

POSITIVITY OF THE DENSITY FOR THE STOCHASTIC WAVE EQUATION IN TWO SPATIAL DIMENSIONS^{*,**}

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Abstract. We consider the random vector $u(t, \underline{x}) = (u(t, x_1), \dots, u(t, x_d))$, where $t > 0$, x_1, \dots, x_d are distinct points of \mathbb{R}^2 and u denotes the stochastic process solution to a stochastic wave equation driven by a noise white in time and correlated in space. In a recent paper by Millet and Sanz–Solé [10], sufficient conditions are given ensuring existence and smoothness of density for $u(t, \underline{x})$. We study here the positivity of such density. Using techniques developed in [1] (see also [9]) based on Analysis on an abstract Wiener space, we characterize the set of points $y \in \mathbb{R}^d$ where the density is positive and we prove that, under suitable assumptions, this set is \mathbb{R}^d .

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1. INTRODUCTION

Consider the stochastic wave equation with two-dimensional spatial variable

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - \Delta \right) u(t, x) &= \sigma(u(t, x)) F(dt, dx) + b(u(t, x)), \\ u(0, x) &= u_0(x), \\ \frac{\partial u}{\partial t}(0, x) &= v_0(x) \end{aligned} \tag{1.1}$$

$(t, x) \in [0, \infty[\times \mathbb{R}^2$.

The coefficients σ and b are \mathcal{C}^∞ functions with bounded derivatives of any order $i \geq 1$. The noise $F(t, x)$ is supposed to be a martingale measure (in the sense given by Walsh in [16]) obtained as the extension of a centered Gaussian field $\{F(\varphi), \varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^2)\}$ with covariance

$$E(F(\varphi) F(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \varphi(t, x) f(|x - y|) \psi(t, y). \tag{1.2}$$

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Assume that $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous on $]0, \infty[$. Moreover, we suppose that the measure $\Gamma(dx) = f(x)dx$ on \mathbb{R}^2 is a nonnegative tempered distribution. The initial conditions u_0, v_0 , are real functions satisfying some conditions that will be determined later.

We give a meaning to the formal expression (1.2) using its mild formulation, as follows. Let \mathcal{H} be the completion of the inner-product space of measurable functions $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy |\varphi(x)| f(|x-y|) |\varphi(y)| < \infty$, endowed with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \varphi(x) f(|x-y|) \psi(y)$. Denote by $\{e_j, j \geq 0\}$ a CONS of functions of \mathcal{H} . Then $\{W_j(t) = \int_0^t \int_{\mathbb{R}^2} e_j(x) F(ds, dx), j \geq 0\}$ is a sequence of independent standard Brownian motions.

Let $S(t, x)$, $(t, x) \in [0, \infty[\times \mathbb{R}^2$, be the fundamental solution of $(\frac{\partial^2}{\partial t^2} - \Delta)g = 0$. That is,

$$S(t, x) = \frac{1}{2\pi} (t^2 - |x|^2)^{-1/2} 1_{\{|x| < t\}}.$$

Set

$$X^0(t, x) = \int_{\mathbb{R}^2} S(t, x-y) v_0(y) dy + \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^2} S(t, x-y) u_0(y) dy \right).$$

By a solution of equation (1.2) we mean a stochastic process $\{u(t, x), (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2\}$ satisfying

$$\begin{aligned} u(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \langle S(t-s, x-*) \sigma(u(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(u(s, y)) ds dy. \end{aligned} \tag{1.3}$$

In [10] we studied the existence and uniqueness of solution of (1.3) (see also [4]). We also addressed the smoothness of $u(t, x)$, at a fixed point $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, in the sense of Malliavin Calculus, and the existence and smoothness of density for the probability law on \mathbb{R}^d of

$$u(t, \underline{x}) := (u(t, x_1), \dots, u(t, x_d)),$$

with $t > 0$ and x_1, \dots, x_d distinct points of \mathbb{R}^2 .

This last result has been obtained under the following set of assumptions

- (i) there exist $a_1 \geq a_2 > 0$ such that $2(1+a_2)(a_1-a_2) < a_2 \leq a_1 < 2$, positive constants C_1 and C_2 such that for $t \in [0, T]$,

$$C_1 t^{a_1} \leq \int_0^t y f(y) \ln \left(1 + \frac{t}{y} \right) dy \leq C_2 t^{a_2};$$

- (ii) $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 , bounded, with $a_2/2(1+a_2)$ -Hölder continuous partial derivatives, $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ and there exists $q_0 \in]4, +\infty]$ such that $|v_0| + |\nabla u_0| \in L^{q_0}(\mathbb{R}^2)$;
 (iii) σ and b are \mathcal{C}^∞ with bounded derivatives of any order $i \geq 1$;
 (iv) there exists $a > 0$ such that $|\sigma(u(t, x_j))| \geq a$, for any $j = 1, \dots, d$, a.s.

These conditions are satisfied for instance by the function $f(x) = x^{-\alpha}$, $0 < \alpha < 2$, with $a_1 = a_2 = 2 - \alpha$.

In Millet and Morien in [8] there is a slight improvement of the previous result. These authors show that in the above-quoted set of hypothesis, assumptions (i) and (ii) can be replaced by the weaker ones:

- (i') there exist $0 < a_2 \leq a_1 < 2$ such that $2(a_1 - a_2) < a_2 \wedge 1$, positive constant C_1 such that for $t \in [0, T]$,

$$C_1 t^{a_1} \leq \int_0^t y f(y) \ln \left(1 + \frac{t}{y} \right) dy \tag{1.4}$$

and

$$\int_{0^+} y^{1-a_2} f(y) dy < \infty; \quad (1.5)$$

(ii') $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 , bounded, with $\frac{1}{2}(a_2 \wedge 1)$ -Hölder continuous partial derivatives, $\nabla u_0 \in L^{q_1}(\mathbb{R}^2)$ for some $q_1 > 2$; $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ belongs to $L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in [4 \vee \frac{2}{1-(a_2 \wedge 1)}, \infty]$.

Recently, in [6] a related problem has been studied. The results apply in particular to equations like (1.2) driven by a centered Gaussian noise with covariance given by

$$E(F(\varphi) F(\psi)) = \int_{\mathbb{R}_+} dt \int_{\mathbb{R}^2} \Gamma(dx) (\varphi(s, \cdot) * \tilde{\psi}(s, \cdot))(x), \quad (1.6)$$

where Γ is a non-negative, non-negative definite tempered measure, the symbol $*$ means the convolution and $\tilde{\psi}(s, x) = \psi(s, -x)$. Notice that if $\Gamma(dx) = f(|x|)dx$ then (1.6) reduces to (1.2). It is proved that for $d = 1$ the following set of assumptions yield the existence and smoothness of density for the law of $u(t, x)$, with $t > 0$ and $x \in \mathbb{R}^2$: the preceding condition (iii)

(iv') there exists $C > 0$ such that $\inf\{|\sigma(z)|, z \in \mathbb{R}\} \geq C$;

(v) there exists $\eta \in (0, \frac{3}{4})$ such that

$$\int_{\mathbb{R}^2} \frac{\mu(d\xi)}{(1 + |\xi|^2)^\eta} < \infty, \quad (1.7)$$

where μ is the spectral measure of Γ .

In the particular case $\Gamma(dx) = f(|x|)dx$ the above condition (1.7) implies property (1.5) of (i') with $a_2 = 2(1 - \eta) \in (\frac{1}{2}, 2)$ (see for instance [5, 15]).

Denote by $p_{t, \underline{x}}(y)$ the density of $u(t, \underline{x})$ at $y \in \mathbb{R}^d$. The purpose of this paper is to prove the next statement.

Theorem 1. *Assume that the conditions (i', ii', iii) and (iv') hold. Suppose in addition that the functional*

$$J(\varphi, \psi) = \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy \varphi(x) f(|x - y|) \psi(y), \quad \varphi, \psi \in \mathcal{D}(\mathbb{R}^2), \quad (1.8)$$

is positive. Then, for any $y \in \mathbb{R}^d$, $p_{t, \underline{x}}(y)$ is strictly positive.

In the remaining of this section we give a brief description of the method we have used to approach this problem.

Let $T > 0$ be fixed. The reproducing kernel Hilbert space of the Gaussian process $\{(W_j(t))_{j \geq 0}, t \in [0, T]\}$ is the set H_T of functions $h : [0, T] \rightarrow \mathbb{R}^N$ such that $\sum_{j \geq 0} \int_0^T |h_j(s)|^2 ds < \infty$. For any $h \in H_T$, let $\{\Phi^h(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$ be the solution of

$$\begin{aligned} \Phi^h(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \langle S(t-s, x - *) \sigma(\Phi^h(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \\ &+ \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(\Phi^h(s, y)) ds dy. \end{aligned} \quad (1.9)$$

This process is called the skeleton of $\{u(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$. The function $h \in H_T \rightarrow \Phi^h(t, x) \in \mathbb{R}$ is Fréchet differentiable (see the Appendix). Set $\Phi^h(t, \underline{x}) = (\Phi^h(t, x_1), \dots, \Phi^h(t, x_d))$. Denote by $\gamma_{\Phi^h(t, \underline{x})}$ the $d \times d$ matrix whose entries are $\langle \bar{D}\Phi^h(t, x_i), \bar{D}\Phi^h(t, x_j) \rangle_{H_T}, i, j = 1, \dots, d$, where \bar{D} means the Fréchet derivative operator. It is called the deterministic Malliavin matrix.

Then we proceed in two steps:

Step 1. Assume (i', ii', iii) and (iv). We prove the equivalence between the next two properties on $y \in \mathbb{R}^d$: (a) $p_{t,\underline{x}}(y) > 0$ and (b) there exists $h \in H_T$ such that $\Phi^h(t, \underline{x}) = y$ and $\det \gamma_{\Phi^h(t, \underline{x})} > 0$.

Step 2. Suppose (i', ii', iii) and (iv'). Then, if the functional $J(\varphi, \psi)$ defined in (1.8) is positive, any $y \in \mathbb{R}^d$ satisfies (b).

For diffusion processes satisfying some non-degeneracy requirements the equivalence between the analogue of properties (a, b) above has been proved in [3].

A characterization of the points of positive density, analogous to the equivalence between (a) and (b), for functionals defined in an abstract Wiener space has been developed in [1] and then applied to diffusions.

A similar general setting, more close to the ideas of [3], has been presented in [14]. These abstract formulations allow to analyze many interesting examples in the infinite dimensional case, like solutions of stochastic partial differential equations (see, for instance [2, 9]).

We achieve the goal quoted in Step 1 proving first a weaker (localized) version of the criterium given in [14]. Step 2 requires to solve an inverse problem and also requires a careful analysis of the matrix $\gamma_{\Phi^h(t, \underline{x})}$. The positivity of the functional $J(\varphi, \psi)$ is used in the study of the first question, the second one is carried out by exploiting assumption (i').

The programme of the paper is as follows. In Section 2 we precise the tools used in the above-mentioned Step 1. For the proof of (b) \Rightarrow (a) we use Proposition 4.2.2 in [14]. The proof of (a) \Rightarrow (b) needs the weaker version of Proposition 4.2.1 in [14] stated as Proposition 2. Section 3 is devoted to complete Step 1. We give an approximation result on a class of evolution equations which includes (1.3) and (1.9). This ensures the validity of the hypothesis needed to apply the criterium established in Section 2; but it has its own interest. Finally, we devote Section 4 to the proof of Step 2.

2. POINTS OF POSITIVE DENSITY OF FUNCTIONALS DEFINED ON AN ABSTRACT WIENER SPACE

We devote this section to set up the method of the proof of the first step of Theorem 1 in Section 1. We follow the approach of [14]; however some modifications are needed.

For the sake of understanding we start by giving some basic notions and facts on Malliavin Calculus and refer the reader to [13, 14] for a complete presentation of this topic.

