# ON EXTREME VALUE THEORY FOR GROUP STATIONARY GAUSSIAN PROCESSES 

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#### Abstract

We study extreme value theory of right stationary Gaussian processes with parameters in open subsets with compact closure of (not necessarily Abelian) locally compact topological groups. Even when specialized to Euclidian space our result extend results on extremes of stationary Gaussian processes and fields in the literature by means of requiring weaker technical conditions as well as by means of the fact that group stationary processes need not be stationary in the usual sense (that is, with respect to addition as group operation).


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## 1. Introduction

Let $\{X(t)\}_{t \in T}$ be a standardized (having mean zero and variance one) right stationary separable Gaussian process defined on a (not necessarily Abelian) locally compact separable metric topological group $T=(T, \bullet, \varrho)$ with group operation - and metric $\varrho$. We assume that $X$ is continuous in the sense of squared mean ( $\mathbb{L}^{2}$-) convergence. Here right stationarity means that the covariance $\mathbf{E}\{X(s) X(t)\}$ is invariant under time shifts from the right. Hence there exists a one-parameter covariance function $r: T \rightarrow[-1,1]$ given by $r(t):=\mathbf{E}\{X(I) X(t)\}$ where $I$ is the identity element of the group $(T, \bullet)$ such that

$$
\mathbf{E}\{X(s) X(t)\}=\mathbf{E}\left\{X(I) X\left(t \bullet s^{-1}\right)\right\}=r\left(t \bullet s^{-1}\right) \quad \text { for } s, t \in T .
$$

In addition to the original metric $\varrho$ on $T$ we also consider the covariance pseudo-metric $\rho$ on $T$ given by

$$
\begin{equation*}
\rho(s, t)=\rho\left(I, t \bullet s^{-1}\right):=\sqrt{\mathbf{E}\left\{(X(t)-X(s))^{2}\right\}}=\sqrt{2-2 r\left(t \bullet s^{-1}\right)} \quad \text { for } s, t \in T \tag{1.1}
\end{equation*}
$$

where $s^{-1}$ denotes the group inverse of $s$ (that makes $s \bullet s^{-1}=s^{-1} \bullet s=I$ ). Note that the $\rho$-topology is weaker than the $\varrho$-topology by the assumed continuity of $X$.

Write $S_{\varepsilon}:=\{t \in T: \rho(S, t)<\varepsilon\}$ for $S \subseteq T$ and $\varepsilon>0$, where $\rho(S, t)=\inf \{\rho(s, t): s \in S\}$. We will study the asymptotic behaviour of the tail probability $\mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\}$ as $u \rightarrow \infty$ for a non-empty $\varrho$-open set $K \subseteq T$

[^0]such that $K_{\varepsilon}$ has $\varrho$-compact closure $\operatorname{clos}\left(K_{\varepsilon}\right)$ for some $\varepsilon>0$ and $r\left(t \bullet s^{-1}\right)<1$ for $s, t \in \operatorname{clos}\left(K_{\varepsilon}\right)$ [making $\rho$ a metric on $\left.\operatorname{clos}\left(K_{\varepsilon}\right)\right]$.

Now a $\rho$-convergent sequence $\left\{t_{i}\right\}_{i=1}^{\infty}$ in $\operatorname{clos}\left(K_{\varepsilon}\right)$ must be $\varrho$-convergent since if $\rho\left(t_{i}, t\right) \rightarrow 0$, then any $\varrho$ convergent subsequence $\left\{t_{i_{k}}\right\}$ with limit $\hat{t}$ say must satisfy $\rho(\hat{t}, t)=0$ (as $\rho$ is weaker than $\varrho$ ) so that every $\varrho$ convergent subsequence $\left\{t_{i_{k}}\right\}$ of $\left\{t_{i}\right\} \varrho$-converges to $t$, proving that the $\varrho$-topology is weaker than the $\rho$-topology. Hence the $\varrho$-topolgy and $\rho$-topolgy on $\operatorname{clos}\left(K_{\varepsilon}\right)$ coincide.

Let $\mu$ denote a right Haar measure on $T$ [satisfying $\mu(S \bullet t)=\mu(S)$ for $S \subseteq T$ and $t \in T$ ] and put $q(u)=$ $\mu\left(\{I\}_{1 / u}\right)$ for $u>0$. Further, let $\bar{\Phi}(u)=\mathbf{P}\{\mathrm{N}(0,1)>u\}$ for $u \in \mathbb{R}$

Our main results are Theorems 1.1-1.3. In Section 2 we will give several examples of application of Theorems 1.2 and 1.3 that illustrate their usefulness. In connection with these examples we also make several bibliographical notes. In Section 3 we prove three technical lemmas on covering numbers that are needed for the proof of Theorems 1.1 and 1.2. Finally, in Section 4-6 we prove Theorems 1.1-1.3, respectively.

Theorem 1.1. With the above notation and assumptions, assume in addition that the following limit superior is finite

$$
\begin{equation*}
M:=\limsup _{u \rightarrow \infty} \int_{t \in\{I\}_{2 / u}} \mathbf{P}\{X(t)>u \mid X(I)>u\} \frac{\mathrm{d} \mu(t)}{q(u)}<\infty \tag{1.2}
\end{equation*}
$$

Then $\{X(t)\}_{t \in K}$ has an a.s. continuous version and it holds that

$$
\begin{equation*}
0<\liminf _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} \leq \limsup _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\}<\infty \tag{1.3}
\end{equation*}
$$

Theorem 1.2. With the above notation and assumptions, assume in addition that the following limit exists (and is finite) for every choice of $N, n \in \mathbb{N}$ :

$$
\begin{equation*}
m(N, n):=\lim _{u \rightarrow \infty} \int_{t \in\left(\{I\}_{N / u}\right)^{n}} \mathbf{P}\left\{\bigcap_{i=1}^{n}\left\{X\left(t_{i}\right)>u\right\} \mid X(I)>u\right\} \frac{\mathrm{d} \mu^{n}(t)}{q(u)^{n}} \tag{1.4}
\end{equation*}
$$

Then there exists a sequence of probability distribution functions $\left\{G_{N}\right\}_{N=1}^{\infty}$ on $(0, \infty)$ such that $G_{N}$ has n-th moment $m(N, n)$ for $N, n \in \mathbb{N}$. Further, the limit $G(x)=\lim _{N \rightarrow \infty} G_{N}(x)$ exists for $x>0$ as do the limit

$$
\begin{equation*}
H:=\lim _{x \downarrow 0} \int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y} \tag{1.5}
\end{equation*}
$$

with value $H \in(0, \infty)$ and

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\}=\mu(K) H \tag{1.6}
\end{equation*}
$$

Theorem 1.3. Let $V_{u}: T \rightarrow T$ be a measurable function such that $V_{u} I=I$ and $\{I\}_{\delta / u} \subseteq V_{u} T$ for $u>0$ sufficiently large for any choice of $\delta>0$. Then (1.4) holds if there exist measurable functions $\lambda: T \rightarrow[0, \infty)$ and $\psi: T^{2} \rightarrow[0, \infty)$ such that

$$
\begin{align*}
\mathfrak{O}_{\delta}:= & \{t \in T: \psi(I, t)<\delta\} \quad \text { has compact closure for } \delta>0  \tag{1.7}\\
& \lim _{u \rightarrow \infty} u \rho\left(V_{u} s, V_{u} t\right)=\psi(s, t) \quad \text { for } s, t \in T \tag{1.8}
\end{align*}
$$

$$
\begin{gather*}
\lim _{u \rightarrow \infty} \mu\left(\left\{t \in T: u \rho\left(I, V_{u} t\right)<\delta\right\} \Delta \mathfrak{O}_{\delta}\right)=0 \quad \text { for } \delta>0,  \tag{1.9}\\
\mu \circ V_{u}^{-1} \text { is absolutely continuous with respect to } \mu \quad \text { for } u>0 \quad \text { sufficiently large, }  \tag{1.10}\\
\lim _{u \rightarrow \infty} q(u) \frac{\mathrm{d}\left(\mu \circ V_{u}^{-1}\right)}{\mathrm{d} \mu}\left(V_{u} t\right)=\lambda(t) \text { a.e. ( } \mu \text { ) for } t \in T,  \tag{1.11}\\
\liminf _{u \rightarrow \infty} q(u) \underset{t \in\{I\}_{\delta / u}}{\operatorname{ess} \inf } \frac{\mathrm{~d}\left(\mu \circ V_{u}^{-1}\right)}{\mathrm{d} \mu}(t)>0 \quad \text { for } \delta>0 . \tag{1.12}
\end{gather*}
$$

The requirement (1.4) is an abstract (topological group) version of convergence of suitably normalized finite dimensional distributions that is typically assumed in works on extreme value theory for stochastic processes on Euclidian space as e.g., Condition A in Albin [3].

Argubly, our expression (1.5) for the constant $H$ is somewhat less explicit than corresponding expressions in the literature where Euclidian space structure is used in the proofs. However, except in very special cases the value of $H$ cannot be calculated explicitely anyway so one might argue that there is not terribly much lost from this.

Our setting with a topological group parameter space is the natural one for a continuous stationary process. Except for the group structure it is basically the same setting as was first used by Dudley [9] and Fernique [11] to study continuity and boundedness of Gaussian processes and later by e.g., Adler [2] and Samorodnitsky [18] to obtain upper and lower bounds on Gaussian extrema. Our contribution is to obtain sharp asymptotic estimates for extremes and to that end we have to assume stationarity which in turn requires a group structure. Our proofs require no Euclidian structure and rely on somewhat sharpened estimation techniques in line with what the above authors utilize together with an adaption of sojourn techniques developed by Berman in an array of papers, see [5] for a survey. Although Berman himself uses sojourns only in Euclidian settings they are not really thus restricted but their natural setting is argubly rather that of a topological group with a Haar sojourn time measure. However, the discrete approximation techniques that have origin in Pickands' papers [15, 16] and that is more commonly used in works on extreme value theory for stochastic processes do very much require the Euclidian space structure.

## 2. ExAMPLES OF APPLICATION

Our first four examples are Euclidian. Some of them are well-known results from the literature while others are not but in the latter case we do not in anyway claim that they cannot be established by clever usage of what is already known together with tools readily available to an expert in Gaussian extremes. Our point is instead that all our quite diverse examples come from application of the one and same Theorems 1.2 and 1.3. They do not require anything more than quite simple verifications of the conditions (1.7)-(1.12) of a purely computational character.

