**Lp-solutions of Backward Doubly Stochastic Differential Equations with Stochastic Lipschitz Condition and p ∈ (1, 2)**

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**Abstract.** We study backward doubly stochastic differential equations where the coefficients satisfy stochastic Lipschitz condition. We prove the existence and uniqueness of the solution in $L^p$ with $p \in (1, 2)$.

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1. Introduction

Backward doubly stochastic differential equations (BDSDEs in short) are equations driven by two independent Brownian motions, i.e., equations which involve both a standard forward stochastic integral $dW_t$ and a backward stochastic Kunita-Itô integral $dB_t$:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad t \in [0, T],$$

where $\xi$ is a random variable called the terminal condition, $f$ and $g$ are the coefficients (also called generators) and $(Y, Z)$ are the unknown processes that we study the existence under certain conditions on the data $(\xi, f, g)$. This kind of equations, in the nonlinear case, has been introduced by Pardoux and Peng [1]. They obtained the first result on the existence and uniqueness of solution in $L^p, p \geq 2$ with Lipschitz coefficients. Recently, Aman [2] replaced the Lipschitz condition on $f$ in the variable $y$ from [1] with a monotone one and provided the existence and uniqueness of the solution for BDSDEs (1.1) in $L^p, p \in (1, 2)$.

More recently, Owo [3] proved the existence and uniqueness of the solution for BDSDEs (1.1), when the coefficients $f$ and $g$ are stochastic Lipschitz continuous, i.e., the constants of Lipschitz in [1, 2] are replaced with stochastic ones. However the solution in Owo [3] is taken in $L^2$ space. This limits the scope for several applications. For example, let $T = 1$ and suppose that the terminal condition is given by $\xi = e^{\left(\frac{W_2 - W_1}{p}\right)}1_{\{W_1 > p\}}$ for some $p \in (1, 2)$. A simple calculation of the expectation of $|\xi|^2$ and $|\xi|^p$ for $p \in (1, 2)$, yields that

$$E(|\xi|^2) = +\infty \quad \text{and} \quad E(|\xi|^p) = \frac{1}{\sqrt{2\pi p}} e^{(-p^2)} < +\infty.$$
So that the existence result in Owo [3] can not be applied to solve the above BDSDE with such a terminal condition \( \xi \). To correct this shortcoming, we study in this paper, the \( L^p \)-solution with \( p \in (1, 2) \) for BDSDEs with stochastic Lipschitz coefficients. Our work provides an extension of result obtained in \( L^p \), \( p \in (1, 2) \) by J. Wang et al. [4] for BSDEs with a stochastic Lipschitz coefficient, that is when \( g \equiv 0 \).

The paper is organized as follows. In Section 2, we introduce some preliminaries including some notations and some spaces. In Section 3, some useful \textit{a priori} estimates are given. Section 4 is devoted to the main result, \textit{i.e.}, the existence and uniqueness solution in \( L^p \) with \( p \in (1, 2) \).

\section{2. Preliminaries}

The standard inner product of \( \mathbb{R}^k \) is denoted by \( \langle \cdot, \cdot \rangle \) and the Euclidean norm by \( \| \cdot \| \).

A norm on \( \mathbb{R}^{d \times k} \) is defined by \( \sqrt{Tr(zz^*)} \), where \( z^* \) is the the transpose of \( z \). We will also denote this norm by \( | \cdot | \).

Let \(( \Omega, \mathcal{F}, \mathbb{P} )\) be a probability space and \( T \) be a fixed final time.

Throughout this paper \( \{ W_t : 0 \leq t \leq T \} \) and \( \{ B_t : 0 \leq t \leq T \} \) will denote two independent Brownian motions, with values in \( \mathbb{R}^d \) and \( \mathbb{R}^l \), respectively.