Let (Ω, H, P) be an abstract Wiener space. For any $h \in H$ we denote by $W(h)$ the Itô-Wiener integral. Let \mathcal{S} be the class of cylindrical Wiener functionals, that is, the set of random vectors of the form

$$F = f(W(h_1), \dots, W(h_n)), \quad (2.1)$$

with $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$, $h_1, \dots, h_n \in H$. For F as in (2.1) the Malliavin derivative is the H -valued random variable defined by

$$DF = \sum_{i=1}^n \frac{\partial f}{\partial x_i} (W(h_1), \dots, W(h_n)) h_i.$$

For any integer $k \geq 1$ and any real number $p \in [1, \infty)$ we define $\mathbb{D}^{k,p}$ as the completion of \mathcal{S} with respect to the norm

$$\|F\|_{k,p} = \left(E(|F|^p) + \sum_{j=1}^k E(\|D^j F\|_{H^{\otimes j}}^p) \right)^{1/p}.$$

Suppose that $F = (F^1, \dots, F^d)$ is a random vector whose components belong to $\mathbb{D}^{1,2}$. The Malliavin matrix of F is the $d \times d$ matrix with entries $\langle DF^i, DF^j \rangle_H$, $i, j = 1, \dots, d$; it is denoted by γ_F .

Set $\mathbb{D}^\infty = \cap_{k,p} \mathbb{D}^{k,p}$. A random vector F is nondegenerate if $F \in \mathbb{D}^\infty(\mathbb{R}^d)$, that means, all the components of F belong to \mathbb{D}^∞ , and moreover $\det(\gamma_F)^{-1} > 0$. It is well-known that the law of a nondegenerate random vector has a \mathcal{C}^∞ density with respect to the Lebesgue measure.

Let $\Phi : H \rightarrow \mathbb{R}^d$ be Fréchet differentiable. By analogy with the random case we define $\gamma_{\Phi(h)}$ as the matrix with entries $\langle \bar{D}\Phi^i(h), \bar{D}\Phi^j(h) \rangle_H$, $i, j = 1, \dots, d$; it is called (after Bismut) the deterministic Malliavin matrix.

For any $y \in \mathbb{R}^d$ we consider the following properties:

- (A) the density p of the random vector F is strictly positive at y ;
- (B) there exists $h \in H$ with $\Phi(h) = y$ and $\det \gamma_{\Phi(h)} > 0$.

For elements $h_1, \dots, h_d \in H$, $z \in \mathbb{R}^d$, set $\underline{h} = (h_1, \dots, h_d)$ and define

$$(T_z W)(h) = W(h) + \sum_{j=1}^d z_j \langle h, h_j \rangle_H, \quad h \in H.$$

Moreover, for $p \in [1, \infty)$, $k \geq 0$, set

$$R_{\underline{h}, k, p} F = \int_{\{|z| \leq 1\}} \|(D^k F)(T_z W)\|_{H^{\otimes k}}^p dz.$$

We next quote Proposition 4.2.2 of [14].

Proposition 1. *Let F be a nondegenerate random vector, $\Phi : H \rightarrow \mathbb{R}^d$ be a Fréchet continuously differentiable mapping. Fix $h \in H$ and assume that there exists a sequence of measurable, absolutely continuous transformations $T_n^h : \Omega \rightarrow \Omega$, $n \geq 0$, such that, for every $\varepsilon > 0$, $k = 0, 1, 2, 3$ and some $p > d$,*

$$\lim_{n \rightarrow \infty} P \left\{ |F \circ T_n^h - \Phi(h)| > \varepsilon \right\} = 0, \quad (2.2)$$

$$\lim_{n \rightarrow \infty} P \left\{ \|(DF) \circ T_n^h - (\bar{D}\Phi)(h)\|_H > \varepsilon \right\} = 0, \quad (2.3)$$

$$\lim_{M \rightarrow \infty} \sup_n P \left\{ (R_{\bar{D}\Phi(h), k, p} F) \circ T_n^h > M \right\} = 0. \quad (2.4)$$

Then (B) implies (A).

In order to set up the conditions ensuring (A) \Rightarrow (B) we need a localized version of Proposition 4.2.1 in [14]. In fact, the Wiener functional $u(t, \underline{x})$ does not satisfy the convergence assumption needed to apply this proposition. A particular localization on Ω is required. This leads to a convergence in probability on $\mathbb{D}^\infty(\mathbb{R}^d)$.

Let $(H_n)_{n \geq 1}$ be an increasing sequence of finite dimensional subspaces of H such that $\cup_{n \geq 1} H_n$ is dense in H . Let $W^n : \Omega \rightarrow H_n$ be a sequence of random variables belonging to \mathbb{D}^∞ . We introduce a localizing sequence, as follows. Let

$$A_n^\gamma = \left\{ \omega : \|W^n(\omega)\|_H^2 \leq \gamma C(n) \right\}, \quad n \in \mathbb{N}, \quad \gamma \in (0, \infty),$$

where $\{C(n), n \geq 1\}$ is an increasing sequence such that

$$\lim_{n \rightarrow \infty} P \left((A_n^{\gamma_0})^c \right) = 0,$$

for some $\gamma_0 > 0$.

Let $\Psi^\gamma : \mathbb{R}_+ \rightarrow [0, 1]$ be a C^∞ function with bounded derivatives of any order, such that

$$\Psi^\gamma(x) = \begin{cases} 1, & \text{if } x < \gamma \\ 0, & \text{if } x > 2\gamma. \end{cases} \quad (2.5)$$

Set $G^{\gamma,n}(\omega) = \Psi^\gamma\left(\frac{\|W^n(\omega)\|_H^2}{C(n)}\right)$. Notice that

$$\mathbb{1}_{A_n^\gamma} \leq G^{\gamma,n} \leq \mathbb{1}_{A_n^{2\gamma}}. \quad (2.6)$$

Assume that $G^{\gamma,n} \in \mathbb{D}^\infty$ uniformly in n . That means, for all integer $k \geq 1$ and $p \in [1, \infty)$,

$$\sup_{n \geq 1} \|G^{\gamma,n}\|_{k,p} < +\infty. \quad (2.7)$$

Proposition 2. *Let $F : \Omega \rightarrow \mathbb{R}^d$ be a nondegenerate random vector, $\Phi : H \rightarrow \mathbb{R}^d$ be infinitely Fréchet differentiable such that*

(H1) $\Phi(W^n) \in \mathbb{D}^\infty(\mathbb{R}^d)$, for any $n \geq 1$, and

$$\lim_{n \rightarrow \infty} E\left(\|D^k(\Phi(W^n) - F)\|_{H_T^{\otimes k}}^p \mathbb{1}_{A_n^\gamma}\right) = 0,$$

for any $k \geq 1$, $p \in [1, \infty)$, $\gamma \geq \gamma_0$.

Then, for each $y \in \mathbb{R}^d$, (A) implies (B).

Proof. The arguments are similar to those of Proposition 4.2.1 [14] (see also [3] and [1]) with an additional ingredient of localization.

Let f be any continuous positive function with compact support $[a^0, b^0]$ containing y ; then

$$c := \mathbb{E}(f(F)) > 0.$$

Fix $\gamma \geq \gamma_0$, $\varepsilon \in]0, c[$; let $n_0 \in \mathbb{N}$ be such that $\mathbb{P}\left((A_n^\gamma)^c\right) < \frac{\varepsilon}{\|f\|_\infty}$ for any $n \geq n_0$. Then, $0 < \mathbb{E}(f(F)) \leq \mathbb{E}\left(f(F)\mathbb{1}_{A_n^\gamma}\right) + \varepsilon$ and consequently, $\mathbb{E}\left(f(F)\mathbb{1}_{A_n^\gamma}\right) > 0$, for any $n \geq n_0$. Therefore (2.6) implies that $\mathbb{E}\left(f(F)G^{\gamma,n}\right) > 0$ for $\gamma > \gamma_0$, $n \geq n_0$.

For every $M \geq 1$, let $\alpha_M \in C_b^\infty(\mathbb{R})$ be such that $0 \leq \alpha_M \leq 1$, $\alpha_M(x) = 0$ if $|x| \leq \frac{1}{M}$ and $\alpha_M(x) = 1$ if $|x| \geq \frac{2}{M}$. Since F is nondegenerate we have that $\lim_{M \rightarrow +\infty} \alpha_M(\det \gamma_F) = 1$, a.s. Consequently, $0 < \mathbb{E}\left(f(F)G^{\gamma,n}\right) = \lim_{M \rightarrow +\infty} \mathbb{E}\left(f(F)G^{\gamma,n}\alpha_M(\det \gamma_F)\right)$ and there exists a positive integer M such that $\mathbb{E}\left(f(F)G^{\gamma,n}\alpha_M(\det \gamma_F)\right) > 0$.

We want to prove that for $\gamma \geq \gamma_0$

$$\lim_{n \rightarrow +\infty} \left| \mathbb{E}\left(\left(f(F)\alpha_M(\det \gamma_F) - f(\Phi(W^n))\alpha_M(\det \gamma_{\Phi(W^n)})\right)G^{\gamma,n}\right) \right| = 0. \quad (2.8)$$

This will imply the existence of a positive integer n_0 such that

$$\mathbb{E}\left(f(\Phi(W^n))G^{\gamma,n}\alpha_M(\det \gamma_{\Phi(W^n)})\right) > 0, \quad (2.9)$$

for any $n \geq n_0$. Let $\bar{f}(x_1, \dots, x_d) = \int_{a_1^0}^{x_1} \dots \int_{a_d^0}^{x_d} f(u_1, \dots, u_d) du_1 \dots du_d$, $a^0 = (a_1^0, \dots, a_d^0)$.

The integration by parts formula of the Malliavin Calculus implies that

$$\begin{aligned} T_n &:= \mathbb{E} \left[(f(F) \alpha_M(\det \gamma_F) - f(\Phi(W^n)) \alpha_M(\det \gamma_{\Phi(W^n)})) G^{\gamma, n} \right] \\ &= \mathbb{E} \left[\bar{f}(F) H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F)) \right. \\ &\quad \left. - \bar{f}(\Phi(W^n)) H_{\{1, \dots, d\}}(\Phi(W^n), G^{\gamma, n} \alpha_M(\det \gamma_{\Phi(W^n)})) \right], \end{aligned}$$

where, for any multiindex $\alpha = (\alpha_1, \dots, \alpha_k) \in \{1, \dots, d\}^k$, H_α are random variables defined recursively as follows. If U is a nondegenerate functional and $V \in \mathbb{D}^\infty$ then $H_{\{i\}}(U, V) = \sum_{j=1}^d \delta(V(\gamma_U^{-1})^{ij} D U^j)$ and $H_\alpha(U, V) = H_{\{\alpha_k\}}(U, H_{\{\alpha_1, \dots, \alpha_{k-1}\}}(U, V))$, where δ is the adjoint operator of D which is, as D , a local operator. δ is also called the Skorohod integral.