Our findings are primarily intended to be used for processes with a "continuous parameter". However, we think that it is worthwhile to shortly demonstrate that our findings also cover the discrete parameter case.

Example 2.1 (Discrete groups). If $\rho$ is discrete, that is, if $\{I\}_{\delta}=\{I\}$ for $\delta>0$ small enough, then the Haar measure $\mu$ is the counting measure and a set $K \subseteq T$ is open with compact closure if and only if $K$ is finite. Clearly, all limits in (1.4) are 1 , so that $G_{N}$ is an atom at $1, H=1$ and Theorem 1.2 gives

$$
\begin{equation*}
\mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} \sim(\# K) \bar{\Phi}(u) \quad \text { as } u \rightarrow \infty \tag{2.1}
\end{equation*}
$$

This applies, e.g., to a standardized time discrete stationary (in the usual sense) Gaussian process $\{X(t)\}_{t \in \mathbb{Z}}$ with $r(k)<1$ for $k \neq 0$.

Of course, it is very easy to derive the findings of Example 2.1 by a page or so of direct calculations. The point by still including the example is that we want to show that we can cover both the discrete and the continuous in a common treatment.

Our next example shows how a landmark result by Pickands $[15,16]$ from 1969 can be recovered quite easily from Theorems 1.2 and 1.3. See also, for example, [4] and Chapter 12 from [13] for more information about Pickands' result.

Example 2.2 (Pickands' $H_{\alpha}$ ). Take $(T, \bullet, \varrho)=(\mathbb{R},+,|\cdot|)$ and let $r$ satisfy

$$
\begin{equation*}
r(t)<1 \quad \text { for } t \in[-h, h] \backslash\{0\} \quad \text { and } \quad r(t)=1-|t|^{\alpha}-\mathrm{o}\left(|t|^{\alpha}\right) \text { as } t \rightarrow 0 \tag{2.2}
\end{equation*}
$$

for some constants $\alpha \in(0,2]$ and $h>0$. Now $\mu$ is the Lebesgue measure on $\mathbb{R}$ and it is quick work to check that (1.7)-(1.12) hold for $V_{u} t=u^{-2 / \alpha} t$ with $\varphi(s, t)=\sqrt{2|t-s|^{\alpha}}, \lambda(t)=2^{1-1 / \alpha}$ (not depending on $t \in T$ ), $\mathfrak{O}_{\delta}=\left(-\left(\delta^{2} / 2\right)^{1 / \alpha},\left(\delta^{2} / 2\right)^{1 / \alpha}\right)$ and $q(u) \sim 2^{1-1 / \alpha} u^{-2 / \alpha}$ as $u \rightarrow \infty$. And so Theorems 1.2 and 1.3 give

$$
\mathbf{P}\left\{\sup _{t \in[0, h]} X(t)>u\right\} \sim h H_{\alpha} u^{2 / \alpha} \bar{\Phi}(u) \quad \text { as } u \rightarrow \infty
$$

for some constant $H_{\alpha}>0$. This is he famous Pickands' constant, the only values of which are known are $H_{1}=1$ and $H_{2}=1 / \sqrt{\pi}$. However, starting as late as this century, there has appeared both theoretical bounds and numerical estimates for $H_{\alpha}$, see e.g., Burnecki and Michna [8].

Next we show how a result on extremes of Rayleigh processes by Albin ([3], Thm. 9) can be derived from Theorems 1.2 and 1.3. Extreme value studies of Rayleigh processes origin in Sharpe [20]. The way in which their extreme behaviour are obtain from that of Gaussian fields in our example was also used by Lindgren [14] and Piterbarg [17].
Example 2.3 (Rayleigh processes). Write $\langle x \mid y\rangle=\sum_{i=1}^{n} x_{i} y_{i}$ for $x, y \in \mathbb{R}^{n}$ and $S^{n-1}=\left\{x \in \mathbb{R}^{n}:\langle x \mid x\rangle=1\right\}$. Let $R_{x}$ be the rotation that maps $(1,0, \ldots, 0) \in S^{n-1}$ to $x \in S^{n-1}$ and let $S^{n-1}$ have group operation $x \star y=$ $R_{x} R_{y}(1,0, \ldots, 0)$. Take $T=\mathbb{R} \times S^{n-1}$ with group operation $(s, x) \bullet(t, y)=(s+t, x \star y)$ and metric $\varrho$ the Euclidian distance on $\mathbb{R}^{n+1}$. Let $Y_{1}, \ldots, Y_{n}$ be independent copies of the process $X$ considered in Example 2.2 with common covariance function $r$ satisfying (2.2). Take $\{X(t, x)\}_{(t, x) \in \mathbb{R} \times S^{n-1}}$ to be given by $X(t, x)=\langle x, Y(t)\rangle$ where $Y(t)=\left(Y_{1}(t), \ldots, Y_{n}(t)\right)$. Then $X$ has covariance function $\mathbf{E}\{X(s, x) X(t, y)\}=r(t-s)\langle x \mid y\rangle$ and is continuous in squared mean and right stationary. In this example we will study extremes of the Rayleigh process $|Y|$.

Employing spherical coordinates we can represent any $x \in S^{n-1}$ as

$$
x=x(\phi)=\left(\cos \left(\phi_{1}\right), \sin \left(\phi_{1}\right) \cos \left(\phi_{2}\right), \ldots,\left(\prod_{i=1}^{n-2} \sin \left(\phi_{i}\right)\right) \cos \left(\phi_{n-1}\right), \prod_{i=1}^{n-1} \sin \left(\phi_{i}\right)\right)
$$

for some $\phi=\left(\phi_{1}, \ldots, \phi_{n-1}\right) \in[0,2 \pi) \times[0, \pi] \times \ldots \times[0, \pi]$ with Haar measure

$$
\mathrm{d} \mu(t, x(\phi))=\left(\prod_{i=1}^{n-2} \sin ^{n-i-1}\left(\phi_{i}\right)\right) \mathrm{d} t \mathrm{~d} \phi_{1} \ldots \mathrm{~d} \phi_{n-1}
$$

Taking $V_{u}(t, x(\phi))=\left(u^{-2 / \alpha} t, x\left(\phi_{1} / u, \phi_{2}, \ldots, \phi_{n-1}\right)\right)$ we have

$$
\left(\mathrm{d}\left(\mu \circ V_{u}^{-1}\right) / \mathrm{d} \mu\right)(t, x(\phi))=u^{2 / \alpha+1} \sin ^{n-2}\left(\phi_{1} u\right) / \sin ^{n-2}\left(\phi_{1}\right)
$$

so that

$$
\left(\mathrm{d}\left(\mu \circ V_{u}^{-1}\right) / \mathrm{d} \mu\right)\left(V_{u}(t, x(\phi))\right) \sim u^{2 / \alpha+n-1} \sin ^{n-2}\left(\phi_{1}\right) / \phi_{1}^{n-2} \quad \text { as } u \rightarrow \infty
$$

It is also a straightforward matter to see that (1.7)-(1.12) hold with

$$
\psi((s, x(\phi)),(t, y(\theta)))=\sqrt{2 C|t-s|^{\alpha}+\phi_{1}^{2}+\theta_{1}^{2}-\phi_{1} \theta_{1} \cos \left(\phi_{2}\right) \cos \left(\theta_{2}\right)}
$$

$\mathfrak{O}_{\delta}=\left\{(t, x(\phi)) \in \mathbb{R} \times S^{n-1}: \sqrt{2 C|t|^{\alpha}+\phi_{1}^{2}}<\delta\right\}$ and

$$
q(u) \sim u^{2 / \alpha+n-1} \int_{\left\{(t, x(\phi)) \in \mathbb{R} \times S^{n-1}: 2 C|t|^{\alpha}+\phi_{1}^{2}<1\right\}} \phi_{1}^{n-2}\left(\prod_{i=2}^{n-2} \sin ^{n-i-1}\left(\phi_{i}\right)\right) \mathrm{d} t \mathrm{~d} \phi_{1} \ldots \mathrm{~d} \phi_{n-1}
$$

as $u \rightarrow \infty$. Using the elementary fact that $\sup _{(t, x) \in[0, h] \times S^{n-1}} X(t, x)=\sup _{t \in[0, h]}|Y(t)|$ together with Theorems 1.2 and 1.3 we may now conclude that the limit

$$
\lim _{u \rightarrow \infty} \frac{u^{1-n-2 / \alpha}}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{(t, x) \in[0, h] \times S^{n-1}} X(t, x)>u\right\}=\lim _{u \rightarrow \infty} \frac{u^{1-n-2 / \alpha}}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in[0, h]}|Y(t)|>u\right\}
$$

exists and is strictly positive and finite.
There exist quite complicated random field extensions of the fundamental result by Pickands discussed in Example 2.2. These extensions are due to Adler ([1], pp. 162-65), Bickel and Rosenblatt [6] and Piterbarg ([17], Sect. 7). In the next example we give further extensions of (the stationary process versions of) these extensions. While our extensions should in no way be hard to visualize being valid by an expert, it is interesting that their proofs come more or less just by inspection of that Theorem 1.3 applies while proofs in the literature are very technically complicated.

