta$, thus,

$$\begin{aligned} a\alpha\beta\mathrm{e}^{(\beta-\delta)(t-T)} - (b\beta + a\alpha)(\beta-\delta) - a\alpha\delta &< a\alpha\beta - (b\beta + a\alpha)(\beta-\delta) - a\alpha\delta \\ &= -(\beta-\delta)b\beta < 0, \end{aligned}$$

which implies $\dot{\lambda}_1(t) < 0$.

When $\beta < \delta$, it can be found that $\mathrm{e}^{(\beta-\delta)(t-T)} > 1$, so,

$$\begin{aligned} a\alpha\beta\mathrm{e}^{(\beta-\delta)(t-T)} - (b\beta + a\alpha)(\beta-\delta) - a\alpha\delta &> a\alpha\beta - (b\beta + a\alpha)(\beta-\delta) - a\alpha\delta \\ &= -(\beta-\delta)b\beta > 0, \end{aligned}$$

which means $\dot{\lambda}_1(t) < 0$. As a result, $\lambda_1(t)$ is monotonously decreasing.

Proof of Corollary 3.2. Here, we omit the time argument and superscript for simplicity.

1) Since $u = \frac{\lambda_1}{c}$ and $\dot{\lambda}_1 < 0$, it is easy to find that $\dot{u} < 0$. Furthermore, the second-order derivations of u with respect to t is calculated in the following:

$$\ddot{u} = \frac{\ddot{\lambda}_1}{c} = \frac{(p-s)\mathrm{e}^{\delta(t-T)}}{c\beta(\beta-\delta)}\left(a\alpha\beta^2\mathrm{e}^{(\beta-\delta)(t-T)} - \delta(b\beta + a\alpha)(\beta-\delta) - a\alpha\delta^2\right),$$

when $\beta > \delta$, it can be generated that $\mathrm{e}^{(\beta-\delta)(t-T)} < 1$, and

$$\begin{aligned} a\alpha\beta^2\mathrm{e}^{(\beta-\delta)(t-T)} - \delta(b\beta + a\alpha)(\beta-\delta) - a\alpha\delta^2 &< a\alpha\beta^2 - \delta(b\beta + a\alpha)(\beta-\delta) - a\alpha\delta^2 \\ &= \beta(\beta-\delta)(a\alpha - b\delta), \end{aligned} \tag{A.1}$$

thus,

$$\ddot{u} < (p-s)\mathrm{e}^{\delta(t-T)}(a\alpha - b\delta),$$

therefore, $\ddot{u} < 0$ when $\beta > \delta$ and $a\alpha < b\delta$.

When $\beta < \delta$, it can be seen that $\mathrm{e}^{(\beta-\delta)(t-T)} > 1$, and

$$a\alpha\beta^2\mathrm{e}^{(\beta-\delta)(t-T)} - \delta(b\beta + a\alpha)(\beta-\delta) - a\alpha\delta^2 > \beta(\beta-\delta)(a\alpha - b\delta),$$

thus,

$$\ddot{u} < (p-s)\mathrm{e}^{\delta(t-T)}(a\alpha - b\delta),$$

therefore, $\ddot{u} < 0$ when $\beta < \delta$ and $a\alpha < b\delta$.

In conclusion, u is a concave decreasing function if $a\alpha < b\delta$.

2) Differentiating u with respect to parameter b yields

$$\frac{\partial u}{\partial b} = \frac{1}{c\delta}\left(1 - \mathrm{e}^{\delta(t-T)}\right).$$

Obviously $\frac{\partial u}{\partial b} > 0$, which indicates u is increasing in b. In addition, u is inversely proportional to parameter c, therefore, u is decreasing in c.

3) Differentiating u with respect to a, α, β and δ respectively, yields

$$\frac{\partial u}{\partial a} = \frac{(p-s)}{c\beta}\left(\frac{\alpha}{\delta}(1-\mathrm{e}^{\delta(t-T)}) + \frac{\alpha}{\beta-\delta}\left(\mathrm{e}^{\beta(t-T)} - \mathrm{e}^{\delta(t-T)}\right)\right), \tag{A.2}$$

$$\frac{\partial u}{\partial \alpha} = \frac{(p-s)}{c\beta}\left(\frac{a}{\delta}(1-\mathrm{e}^{\delta(t-T)}) + \frac{a}{\beta-\delta}\left(\mathrm{e}^{\beta(t-T)} - \mathrm{e}^{\delta(t-T)}\right)\right), \tag{A.3}$$