Since $G^{\gamma, n} = 0$ on $(A_n^{2\gamma})^c$,

$$H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F)) = \mathbb{1}_{A_n^{2\gamma}} H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F))$$

and

$$H_{\{1, \dots, d\}}(\Phi(W^n), G^{\gamma, n} \alpha_M(\det \gamma_{\Phi(W^n)})) = \mathbb{1}_{A_n^{2\gamma}} H_{\{1, \dots, d\}}(\Phi(W^n), G^{\gamma, n} \alpha_M(\det \gamma_{\Phi(W^n)})).$$

Consequently $|T_n| \leq T_n^1 + T_n^2$ with

$$T_n^1 = \left| \mathbb{E} \left((\bar{f}(F) - \bar{f}(\Phi(W^n))) \mathbb{1}_{A_n^{2\gamma}} H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F)) \right) \right|$$

and

$$\begin{aligned} T_n^2 &= \left| \mathbb{E} \left(\bar{f}(\Phi(W^n)) \mathbb{1}_{A_n^{2\gamma}} \left[H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F)) \right. \right. \right. \\ &\quad \left. \left. \left. - H_{\{1, \dots, d\}}(\Phi(W^n), G^{\gamma, n} \alpha_M(\det \gamma_{\Phi(W^n)})) \right] \right) \right|. \end{aligned}$$

Let us first check that $\lim_{n \rightarrow +\infty} T_n^1 = 0$. Indeed, $T_n^1 \leq T_n^{1,1} \times T_n^{1,2}$ where

$$\begin{aligned} T_n^{1,1} &= \left(\mathbb{E} (|\bar{f}(F) - \bar{f}(\Phi(W^n))|^p \mathbb{1}_{A_n^{2\gamma}}) \right)^{1/p} \\ T_n^{1,2} &= \left\| H_{\{1, \dots, d\}}(F, G^{\gamma, n} \alpha_M(\det \gamma_F)) \right\|_{L^q(\Omega)} \end{aligned}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Assumption (H1) implies that

$$\overline{\lim}_n T_n^{1,1} \leq \|f\|_\infty \lim_{n \rightarrow +\infty} \left(\mathbb{E} (|F - \Phi(W^n)|^p \mathbb{1}_{A_n^{2\gamma}}) \right)^{1/p} = 0.$$

Moreover, $\sup_n T_n^{1,2} < +\infty$. Indeed, Proposition 3.2.2 in [14] yields:

$$T_n^{1,2} \leq C(q, d) \left(|\gamma_F^{-1}|_k^a \|F\|_{b,c}^{a'} \|G^{\gamma, n} \alpha_M(\det \gamma_F)\|_{b',c'}^{a''} \right),$$

for some positive real numbers $k, a, c, a', c', a'' > 1$ and positive integers b, b' .

The properties on $G^{\gamma,n}$ imply that $\sup_n T_n^{1,2} < +\infty$. We now prove that $\lim_{n \rightarrow +\infty} T_n^2 = 0$. For any $p > 1$,

$$T_n^2 \leq \|\bar{f}\|_\infty \mathbb{E} \left| \mathbb{1}_{A_n^{2\gamma}} \left[H_{\{1,\dots,d\}}(F, G^{\gamma,n} \alpha_M(\det \gamma_F)) - H_{\{1,\dots,d\}}(\Phi(W^n), G^{\gamma,n} \alpha_M(\det \gamma_{\Phi(W^n)})) \right] \right|.$$

The L^p -inequalities for the Skorohod integral yield

$$T_n^2 \leq C(d) \|\bar{f}\|_\infty \mathbb{E} \left| \mathbb{1}_{A_n^{2\gamma}} \left(\|F - \Phi(W^n)\|_{e,r} + \|G^{\gamma,n}(\alpha_M(\det \gamma_F)) - \alpha_M(\det \gamma_{\Phi(W^n)})\|_{e',r'} \right) \right|,$$

for some $r, r' > 1$, $e, e' \in \mathbb{N}$ (see [12] for the one dimensional case).

Thus $\lim_{n \rightarrow \infty} T_n^2 = 0$. This finishes the proof of (2.8) and therefore that of (2.9).

Next we consider a function $\beta_K : \mathbb{R} \rightarrow [0, 1]$, \mathcal{C}^∞ , such that $\beta_K(x) = 1$ if $|x| \leq K$ and $\beta_K(x) = 0$ if $|x| \leq K+1$. Since $\|W^n\|_H^2$ is finite a.s., it is clear that for any fixed $n \geq 1$, $\beta_K(\|W^n\|_H^2)$ converges to 1 a.s. as $K \rightarrow \infty$.

Notice that since the functions f and α_M are positive and the random variable $G^{\gamma,n}$ is positive and bounded by 1, the inequality (2.9) implies that for any $n \geq n_0$,

$$\mathbb{E} \left(f(\Phi(W^n)) \alpha_M(\det \gamma_{\Phi(W^n)}) \right) > 0. \quad (2.10)$$

Thus, for any fixed $n \geq n_0$ there exists $K_0(n)$ such that for any $K \geq K_0(n)$,

$$\mathbb{E} \left(f(\Phi(W^n)) \alpha_M(\det \gamma_{\Phi(W^n)}) \beta_K(\|W^n\|_H^2) \right) > 0. \quad (2.11)$$

This yields, for any $K \geq K_0(n)$ and any $\varepsilon > 0$

$$\mathbb{P} \left(|\Phi(W^n) - y| < \varepsilon, |\det \gamma_{\Phi(W^n)}| \geq \frac{1}{M}, \|W^n\|_H^2 \leq K+1 \right) > 0. \quad (2.12)$$

Indeed, otherwise, if for some $K \geq K_0(n)$ and some $\varepsilon > 0$

$$\mathbb{P} \left(|\Phi(W^n) - y| < \varepsilon, |\det \gamma_{\Phi(W^n)}| \geq \frac{1}{M}, \|W^n\|_H^2 \leq K+1 \right) = 0, \quad (2.13)$$

one could find a function f bounded, positive and continuous such that $y \in \text{supp } f$ and

$$\mathbb{E} \left(f(\Phi(W^n)) \alpha_M(\det \gamma_{\Phi(W^n)}) \beta_K(\|W^n\|_H^2) \right) = 0, \quad (2.14)$$

because $\mathbb{1}_{\{|\det \gamma_{\Phi(W^n)}| \geq \frac{1}{M}\}} \geq \alpha_M(\det \gamma_{\Phi(W^n)})$ and $\mathbb{1}_{\{\|W^n\|_H^2 \leq K+1\}} \geq \beta_K(\|W^n\|_H^2)$. This contradicts (2.11). Let $\bar{k} = K_0(n) + 1$. Then from (2.12) we can find a sequence of elements $h_m \in H_n$ such that for any $m \geq 1$, $|\Phi(h_m) - y| < \frac{1}{m}$, $\|h_m\|_H^2 \leq \bar{k}$ and $|\det \gamma_{\Phi(h_m)}| \geq \frac{1}{M}$.

The compactness of bounded and closed sets in H_n implies that we can select a subsequence converging to some element $h \in H$ which verifies $\Phi(h) = y$ and $|\det \gamma_{\Phi(h)}| > 0$. That means (B) holds. \square

In this article we shall apply the preceding results to the following particular case. Fix any $T > 0$. Consider a sequence $\{W_j(t), t \in [0, T]\}$, $j \geq 0$, of standard Wiener processes and let (Ω, H_T, P) be the associated Wiener space. That means $\Omega = \mathcal{C}([0, T]; \mathbb{R}^{\mathbb{N}})$, H_T is the Hilbert space $L^2([0, T]; \mathbb{R}^{\mathbb{N}})$ and P is the Wiener measure on Ω . The random vector F will be $u(t, \underline{x}) = (u(t, x_1), \dots, u(t, x_d))$, the solution to the wave equation (1.3) at different points $(t, x_1), \dots, (t, x_d)$ of $[0, T] \times \mathbb{R}^2$. The functional $\Phi : H \rightarrow \mathbb{R}^d$ will be the skeleton $\Phi^h(t, \underline{x}) = (\Phi^h(t, x_1), \dots, \Phi^h(t, x_d))$ defined in (1.9).

Let us precise which are the sequences $(T_n^h)_{n \geq 0}$ and $(W^n)_{n \geq 0}$ in the above Propositions 1 and 2.

For any fixed $n \in \mathbb{N}$ we denote by Δ_i the interval $[\frac{iT}{2^n}, \frac{(i+1)T}{2^n})$ and write $W_j(\Delta_i)$ for the increment $W_j(\frac{(i+1)T}{2^n}) - W_j(\frac{iT}{2^n})$.

Let $W^n = (W_j^n = \int_0^\cdot \dot{W}_j^n(s) ds, j \in \mathbb{N})$ be as in [11], that is, $\dot{W}_j^n = 0$ if $j > n$ and, for $0 \leq j \leq n$,

$$\dot{W}_j^n(t) = \begin{cases} \sum_{i=1}^{2^n} 2^n T^{-1} W_j(\Delta_{i-1}) 1_{\Delta_i}(t), & \text{if } t \in [2^{-n}T, T], \\ 0, & \text{if } t \in [0, 2^{-n}T]. \end{cases}$$

Notice that for each $n \in \mathbb{N}$, $(\dot{W}_j^n, 0 \leq j \leq n)$ belongs to the finite dimensional subspace of H_T generated by $\{2^n T^{-1} 1_{\Delta_i}, i = 1, \dots, 2^n - 1\}$. We identify $(\dot{W}_j^n, 0 \leq j \leq n)$ with an element of H_T by putting $\dot{W}_j^n \equiv 0$ for $j > n$.

For any $h \in H_T$ define

$$T_n^h(\omega) = \left(W_j - W_j^n + \int_0^\cdot h_j(s) ds, j \geq 0 \right). \quad (2.15)$$

Girsanov's theorem yields that $P \circ (T_n^h)^{-1} \ll P$.

Finally, the localizing sequence $A_n^\gamma, n \geq 1$ of Proposition 2 is defined as follows. Fix $\gamma > 2T \ln 2, t \in [0, T]$; then

$$A_n^\gamma(t) = \left\{ \|W^n \mathbf{1}_{[0,t]}\|_{H_T}^2 \leq \gamma n^2 2^n T^{-1} \right\}. \quad (2.16)$$

Clearly the sets $A_n^\gamma(t)$ decrease in t and increase in γ . Moreover,

$$\lim_{n \rightarrow \infty} P(A_n^\gamma(T)^c) = 0. \quad (2.17)$$

Indeed,

$$\begin{aligned} P(A_n^\gamma(T)^c) &\leq \sum_{j=1}^n \sum_{i=0}^{2^n-1} P(|W_j(\Delta_i)|^2 \geq \gamma n 2^{-n}) \\ &\leq n 2^n P(|Z| \geq \sqrt{\gamma} n^{\frac{1}{2}} T^{-\frac{1}{2}}) \leq \frac{n 2^n T^{\frac{1}{2}}}{\sqrt{\gamma} n^{\frac{1}{2}}} \exp\left(-\frac{\gamma n}{2T}\right) \\ &\leq \gamma^{-\frac{1}{2}} n^{\frac{1}{2}} T^{\frac{1}{2}} \exp\left(-n\left(\frac{\gamma}{2T} - \log 2\right)\right), \end{aligned}$$

where Z is a standard Gaussian random variable. Thus (2.17) holds true.

A simple computation shows that for any $0 \leq t < t' \leq T, p \in [2, \infty)$ and $[t, t'] \subset \Delta_i$ for some i ,

$$E(\|W^n \mathbf{1}_{[t,t']}\|_{H_T}^p) \leq n^{\frac{p}{2}}. \quad (2.18)$$

In fact,

$$\begin{aligned} E(\|W^n 1_{[t,t']}\|_{H_T}^p) &= E\left(\left(\sum_{j=1}^n 2^{2n} T^{-2} (W_j(\Delta_{i-1}))^2 (t-t')\right)^{\frac{p}{2}}\right) \\ &\leq 2^{np} T^{-p} (t-t')^{\frac{p}{2}} E\left(\sum_{j=1}^n (W_j(\Delta_{i-1}))^2\right)^{\frac{p}{2}} \\ &\leq n^{\frac{p}{2}}. \end{aligned}$$

Let

$$G^{\gamma,n}(\omega) = \Psi^\gamma\left(\frac{\|W^n\|_{H_T}^2}{n^2 2^n T^{-1}}\right) = \Psi^\gamma\left(n^{-2} \sum_{j=1}^n \sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2\right), \quad (2.19)$$

$n \geq 1$, where Ψ^γ is given by (2.5). This sequence satisfies property (2.7). Indeed, this can be proved by direct computations, as follows.