Example 2.4 (Fields on $\left.\mathbb{R}^{n}\right)$. Take $(T, \bullet, \varrho)=\left(\mathbb{R}^{n},+,|\cdot|\right)$ where $n=k_{1}+\ldots+k_{L}$ for some integers $k_{1}, \ldots, k_{L} \geq 1$. Let $\varrho_{1}, \ldots, \varrho_{L}:(0, \infty) \rightarrow(0, \infty)$ be measurable functions such that $\lim _{t \downarrow 0} \varrho_{\ell}(x t) / \varrho_{\ell}(t)=x^{\alpha_{\ell}}$ for $x>0$, for some constants $\alpha_{1}, \ldots, \alpha_{L} \in(0,2]$ (that is, $\varrho_{1}, \ldots, \varrho_{L}$ are regularly varying functions at $0^{+}$). Let there exist probability measures $m_{\ell}$ on $S^{k_{\ell}-1}$ such that

$$
Q:=\mu\left(\left\{\left(t^{1}, \ldots, t^{L}\right) \in \mathbb{R}^{k_{1}} \times \ldots \times \mathbb{R}^{k_{L}}: \sum_{\ell=1}^{L} \int_{S^{k_{\ell}-1}}\left|\left\langle t^{\ell}, \theta\right\rangle\right|^{\alpha_{\ell}} \mathrm{d} m_{\ell}(\theta)<1\right\}\right)<\infty
$$

and

$$
\left.1-r(t) \sim \sum_{\ell=1}^{L} \varrho_{\ell}\left(\left|t^{\ell}\right|\right) \int_{S^{k} \ell_{\ell}-1}\left|\left\langle t^{\ell} /\right| t^{\ell}\right|, \theta\right\rangle\left.\right|^{\alpha_{\ell}} \mathrm{d} m_{\ell}(\theta) \quad \text { as } \quad|t| \rightarrow 0
$$

Writing $\varrho_{\ell}^{\leftarrow}(t)=\inf \left\{s>0: \varrho_{\ell}(s) \geq t\right\}$ (that is, the left inverse of $\varrho_{\ell}$ ) we have

$$
\lim _{t \downarrow 0} \varrho_{\ell}^{\leftarrow}(x t) / \varrho_{\ell}^{\overleftarrow{ }}(t)=x^{1 / \alpha_{\ell}} \quad \text { for } \quad x>0 \quad \text { and } \quad \lim _{t \downarrow 0} \varrho_{\ell}\left(\varrho_{\ell}^{\leftarrow}(t)\right) / t=1
$$

see, e.g., ([7], Thm. 1.5.12). From this we see that (1.7)-(1.12) hold for $\left(V_{u} t\right)_{i}=\varrho_{\ell}\left(u^{-2}\right) t_{i}$ for $i \in\left\{1+k_{1}+\ldots+\right.$ $\left.k_{\ell-1}, \ldots, k_{1}+\ldots+k_{\ell}\right\}$ and $\ell \in\{1, \ldots, L\}$ (where $k_{0}:=0$ ) with $\psi(s, t)=\sqrt{\left.2 \sum_{\ell=1}^{L} \int_{S^{k}-1}|\langle | t-s|^{\ell}, \theta\right\rangle\left.\right|^{\alpha_{\ell}} \mathrm{d} m_{\ell}(\theta)}$, $\lambda(t)=Q$ and $q(u) \sim Q \prod_{\ell=1}^{L} \varrho_{\ell}^{\overleftarrow{( }}\left(u^{-2}\right)^{-k_{\ell}}$ as $u \rightarrow \infty$. For any non-empty open $K \subset \mathbb{R}^{n}$ with compact closure Theorems 1.2 and 1.3 therefore apply to show that the limit

$$
H=\lim _{u \rightarrow \infty} \frac{\prod_{\ell=1}^{L} \varrho_{\ell} \overleftarrow{( }\left(u^{-2}\right)^{k_{\ell}}}{\mu(K) \bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\}
$$

exists and is strictly positive and finite and independent of the choice of $K$.
Our findings apply to extend the highly abstract finding of Evans [10] on extremes of Gaussian process with local field type parameters. As the mathematics for this are somewhat unusual in more applied type probability papers like ours we omit these applications. But would still like to point out that they offer a possibility to move further away from the Euclidian setting than we actually do in our remaining examples.

So next instead comes a toy example of a process that is group stationary but not stationary in the ordinary sense (with respect to Euclidian addition).

Example $2.5\left(\mathbb{Z}_{m} \times((0, \infty), \cdot)\right)$. Take $T=\mathbb{Z}_{m} \times(0, \infty)=\{0,1, \ldots, m-1\} \times(0, \infty)$ with group operation $(i, s) \bullet(j, t)=\left(i+_{m} j, s+t\right)$ where $+_{m}$ is addition modulo $m$. Let the metric be $\varrho((i, s),(j, t))=\sqrt{\delta_{i-j}^{2}+|s-t|^{2}}$ (with $\delta_{i-j}$ as in Example 2.1), so that $\mu$ is the product measure of the counting measure and $\mathrm{d} t / t$ (with obvious notation). Pick a number $p \in(0,1)$ and take $X(i, t)=p Y(i)+\sqrt{1-p^{2}} Z(t)$ for $(i, t) \in \mathbb{Z}_{m} \times \mathbb{R}^{+}$, where $\{Y(i)\}_{i \in \mathbb{Z}_{m}}$ and $\{Z(t)\}_{t>0}$ are independent standardized stationary (with respect to their group operation addition) Gaussian processes such that $\mathbf{E}\{Y(i) Y(j)\}<1$ for $i \neq j$ and $\mathbf{E}\{Z(s) Z(t)\}=\left((s / t)^{2 H}+(t / s)^{2 H}-|\sqrt{t / s}-\sqrt{s / t}|^{2 H}\right)$ for an $H \in(0,1]$ [which is to say that $Z(t)=t^{-H} B^{H}(t)$ where $B^{H}(t)$ is fractional Brownian motion with index of self-similarity $H$ ].

It is not hard to see that (1.7)-(1.12) hold for $V_{u}(i, t)=\left(0,1+u^{-1 / H} t\right)$ giving $q(u) \sim C u^{-1 / H}$ for some constant $C>0$ so that

$$
\begin{equation*}
\mathbf{P}\left\{\sup _{(i, t) \in \mathbb{Z}_{m} \times\left[1, \mathrm{e}^{h}\right]} X(i, t)>u\right\} \sim m h H u^{1 / H} \bar{\Phi}(u) \quad \text { as } u \rightarrow \infty \tag{2.3}
\end{equation*}
$$

for some finite constant $H>0$ (that with some additional inspection can be seen not to depend on $m$ or $h$ ).
Clearly, it is not hard for an expert to establish Example 2.5 by usage of known results and a Bonferroni type of estimate together with the fact that the $Y$ - and $Z$-processes cooperate to give extreme values [making extremes of $X$ a factor $m$ times more likely than those of $\left.\{X(1, t)\}_{t>0}\right]$. But our point is that this example is already included in the range of our results and require no additional manipulations.

Now let $W$ be set-indexed Brownian motion on $T$, that is, a Gaussian process on measurable subsets of $T$ with $\mathbf{E}\{W(A)\}=0$ and $\mathbf{E}\{W(A) W(B)\}=\mu(A \cap B)$. The process $W$ can be viewed as a so called independently scattered Gaussian random measure on $T$ and a corresponding random integral

$$
\int_{T} f \mathrm{~d} W=\int_{t \in T} f(t) \mathrm{d} W(t)=\int_{t \in T} f(t) W(\mathrm{~d} t)
$$

is well-known to be well-defined for $f \in \mathbb{L}^{2}(T, \mu)$, see e.g., Samorodnitsky and Taqqu ([19], Chap. 3) (albeit the theory for the Gaussian integral can be developed easier than in their more general setting of integrals with respect to $\alpha$-stable random measures).

An easy way to find a whole bunch of standardized right stationary Gaussian processes on $T$ is to consider the moving average process

$$
\begin{equation*}
\{X(t)\}_{t \in T}=\left\{\int_{r \in T} f\left(t \bullet r^{-1}\right) \mathrm{d} W(r)\right\}_{t \in T} \tag{2.4}
\end{equation*}
$$

for $f \in \mathbb{L}^{2}(T, \mu)$ with $\|f\|_{\mathbb{L}^{2}(T, \mu)}=1$. For this process we have $\mathbf{E}\{X(t)\}=0$ and

$$
\begin{equation*}
\mathbf{E}\{X(s) X(t)\}=\int_{r \in T} f\left(s \bullet r^{-1}\right) f\left(t \bullet r^{-1}\right) \mathrm{d} \mu(r)=\int_{r \in T} f\left(r^{-1}\right) f\left(t \bullet s^{-1} \bullet r^{-1}\right) \mathrm{d} \mu(r) \tag{2.5}
\end{equation*}
$$

We now consider a Gaussian moving average processes as defined in the previous paragraph with parameter in the (non-Abelian) topological group of affine maps.

Example $2.6(\operatorname{Aff}(\mathbb{R}))$. The space $\operatorname{Aff}(\mathbb{R})$ of affine maps $\mathbb{R} \ni x \rightarrow a x+b \in \mathbb{R}$ with $(a, b) \in(0, \infty) \times \mathbb{R}$ is a group with group operation composition $(a, b) \bullet(c, d)=(a c, b+a d)$. We endow $(\operatorname{Aff}(\mathbb{R}, \bullet)$ with the Euclidian distance $\varrho$ [between $(a, b)$ and $(c, d)$ seen as members of $\mathbb{R}^{2}$ ] which makes it a locally compact topological group with right Haar measure $\mathrm{d} \mu(a, b)=\mathrm{d} a \mathrm{~d} b / a, I=(1,0)$ and $(a, b)^{-1}=(1 / a,-b / a)$.

Consider a standardized right stationary Gaussian moving processes on $T=\operatorname{Aff}(\mathbb{R})$ given by (2.4) so that by (2.5) (and as $f$ square-integrates to 1 ) we have

$$
\begin{equation*}
\rho(I,(a, b))^{2}=2 r(I)-2 r((a, b))=2-2 \int_{t \in T} f\left(t^{-1}\right) f\left((a, b) \bullet t^{-1}\right) \mathrm{d} \mu(t) \tag{2.6}
\end{equation*}
$$

To proceed from here we do (perhars not suprisingly) assume that

$$
\begin{equation*}
\rho(I,(1+x, y))^{2}=C|x|^{\alpha}+D|y|^{\beta}+o\left(|x|^{\alpha}\right)+o\left(|y|^{\beta}\right) \quad \text { as }(x, y) \rightarrow(0,0) \tag{2.7}
\end{equation*}
$$

for some constants $C, D>0$ and $\alpha, \beta \in(0,2]$. Clearly, many a common function $f$ selected to be used in (2.4)(2.6) will yield to (2.7). (More general conditions as in Exemple 2.4 could be imposed but wouldn't really add anything to our story.)