$$\frac{\partial u}{\partial \beta} = (p-s)\frac{a\alpha}{c\beta}\left(\frac{\mathrm{e}^{\delta(t-T)}-1}{\delta\beta} + \frac{\mathrm{e}^{\delta(t-T)} - \mathrm{e}^{\beta(t-T)}}{\beta(\beta-\delta)^2}(2\beta-\delta) + \frac{\mathrm{e}^{\beta(t-T)}(t-T)}{\beta-\delta}\right), \tag{A.4}$$

$$\frac{\partial u}{\partial \delta} = \frac{(p-s)}{c\beta}\left(\frac{(b\beta+a\alpha)}{\delta^2}\left(\mathrm{e}^{\delta(t-T)}-1\right) + \frac{a\alpha}{(\beta-\delta)^2}\left(\mathrm{e}^{\beta(t-T)} - \mathrm{e}^{\delta(t-T)}\right)\right. \tag{A.5}$$
$$\left. - \left(\frac{b\beta+a\alpha}{\delta} + \frac{a\alpha}{\beta-\delta}\right)\mathrm{e}^{\delta(t-T)}(t-T)\right).$$

For (A.2), according to the differential mean value theorem, we get that $1 - \mathrm{e}^{\delta(t-T)} = -\delta(t-T)\mathrm{e}^{\mu_1(t-T)}$ and $\mathrm{e}^{\beta(t-T)} - \mathrm{e}^{\delta(t-T)} = (\beta-\delta)(t-T)\mathrm{e}^{\mu_2(t-T)}$, where $\mu_1 \in (0,\delta)$ and $\mu_2 \in (\beta,\delta)$ if $\delta > \beta$, otherwise, $\mu_2 \in (\delta,\beta)$. Consequently,

$$\frac{\partial u}{\partial a} = \frac{(p-s)\alpha(t-T)}{c\beta}\left(\mathrm{e}^{\mu_2(t-T)} - \mathrm{e}^{\mu_1(t-T)}\right).$$

If $\beta > \delta$, it is easy to see that $\mu_2 > \mu_1$ and $\mathrm{e}^{\mu_2(t-T)} < \mathrm{e}^{\mu_1(t-T)}$, thus $\frac{\partial u}{\partial a} > 0$, namely, u is increasing in a if $\beta > \delta$.

For (A.3), by virtue of the method above, we can also get that $\frac{\partial u}{\partial \alpha} > 0$ if $\beta > \delta$, namely, u is increasing in α if $\beta > \delta$.

For (A.4), it is easy to verify that $\frac{\partial u}{\partial \beta} < 0$ if $\beta > \delta$.

For (A.5), if $\beta > \delta$, it is easy to find that $\frac{\partial u}{\partial \delta} < 0$, in a word, u is decreasing in δ if $\beta > \delta$.

Proof of Proposition 3.3.

When $\beta = \delta$, adjoint variable λ_1 can be rewritten as

$$\lambda_1(t) = (p-s)\left(\frac{b\beta + a\alpha + a\alpha\beta\mathrm{e}^{\beta(t-T)}t - (b\beta + a\alpha + a\alpha\beta T)\mathrm{e}^{\beta(t-T)}}{\beta^2}\right)\cdot \tag{A.6}$$

Differentiating (A.6) with respect to t, yields

$$\dot{\lambda}_1(t) = (p-s)\mathrm{e}^{\beta(t-T)}\left(a\alpha\beta(t-T)-b\right) < 0.$$

Hence, $\lambda_1(t)$ is monotonously decreasing when $\beta = \delta$.

Proof of Proposition 4.1.

When the planning horizon is infinite, the Hamiltonian function is the same as (3.2), and the optimal conditions are as follows

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u = \frac{\lambda_1}{c}, \tag{A.7}$$

and

$$\begin{aligned} \dot{\lambda}_1(t) &= \rho\lambda_1 - \frac{\partial H}{\partial G}, \quad \lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_1(t) = 0, \\ \dot{\lambda}_2(t) &= \rho\lambda_2 - \frac{\partial H}{\partial r}, \quad \lim_{t\to\infty} \mathrm{e}^{-\rho t}\lambda_1(t) = 0. \end{aligned} \tag{A.8}$$

Solving the equation (A.8), yields

$$\begin{aligned} \lambda_1 &= (p-s)\left(\frac{b}{\delta+\rho} + \frac{a\alpha}{(\beta+\rho)(\delta+\rho)}\right), \\ \lambda_2 &= (p-s)\frac{a}{\beta+\rho}\cdot \end{aligned}$$

Therefore, the optimal advertising strategy is a constant given in (4.2).

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