Let $k \in \mathbb{N}$ and fix a set $B_k = \{\alpha_i = (r_i, j_i) \in \mathbb{R}_+ \times \mathbb{N}, i = 1, \dots, k\}$. Let $\alpha = (\alpha_1, \dots, \alpha_k)$ and denote by P_m the set of partitions of B_k consisting of m disjoint subsets p_1, \dots, p_m , $m = 1, \dots, k$; set $|p_i| = \text{card } p_i$. Let Y be any random variable in $\mathbb{D}^{k,2}$, $k \geq 1$, g be a real \mathcal{C}^k function with bounded derivatives up to order k . Leibniz's rule for Malliavin's derivatives yields

$$D_\alpha^k(g(Y)) = \sum_{m=1}^k \sum_{P_m} c_m g^{(m)}(Y) \prod_{i=1}^m D_{p_i}^{|p_i|} Y, \quad (2.20)$$

with some positive coefficients c_m , $m = 1, \dots, k$, $c_1 = 1$. We want to apply this formula to $g = \Psi^\gamma$ and $Y = n^{-2} \sum_{j=1}^n \sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2$, $n \geq 1$. Notice that these random variables have null components on the n -th Wiener chaos for $n \geq 3$. Hence, it suffices to prove (2.7) for $k = 0, 1, 2$ and $p \in [1, \infty)$. For these values of k , the order of the derivatives in the right hand-side of (2.20) are clearly less or equal to 2.

For $k = 0$, the result is obvious, since Ψ^γ is bounded by 1. Set

$$F_n = n^{-2} \sum_{j=1}^n \sum_{i=1}^{2^n} W_j(\Delta_{i-1})^2,$$

then, $D_{r,j} F_n = 0$, if $j > n$ and for $j \leq n$

$$D_{r,j} F_n = n^{-2} \sum_{i=1}^{2^n} 2W_j(\Delta_{i-1}) \mathbb{1}_{\Delta_{i-1}}(r).$$

Furthermore, $D_{(r_1, j_1)(r_2, j_2)}^2 F_n = 0$, if $j_1 > n$ or $j_1 \leq n$ but $j_1 \neq j_2$ and, for $j_1 = j_2 \leq n$,

$$D_{(r_1, j_1)(r_2, j_2)}^2 F_n = n^{-2} \sum_{i=1}^{2^n} 2\mathbb{1}_{\Delta_{i-1}}(r_1) \mathbb{1}_{\Delta_{i-1}}(r_2).$$

Applying Hölder's inequality it is easy to check that

$$E \left(\sum_{j \in \mathbb{N}} \int_0^T dr |D_{r,j} F_n|^2 \right)^{\frac{p}{2}} \leq C n^{-\frac{3}{2}p} 2^{p(-\frac{n}{2}+1)}.$$

Moreover, a direct computation shows that

$$E \left(\sum_{j_1, j_2 \in \mathbb{N}} \int_0^T dr_1 \int_0^T dr_2 |D_{(r_1, j_1)(r_2, j_2)}^2 F_n|^2 \right)^{\frac{p}{2}} \leq C n^{-\frac{3}{2}p} 2^{p(-\frac{n}{2}+1)}.$$

Thus, for any $p \in [1, \infty)$ and $k = 0, 1, 2$, we have that $\sup_n \|F_n\|_{k,p} < \infty$. Then, by means of (2.20) we complete the proof of (2.7) for the sequence of random variables defined in (2.19).

3. APPROXIMATION IN PROBABILITY IN \mathbb{D}^∞ . CHARACTERIZATION OF POINTS OF POSITIVE DENSITY

In this section we present an approximation result for a class of evolution equations which include as particular cases (1.3) and (1.9). This general setting allows to check the validity of the assumptions of Propositions 1 and 2 for the Wiener functional $F = u(t, \underline{x})$.

Let $A, B, G, b : \mathbb{R} \rightarrow \mathbb{R}$, $h \in H_T$. Consider the evolution equations

$$\begin{aligned} X_n(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \left\{ \langle S(t-s, x-*) A(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \right. \\ &\quad + \langle S(t-s, x-*) B(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} \dot{W}_j^n(s) ds \\ &\quad + \left. \langle S(t-s, x-*) G(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \right\} \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(X_n(s, y)) ds dy, \end{aligned} \tag{3.1}$$

$$\begin{aligned} X(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \left\{ \langle S(t-s, x-*) (A+B)(X(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \right. \\ &\quad + \left. \langle S(t-s, x-*) G(X(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \right\} \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(X(s, y)) ds dy. \end{aligned} \tag{3.2}$$

The existence and uniqueness of solution for equations (3.1) and (3.2) have been addressed in [11]. Notice that the processes X_n and X depend on $h \in H_T$.

We introduce the following set of assumptions (see (1.5) and (ii') in Sect. 1).

There exists $\beta_0 \in]0, 2[$ such that

- (C1) $\int_{0^+} r^{1-\beta_0} f(r) dr < +\infty$;
- (C2) $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class \mathcal{C}^1 , bounded, with $\frac{1}{2}(\beta_0 \wedge 1)$ -Hölder continuous partial derivatives, $\nabla u_0 \in L^{q_1}(\mathbb{R}^2)$ for some $q_1 > 2$; $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ belongs to $L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in]4 \vee \frac{2}{1-(\beta_0 \wedge 1)}, \infty]$;
- (C3) the coefficients A, B, G, b of equation (3.1) (and (3.2)) are \mathcal{C}^∞ functions with bounded derivatives of any order $k \geq 1$.

Conditions (C1–C3) ensure that the trajectories of the solution of (1.3) are β -Hölder continuous in (t, x) for $\beta < \frac{1}{2}(\beta_0 \wedge 1)$ (see Th. 2.2 in [8] and Prop. 1.4 in [10]).

For any separable Hilbert space \mathbb{E} we denote by $H_T(\mathbb{E})$ the Hilbert space of functions $g : [0, T] \rightarrow \mathbb{E}^{\mathbb{N}}$ such that $\sum_{j \geq 0} \int_0^T \|g_j(s)\|_{\mathbb{E}}^2 ds < \infty$. Notice that $H_T(\mathbb{R}) = H_T$. In this section we will deal with $\mathbb{E} = \mathbb{R}^d$ and $\mathbb{E} = \mathcal{H}$, where \mathcal{H} has been defined in the introduction.

We now state the main technical result of this section. We shall see later in Theorem 3 that by a particular choice of the coefficients A, B, G of equations (3.1) and (3.2), the next Theorem 2 allows to complete the first step of our programme.

Theorem 2. *Assume (C1–C3). Then, for any $p \in (1, \infty)$, $k \in \mathbb{Z}_+$, $\gamma > 2T \ln 2$ and every compact set $K \subset \mathbb{R}^2$,*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\underline{x} \in K^d} E \left(\|D^k(X_n(t, \underline{x}) - X(t, \underline{x}))\|_{H_T(\mathbb{R}^d) \otimes^k}^p \mathbb{1}_{A_n^\gamma(t)} \right) = 0, \quad (3.3)$$

where, for $k = 0$, $D^0 = Id$ and $\|\cdot\|_{H_T(\mathbb{R}^d) \otimes^0}$ is the Euclidean norm in \mathbb{R}^d . Moreover, the convergence in (3.3) is uniform in h on bounded sets of H_T .

This theorem provides an extension of Proposition 2.3 in [11], where the convergence stated in (3.3) has been proved in the L^p norm. Actually, in this proposition the localizing sequence is given by $\bar{A}_n^\gamma(t) = \left\{ \sup_{0 \leq j \leq n} \sup_{0 \leq i \leq ([2^n t T^{-1}] - 1)_+} |W_j(\Delta_i)| \leq \sqrt{\gamma} 2^{-n/2} n^{1/2} \right\}$, which is included in $A_n^\gamma(t)$ and also satisfies $\lim_{n \rightarrow \infty} P((\bar{A}_n^\gamma(T))^c) = 0$ (see Lem. 2.1 in [11]). However looking carefully at the proof we realize that only two facts concerning the localization are needed: (a) $E(\|W^n\|_{H_T}^p \mathbb{1}_{\bar{A}_n^\gamma(t)}) \leq C n^p 2^{\frac{np}{2}}$ and (b) $E(\|W^n \mathbb{1}_{[t_n, t]}\|_{H_T}^p \mathbb{1}_{\bar{A}_n^\gamma(t)}) \leq C n^p$. Here we have used the following notation: for any $n \geq 1$, $t \in [0, T]$, we set $t_n = \max\{k 2^{-n} T; k = 1, \dots, 2^n - 1 : k 2^{-n} T \leq t\}$, $t_n = (t_n - 2^{-n} T) \vee 0$. Property (a) is clearly true with $A_n^\gamma(t)$ instead of $\bar{A}_n^\gamma(t)$, by the very definition of $A_n^\gamma(t)$. Property (b) is a trivial consequence of (2.18).

The proof in the \mathbb{D}^∞ convergence consists in a quite long and tricky exercise with almost no new ideas. For this reason we do not give a detailed proof. Instead, we draw an outline and also state the technical lemmas which are needed. With these ingredients we provide the readers interested in a complete proof with the main guidelines to check by themselves this extension.

Let us introduce some additional notation.

$$\begin{aligned} X_n^-(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^{t_n} \left\{ \langle S(t-s, x - *) A(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \right. \\ &\quad + \langle S(t-s, x - *) B(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} \dot{W}_j^n(s) ds \\ &\quad \left. + \langle S(t-s, x - *) G(X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \right\} \\ &\quad + \int_0^{t_n} \int_{\mathbb{R}^2} S(t-s, x-y) b(X_n(s, y)) ds dy, \end{aligned} \quad (3.4)$$

$$\begin{aligned} X^-(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^{t_n} \left\{ \langle S(t-s, x - *) (A+B)(X(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \right. \\ &\quad \left. + \langle S(t-s, x - *) G(X(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \right\} \\ &\quad + \int_0^{t_n} \int_{\mathbb{R}^2} S(t-s, x-y) b(X(s, y)) ds dy. \end{aligned} \quad (3.5)$$

Notice that, although it is not explicit in the notation, X^- depends on n .

In Section 2 of [10] we have proved that, if $G = 0$, $X(t, x) \in \mathbb{D}^\infty$, for any $t \in [0, T]$, $x \in \mathbb{R}^2$. A slight modification of the proof allows to establish the same result for G satisfying the assumptions of Theorem 2 as well as for $X_n(t, x)$, $X_n^-(t, x)$ and $X^-(t, x)$, respectively. Moreover, for any $k \in \mathbb{Z}_+$ and $p \in [1, \infty)$,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k X(t, x)\|_{L^p(\Omega; H_T^{\otimes k})} < \infty. \quad (3.6)$$

We also need the explicit form of the evolution equations satisfied by the Malliavin derivatives of these processes. To this end we consider the chain rule given in (2.20). Let $\Delta_\alpha(g, Y) = D_\alpha^k(g(Y)) - g'(Y) D_\alpha^k Y$ and $\Delta(g, Y)$ be the stochastic process with components $\Delta_\alpha(g, Y)$. For any $r_1, \dots, r_k \in \mathbb{R}$ we define $\bigvee_i r_i = \max(r_1, \dots, r_k)$ and for $\alpha_i = (r_i, j_i) \in \mathbb{R}_+ \times \mathbb{N}$, $\hat{\alpha}_i = (\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k)$.