Taking $V_{u}(a, b)=\left(1+u^{-2 / \alpha}(a-1), u^{-2 / \beta} b\right)$ it is easy to see that (2.7) gives

$$
u \rho\left(V_{u}(a, b), V_{u}(c, d)\right)=u \rho\left(I, V_{u}(c, d) \bullet V_{u}(a, b)^{-1}\right) \rightarrow \sqrt{C|c-a|^{\alpha}+D|d-b|^{\beta}}
$$

as $u \rightarrow \infty$, where the right-hand side is our $\psi((a, b),(c, d))$ in (1.8). In a similar fashion we readily see that all other conditions of Theorem 1.3 hold with

$$
q(u) \sim\left(\int_{\left\{(x, y) \in \mathbb{R}^{2}: C|x|^{\alpha}+D|y|^{\beta}<1\right\}} d x d y\right) u^{-2 / \alpha-2 / \beta}:=Q u^{-2 / \alpha-2 / \beta}
$$

as $u \rightarrow \infty$ and

$$
\mu \circ V_{u}^{-1}(A)=\int_{\left\{(a, b) \in(0, \infty) \times \mathbb{R}: V_{u}((a, b)) \in A\right\}} \frac{\mathrm{d} a \mathrm{~d} b}{a}=\int_{(\hat{a}, \hat{b}) \in A} u^{2 / \alpha+2 / \beta} \frac{\mathrm{d} \hat{a} \mathrm{~d} \hat{b}}{1+u^{2 / \alpha}(\hat{a}-1)},
$$

giving

$$
q(u) \frac{\mathrm{d}\left(\mu \circ V_{u}^{-1}\right)}{\mathrm{d} \mu}\left(V_{u}(a, b)\right)=\left.\frac{Q \hat{a}}{1+u^{2 / \alpha}(\hat{a}-1)}\right|_{(\hat{a}, \hat{b})=V_{u}(a, b)} \rightarrow \frac{Q}{a}:=\lambda(a, b)
$$

as $u \rightarrow \infty$. And so we have (1.6) with $q(u)$ as above.
One might also consider Gaussian moving average processes on, e.g., the space $\mathrm{SL}_{2}(\mathbb{R})$ of 2-dimensional matrices with unit determinant. Recall that each $G \in \mathrm{SL}_{2}(\mathbb{R})$ has a unique so called Iwasawa decomposition as the product of three matrices

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=\left(\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right)\left(\begin{array}{cc}
x & 0 \\
0 & 1 / x
\end{array}\right)\left(\begin{array}{ll}
1 & r \\
0 & 1
\end{array}\right):=G(\theta, x, r)
$$

for a unique choice of $(\theta, x, r) \in(-\pi, \pi] \times(0, \infty) \times \mathbb{R}$ given by

$$
x=\sqrt{a^{2}+c^{2}}, \quad \cos (\theta)=a / x, \quad \sin (\theta)=c / x \quad \text { and } \quad r=(a b+c d) / x^{2}
$$

with corresponding right Haar measure $\mathrm{d} \theta \mathrm{d} x \mathrm{~d} r / x$ and $I=(0,1,0)$, see, e.g., [12].
Example 2.7 $\left(\mathrm{SL}_{2}(\mathbb{R})\right)$. Clearly, for "nice functions" $f \in T=\mathbb{L}^{2}\left(\mathrm{SL}_{2}(\mathbb{R}), \mu\right)$ with $\|f\|_{\mathbb{L}^{2}\left(\mathrm{SL}_{2}(\mathbb{R}), \mu\right)}=1$ the process (2.4) with auto-correlation (2.5) will satisfy

$$
\rho(I, G(\theta, 1+x, r))^{2}=2-2 \int_{t \in T} f\left(t^{-1}\right) f\left(G(\theta, 1+x, r) \bullet t^{-1}\right) \mathrm{d} \mu(t)=\left(\begin{array}{lll}
\theta & x & r
\end{array}\right) Q\left(\begin{array}{l}
\theta \\
x \\
r
\end{array}\right)+o\left(\|(\theta, x, r)\|^{2}\right)
$$

for some positive definite $3 \mid 3$-matrix $Q$. On the other hand one can establish (employinging, e.g., Mathematica) that

$$
G(\theta, 1+x, r) \bullet G(\phi, 1+y, s)^{-1}=G\left(\theta-\phi+O\left(\|(\theta, x, r)\|^{2}\right), 1+x-y+O\left(\|(\theta, x, r)\|^{2}\right), r-s+O\left(\|(\theta, x, r)\|^{2}\right)\right)
$$

as $\|(\theta, x, r)\| \rightarrow 0$. Taking $V_{u}(\theta, x, r)=\left(u^{-1} \theta, 1+u^{-1}(x-1), u^{-1} r\right)$ we thus get

$$
u \rho\left(G\left(V_{u}(\theta, x, r)\right), G\left(V_{u}(\phi, y, s)\right)\right) \rightarrow \sqrt{(\phi-\theta y-x s-r) Q(\phi-\theta y-x s-r)^{t}}
$$

as $u \rightarrow \infty$, where the right-hand side is our $\psi$ in (1.8). In a similar fashion we readily see that all other conditions of Theorem 1.3 hold with

$$
q(u) \sim\left(\int_{\left\{(a, b, c) \in \mathbb{R}^{3}:(a b c) Q(a b c)^{t}<1\right\}} \mathrm{d} a \mathrm{~d} b \mathrm{~d} c\right) u^{-3}:=R u^{-3}
$$

as $u \rightarrow \infty$ and

$$
\mu \circ V_{u}^{-1}(A)=\int_{\left\{(\theta, x, r) \in(-\pi, \pi] \times(0, \infty) \times \mathbb{R}: V_{u}(\theta, x, r) \in A\right\}} \frac{\mathrm{d} \theta \mathrm{~d} x \mathrm{~d} r}{x}=\int_{(a, b, c) \in A} \frac{u^{-3} \mathrm{~d} a \mathrm{~d} b \mathrm{~d} c}{1+u(b-1)},
$$

giving

$$
q(u) \frac{\mathrm{d}\left(\mu \circ V_{u}^{-1}\right)}{\mathrm{d} \mu}\left(V_{u}(\theta, x, r)\right)=\left.\frac{R b}{1+u(b-1)}\right|_{(a, b, c)=V_{u}(\theta, x, r)} \rightarrow \frac{R}{x}:=\lambda(\theta, x, r)
$$

as $u \rightarrow \infty$. And so we have (1.6) with $q(u)$ as above.

## 3. Three lemmas on covering numbers

In Sections 3 and 4 we make use of estimations techniques originally developed by Dudley [9] and Fernique [11] to study continuity and boundedness of Gaussian processes on abstract parameter spaces and later by e.g., Adler [2] and Samorodnitsky [18] to obtain upper and lower bounds on Gaussian extrema.

In the proofs of Theorems 1.1 and 1.2 we will measure sizes of ( $\rho$-totally bounded) sets $S \subseteq T$ in terms of how many open $\rho$-balls of radius $\varepsilon>0$ they can be covered with

$$
\mathcal{N}(S ; \varepsilon)=\inf \left\{n \in \mathbb{N}: t_{1}, \ldots, t_{n} \in S \text { and } \rho\left(\left\{t_{1}, \ldots, t_{n}\right\}, t\right)<\varepsilon \text { for each } t \in S\right\}
$$

A closely related quantity is how many points that mutually are at least a $\rho$-distance $\varepsilon>0$ a part that can be contained in $S$

$$
\mathcal{M}(S ; \varepsilon)=\sup \left\{m \in \mathbb{N}: s_{1}, \ldots, s_{m} \in S \text { and } \rho\left(s_{i}, s_{j}\right) \geq \varepsilon \text { for } i \neq j\right\}
$$

The following basic relation between the covering numbers $\mathcal{N}(S, \varepsilon)$ and $\mathcal{M}(S, \varepsilon)$ is true for any metric $\rho$ on $T$ [and not only for our choice of it to be given by (1.1)]:

Lemma 3.1. For a non-empty $\rho$-totally bounded set $S \subseteq T$ we have

$$
\begin{equation*}
\mathcal{N}(S ; \varepsilon) \leq \mathcal{M}(S ; \varepsilon) \leq \mathcal{N}(S ; \varepsilon / 2) \quad \text { for } \varepsilon>0 \tag{3.1}
\end{equation*}
$$

Proof. To prove the first inequality in (3.1) we take $s_{1}, \ldots, s_{m} \in S$ with $m=\mathcal{M}(S ; \varepsilon)$ and $\rho\left(s_{i}, s_{j}\right) \geq \varepsilon$ for $i \neq j$. Then we cannot have $\rho\left(\left\{s_{1}, \ldots, s_{m}\right\}, t\right) \geq \varepsilon$ for any $t \in S$ as that would imply that $\mathcal{M}(S ; \varepsilon) \geq m+1$. Hence we have $\rho\left(\left\{s_{1}, \ldots, s_{m}\right\}, t\right)<\varepsilon$ for each $t \in S$ so that $\mathcal{N}(S ; \varepsilon) \leq m$.

To prove the second inequality in (3.1) we take $t_{1}, \ldots, t_{n} \in S$ with $n=\mathcal{N}(S ; \varepsilon / 2)$ and $\rho\left(\left\{t_{1}, \ldots, t_{n}\right\}, t\right)<\varepsilon / 2$ for every $t \in S$. Then, with $s_{1}, \ldots, s_{m} \in S$ choosen as in the previous paragraph, for any given $t_{j}$ we can only have $\rho\left(s_{i}, t_{j}\right)<\varepsilon / 2$ for at most one $s_{i}$ because if $\rho\left(s_{i_{1}}, t_{j}\right), \rho\left(s_{i_{2}}, t_{j}\right)<\varepsilon / 2$ for $i_{1} \neq i_{2}$ we get $\rho\left(s_{i_{1}}, s_{i_{2}}\right)<\varepsilon$ by the triangle inequality, which is a contradiction. Hence we have $\mathcal{M}(S ; \varepsilon) \leq \mathcal{N}(S ; \varepsilon / 2)$.

We will need quite detailed quantitative information about how the covering numbers $\mathcal{N}(S, \varepsilon)$ and $\mathcal{M}(S, \varepsilon)$ depend on the radius of $S$. This information is established in the following two lemmas and rely crucially on the assumption (1.2) in Theorem 1.1.