Let \( \mathcal{N} \) denote the class of \( \mathbb{P} \)-null sets of \( \mathcal{F} \). For each \( t \in [0, T] \), we define

\[ \mathcal{F}_t \overset{\Delta}{=} \mathcal{F}_t^W \vee \mathcal{F}_t^B, \]

where for any process \( \{ \eta_t : t \geq 0 \} \); \( \mathcal{F}_s^\eta = \sigma \{ \eta_r - \eta_s : s \leq r \leq t \} \vee \mathcal{N} \) and \( \mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta. \)

Note that \( \{ \mathcal{F}_t^W, t \in [0, T] \} \) is an increasing filtration and \( \{ \mathcal{F}_t^B, t \in [0, T] \} \) is a decreasing filtration, and the collection \( \{ \mathcal{F}_t, t \in [0, T] \} \) is neither increasing nor decreasing, so it does not constitute a filtration.

For every random process \( (a(t))_{t \geq 0} \) with positive values, such that \( a(t) \) is \( \mathcal{F}_t^W \)-measurable for a.e \( t \geq 0 \), we define an increasing process \( (A(t))_{t \geq 0} \) by setting \( A(t) = \int_0^t a^2(s)ds \).

For \( p > 1 \) and \( \beta > 0 \), we denote by:

- \( \mathcal{H}^p_\beta(a, T, \mathbb{R}^n) \) the set of jointly measurable processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^n \), such that \( \varphi(t) \) is \( \mathcal{F}_t \)-measurable, for a.e. \( t \in [0, T] \), with \( \| \varphi \|^p_{\mathcal{H}^p_\beta} = \mathbb{E} \left[ \left( \int_0^T e^{\beta A(t)} |\varphi(t)|^2 dt \right)^{\frac{p}{2}} \right] < \infty. \)
- \( \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^n) \) the set of jointly measurable processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^n \), such that \( \varphi(t) \) is \( \mathcal{F}_t \)-measurable, for a.e. \( t \in [0, T] \), with \( \| \varphi \|^p_{\mathcal{H}^{p, a}_\beta} = \mathbb{E} \left[ \int_0^T a^2(t)e^{\beta A(t)} |\varphi(t)|^p dt \right] < \infty. \)
- \( \mathcal{S}^p_\beta(a, T, \mathbb{R}^n) \) the set of jointly measurable continuous processes \( \varphi : \Omega \times [0, T] \rightarrow \mathbb{R}^n \), such that \( \varphi(t) \) is \( \mathcal{F}_t \)-measurable, for any \( t \in [0, T] \), with \( \| \varphi \|^p_{\mathcal{S}^p_\beta} = \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |\varphi(t)|^p \right] < \infty. \)

Note that the space \( \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^k) \) (resp. \( \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^{k \times d}) \)) with the norm \( \| \cdot \|_{\mathcal{H}^{p, a}_\beta} \) (resp. \( \| \cdot \|_{\mathcal{H}^{p, a}_\beta} \)) is a Banach space. So is the space

\[ \mathcal{M}^p_\beta(a, T) = \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^k) \times \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^{k \times d}), \]

with the norm \( \| (Y, Z) \|^p_{\mathcal{M}^p_\beta} = \| Y \|^p_{\mathcal{H}^{p, a}_\beta} + \| Z \|^p_{\mathcal{H}^{p, a}_\beta}. \) Also is the space

\[ \mathcal{M}^p_{\beta, c}(a, T) = \left( \mathcal{S}^p_\beta(a, T, \mathbb{R}^k) \cap \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^k) \right) \times \mathcal{H}^{p, a}_\beta(a, T, \mathbb{R}^{k \times d}), \]

with the norm \( \| (Y, Z) \|^p_{\mathcal{M}^p_{\beta, c}} = \| Y \|^p_{\mathcal{S}^p_\beta} + \| Y \|^p_{\mathcal{H}^{p, a}_\beta} + \| Z \|^p_{\mathcal{H}^{p, a}_\beta}. \)
Throughout the paper, the coefficients $f: \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^k$ and $g: \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \to \mathbb{R}^{k \times l}$, and the terminal value $\xi: \Omega \to \mathbb{R}^k$ satisfy the following assumptions, for $\beta > 0$:

(\textbf{H1}) $f$ and $g$ are jointly measurable, and there exist three nonnegative processes $\{r(t): t \in [0,T]\}$, $\{\theta(t): t \in [0,T]\}$ and a constant $0 < \alpha < 1$, such that:

(i) for a.e. $t \in [0,T]$, $r(t)$, $\theta(t)$ and $v(t)$ are $\mathcal{F}_t^W$-measurable;

(ii) for all $t \in [0,T]$ and all $(y, z), (y', z') \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$,

\[
\begin{align*}
&\left| f(t, y, z) - f(t, y', z') \right| \leq r(t) \left| y - y' \right| + \theta(t) \left| z - z' \right| \\
&\left| g(t, y, z) - g(t, y', z') \right|^2 \leq v(t) \left| y - y' \right|^2 + \alpha \left| z - z' \right|^2.
\end{align*}
\]

(\textbf{H2}) For all $t \in [0,T]$, $a^2(t) = r(t) + \theta^2(t) + v(t) > 0$, with $A(T) < L$, $\mathbb{P}$-a.s., where $L$ is a positive constant.

(\textbf{H3})

(i) $\xi$ is a $\mathcal{F}_T$-measurable random variable, such that $\mathbb{E}\left[ e^{\frac{\beta}{2} A(T)} | \xi |^p \right] < +\infty$;

(ii) for a.e. $t \in [0,T]$ and any $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d}$, $f(t, y, z)$ and $g(t, y, z)$ are $\mathcal{F}_t$-measurable, such that $\mathbb{E}\left[ \left( \int_0^T e^{\beta A(s)} \frac{|f_s|^2}{a^2(s)} ds \right)^{\frac{p}{2}} + \left( \int_0^T e^{\beta A(s)} \frac{|g_s|^2}{a^2(s)} ds \right)^{\frac{p}{2}} \right] < +\infty$, where $f_s = f(s, 0, 0)$ and $g_s = g(s, 0, 0)$.

\textbf{Definition 2.1.} A solution of BDSDE (1.1) is a pair of progressively measurable processes $(Y, Z): \Omega \times [0,T] \to \mathbb{R}^k \times \mathbb{R}^{k \times d}$ such that $\mathbb{P}$-a.s., $t \mapsto f(t, Y_t, Z_t)$ belongs to $L^1(0,T)$, $t \mapsto g(t, Y_t, Z_t)$ and $t \mapsto Z_t$ belong to $L^2(0,T)$ and satisfy equation (1.1).

Moreover, let $\beta > 0$ and let $\alpha$ be an $\mathcal{F}_T$-adapted process, a solution $(Y, Z)$ is said to be an $(\alpha, \beta)$-solution of the BDSDE (1.1) if $\mathbb{P}$-a.s., $t \mapsto e^{\frac{\beta}{2} A(t)} f(t, Y_t, Z_t)$ and $t \mapsto a^2(t) e^{\frac{\beta}{2} A(t)} Y_t$ belong to $L^1(0,T)$, $t \mapsto e^{\frac{\beta}{2} A(t)} g(t, Y_t, Z_t)$ and $t \mapsto e^{\frac{\beta}{2} A(t)} Z_t$ belong to $L^2(0,T)$.

For $p > 1$, a solution is said to be an $L^p$-solution if we have, moreover $(Y, Z) \in \mathcal{M}^p_{\alpha, \beta}(a, T)$.

\textbf{Remark 2.2.} Because of assumption (\textbf{H2}), the space $\mathcal{M}^p_{\alpha, \beta}(a, T)$ does not depend anymore on $\beta$.

Under assumptions (\textbf{H1})—(\textbf{H3}), as we can see in the following Lemma, for $p > 1$, any $L^p$-solution in the sense of definition 2.1, is an $(\alpha, \beta)$-solution.

\textbf{Lemma 2.3.} For $p > 1$, if $(Y, Z) \in \mathcal{M}^p_{\alpha, \beta}(a, T)$ and (\textbf{H1})—(\textbf{H3}) hold, then $t \mapsto e^{\frac{\beta}{2} A(t)} f(t, Y_t, Z_t)$ and $t \mapsto a^2(t) e^{\frac{\beta}{2} A(t)} Y_t$ belong to $L^1(0,T)$, $t \mapsto e^{\frac{\beta}{2} A(t)} g(t, Y_t, Z_t)$ and $t \mapsto e^{\frac{\beta}{2} A(t)} Z_t$ belong to $L^2(0,T)$, $\mathbb{P}$-a.s.