Then, following the proof of Theorem 2.2 in [10], we obtain

$$\begin{aligned} D_\alpha^k X_n(t, x) &= \sum_{i=1}^k \langle S(t - r_i, x - *) D_{\hat{\alpha}_i}^{k-1}(A(X_n(r_i, *))), e_{j_i}(\cdot) \rangle_{\mathcal{H}} \\ &+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(A, X_n(s, *)), e_j(\cdot) \rangle_{\mathcal{H}} W_j(ds) \\ &+ \sum_{i=1}^k \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) D_{\hat{\alpha}_i}^{k-1}(B(X_n(s, *))), e_{j_i}(\cdot) \rangle_{\mathcal{H}} \\ &\times \sum_{i=1}^{2^n} 2^n T^{-1} \times 1_{\Delta_{i-1}}(r_i) \times 1_{\Delta_i}(s) ds \\ &+ \sum_{j=0}^n \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(B, X_n(s, *)), e_j(\cdot) \rangle_{\mathcal{H}} \dot{W}_j^n(s) ds \\ &+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(G, X_n(s, *)), e_j(\cdot) \rangle_{\mathcal{H}} h_j(s) ds \\ &+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) \Delta_\alpha(b, X_n(s, y)) ds dy \\ &+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) A'(X_n(s, *)), D_\alpha^k X_n(s, *), e_j(\cdot) \rangle_{\mathcal{H}} W_j(ds) \\ &+ \sum_{j=0}^n \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) B'(X_n(s, *)), D_\alpha^k X_n(s, *), e_j(\cdot) \rangle_{\mathcal{H}} \dot{W}_j^n(s) ds \\ &+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) G'(X_n(s, *)), D_\alpha^k X_n(s, *), e_j(\cdot) \rangle_{\mathcal{H}} h_j(s) ds \\ &+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) b'(X_n(s, y)) D_\alpha^k X_n(s, y) ds dy \end{aligned} \quad (3.7)$$

and similarly,

$$\begin{aligned}
D_\alpha^k X(t, x) &= \sum_{i=1}^k \langle S(t - r_i, x - *) D_{\hat{\alpha}_i}^{k-1}((A + B)(X(r_i, *))), e_{j_i}(*) \rangle_{\mathcal{H}} \\
&+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(A + B, X(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \\
&+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(G, X(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \\
&+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) \Delta_\alpha(b, X(s, y)) ds dy \\
&+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) (A + B)'(X(s, *)) D_\alpha^k X(s, *), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \\
&+ \sum_{j \geq 0} \int_{\bigvee_i r_i}^t \langle S(t - s, x - *) G'(X(s, *)) D_\alpha^k X(s, *), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \\
&+ \int_{\bigvee_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) b'(X(s, y)) D_\alpha^k X(s, y) ds dy, \tag{3.8}
\end{aligned}$$

in the case where $\bigvee_i r_i \leq t$. Otherwise, $D_\alpha^k X_n(t, x) = D_\alpha^k X(t, x) = 0$.

The equations satisfied by $D_\alpha^k X_n^-(t, x)$ (resp. $D_\alpha^k X^-(t, x)$) are obtained substituting in (3.7) (resp. in (3.8)) the upper bound in the integral by t_n and multiplying the first term of the right hand-side of (3.7) (resp. of (3.8)) by $1_{[0, t_n]}(r_i)$.

Outline of the proof of Theorem 2. We shall apply induction on k and assume that $d = 1$. The proof for $d > 1$ is analogous. For $k = 0$, the convergence (3.3) has been proved in Proposition 2.3 of [11].

Set

$$D_\alpha^k (X_n(t, x) - X(t, x)) = \sum_{i=1}^8 V_{i, \alpha}^n(t, x),$$

where

$$\begin{aligned}
V_{1, \alpha}^n(t, x) &= \sum_{i=1}^k \langle S(t - r_i, x - *) [D_{\hat{\alpha}_i}^{k-1}(A(X_n(r_i, *))) - D_{\hat{\alpha}_i}^{k-1}(A(X(r_i, *)))], \\
&e_{j_i}(*) \rangle_{\mathcal{H}} \\
V_{2, \alpha}^n(t, x) &= \sum_{i=1}^k \int_0^t \langle S(t - s, x - *) D_{\hat{\alpha}_i}^{k-1}(B(X_n(s, *))), e_{j_i}(*) \rangle_{\mathcal{H}} \\
&\times \sum_{\ell=1}^{2^n} 2^n T^{-1} 1_{\Delta_{\ell-1}}(r_i) 1_{\Delta_\ell}(s) ds - \langle S(t - r_i, x - *) D_{\hat{\alpha}_i}^{k-1}(B(X(r_i, *))), \\
&e_{j_i}(*) \rangle_{\mathcal{H}},
\end{aligned}$$

$$\begin{aligned}
V_{3,\alpha}^n(t,x) &= \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) [\Delta_\alpha(G, X_n(s, *)) - \Delta_\alpha(G, X(s, *))], e_j(*) \rangle_{\mathcal{H}} \\
&\quad \times h_j(s) \, ds, \\
V_{4,\alpha}^n(t,x) &= \int_{\mathbb{V}_{r_i}}^t \int_{\mathbb{R}^2} S(t-s, x-y) (\Delta_\alpha(b, X_n(s, y)) - \Delta_\alpha(b, X(s, y))) \\
&\quad \times \, ds \, dy, \\
V_{5,\alpha}^n(t,x) &= \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) [G'(X_n(s, *)) D_\alpha^k X_n(s, *) - G'(X(s, x))] \\
&\quad \times D_\alpha^k X(s, *)], e_j(*) \rangle_{\mathcal{H}} h_j(s) \, ds, \\
V_{6,\alpha}^n(t,x) &= \int_{\mathbb{V}_{r_i}}^t \int_{\mathbb{R}^2} S(t-s, x-y) [b'(X_n(s, y)) D_\alpha^k X_n(s, y) - b'(X(s, y)) D_\alpha^k X(s, y)] \\
&\quad \times \, ds \, dy, \\
V_{7,\alpha}^n(t,x) &= \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) \Delta_\alpha(A, X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \\
&\quad + \sum_{j=0}^n \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) \Delta_\alpha(B, X_n(s, *)), e_j(*) \rangle_{\mathcal{H}} \dot{W}_j^n(s) \, ds \\
&\quad - \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) \Delta_\alpha(A+B, X(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds), \\
V_{8,\alpha}^n(t,x) &= \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) A'(X_n(s, *)) D_\alpha^k X_n(s, *), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \\
&\quad + \sum_{j=0}^n \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) B'(X_n(s, *)) D_\alpha^k X_n(s, *), e_j(*) \rangle_{\mathcal{H}} \dot{W}_j^n(s) \, ds \\
&\quad - \sum_{j \geq 0} \int_{\mathbb{V}_{r_i}}^t \langle S(t-s, x-*) (A+B)'(X(s, *)) D_\alpha^k X(s, *), e_j(*) \rangle_{\mathcal{H}} W_j(ds).
\end{aligned}$$

Let $U_i^n(t, x) = E\left(\|V_i^n(t, x)\|_{H_T^{\otimes k}}^p 1_{A_n^\gamma(t)}\right)$, $i = 1, \dots, 8$.

We have to prove that

$$\lim_{n \rightarrow \infty} \sup_{(t,x) \in K_m^T} U_\rho^n(t, x) = 0, \quad (3.9)$$

for any $\rho = 1, 2, 3, 4, 7$, and

$$U_\rho^n(t, x) \leq C \int_{\mathbb{V}_{r_i}}^t \sup_{(u,x) \in K_{m+T}^s} E\left(\|D^k(X_n(u, x) - X(u, x))\|_{H_T^{\otimes k}}^p 1_{A_n^\gamma(u)}\right) \, ds, \quad (3.10)$$

for $\rho = 5, 6, 8$, where $K_m^t = [0, t] \times \{x \in \mathbb{R}^2 : \|x\| \leq m\}$, $t \in [0, T]$ and $m \in \mathbb{N}$, and we are assuming that (3.3) holds up to the order of derivation $k-1$. This can be done using the same ideas as in the proof of the above mentioned Proposition 2.3 in [11] taking into account the results given in the next Lemmas 1 to 3. Basically, the proof of (3.9) for $\rho = 1, 2, 3, 4$ follows easily from the induction hypothesis, the proof for $\rho = 2$ uses Lemma 3 and induction; the proof of (3.10) for $\rho = 5, 6$ uses the boundedness and Lipschitz properties of the coefficients.

Notice that the structure of $U_\rho^n(t, x)$ for $\rho = 7, 8$ is similar. These terms are the most difficult to study. However the ideas used in Proposition 2.3 in [11] can be adapted using the additional ingredient of induction for the term corresponding to $\rho = 7$. Once (3.9) and (3.10) have been established, the proof of the theorem is completed by means of a Gronwall-type argument.

The next statements correspond to the appropriate extensions of Lemmas 2.2–2.4 in [11], they are proved by induction on the derivative order k . The third one is related with the result stated in Proposition 2.2 in [11], we give a complete proof of it; as a by-product of this lemma we obtain Hölder continuity in time of the Malliavin derivative process $D^k X(t, x), t \geq 0$.

Lemma 1. *We assume the hypothesis of Theorem 2. Then, for any $c \geq 0, k \in \mathbb{Z}_+, p \in [1, \infty)$, every integer $n \geq 1$ and $0 < \beta < \beta_0 \wedge 1$,*

$$\sup_{\|h\|_{H_T} \leq c} \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k(X(t, x) - X^-(t, x))\|_{L^p(\Omega; H_T^{\otimes k})} \leq C 2^{-n \frac{\beta+1}{2}}.$$

Lemma 2. *We suppose that the assumptions of Theorem 2 are satisfied. For any $k \in \mathbb{Z}_+, p \in [1, \infty), \gamma > 2T \ln 2, c \geq 0$, every integer $n \geq 1$ and $0 < \beta < \beta_0 \wedge 1$,*

$$\begin{aligned} & \sup_{\|h\|_{H_T} \leq c} \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k(X_n(t, x) - X_n^-(t, x)) 1_{A_n^\gamma(t)}\|_{L^p(\Omega; H_T^{\otimes k})} \\ & \leq C n 2^{-n \frac{1+\beta}{2}} \left(1 + \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k(X_n(t, x)) 1_{A_n^\gamma(t)}\|_{L^p(\Omega; H_T^{\otimes k})} \right), \\ & \sup_{\|h\|_{H_T} \leq c} \sup_{n \geq 1} \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k(X_n(t, x) + X_n^-(t, x)) 1_{A_n^\gamma(t)}\|_{L^p(\Omega; H_T^{\otimes k})} < \infty. \end{aligned} \quad (3.11)$$

Consequently,

$$\sup_{\|h\|_{H_T} \leq c} \sup_{(t,x) \in [0,T] \times \mathbb{R}^2} \|D^k(X_n(t, x) - X_n^-(t, x)) 1_{A_n^\gamma(t)}\|_{L^p(\Omega; H_T^{\otimes k})} \leq C n 2^{-n \frac{1+\beta}{2}}.$$

Lemma 3. *We assume the hypothesis of Theorem 2. Then, for any $c \geq 0, p \in [1, \infty), 0 \leq t \leq t' \leq T, k \in \mathbb{Z}_+, \alpha \in (0, \frac{1}{2}(\beta_0 \wedge 1))$, we have*

$$\sup_{\|h\|_{H_T} \leq c} \|D^k(X(t, x) - X(t', x))\|_{L^p(\Omega; H_T^{\otimes k})} \leq C |t - t'|^\alpha. \quad (3.12)$$

Proof. Fix $0 \leq t \leq t' \leq T, x \in \mathbb{R}^2$; set $\gamma(t, t', x; s, y) = S(t - s, x - y) - S(t' - s, x - y), (s, y) \in [0, T] \times \mathbb{R}^2$. Notice that $(s, y, z) \mapsto \gamma(t, t', x; s, y) f(|y - z|) \gamma(t, t', x; s, z)$ defines a density on $[0, T] \times \mathbb{R}^2 \times \mathbb{R}^2$.