Lemma 3.2. Under the hypothesis of Theorem 1.1 there exist constants $C>1, \delta_{0} \in(0,1]$ and $c>0$ such that

$$
\begin{equation*}
\mu\left(\{I\}_{\varepsilon}\right) \leq \frac{C}{\mathcal{N}(K ; \varepsilon)} \leq C^{2} \mu\left(\{I\}_{\varepsilon}\right) \leq C^{3} \lambda^{c} \mu\left(\{I\}_{\varepsilon / \lambda}\right) \tag{3.2}
\end{equation*}
$$

for $0<\varepsilon \leq \delta \leq \delta_{0}$ and $\lambda \geq 1$. Moreover, we have

$$
\begin{equation*}
\underset{\varepsilon \downarrow 0}{\limsup } \frac{\mathcal{M}(S ; \varepsilon)}{\mathcal{M}(K ; \varepsilon)} \rightarrow 0 \quad \text { as } \mu(S) \downarrow 0 \tag{3.3}
\end{equation*}
$$

Proof. To prove the last inequality in (3.2) we note that $r(t)=1-\rho(I, t)^{2} / 2 \geq 1-2 / u^{2} \geq 1 /\left(1+4 / u^{2}\right)>0$ (with the last inequality being elementary) for $t \in\{I\}_{2 / u}$ and $u \geq 2$. As $r(t) \leq r(I) \leq 1$ and $X(I)$ is independent of $X(t)-r(t) X(I)$ it follows that

$$
\mathbf{P}\{X(t)>u \mid X(I)>u\} \geq \frac{\mathbf{P}\{X(t)-r(t) X(I)>0, r(t) X(I)>u\}}{\bar{\Phi}(u)}=\frac{\bar{\Phi}(u / r(t))}{2 \bar{\Phi}(u)} \geq \frac{\bar{\Phi}\left(u\left(1+4 / u^{2}\right)\right)}{2 \bar{\Phi}(u)}
$$

for $t \in\{I\}_{2 / u}$ and $u \geq 2$. From this in turn, together with (1.2) we conclude that

$$
\limsup _{u \rightarrow \infty} \frac{\mu\left(\{I\}_{2 / u}\right)}{\mu\left(\{I\}_{1 / u}\right)} \leq \limsup _{u \rightarrow \infty} \sup _{t \in\{I\}_{2 / u}} \frac{m(2,1)}{\mathbf{P}\{X(t)>u \mid X(I)>u\}} \leq \limsup _{u \rightarrow \infty} \frac{2 M \bar{\Phi}(u)}{\bar{\Phi}\left(u\left(1+4 / u^{2}\right)\right)}=2 M \mathrm{e}^{4}
$$

where we made use of the elementary fact that

$$
\begin{equation*}
\bar{\Phi}(u) \sim \mathrm{e}^{-u^{2} / 2} /(\sqrt{2 \pi} u) \quad \text { as } u \rightarrow \infty \tag{3.4}
\end{equation*}
$$

(cf. e.g., [13], Eq. 1.5.4) to obtain the last equality. As $f(x)=\mu\left(\{I\}_{1 / x}\right)$ obviously is a non-inreasing function of $x \in(0, \infty)$, it follows from well-known basic facts from the theory of regular variation (and is not very hard to verify oneself) that $f(x)$ is a function of so called bounded decrease as $x \rightarrow \infty$ (cf.e.g., ([7], Sects. 2.0 and 2.1) and therefore satisfies the right inequality in (3.2) (cf. e.g., [7], Prop. 2.2.1).

The middle inequality in (3.2) follows from the obvious fact that

$$
\begin{equation*}
\mu(K) \leq \mathcal{N}(K ; \varepsilon) \mu\left(\{I\}_{\varepsilon}\right) \quad \text { for } \varepsilon>0 \tag{3.5}
\end{equation*}
$$

To prove the first inequality in (3.2) we note that for an open set $S \subseteq K$ the left inequality in (3.1) together with the right inequality in (3.2) show that

$$
\begin{equation*}
\infty>\mu\left(K_{\varepsilon_{0}}\right) \geq \mu\left(S_{\varepsilon_{0}}\right) \geq \mathcal{M}(S ; \varepsilon) \mu\left(\{I\}_{\varepsilon / 2}\right) \geq \mathcal{N}(S ; \varepsilon) \mu\left(\{I\}_{\varepsilon / 2}\right) \geq \frac{\mathcal{N}(S ; \varepsilon) \mu\left(\{I\}_{\varepsilon}\right)}{C 2^{c}} \tag{3.6}
\end{equation*}
$$

for $\varepsilon \in\left(0, \varepsilon_{0}\right]$ for $\varepsilon_{0}>0$ small enough. With $S=K$ this gives the left inequality in (3.2).
To prove (3.3) we use (3.1) and (3.2) together with (3.5) and (3.6) to obtain

$$
\limsup _{\varepsilon \downarrow 0} \frac{\mathcal{M}(S ; \varepsilon)}{\mathcal{M}(K ; \varepsilon)} \leq \limsup _{\varepsilon \downarrow 0} \frac{\mathcal{N}(S ; \varepsilon / 2)}{\mathcal{N}(K ; \varepsilon)} \leq \frac{C^{2} 2^{2 c} \mu\left(S_{\varepsilon_{0}}\right)}{\mu(K)} \rightarrow 0 \quad \text { as } \mu(S) \downarrow 0 \text { and } \varepsilon_{0} \downarrow 0
$$

(in that order), by the dominated convergence theorem.
Lemma 3.3. Under the hypothesis of Theorem 1.1 there exist constants $C>1, \delta_{0} \in(0,1]$ and $c>0$ such that

$$
\begin{equation*}
\frac{\mathcal{N}\left(\{I\}_{\delta} ; \varepsilon / \lambda\right)}{C \lambda^{c}} \leq \mathcal{N}\left(\{I\}_{\delta} ; \varepsilon\right) \leq C(\delta / \varepsilon)^{c} \quad \text { for } 0<\varepsilon \leq \delta \leq \delta_{0} \quad \text { and } \lambda \geq 1 \tag{3.7}
\end{equation*}
$$

Proof. We start by noting the following version of (3.6):

$$
\begin{equation*}
C 2^{c} \mu\left(\{I\}_{\delta}\right) \geq \mu\left(\{I\}_{2 \delta}\right) \geq \mu\left(\{I\}_{\delta+\varepsilon / 2}\right) \geq \mathcal{M}\left(\{I\}_{\delta} ; \varepsilon\right) \mu\left(\{I\}_{\varepsilon / 2}\right) \geq \frac{\mathcal{N}\left(\{I\}_{\delta} ; \varepsilon\right) \mu\left(\{I\}_{\varepsilon}\right)}{C 2^{c}} \tag{3.8}
\end{equation*}
$$

for $0<\varepsilon \leq \delta \leq \delta_{0}$ for $\delta_{0}>0$ small enough. Using this together with the right inequality in (3.2) we get the right inequality in (3.7). Further we may use (3.8) with $\varepsilon$ replaced by $\varepsilon / \lambda$ together with the obvious fact that $\mu\left(\{I\}_{\delta}\right) \leq \mathcal{N}\left(\{I\}_{\delta} ; \varepsilon\right) \mu\left(\{I\}_{\varepsilon}\right)$ and the right inequality in (3.2) to obtain

$$
\frac{\mathcal{N}\left(\{I\}_{\delta} ; \varepsilon\right)}{\mathcal{N}\left(\{I\}_{\delta} ; \varepsilon / \lambda\right)} \geq \frac{\mu\left(\{I\}_{\varepsilon / \lambda}\right)}{C^{2} 2^{2 c} \mu\left(\{I\}_{\varepsilon}\right)} \geq \frac{1}{C^{3} 2^{2 c} \lambda^{c}} \quad \text { for } 0<\varepsilon \leq \delta \leq \delta_{0} \text { and } \lambda \geq 1
$$

which in turn is the left inequality in (3.7).

## 4. Proof of Theorem 1.1

Proof of Theorem 1.1. As basic geometry together with (3.7) give

$$
\begin{equation*}
\mathcal{N}(K ; \varepsilon) \leq \mathcal{N}\left(K ; \delta_{0}\right) \mathcal{N}\left(\{I\}_{\delta_{0}} ; \varepsilon\right) \leq C \mathcal{N}\left(K ; \delta_{0}\right)\left(\delta_{0} / \varepsilon\right)^{c} \quad \text { for } 0<\varepsilon \leq \delta_{0} \tag{4.1}
\end{equation*}
$$

it follows that

$$
\int_{0}^{1} \sqrt{\ln (\mathcal{N}(K ; \varepsilon))} d \varepsilon \leq \sqrt{\ln \left(\mathcal{N}\left(K ; \delta_{0}\right)\right)}+\int_{0}^{\delta_{0}} \sqrt{\ln \left(C \mathcal{N}\left(K ; \delta_{0}\right) \delta_{0}^{c}\right)-c \ln (\varepsilon)} d \varepsilon<\infty
$$

This implies that $\{X(t)\}_{t \in K}$ has an a.s. continuous version, see e.g., ([2], Thm. 1.1).
From the elementary inequality

$$
\sqrt{1-\theta} \bar{\Phi}(u / \sqrt{1-\theta}) \leq \mathrm{e}^{-\theta u^{2} / 2} \bar{\Phi}(u) \text { for } \theta \in[0,1) \text { and } u>0
$$

we get

$$
\begin{equation*}
\frac{\mathbf{P}\{X(s)>u, X(t)>u\}}{\bar{\Phi}(u)} \leq \frac{\mathbf{P}\{X(s)+X(t)>2 u\}}{\bar{\Phi}(u)}=\frac{\bar{\Phi}\left(u / \sqrt{1-\rho(s, t)^{2} / 4}\right)}{\bar{\Phi}(u)}=\frac{\bar{\Phi}\left(u / \sqrt{1-\epsilon^{2} / 4}\right)}{\bar{\Phi}(u)} \leq \frac{\mathrm{e}^{-\epsilon^{2} u^{2} / 8}}{\sqrt{1-\epsilon^{2} / 4}} \tag{4.2}
\end{equation*}
$$

for $\epsilon^{2} \leq \rho(s, t)^{2}<2$ and $u>0$. Taking $\mathfrak{M} \subseteq K$ with $\rho(\mathfrak{M} \backslash\{t\}, t) \geq n / u$ for $t \in \mathfrak{M}$ and $\# \mathfrak{M}=\mathcal{M}(K ; n / u)$ we have $1 /\left(C^{2} n^{c} q(u)\right) \leq \# \mathfrak{M} \leq C / q(u)$ for $n \geq 2$ and $u>0$ large enough by (3.1) and (3.2), while $\mathcal{M}\left(\{I\}_{k / u} ; n / u\right) \leq$ $C(k /(2 n))^{c}$ for $k \geq n$ and $u>0$ large enough by (3.1) and (3.7). Further, we have $q(u)^{-1} \leq C^{2} \mathcal{N}\left(K ; \delta_{0}\right) \delta_{0}^{c} u^{c}$ for $u>0$ large enough by (3.2) and (4.1). Hence we may use Bonferroni's inequality and (4.2) to obtain

$$
\begin{align*}
& \liminf _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in \mathfrak{M}} X(t)>u\right\} \\
& \geq \liminf _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)}\left((\# \mathfrak{M}) \bar{\Phi}(u)-\sum_{\mathfrak{M} \ni s \neq t \in \mathfrak{M}} \mathbf{P}\{X(s)>u, X(t)>u\}\right) \\
& \geq \liminf _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)}\left((\# \mathfrak{M}) \bar{\Phi}(u)-\sum_{s \in \mathfrak{M}} \sum_{k=n}^{\infty} \sum_{t \in \mathfrak{M} \cap\{s\}_{(k+1) / u \backslash\{s\}_{k / u}}} \mathbf{P}\{X(s)>u, X(t)>u\}\right) \\
& \geq \liminf _{u \rightarrow \infty} q(u)(\# \mathfrak{M})\left(1-\sum_{k=n}^{\lfloor\epsilon u\rfloor} \mathcal{M}\left(\{I\}_{(k+1) / u} ; n / u\right) \frac{\bar{\Phi}\left(u / \sqrt{1-k^{2} /\left(4 u^{2}\right)}\right)}{\bar{\Phi}(u)}-(\# \mathfrak{M}) \frac{\bar{\Phi}\left(u / \sqrt{1-\epsilon^{2} / 4}\right)}{\bar{\Phi}(u)}\right) \\
& \geq \frac{1}{C^{2} n^{c}}\left(1-\sum_{k=n}^{\infty} \frac{C((k+1) /(2 n))^{c} \mathrm{e}^{-k^{2} / 8}}{\sqrt{1-\epsilon^{2} / 4}}-\limsup _{u \rightarrow \infty} \frac{C^{3} \mathcal{N}\left(K ; \delta_{0}\right) \delta_{0}^{c} u^{c} \mathrm{e}^{-\epsilon^{2} u^{2} / 8}}{\sqrt{1-\epsilon^{2} / 4}}\right) \\
& >0 \quad \text { for } n \text { large enough. } \tag{4.3}
\end{align*}
$$

This establishes the left inequality in (1.3).
Employing the elementary inequalities

$$
1 / \sqrt{1+x} \geq 1-x / 2 \quad \text { for } x \geq 0 \quad \text { and } \quad \bar{\Phi}(u+x) \leq \mathrm{e}^{-u x} \bar{\Phi}(u) \quad \text { for } x, u \geq 0
$$

it follows that

$$
\begin{align*}
\mathbf{P}\{X(t)>\hat{u}+\nu / u, X(s) \leq \hat{u}\} & \leq \mathbf{P}\{(1+x) X(t)-x X(s)>u+x \nu / u\} \\
& =\bar{\Phi}\left(\frac{u+x \nu / u}{\sqrt{1+\left(x+x^{2}\right) \rho(s, t)^{2}}}\right) \\
& \leq \bar{\Phi}\left(u+\frac{x \nu}{u}-\left(u+\frac{x \nu}{u}\right) \frac{\left(x+x^{2}\right) \rho(s, t)^{2}}{2}\right) \\
& =\bar{\Phi}\left(u+\frac{x \nu}{2 u}+\frac{x}{2 u}\left(\nu-\left(u^{2}+x \nu\right)(1+x) \rho(s, t)^{2}\right)\right) \\
& \leq \mathrm{e}^{-x \nu / 2} \bar{\Phi}(u) \tag{4.4}
\end{align*}
$$

for $x \geq 0, \nu \geq\left(u^{2}+x \nu\right)(1+x) \rho(s, t)^{2}$ and $\hat{u} \geq u \geq 0$. Now pick constants $\delta, \lambda \in(0,1)$ and let $\mathfrak{N}=\bigcup_{n=0}^{\infty} \mathfrak{N}_{n}$ where $\mathfrak{N}_{n} \subseteq\{I\}_{a / u}$ satisfies $\rho\left(\mathfrak{N}_{n}, t\right)<a \lambda^{n} / u$ for $t \in\{I\}_{a / u}$ and $\# \mathfrak{N}_{n}=\mathcal{N}\left(\{I\}_{a / u} ; a \lambda^{n} / u\right)$ with $\mathfrak{N}_{0}=\{I\}$.

Taking $\nu_{n}=\delta \lambda^{n / 2}(1-\sqrt{\lambda})$ and $x=\lambda^{-n} / \sqrt{a}$ we have $\sum_{n=0}^{\infty} \nu_{n}=\delta$ and $\nu_{n} \geq\left(u^{2}+x \nu_{n}\right)(1+x) \rho(s, t)^{2}$ for $\rho(s, t) \leq a \lambda^{n-1} / u, u>0$ large enough and $a>0$ small enough. As $\mathfrak{N}$ is dense in $\{I\}_{a / u}$, (3.7) and (4.4) give

$$
\begin{align*}
& \mathbf{P}\left\{\sup _{t \in\{I\}_{a / u}} X(t)>u+\delta / u, X(I) \leq u\right\} \\
& \leq \mathbf{P}\left\{\bigcup_{n=1}^{\infty}\left[\bigcup_{L \in \mathfrak{N}_{n}}\left\{X(t)>u+\left(\sum_{\ell=1}^{n} \nu_{\ell}\right) / u\right\} \cap \bigcap_{k=0}^{n-1} \bigcap_{s \in \mathfrak{N}_{k}}\left\{X(s) \leq u+\left(\sum_{\ell=1}^{k} \nu_{\ell}\right) / u\right\}\right]\right\} \\
& \leq \sum_{n=1}^{\infty} \mathbf{P}\left\{\bigcup_{t \in \mathfrak{N}_{n}}\left\{X(t)>u+\left(\sum_{\ell=1}^{n} \nu_{\ell}\right) / u\right\} \cap \bigcap_{s \in \mathfrak{N}_{n-1}}\left\{X(s) \leq u+\left(\sum_{\ell=1}^{n-1} \nu_{\ell}\right) / u\right\}\right\} \\
& \leq \sum_{n=1}^{\infty} \sum_{t \in \mathfrak{N}_{n}} \inf _{n \in \mathfrak{N}_{n-1}} \mathbf{P}\left\{X(t)>u+\left(\nu_{n}+\sum_{\ell=1}^{n-1} \nu_{\ell}\right) / u, X(s) \leq u+\left(\sum_{\ell=1}^{n-1} \nu_{\ell}\right) / u\right\} \\
& \leq \sum_{n=1}^{\infty} \mathcal{N}\left(\{I\}_{a / u} ; a \lambda^{n} / u\right) \mathrm{e}^{-x \nu_{n} / 2} \bar{\Phi}(u) \\
& \leq \sum_{n=1}^{\infty} C \lambda^{-c n} \exp \left\{-\frac{\delta(1-\sqrt{\lambda})}{2 \sqrt{a} \lambda^{n / 2}}\right\} \bar{\Phi}(u) \\
& =o\left(a^{c}\right) \bar{\Phi}(u) \text { as } a \downarrow 0 \text { for } u>0 \text { large enough. } \tag{4.5}
\end{align*}
$$

Now take $\mathfrak{M} \subseteq K$ with $\rho(\mathfrak{M} \backslash\{t\}, t) \geq a / u$ for $t \in \mathfrak{M}$ and $\# \mathfrak{M}=\mathcal{M}(K ; a / u)$. Note that $\# \mathfrak{M} \leq C^{2}(2 / a)^{c} / q(u)$ for $a \in(0,2]$ and $u>0$ large enough by (3.1) and (3.2) and that $\rho(\mathfrak{M}, t)<a / u$ for every $t \in K$ [as otherwise we would have $\# \mathfrak{M}<\mathcal{M}(K ; a / u)]$. Hence we may employ the fact that $\bar{\Phi}(u+\delta / u) \sim \mathrm{e}^{-\delta} \bar{\Phi}(u)$ as $u \rightarrow \infty$ [by (3.4)] together with monotonicity of $q$ and (4.5) to see that

$$
\begin{align*}
\limsup _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} & =\limsup _{u \rightarrow \infty} \frac{q(u+\delta / u)}{\bar{\Phi}(u+\delta / u} \mathbf{P}\left\{\sup _{t \in K} X(t)>u+\delta / u\right\} \\
& \leq \mathrm{e}^{\delta} \limsup _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u+\delta / u\right\} \\
& \leq \mathrm{e}^{\delta} \limsup _{u \rightarrow \infty} \frac{q(u)}{\bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u+\delta / u \text { or } \sup _{s \in \mathfrak{M}} X(s)>u\right\} \\
& \leq \mathrm{e}^{\delta} \limsup _{u \rightarrow \infty} \frac{q(u)(\# \mathfrak{M})}{\bar{\Phi}(u)}\left(\mathbf{P}\left\{\sup _{t \in\{I\}_{a / u}} X(t)>u+\delta / u, X(I) \leq u\right\}+\bar{\Phi}(u)\right) \\
& \leq C^{2} \mathrm{e}^{\delta}(a / 2)^{-c}\left(\mathrm{o}\left(a^{c}\right)+1\right) \quad \text { for } a>0 \text { small enough and } \delta>0 . \tag{4.6}
\end{align*}
$$

This establishes the right inequality in (1.3).

## 5. Proof of Theorem 1.2

In Section 5 we employ an adaption of the sojourn approach to extremes developed by Berman in an array of papers. Although Berman uses sojourns only in Euclidian settings they do not really have that restriction but their natural setting is argubly rather that of a topological group with a Haar sojourn time measure. This is crucial for us as that is not the case with the more commonly used discrete approximation techniques that origin in Pickands fundamental papers.