We first prove (3.12) for $k = 0$. For any fixed $p \in [1, \infty)$ we set

$$E(|X(t, x) - X(t', x)|^p) \leq C \sum_{i=1}^4 R_i(t, t'; x),$$

with

$$\begin{aligned} R_1(t, t'; x) &= |X^0(t, x) - X^0(t', x)|^p, \\ R_2(t, t'; x) &= E \left(\left| \sum_{j \geq 0} \int_0^T \langle \gamma(t, t', x; s, *) (A + B)(X(s, *)), e_j(*) \rangle_{\mathcal{H}} W_j(ds) \right|^p \right), \\ R_3(t, t'; x) &= E \left(\left| \sum_{j \geq 0} \int_0^T \langle \gamma(t, t', x; s, *) G(X(s, *)), e_j(*) \rangle_{\mathcal{H}} h_j(s) ds \right|^p \right), \\ R_4(t, t'; x) &= E \left(\left| \int_0^T \gamma(t, t', x; s, y) b(X(s, y)) ds dy \right|^p \right). \end{aligned}$$

In the proof of Proposition 1.4 in [10] we have established that

$$R_1(t, t'; x) \leq C \left\{ \|v_0\|_{q_0}^p |t - t'|^{p(\frac{1}{q} - \frac{1}{2})} + |t - t'|^{\frac{p}{2}(\beta_0 \wedge 1)} \right\},$$

where $\frac{1}{q} + \frac{1}{q_0} = 1$ and we have used (C2). The restriction on q_0 yields, for $\alpha \in (0, \frac{1}{2}(\beta_0 \wedge 1))$,

$$R_1(t, t'; x) \leq C |t - t'|^{\alpha p}. \quad (3.13)$$

Burkholder's inequality, then Fubini's theorem and finally Hölder's inequality with respect to the measure whose density has been described at the beginning of the proof, yield

$$R_2(t, t'; x) \leq C \|\gamma(t - t', x; \cdot, *)\|_{L^2([0, T]; \mathcal{H})}^p \sup_{(s, x) \in [0, T] \times \mathbb{R}^2} E(|(A + B)(X(s, x))|^p).$$

In [11], Lemma 2.5 (see the proof of (2.42) with $x = \bar{x}$), it is proved that

$$\|\gamma(t, t', x, \cdot, *)\|_{L^2([0, T]; \mathcal{H})}^2 \leq \mu_{t, t'-t} + \tilde{\mu}_{t, t'-t} + 2(\mu_{t, t'-t} \tilde{\mu}_{t, t'-t})^{1/2},$$

with $\mu_{t, h}, \tilde{\mu}_{t, h}$ are defined in (A.2, A.3), respectively. The bound (A.5) yields

$$R_2(t, t'; x) \leq C(|t - t'|^{\alpha p}), \quad (3.14)$$

with $\alpha \in (0, \frac{1}{2}(\beta_0 \wedge 1))$.

The same bound is obtained for the remaining terms $R_i(t, t'; x)$, $i = 3, 4$, following similar ideas.

The proof for any integer $k \geq 1$ is completed using induction on k . \square

We continue by setting the additional ingredients needed in the application of Proposition 1 in our example.

Lemma 4. *Let $\underline{z} = (z_1, \dots, z_d)$. For any integer $k \geq 0$ and any $h \in H_T$ set*

$$\rho_n^{k, h, \underline{z}}(t, \underline{x})(\omega) = D^k u(t, \underline{x}) \left(W - W^n + \int_0^t \left(h + \sum_{i=1}^d z_i \bar{D} \Phi^h(t, x_i) \right) (s) ds \right),$$

where \bar{D} denotes the Fréchet derivative operator. Then, for any $p \geq 1$,

$$\lim_{M \rightarrow \infty} \sup_n P \left\{ \int_{|\underline{z}| \leq 1} \|\rho_n^{k, h, \underline{z}}(t, \underline{x})\|_{H_T^{\otimes k}}^p dz > M \right\} = 0.$$

Proof. Fix $\varepsilon > 0$. By property (2.17) there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ and $\gamma > 2T \ell n 2$, $P(A_n^\gamma(t)^c) < \varepsilon$. Then, Chebychev's inequality yields, for any $n \geq n_0$,

$$P \left\{ \int_{|\underline{z}| \leq 1} \|\rho_n^{k,h,\underline{z}}(t, \underline{x})\|_{H^{\otimes k}}^p dz > M \right\} \leq \varepsilon + \frac{1}{M} \sup_{|\underline{z}| \leq 1} E \left(\|\rho_n^{k,h,\underline{z}}(t, \underline{x})\|_{H_T^{\otimes k}}^p 1_{A_n^\gamma(t)} \right).$$

Thus, it suffices to check that

$$\sup_n \sup_{|\underline{z}| \leq 1} E \left(\|\rho_n^{k,h,\underline{z}}(t, \underline{x})\|_{H_T^{\otimes k}}^p 1_{A_n^\gamma(t)} \right) \leq C < \infty. \quad (3.15)$$

The equation satisfied by $D^k u(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^2$, is obtained from equation (3.8) with $A = G = 0$, $B = \sigma$. Set $\tilde{W}^{n,h,\underline{z}} = W - W^n + \int_0^t (h + \sum_{i=1}^d z_i \bar{D} \Phi^h(t, x_i))(s) ds$; then, each component of the d -dimensional random vector $\rho_n^{k,h,\underline{z}}(t, \underline{x})$ satisfies the following equation, where for simplicity we have omitted the index of the variable x :

$$\begin{aligned} \rho_{n,\alpha}^{k,h,\underline{z}}(t, x) &= \sum_{i=1}^k \langle D_{\hat{\alpha}_i}^{k-1}(\sigma(u(r_i, *))) (\tilde{W}^{n,h,\underline{z}}) S(t - r_i, x - *), e_{j_i}(\cdot) \rangle_{\mathcal{H}} \\ &+ \sum_{j \geq 0} \int_{V_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(\sigma, u(s, *)) (\tilde{W}^{n,h,\underline{z}}), e_j(\cdot) \rangle_{\mathcal{H}} W_j(ds) \\ &+ \sum_{j \geq 0} \int_{V_i r_i}^t \langle S(t - s, x - *) \Delta_\alpha(\sigma, u(s, *)) (\tilde{W}^{n,h,\underline{z}}), e_j(\cdot) \rangle_{\mathcal{H}} (h_j(s) \\ &- \dot{W}_j^n(s) + \sum_{i=1}^d z_i \bar{D}_{s,j} \Phi^h(t, x_i)) ds \\ &+ \int_{V_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) \Delta_\alpha(b, u(s, y)) (\tilde{W}^{n,h,\underline{z}}) ds dy \\ &+ \sum_{j \geq 0} \int_{V_i r_i}^t \langle S(t - s, x - *) \sigma'(\rho_n^{0,h,\underline{z}}(s, *)) \rho_{n,\alpha}^{k,h,\underline{z}}(s, *), e_j(\cdot) \rangle_{\mathcal{H}} W_j(ds) \\ &+ \sum_{j \geq 0} \int_{V_i r_i}^t \langle S(t - s, x - *) \sigma'(\rho_n^{0,h,\underline{z}}(s, *)) \rho_{n,\alpha}^{k,h,\underline{z}}(s, *), e_j(\cdot) \rangle_{\mathcal{H}} (h_j(s) \\ &- \dot{W}_j^n(s) + \sum_{i=1}^d z_i \bar{D}_{s,j} \Phi^h(t, x_i)) ds \\ &+ \int_{V_i r_i}^t \int_{\mathbb{R}^2} S(t - s, x - y) b'(\rho_n^{0,h,\underline{z}}(s, y)) \rho_{n,\alpha}^{k,h,\underline{z}}(s, y) ds dy. \end{aligned} \quad (3.16)$$

Notice that, by the chain rule (2.20)

$$D_{\hat{\alpha}_i}^{k-1}(\sigma(u(s, x))) (\tilde{W}^{n,h,\underline{z}}) = \sum_{m=1}^{k-1} \sum_{P_m} c_m \sigma^{(m)}(\rho_n^{0,h,\underline{z}}(s, x)) \prod_{j=1}^m \rho_{n,p_i}^{|p_j|,h,\underline{z}}(s, x),$$

where if $\alpha = (\alpha_1, \dots, \alpha_k)$, P_m is the set of partitions of the set $\{(\alpha_1, \dots, \alpha_{i-1}, \alpha_{i+1}, \dots, \alpha_k) \in (\mathbb{R}_+ \times \mathbb{N})^{k-1}\}$ consisting of m disjoint subsets p_1, \dots, p_m , $m = 1, \dots, k-1$. Moreover, for $g = \sigma, b$,

$$\Delta_\alpha(g, u(s, x)) (\tilde{W}^{n,h,\underline{z}}) = D_\alpha^k(g(u(s, x))) (\tilde{W}^{n,h,\underline{z}}) - g'(\rho_n^{0,h,\underline{z}}(s, x)) \rho_{n,\alpha}^{k,h,\underline{z}}(s, x).$$

Therefore equation (3.16) has the same structure than equation (3.7) with h_j replaced by $h_j + \sum_{i=1}^d z_i \bar{D}_{\cdot,j} \Phi^h(t, x_i)$. We point out in the Appendix that, for any $a \geq 0$, $(t, x) \in [0, T] \times \mathbb{R}^2$, $\sup_{\|h\|_{H_T} \leq a} \|\bar{D} \Phi^h(t, x)\|_{H_T} \leq C < \infty$. Thus, Lemma 2 (see (3.11)) shows (3.15) and completes this proof. \square

Remark 1. Consider the transformation defined in (2.15), $T_n^h(\omega) = (W_j - W_j^n + \int_0^{\cdot} h_j(s) ds)_{j \geq 0}$, $h \in H_T$. Using the notation introduced in Lemma 4 we have that $u(t, x) (T_n^h) = \rho_n^{0,h,0}(t, x)$ and $D u(t, x) (T_n^h) = \rho_n^{1,h,0}(t, x)$. Hence (3.16) yields

$$\begin{aligned} \rho_{n,\alpha}^{1,h,0}(t, x) &= \langle S(t-r, x-*) \sigma(\rho_n^{0,h,0}(r, *)), e_j(*) \rangle_{\mathcal{H}} \\ &+ \sum_{k \geq 0} \int_r^t \langle S(t-s, x-*) \sigma'(\rho_n^{0,h,0}(r, *)), \rho_{n,\alpha}^{1,h,0}(s, *) e_k(*) \rangle_{\mathcal{H}} \\ &\times \{W_k(ds) - \dot{W}_k^n(s) ds + h_k(s) ds\} \\ &+ \int_r^t \int_{\mathbb{R}^2} S(t-s, x-y) b'(\rho_n^{0,h,0}(s, y)) \rho_{n,\alpha}^{1,h,0}(s, y) ds dy, \end{aligned}$$

with $\alpha = (r, j) \in [0, T] \times \mathbb{N}$.

Then, Theorem 2 with $k = 1$ and the coefficients $A = -B = G = \sigma$ yields

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\underline{x} \in K^d} E \left(\|(D u(t, \underline{x})) \circ T_n^h - \bar{D} \Phi^h(t, \underline{x})\|_{H_T(\mathbb{R}^d)}^p 1_{A_n^\gamma(t)} \right) = 0. \quad (3.17)$$

We conclude this section by completing Step 1 of our programme, as has been described in Section 1. Recall that $p_{t,\underline{x}}(y)$ denotes the density of the random vector $(u(t, x_1), \dots, u(t, x_d))$ at $y \in \mathbb{R}^d$.

Theorem 3. We assume

- (i') there exist $0 < a_2 \leq a_1 < 2$ such that $2(a_1 - a_2) < a_2 \wedge 1$, and a positive constant C_1 such that for $t \in [0, T]$, $C_1 t^{a_1} \leq \int_0^t y f(y) \ln \left(1 + \frac{t}{y}\right) dy$ and $\int_{0^+} y^{1-a_2} f(y) dy < \infty$;
- (ii') $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ is of class C^1 , bounded, with $\frac{1}{2}(a_2 \wedge 1)$ -Hölder continuous partial derivatives, $\nabla u_0 \in L^{q_1}(\mathbb{R}^2)$ for some $q_1 > 2$; $v_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ belongs to $L^{q_0}(\mathbb{R}^2)$ for some $q_0 \in \left]4 \vee \frac{2}{1-(a_2 \wedge 1)}, \infty\right]$;
- (iii) the coefficients σ and b are C^∞ functions with bounded derivatives of any order $i \geq 1$;
- (iv) there exists $a > 0$ such that $|\sigma(u(t, x_j))| \geq a$, for any $j = 1, \dots, d$, a.s.

Then the next two statements on $y \in \mathbb{R}^d$ are equivalent: (a) $p_{t,\underline{x}}(y) > 0$ and (b) there exists $h \in H_T$ such that $\Phi^h(t, \underline{x}) = y$ and $\det \gamma_{\Phi^h(t, \underline{x})} > 0$.

Proof. First we establish (b) \Rightarrow (a). With this purpose, we apply Proposition 1 to $F = (u(t, x_1), \dots, u(t, x_d))$. Let $v_n = u \circ T_n^h$ with T_n^h defined by (2.15). The process $\{v_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$ satisfies the equation

$$\begin{aligned} v_n(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \langle S(t-s, x-*) \sigma(v_n(s, *)), e_j(*) \rangle_{\mathcal{H}} \{W_j(ds) \\ &- \dot{W}_j^n(s) ds + h_j(s) ds\} + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(v_n(s, y)) ds dy, \end{aligned}$$

which is a particular case of equation (3.1) with $A = -B = G = \sigma$. For this choice of coefficients, equation (3.2) coincides with that satisfied by the skeleton $\Phi^h(t, x)$ (see (1.9)). Then, Theorem 2 yields

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T} \sup_{\underline{x} \in K^d} E \left(\| (u \circ T_n^h)(t, \underline{x}) - \Phi^h(t, x) \|^p 1_{A_n^\gamma(t)} \right) = 0,$$

for any $p \in (1, \infty)$, $\gamma > 2T \ell n 2$ and every compact set $K \subset \mathbb{R}^2$. By (2.17) this ensures condition (2.2).

The validity of (2.3) follows from (3.17) and (2.17). Finally (2.4) has been proved in Lemma 4.

Let us now check (a) \Rightarrow (b). Consider the process defined by the evolution equation

$$\begin{aligned} u_n(t, x) &= X^0(t, x) + \sum_{j \geq 0} \int_0^t \langle S(t-s, x - *) \sigma(u_n(s, *)), e_j(*) \rangle_{\mathcal{H}} \dot{W}_j^n(s) ds \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b(u_n(s, y)) ds dy. \end{aligned}$$

This is a particular case of equation (3.1) with $A = G = 0$, $B = \sigma$. Moreover, the process $\{u_n(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$ coincides with the skeleton $\{\Phi^{\dot{W}^n}(t, x), (t, x) \in [0, T] \times \mathbb{R}^2\}$; with this choice of coefficients, equation (3.2) coincides with (1.3). Theorem 2 shows that the assumptions of Proposition 2 are satisfied with $\Phi(h) = \Phi^h(t, \underline{x})$, $F = u(t, \underline{x})$ and the localizing sequence $(A_n^\gamma(t))_{n \in \mathbb{N}}$ defined in (2.16). This completes the proof of the theorem. \square

4. POSITIVITY OF THE DENSITY

The purpose of this section is to analyze under which conditions property (b) of Theorem 3 is satisfied for any $y \in \mathbb{R}^d$, that is, the density $p_{t, \underline{x}}(y)$ is strictly positive everywhere.

Theorem 4. *Suppose that the assumptions (i', ii', iii) of Theorem 3 are satisfied. Moreover, we assume*

(iv') $\{\inf |\sigma(z)|, z \in \mathbb{R}\} > C$, for some constant $C > 0$;

(v) *the covariance functional J defined in (1.8) is positive.*

Then, for any $y \in \mathbb{R}^d$ there exists $h \in H_T$ such that $\Phi^h(t, \underline{x}) = y$ and $\det \gamma_{\Phi^h(t, \underline{x})} > 0$.

Proof. The nondegeneracy of the matrix $\gamma_{\Phi^h(t, \underline{x})}$ has been established in Proposition 3 of the Appendix. Hence it only remains to prove that each $y \in \mathbb{R}^d$ can be reached, with an appropriate choice of $h \in H_T$, through the skeleton.

Set $y = (y_1, \dots, y_d)$ and let $\lambda_1, \dots, \lambda_d \in \mathbb{R}$ satisfying the linear system

$$y_\ell - X^0(t, x_\ell) = \sum_{i=1}^d \lambda_i \gamma_S^{i, \ell}, \quad \ell = 1, \dots, d,$$

where $\gamma_S^{i, \ell} = \langle S(t - \cdot, x_i - *), S(t - \cdot, x_\ell - *) \rangle_{L^2([0, T]; \mathcal{H})}$.

Let $k(s, z) = \sum_{i=1}^d \lambda_i S(t - s, x_i - z)$, $K(s, y) = \int_{\mathbb{R}^2} f(|y - z|) k(s, z) dz$, $s \in [0, T]$, $z, y \in \mathbb{R}^2$. Notice that $k \in L^2([0, T]; \mathcal{H})$ and by Schwarz's inequality

$$\int_0^T ds \int_{\mathbb{R}^2} dy |K(s, y)| S(t - s, x - y) < \infty. \quad (4.1)$$

For any $\tau \in [0, T]$, $x \in \mathbb{R}^2$, set

$$\varphi(\tau, x) = X^0(\tau, x) + \int_0^\tau ds \int_{\mathbb{R}^2} dy K(s, y) S(\tau - s, x - y). \quad (4.2)$$

By construction, $\varphi(t, x_\ell) = y_\ell$, $\ell = 1, \dots, d$.

Let \mathcal{Y} be the linear space consisting of functions $H : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\int_0^T ds \int_{\mathbb{R}^2} dy |H(s, y)| S(t - s, x - y) < \infty$. Consider the linear operator

$$\begin{aligned} \mathcal{T} &= L^2([0, T]; \mathcal{H}) \longrightarrow \mathcal{Y} \\ h &\longmapsto \int_{\mathbb{R}^2} f(|y - z|) h(s, z) dz. \end{aligned}$$

Hypothesis (v) ensures that \mathcal{T} admits an inverse operator \mathcal{T}^{-1} . Indeed, 0 is not an eigenvalue of \mathcal{T} .

Set

$$H_0(s, y) = - \left[b(\varphi(s, y)) - K(s, y) \right] \sigma(\varphi(s, y))^{-1}, \quad (s, y) \in [0, T] \times \mathbb{R}^2. \quad (4.3)$$

It is not difficult to check that $H_0 \in \mathcal{Y}$. Therefore

$$H_0(s, y) = \int_{\mathbb{R}^2} f(|y - z|) h_0(s, z) dz,$$

for some $h_0 \in L^2([0, T]; \mathcal{H})$. Let $h_0^j(s) = \langle h_0(s, *), e_j(*) \rangle_{\mathcal{H}}$, $j \geq 0$. Then $(h_0^j, j \geq 0) \in H_T$.

Substituting the function $K = H_0$ given by (4.3) in (4.2) we conclude that $\Phi^{h_0} \equiv \varphi$, by uniqueness of solution, and consequently $\Phi^{h_0}(t, x_\ell) = y_\ell$, $\ell = 1, \dots, d$, as we wanted to prove. \square

5. APPENDIX

Let $S(t, x)$, $(t, x) \in [0, \infty[\times \mathbb{R}^2$ be the fundamental solution of the stochastic wave equation. We start this section by quoting some notations and results concerning S that have been proved in [10] and [8] and used along the proofs.

For any $t \in [0, T]$, $h \geq 0$, set

$$J(t) = \int_{|y| < |x| < t} \frac{1}{\sqrt{t^2 - |x|^2}} f(|x - y|) \frac{1}{\sqrt{t^2 - |y|^2}} dx dy, \quad (A.1)$$

$$\mu(t) = \int_0^t ds \int_{\mathbb{R}^2} dx \int_{\mathbb{R}^2} dy S(s, x) f(|x - y|) S(s, y) = \frac{1}{2\pi^2} \int_0^t J(s) ds,$$

$$\nu(t) = \frac{1}{2\pi} \int_0^t ds \int_{|x| < s} \frac{dx}{\sqrt{s^2 - |x|^2}} = \frac{t^2}{2},$$

$$\begin{aligned} \mu_{t,h} &= \int_0^t ds \int_{|y| < s} dy \int_{|z| < s} dz [S(s, y) - S(s+h, y)] f(|y - z|) \\ &\quad \times [S(s, z) - S(s+h, z)], \end{aligned} \quad (A.2)$$

$$\tilde{\mu}_{t,h} = \int_0^t ds \int_{s \leq |y| < s+h} dy \int_{s \leq |z| < s+h} dz S(s+h, y) f(|y - z|) S(s+h, z). \quad (A.3)$$

Assume that f satisfies the assumption (C1) of Section 3; then Lemma A.1 in [10] implies

$$J(t) \leq C t^\beta, \quad \mu(t) \leq C t^{\beta+1}, \quad t \in [0, T], \quad \beta < \beta_0. \quad (\text{A.4})$$

Moreover, for h small enough and $0 < \delta < \beta_0 \wedge 1$,

$$\mu_{t,h} + \tilde{\mu}_{t,h} \leq C h^\delta \quad (\text{A.5})$$

(see Lem. A.5 in [10] and [8], Rem. A.6).

For fixed distinct points $x_1, \dots, x_d \in \mathbb{R}^2$, set

$$\gamma_S = \left(\langle S(t - \cdot, x_i - *), S(t - \cdot, x_j - *) \rangle_{L^2([0, T]; \mathcal{H})} \right)_{1 \leq i, j \leq d}.$$

The next lemma provides one of the ingredients in the proofs of Section 4.

Lemma 5. *Assume that there exist constants $C_1 > 0$ and $a_1 \in (0, 2)$ such that for $t \in [0, T]$,*

$$C_1 t^{a_1} \leq \int_0^t y f(y) \ln\left(1 + \frac{t}{y}\right) dy.$$

Then, $\det \gamma_S > 0$.

Proof. It suffices to prove that for any $\underline{v} = (v_1, \dots, v_d) \in \mathbb{R}^d$, $|v| = 1$,

$$\Gamma(s, v) := v^t \gamma_S v = \int_0^t dr \left\| \sum_{i=1}^d v_i S(t-r, x_i - *) \right\|_{\mathcal{H}}^2 > 0.$$

Let $\varepsilon \in (0, t)$ to be determined later. Then $\Gamma(s, v) \geq I_1 - |I_2|$ with

$$\begin{aligned} I_1 &= \sum_{i=1}^d \int_{t-\varepsilon}^t v_i^2 \|S(t-r, x_i - *)\|_{\mathcal{H}}^2 dr = \mu(\varepsilon), \\ I_2 &= \sum_{\substack{i, j=1 \\ i \neq j}}^d \int_{t-\varepsilon}^t v_i v_j \langle S(t-r, x_i - *), S(t-r, x_j - *) \rangle_{\mathcal{H}} dr. \end{aligned}$$

Lemma A.1 [10] together with the assumption of the lemma yield $I_1 \geq C \varepsilon^{a_1+1}$.

Let $m = \inf\{|x_i - x_j|, i \neq j\}$, $M = \sup\{|x_i - x_j|, i \neq j\}$. Then, if $4\varepsilon < m$, for any $y \in \mathbb{R}^2$, $|y - x_i| \leq \varepsilon$, $z \in \mathbb{R}^2$, $|z - x_j| \leq \varepsilon$, we have $\frac{m}{2} \leq |y - z| \leq 2\varepsilon + M$. Therefore $\sup\{f(|y - z|), |y - x_i| \leq \varepsilon, |z - x_j| \leq \varepsilon\} \leq C$. Then

$$\begin{aligned} & \int_{t-\varepsilon}^t \langle S(t-r, x_i - *), S(t-r, x_j - *) \rangle_{\mathcal{H}} dr \\ &= \int_{t-\varepsilon}^t dr \int_{|x_i - y| \leq t-r} dy \int_{|x_j - z| \leq t-r} dz S(t-r, x_i - y) f(|y - z|) S(t-r, x_j - z) \\ &\leq C \int_{t-\varepsilon}^t dr \left(\int_{|x-y| \leq t-r} S(t-r, x-y) dy \right)^2 \leq C \varepsilon^3. \end{aligned}$$

Since $a_1 + 1 < 3$, taking ε small enough we obtain $I_1 - |I_2| > 0$. □

Fix $(t, x) \in [0, T] \times \mathbb{R}^2$ and assume (1.5) and the hypotheses (ii', iii) of Theorem 3. Then the map $h \in H_T \rightarrow \Phi^h(t, x)$ is infinitely Fréchet differentiable. For any integer $k \geq 1$, the k -th order Fréchet derivative $\bar{D}^k \Phi^h(t, x)$ satisfies a differentiable equation like (3.8) with $A = -B = G = \sigma$. In particular, for $k = 1$.

$$\begin{aligned} \bar{D}_{r,j} \Phi^h(t, x) &= \langle S(t-r, x-*) \sigma(\Phi^h(r, *)), e_j(*) \rangle_{\mathcal{H}} \\ &\quad + \sum_{k \geq 0} \int_0^t ds \langle S(t-s, x-*) \sigma'(\Phi^h(s, *)) D_{r,j} \Phi^h(s, *), e_k(*) \rangle_{\mathcal{H}} h_k(s) \\ &\quad + \int_0^t \int_{\mathbb{R}^2} S(t-s, x-y) b'(\Phi^h(s, y)) D_{r,j} \Phi^h(s, y) ds dy, \end{aligned}$$

if $r \leq t$ and $\bar{D}_{r,j} \Phi^h(t, x) = 0$, if $t < r$.