Proof of Theorem 1.2. By Theorem 1.1 we may without loss of generality take $X$ to be a.s. continuous. We will be interested in the so called sojourn time

$$
L_{u}:=\int_{t \in K} \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t)
$$

of $\{X(t)\}_{t \in K}$ above the level $u$. Writing $K_{-\varepsilon}=\left\{t \in K:\{t\}_{\varepsilon} \subseteq K\right\}$ for $\varepsilon>0$ small enough (to make $K_{-\varepsilon}$ nonempty) we may use Fubini's Theorem together with the fact that $\mathbf{E}\left\{L_{u}\right\}=\mu(K) \bar{\Phi}(u)$ to obtain

$$
\begin{align*}
\frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\} & =\frac{1}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\int_{t \in K} \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right) \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t)\right\} \\
& =\int_{t \in K} \mathbf{P}\left\{L_{u} / q(u)>x \mid X(t)>u\right\} \frac{\mathrm{d} \mu(t)}{\mu(K)} \\
& \geq \mathbf{P}\left\{\left.\frac{1}{q(u)} \int_{t \in\{I\}_{N / u}} \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t)>x \right\rvert\, X(I)>u\right\} \frac{\mu\left(K_{-\varepsilon}\right)}{\mu(K)} \tag{5.1}
\end{align*}
$$

for $N \in \mathbb{N}$ and $u>0$ sufficiently large (to make $N / u \leq \varepsilon$ ). Writing $\xrightarrow{\mathrm{d}}$ for weak convergence the method of moments shows that

$$
\begin{equation*}
\left(\left.\frac{1}{q(u)} \int_{t \in\{I\}_{N / u}} \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t) \right\rvert\, X(I)>u\right) \stackrel{\mathrm{d}}{\rightarrow} G_{N} \quad \text { as } u \rightarrow \infty \tag{5.2}
\end{equation*}
$$

since the law on the left has upper endpoint at most $C N^{c}$ by the right inequality in (3.2) with $n$th moment given by the right-hand side of (1.4). Writing $G(x)=\lim _{N \rightarrow \infty} G_{N}(x)$ [where the limit exists as $G_{N}(x)$ is a non-increasing function of $N]$ we may send $\varepsilon \downarrow 0$ and $N \rightarrow \infty$ (in that order) in (5.1) and (5.2) to obtain

$$
\begin{equation*}
\liminf _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\} \geq 1-G(x) \text { for } x>0 \tag{5.3}
\end{equation*}
$$

On the other hand Markov's inequality together with (5.1) and (5.2) show that

$$
\begin{aligned}
& \limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\} \\
& =\limsup _{u \rightarrow \infty} \int_{t \in K} \mathbf{P}\left\{L_{u} / q(u)>x \mid X(t)>u\right\} \frac{\mathrm{d} \mu(t)}{\mu(K)} \\
& \leq \limsup _{u \rightarrow \infty} \mathbf{P}\left\{\left.\frac{1}{q(u)} \int_{t \in\{I\}_{N / u}} \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t)>x-\delta \right\rvert\, X(I)>u\right\} \\
& \quad+\limsup _{u \rightarrow \infty} \sup _{t \in K} \mathbf{P}\left\{\left.\frac{1}{q(u)} \int_{s \in K \backslash\{t\}_{N / u}} \mathbf{1}_{(u, \infty)}(X(s)) \mathrm{d} \mu(s)>\delta \right\rvert\, X(t)>u\right\} \\
& \leq 1-G_{N}(x-2 \delta)+\limsup _{u \rightarrow \infty} \sup _{t \in K} \frac{1}{\delta q(u)} \int_{s \in K \backslash\{t\}_{N / u}} \mathbf{P}\{X(s)>u \mid X(t)>u\} \mathrm{d} \mu(s)
\end{aligned}
$$

for $N \in \mathbb{N}$ and $\delta>0$. Here we may make use of (3.2) and (4.1) together with (4.2) in a similar fashion to that employed to establish (4.3) to obtain

$$
\begin{align*}
& \limsup _{u \rightarrow \infty} \sup _{t \in K} \frac{1}{q(u)} \int_{s \in K \backslash\{t\}_{N / u}} \mathbf{P}\{X(s)>u \mid X(t)>u\} \mathrm{d} \mu(s) \\
& =\limsup _{u \rightarrow \infty} \sup _{t \in K} \frac{1}{q(u)} \sum_{k=N}^{\infty} \int_{s \in K \cap\{t\}_{(k+1) / u} \backslash\{t\}_{k / u}} \mathbf{P}\{X(s)>u \mid X(t)>u\} \mathrm{d} \mu(s) \\
& \leq \limsup _{u \rightarrow \infty} \frac{1}{q(u)} \sum_{k=N}^{\lfloor\epsilon u\rfloor} \mu\left(\{I\}_{(k+1) / u}\right) \frac{\bar{\Phi}\left(u / \sqrt{1-k^{2} /\left(4 u^{2}\right)}\right)}{\bar{\Phi}(u)}+\limsup _{u \rightarrow \infty} \frac{\mu(K)}{q(u)} \frac{\bar{\Phi}\left(u / \sqrt{1-\epsilon^{2} / 4}\right)}{\bar{\Phi}(u)} \\
& \leq \sum_{k=N}^{\infty} \frac{C(k+1)^{c} \mathrm{e}^{-k^{2} / 8}}{\sqrt{1-\epsilon^{2} / 4}}+\limsup _{u \rightarrow \infty} \frac{C^{2} \mu(K) \mathcal{N}\left(K ; \delta_{0}\right) \delta_{0}^{c} u^{c} \mathrm{e}^{-\epsilon^{2} u^{2} / 8}}{\sqrt{1-\epsilon^{2} / 4}} \\
& \rightarrow 0 \text { as } N \rightarrow \infty \text { for } \epsilon>0 \text { small enough. } \tag{5.4}
\end{align*}
$$

Putting things together and sending $N \rightarrow \infty$ and $\delta \downarrow 0$ (in that order) it follows that

$$
\begin{equation*}
\limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\} \leq 1-G\left(x^{-}\right) \quad \text { for } x>0 \tag{5.5}
\end{equation*}
$$

[Note the elementary fact that $G(x)$ must be a non-decreasing function of $x$.]
In order to be able to proceed we have to establish that

$$
\begin{equation*}
\lim _{x \downarrow 0} G(x)=0 \quad \text { and } \quad \lim _{x \rightarrow \infty} G(x)=1 \tag{5.6}
\end{equation*}
$$

To that end we note that elementary considerations give

$$
\mathbf{E}\left\{\left(L_{u} /(q(u)) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\}=x \mathbf{P}\left\{L_{u} / q(u)>x\right\}+\int_{x}^{\infty} \mathbf{P}\left\{L_{u} / q(u)>y\right\} \mathrm{d} y\right.
$$

Hence we may employ (5.5) to obtain the following estimate

$$
\begin{aligned}
\liminf _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} & \geq \limsup _{x \downarrow 0} \liminf _{u \rightarrow \infty} \frac{\int_{0}^{x} \mathbf{P}\left\{L_{u} / q(u)>y\right\} \mathrm{d} y}{x \mathbf{E}\left\{L_{u} / q(u)\right\}} \\
& =\limsup _{x \downarrow 0} \frac{1}{x}\left(1-\limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \int_{x}^{\infty} \mathbf{P}\left\{L_{u} / q(u)>y\right\} \mathrm{d} y\right) \\
& =\limsup _{x \downarrow 0} \frac{1}{x}\left(1-\limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} /(q(u)) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\}\right)\right. \\
& \geq \limsup _{x \downarrow 0} \frac{G\left(x^{-}\right)}{x} .
\end{aligned}
$$

Hence we get $\lim _{x \downarrow 0} G(x)=0$ from the right inequality in (1.3). To see that $\lim _{x \rightarrow \infty} G(x)=1$ we note that (5.3) and together with the Cauchy-Schwarz inequality, Markov's inequality, (1.4) and (5.4) give

$$
\begin{aligned}
& 1-G(x) \\
& \leq \limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\} \\
& =\limsup _{u \rightarrow \infty} \sqrt{\frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right)^{2}\right\}} \sqrt{\frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{L_{u} / q(u)>x\right\}} \\
& =\limsup _{u \rightarrow \infty} \sqrt{\frac{1}{\mu(K) \bar{\Phi}(u)} \int_{(s, t) \in K \times K} \mathbf{P}\{X(s)>u, X(t)>u\} \frac{\mathrm{d} \mu(s) \mathrm{d} \mu(t)}{q(u)}} \frac{1}{\sqrt{x}} \\
& \leq \limsup _{u \rightarrow \infty}\left(\int_{s \in\{I\}_{N / u}} \mathbf{P}\{X(s)>u \mid X(I)>u\} \frac{\mathrm{d} \mu(s)}{q(u)}+\sup _{t \in K} \int_{s \in K \backslash\{t\}_{N / u}} \mathbf{P}\{X(s)>u \mid X(t)>u\} \frac{\mathrm{d} \mu(s)}{q(u)}\right)^{1 / 2} \frac{1}{\sqrt{x}} \\
& \leq \frac{\sqrt{m(N, 1)+\mathrm{o}(1)}}{\sqrt{x}} \text { for } N \text { large } \\
& \rightarrow 0 \text { as } x \rightarrow \infty .
\end{aligned}
$$

Let $\hat{G}(x)=G\left(x^{+}\right)$be the right continuous version of $G$. Then $\hat{G}$ is a probability distribution function on $(0, \infty)$ by (5.6). Moreover, (5.3) and (5.5) show that