It is easy to check that for any $(t, x) \in [0, T] \times \mathbb{R}^2$,

$$\sup_{\|h\|_{H_T} \leq a} \|\bar{D} \Phi^h(t, x)\|_{H_T} \leq C < \infty.$$

We now prove the nondegeneracy of the deterministic Malliavin matrix

$$\gamma_{\Phi^h(t, \underline{x})} = \left(\langle \bar{D} \Phi^h(t, x_i), \bar{D} \Phi^h(t, x_j) \rangle_{H_T} \right)_{1 \leq i, j \leq d}.$$

Proposition 3. *Assume (i', ii', iii) and (iv'). Then $\det \gamma_{\Phi^h(t, \underline{x})}$ is strictly positive.*

Proof. Set $\bar{D}_{r,j} \Phi^h(t, x) = \varphi_{r,j}(t, x) + \psi_{r,j}(t, x)$, with $\varphi_{r,j}(t, x) = \langle S(t-r, x-*) \sigma(\Phi^h(r, *)), e_j(*) \rangle_{\mathcal{H}}$. Let $v \in \mathbb{R}^d$, $|v| = 1$.

The triangle inequality yields

$$v^* \gamma_{\Phi^h(t, \underline{x})} v = \sum_{k=0}^{\infty} \int_0^t dr \left| \sum_{i=1}^d v_i \bar{D}_{r,k} \Phi^h(t, x_i) \right|^2 \geq \frac{1}{2} J_1 - J_2,$$

with

$$\begin{aligned} J_1 &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \left| \sum_{i=1}^d v_i \varphi_{r,k}(t, x_i) \right|^2, \\ J_2 &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \left| \sum_{i=1}^d v_i \psi_{r,k}(t, x_i) \right|^2, \end{aligned}$$

$\gamma \in (0, t)$ to be determined later.

Lower bound for J_1 : We write $J_1 \geq \frac{1}{2} J_{11} - J_{12}$, where

$$\begin{aligned} J_{11} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \left| \sum_{i=1}^d v_i \sigma(\Phi^h(r, x_i)) \langle S(t-r, x_i-*) \sigma(\Phi^h(r, *)), e_k(*) \rangle_{\mathcal{H}} \right|^2, \\ J_{12} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \left| \sum_{i=1}^d v_i \langle S(t-r, x_i-*) \left[\sigma(\Phi^h(r, *)) - \sigma(\Phi^h(r, x_i)) \right], e_k(*) \rangle_{\mathcal{H}} \right|^2. \end{aligned}$$

Furthermore, $J_{11} \geq J_{111} - |J_{112}|$, with

$$\begin{aligned} J_{111} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \sum_{i=1}^d v_i^2 \sigma^2(\Phi^h(r, x_i)) \langle S(t-r, x_i - *) \rangle_{\mathcal{H}}^2, \\ J_{112} &= \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \sum_{\substack{i,j=1 \\ i \neq j}}^d v_i v_j \sigma(\Phi^h(r, x_i)) \sigma(\Phi^h(r, x_j)) \\ &\quad \times \langle S(t-r, x_i - *) \rangle_{\mathcal{H}} \langle S(t-r, x_j - *) \rangle_{\mathcal{H}}. \end{aligned}$$

The lower bound assumption in (i') and (iv') yield

$$J_{111} \geq C \gamma^{1+a_1}. \quad (\text{A.6})$$

Parseval's identity and the growth condition on σ yield

$$|J_{112}| \leq C \sup_{(s,x) \in [0,T] \times \mathbb{R}^2} \left(1 + |\Phi^h(s,x)|\right)^2 \int_{t-\gamma}^t \langle S(t-r, x_i - *) \rangle_{\mathcal{H}} \langle S(t-r, x_j - *) \rangle_{\mathcal{H}} dr,$$

with $i \neq j$. Notice that $\sup_{(s,x) \in [0,T] \times \mathbb{R}^2} |\Phi^h(t,x)| \leq C$,

Therefore, as has been checked in the proof of Lemma 5,

$$|J_{112}| \leq C \gamma^3. \quad (\text{A.7})$$

Then (A.6) and (A.7) yield

$$|J_{11}| \geq C (\gamma^{1+a_1} - \gamma^3). \quad (\text{A.8})$$

Following the same ideas as in the proof of Theorem 2.2 in [8] one can prove that $\sup_{t \in [0,T]} \sup_{|y-z| \leq \xi} (|\Phi^h(t,y) - \Phi^h(t,z)|) \leq C \xi^\alpha$, with $\alpha \in (0, \frac{1}{2} (a_2 \wedge 1))$. Hence

$$\begin{aligned} J_{12} &\leq C \gamma^{2\alpha} \sup \left\{ \int_{t-\gamma}^t \langle S(t-r, x_i - *) \rangle_{\mathcal{H}} \langle S(t-r, x_j - *) \rangle_{\mathcal{H}} dr; i, j = 1, \dots, d \right\} \\ &\leq C \gamma^{2\alpha+1+a_2}. \end{aligned} \quad (\text{A.9})$$

Consequently (A.8) and (A.9) imply

$$J_1 \geq C (\gamma^{1+a_1} - \gamma^3 - \gamma^{2\alpha+1+a_2}). \quad (\text{A.10})$$

Upper bound for J_2 : By the definition of $\psi_{r,k}(t, x)$ and J_2 , we clearly have $J_2 \leq C(J_{21} + J_{22})$ with

$$J_{21} = \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \sum_{i=1}^d v_i^2 \left(\sum_{j=0}^{\infty} \int_r^t \left\langle S(t-s, x_i - *) \sigma'(\Phi^h(s, *)) D_{r,k} \Phi^h(s, *) e_j(*) \right\rangle_{\mathcal{H}} h_j(s) ds \right)^2,$$

$$J_{22} = \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \sum_{i=1}^d v_i^2 \left(\int_r^t ds \int_{\mathbb{R}^2} dy S(t-s, x_i - y) b'(\Phi^h(s, y)) D_{r,k} \Phi^h(s, y) \right)^2.$$

Schwarz's inequality and Parseval's identity ensure

$$J_{21} \leq C \sum_{k=0}^{\infty} \int_{t-\gamma}^t dr \sum_{i=1}^d v_i^2 \int_r^t \left\| S(t-s, x_i - *) \sigma'(\Phi^h(s, *)) D_{r,k} \Phi^h(s, *) \right\|_{\mathcal{H}}^2 ds.$$

Then, applying Fubini's theorem and Schwarz's inequality this term is bounded by

$$C \left(\sup_{(s,y) \in [t-\gamma, t] \times \mathbb{R}^2} \sum_{k=1}^{\infty} \int_{t-\gamma}^t \|D_{r,k} \Phi^h(s, y)\|_{\mathcal{H}}^2 dr \right) \mu(\gamma).$$

Following the proof of Theorem 2.2 in [10] we obtain $J_{21} \leq C \mu(\gamma)^2$. Then, by (A.4),

$$J_{2,1} \leq C \gamma^{2(a_2+1)} \tag{A.11}$$

Jensen's inequality, Fubini's theorem and similar arguments as those used to obtain (A.11) yield

$$J_{22} \leq C \mu(\gamma) \nu(\gamma) \leq C \gamma^{3+a_2}. \tag{A.12}$$

Therefore, by (A.11, A.12)

$$J_2 \leq C \left(\gamma^{2(a_2+1)} + \gamma^{3+a_2} \right). \tag{A.13}$$

Finally (A.10) and (A.13) yield

$$v^* \gamma_{\Phi^h(t, \underline{x})} v \geq C \left(\gamma^{1+a_1} - \gamma^3 - \gamma^{2\alpha+1+a_2} - \gamma^{2(a_2+1)} - \gamma^{3+a_2} \right). \tag{A.14}$$

Set $\gamma = \varepsilon^\delta$, $\delta > 0$, such that

$$\delta(1 + a_1) < 1, \quad 3\delta > 1, \quad \delta(2\alpha + 1 + a_2) > 1, \quad 2\delta(a_2 + 1) > 1, \quad \delta(3 + a_2) > 1,$$

where $\alpha \in (0, \frac{1}{2}(a_2 \wedge 1))$ and a_1, a_2 satisfy the restrictions stated in (i'). It is easy to check that such a choice is possible. Then, the right hand-side of (A.14) is strictly positive. This finishes the proof of the proposition. \square

REFERENCES

- [1] S. Aida, S. Kusuoka and D. Stroock, *On the support of Wiener functionals*, edited by K.D. Elworthy and N. Ikeda, Asymptotic Problems in Probability Theory: Wiener Functionals and Asymptotics. Longman Scient. and Tech., New York, *Pitman Res. Notes in Math. Ser.* **284** (1993) 3-34.

- [2] V. Bally and E. Pardoux, Malliavin Calculus for white-noise driven parabolic spde's. *Potential Anal.* **9** (1998) 27-64.
- [3] G. Ben Arous and R. Léandre, Décroissance exponentielle du noyau de la chaleur sur la diagonale (II). *Probab. Theory Related Fields* **90** (1991) 377-402.
- [4] R. Dalang and N. Frangos, The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26** (1998) 187-212.
- [5] O. Lévêque, *Hyperbolic stochastic partial differential equations driven by boundary noises*. Thèse EPFL, Lausanne, 2452 (2001).
- [6] D. Mrquez-Carreras, M. Mellouk and M. Sarrà, On stochastic partial differential equations with spatially correlated noise: Smoothness of the law. *Stochastic Proc. Appl.* **93** (2001) 269-284.
- [7] M. Métivier, *Semimartingales*. De Gruyter, Berlin (1982).
- [8] A. Millet and P.-L. Morien, On a stochastic wave equation in two dimensions: Regularity of the solution and its density. *Stochastic Proc. Appl.* **86** (2000) 141-162.
- [9] A. Millet and M. Sanz-Solé, Points of positive density for the solution to a hyperbolic spde. *Potential Anal.* **7** (1997) 623-659.
- [10] A. Millet and M. Sanz-Solé, A stochastic wave equations in two space dimension: Smoothness of the law. *Ann. Probab.* **27** (1999) 803-844.
- [11] A. Millet and M. Sanz-Solé, Approximation and support theorem for a two space-dimensional wave equation. *Bernoulli* **6** (2000) 887-915.
- [12] P.-L. Morien, Hölder and Besov regularity of the density for the solution of a white-noise driven parabolic spde. *Bernoulli* **5** (1999) 275-298.
- [13] D. Nualart, *Malliavin Calculus and Related Fields*. Springer-Verlag (1995).
- [14] D. Nualart, Analysis on the Wiener space and anticipating calculus, in *École d'été de Probabilités de Saint-Flour*. Springer-Verlag, *Lecture Notes in Math.* **1690** (1998) 863-901.
- [15] M. Sanz-Solé and M. Sarrà, Path properties of a class of Gaussian processes with applications to spde's, in *Stochastic Processes, Physics and Geometry: New interplays*, edited by F. Gesztesy *et al.* American Mathematical Society, *CMS Conf. Proc.* **28** (2000) 303-316.
- [16] J.B. Walsh, An introduction to stochastic partial differential equations, in *École d'été de Probabilités de Saint-Flour*, edited by P.L. Hennequin. Springer-Verlag, *Lecture Notes in Math.* **1180** (1986) 266-437.