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\}=1-G(x)=1-\hat{G}(x) \text { for } x>0 \tag{5.7}
\end{equation*}
$$

that are continuity points of $G$ and $\hat{G}$. Noting that

$$
\frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{E}\left\{\left(L_{u} / q(u)\right) \mathbf{1}_{(x, \infty)}\left(L_{u} / q(u)\right)\right\}=\frac{\int_{x}^{\infty} y \mathrm{~d} F_{u}(y)}{\int_{0}^{\infty} y \mathrm{~d} F_{u}(y)} \quad \text { where } \quad F_{u}(x)=\mathbf{P}\left\{L_{u} / q(u) \leq x\right\}
$$

we may use (5.7) together with Lemma 1.2.1 of Berman [5] to obtain

$$
\begin{equation*}
\lim _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{L_{u} / q(u)>x\right\}=\int_{x}^{\infty} \frac{d \hat{G}(y)}{y}=\int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y} \quad \text { for } x>0 \tag{5.8}
\end{equation*}
$$

that are continuity points of $G$ and $\hat{G}$.
Picking a sequence of continuity points $\left\{x_{n}\right\}_{n=1}^{\infty}$ of $G$ such that $x_{n} \downarrow 0$ as $n \rightarrow \infty$ we may use (5.8) to obtain

$$
\begin{align*}
\liminf _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} & \geq \lim _{n \rightarrow \infty} \lim _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{L_{u} / q(u)>x_{n}\right\} \\
& =\lim _{n \rightarrow \infty} \int_{x_{n}}^{\infty} \frac{\mathrm{d} G(y)}{y}=\lim _{x \downarrow 0} \int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y} \tag{5.9}
\end{align*}
$$

where the limit on the right-hand side is finite by Theorem 1.1. To establish a matching upper bound we take $\mathfrak{M}$ exactly as in the proof of (4.6) and note that (5.8) and (5.2) together with (5.6), (3.3) and (4.5) give

$$
\begin{align*}
& \limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\} \\
& \leq \underset{\delta \downarrow 0}{\limsup } \limsup _{u \rightarrow \infty} \frac{\mathrm{e}^{\delta} q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{\sup _{t \in K} X(t)>u+\delta / u\right\} \\
& \leq \limsup _{\delta \downarrow 0} \limsup _{a \downarrow 0} \liminf _{n \rightarrow \infty} \limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \\
& \times \mathbf{P}\left\{\sup _{t \in K} X(t)>u+\delta / u \text { or } L_{u} / q(u)>x_{n} \text { or } \sup _{s \in \mathfrak{M}} X(s)>u\right\} \\
& \leq \liminf _{n \rightarrow \infty} \limsup _{u \rightarrow \infty} \frac{q(u)}{\mathbf{E}\left\{L_{u}\right\}} \mathbf{P}\left\{L_{u} / q(u)>x_{n}\right\} \\
& +\limsup \limsup _{n \downarrow 0} \limsup _{n \rightarrow \infty} \limsup _{\varepsilon \downarrow 0} \frac{q(u)}{\mu(K) \bar{\Phi}(u)} \sum_{s \in \mathfrak{M} \cap K_{-\varepsilon}} \mathbf{P}\left\{L_{u} / q(u) \leq x_{n}, X(s)>u\right\} \\
& +\limsup \limsup \limsup _{\varepsilon \downarrow 0} \frac{q(u)}{\mu(K) \bar{\Phi}(u)} \sum_{s \in \mathfrak{M} \backslash K_{-\varepsilon}} \mathbf{P}\{X(s)>u\} \\
& +\limsup _{\delta \downarrow 0} \limsup _{a \downarrow 0} \limsup _{u \rightarrow \infty} \frac{q(u)}{\mu(K) \bar{\Phi}(u)} \sum_{s \in \mathfrak{M}} \mathbf{P}\left\{\sup _{t \in\{s\}_{a / u}} X(t)>u+\delta / u, X(s) \leq u\right\} \\
& \leq \lim _{n \rightarrow \infty} \int_{x_{n}}^{\infty} \frac{\mathrm{d} G(y)}{y} \\
& +\limsup \limsup _{a \downarrow 0} \limsup _{n \rightarrow \infty} \limsup _{N \rightarrow \infty} \frac{q(u)(\# \mathfrak{M})}{\mu(K)} \\
& \times \mathbf{P}\left\{\left.\frac{1}{q(u)} \int_{t \in\{I\}_{N / u}} \mathbf{1}_{(u, \infty)}(X(t)) \mathrm{d} \mu(t) \leq x_{n} \right\rvert\, X(I)>u\right\} \\
& +\underset{a \downarrow 0}{\limsup } \underset{\varepsilon \downarrow 0}{\lim \sup } \limsup _{u \rightarrow \infty} \frac{q(u)(\# \mathfrak{M})}{\mu(K)} \frac{\mathcal{M}\left(K \backslash K_{-\varepsilon} ; a / u\right)}{\mathcal{M}(K ; a / u)} \\
& +\limsup _{\delta \downarrow 0} \limsup _{a \downarrow 0} \limsup _{u \rightarrow \infty} \frac{q(u)(\# \mathfrak{M})}{\mu(K)} \mathbf{P}\left\{\sup _{t \in\{I\}_{a / u}} X(t)>u+\delta / u \mid X(I) \leq u\right\} \\
& =\lim _{x \downarrow 0} \int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y} \\
& +\limsup \sup _{a \downarrow 0} \frac{C^{2}(2 / a)^{c}}{\mu(K)} \limsup _{n \rightarrow \infty} \limsup _{N \rightarrow \infty} G_{N}\left(x_{n}\right) \\
& +\limsup _{a \downarrow 0} \frac{C^{2}(2 / a)^{c}}{\mu(K)} \limsup _{\varepsilon \downarrow 0} \limsup _{u \rightarrow \infty} \frac{\mathcal{M}\left(K \backslash K_{-\varepsilon} ; a / u\right)}{\mathcal{M}(K ; a / u)} \\
& +\limsup _{\delta \downarrow 0} \limsup _{a \downarrow 0} \frac{C^{2}(2 / a)^{c}}{\mu(K)} o\left(a^{c}\right) \\
& =\lim _{x \downarrow 0} \int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y}+0+0+0 . \tag{5.10}
\end{align*}
$$

Putting (5.9) together with (5.10) we conclude that the limits

$$
\lim _{x \downarrow 0} \int_{x}^{\infty} \frac{\mathrm{d} G(y)}{y} \quad \text { and } \quad \lim _{u \rightarrow \infty} \frac{q(u)}{\mu(K) \bar{\Phi}(u)} \mathbf{P}\left\{\sup _{t \in K} X(t)>u\right\}
$$

both exist and that their values coincide. In addition, by Theorem 1.1 that value must be strictly positive and finite. This concludes the proof of Theorem 1.2.

## 6. Proof of Theorem 1.3

Proof of Theorem 1.3. Since $X(t)-r(t) X(I)$ is independent of $X(I),(1.8)$ shows that

$$
\begin{aligned}
& \left.\mathbf{E}\left\{u\left(X\left(V_{u} s\right)-r\left(V_{u} s\right) X(I)\right) u\left(X\left(V_{u} t\right)-r\left(V_{u} t\right) X(I)\right)\right)\right\} \\
& =u^{2}\left[1-\rho\left(V_{u} s, V_{u} t\right)^{2} / 2-\left(1-\rho\left(I, V_{u} s\right)^{2} / 2\right)\left(1-\rho\left(I, V_{u} t\right)^{2} / 2\right)\right] \\
& \rightarrow\left(\psi(I, s)^{2}+\psi(I, t)^{2}-\psi(s, t)^{2}\right) / 2 \quad \text { as } u \rightarrow \infty
\end{aligned}
$$

where the function on the right-hand side must be a covariance function. Further, elementary considerations [using, e.g., (3.4)] show that $(u(X(I)-u) \mid X(I)>u) \rightarrow_{d} \eta$ as $u \rightarrow \infty$ for $\eta$ an exponentially distributed random variable with mean one. This in turn gives

$$
\begin{aligned}
\left(u r\left(V_{u} t\right) X(I)-u \mid X(I)>u\right) & =r\left(V_{u} t\right)(u(X(I)-u) \mid X(I)>u)-u^{2} \rho\left(I, V_{u} t\right)^{2} / 2 \\
& \rightarrow_{d} \eta-\psi(I, t)^{2} / 2 \quad \text { as } u \rightarrow \infty
\end{aligned}
$$

Letting $\{\xi(t)\}_{t \in T}$ denote a zero-mean Gaussian process that is independent of $\eta$ and has covariance function $\mathbf{E}\{\xi(s) \xi(t)\}=\left(\psi(I, s)^{2}+\psi(I, t)^{2}-\psi(s, t)^{2}\right) / 2$ it follows that the finite dimensional distributions of the process $\left\{\left(u\left(X\left(V_{u} t\right)-u\right) \mid X(I)>u\right)\right\}_{t \in K}$ converge weakly to those of $\left\{\xi(t)+\eta-\psi(I, t)^{2} / 2\right\}_{t \in K}$ as $u \rightarrow \infty$. And so we may use (1.7)-(1.12) together with a change variable in the integral on the right-hand side of (1.4) to obtain

$$
\begin{align*}
& \int_{\left(\{I\}_{N / u}\right)^{n}} \mathbf{P}\left\{\bigcap_{i=1}^{n}\left\{X\left(t_{i}\right)>u\right\} \mid X(I)>u\right\} \frac{\mathrm{d} \mu^{n}(t)}{q(u)^{n}} \\
& =\int_{\bigotimes_{i=1}^{n}\left\{t_{i} \in T: u \rho\left(I, V_{u} t_{i}\right)<N\right\}} \mathbf{P}\left\{\bigcap_{i=1}^{n}\left\{u\left(X\left(V_{u} t_{i}\right)-u\right)>0\right\} \mid X(I)>u\right\} \frac{\mathrm{d} \mu^{n}(t)}{\prod_{i=1}^{n} q(u) \frac{\left.\mathrm{d}\left(\mu \circ V_{u}\right)^{-1}\right)}{\mathrm{d} \mu}\left(V_{u} t_{i}\right)} \\
& \rightarrow \int_{\left(\mathfrak{O}_{N}\right)^{n}} \mathbf{P}\left\{\bigcap_{i=1}^{n}\left\{\xi\left(t_{i}\right)+\eta-\psi\left(I, t_{i}\right)^{2} / 2>0\right\}\right\} \frac{\mathrm{d} \mu^{n}(t)}{\prod_{i=1}^{n} \lambda\left(t_{i}\right)} \quad \text { as } u \rightarrow \infty . \tag{6.1}
\end{align*}
$$

And so we have shown that (1.4) holds.

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