\textbf{Proof.} It is obvious that $t \mapsto e^{\frac{\beta}{2} A(t)} Z_t$ belongs to $L^2(0,T)$.

First, for $p \in (1, 2)$, we have

\[
\int_0^T \frac{a^2(s) e^{\beta A(s)} |Y_s|^2}{a^2(s)} ds = \int_0^T \left( e^{(1-\frac{\beta}{2}) A(s)} |Y_s|^2 - p \right) \left( \frac{a^2(s) e^{\frac{\beta}{2} A(s)} |Y_s|^p}{a^2(s)} \right) ds \\
\leq \left( \sup_{0 \leq t \leq T} e^{\frac{\beta}{2} A(t)} |Y_t|^p \right)^{\frac{2}{p}} \left( \int_0^T \frac{a^2(s) e^{\frac{\beta}{2} A(s)} |Y_s|^p}{a^2(s)} ds \right). \tag{2.1}
\]

Next, for $p \geq 2$, we have

\[
\int_0^T \frac{a^2(s) e^{\beta A(s)} |Y_s|^2}{a^2(s)} ds = \int_0^T \left( a^{\frac{2(p-2)}{p}}(s) \right) \left( a^{\frac{\beta}{2} (s)} e^{\beta A(s)} |Y_s|^2 \right) ds \\
\leq \left( \int_0^T a^2(s) ds \right)^{\frac{(p-2)}{p}} \left( \int_0^T a^2(s) e^{\frac{\beta}{2} A(s)} |Y_s|^p ds \right)^{\frac{2}{p}}.
\]
Then, for $p > 1$ and since $(Y, Z) \in \mathcal{M}_{\beta,c}^p (a, T)$, we get that
\[ \int_0^T a^2(s) e^{\beta A(s)} |Y_s|^2 ds < +\infty. \tag{2.2} \]
Therefore,
\[ \int_0^T a^2(s) e^{\frac{1}{2} \beta A(s)} |Y_s| ds \leq \left( \int_0^T a^2(s) ds \right)^{\frac{1}{2}} \left( \int_0^T a^2(s) e^{\beta A(s)} |Y_s|^2 ds \right)^{\frac{1}{2}} < +\infty. \tag{2.3} \]
On the other hand, from the assumptions on $(f, g)$ and noting that $a^2(t) = r(t) + \theta^2(t) + v(t)$ together with (2.2) and (2.3), we get that
\[
\int_0^T e^{\frac{1}{2} \beta A(s)} |f(s, Y_s, Z_s)| ds \leq \int_0^T e^{\frac{1}{2} \beta A(s)} \left( |f_0| + a^2(s) |Y_s| + a(s) |Z_s| \right) ds
\leq \left( \int_0^T a^2(s) ds \right)^{\frac{1}{2}} \left( \int_0^T e^{\beta A(s)} |f_0|^2 a^2(s) ds \right)^{\frac{1}{2}} + \int_0^T a^2(s) e^{\frac{1}{2} \beta A(s)} |Y_s| ds
+ \left( \int_0^T a^2(s) ds \right)^{\frac{1}{2}} \left( \int_0^T e^{\beta A(s)} |Z_s|^2 ds \right)^{\frac{1}{2}} < +\infty,
\]
and
\[
\int_0^T e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 ds \leq 2 \int_0^T a^2(s) e^{\beta A(s)} |Y_s|^2 ds + 2 \alpha \int_0^T e^{\beta A(s)} |Z_s|^2 ds + 2 \int_0^T e^{\beta A(s)} |g_0|^2 ds < +\infty. \tag*{□}
\]
In order to establish a priori estimates of $L^p$-solution of our BDSDE (1.1), we recall the Corollary 2.1 in Aman [2].

**Lemma 2.4.** Let $(Y, Z)$ be a solution of BDSDE (1.1). Then, for any $p \geq 1$ and any $t \in [0, T],$
\[
|Y_t|^p + c(p) \int_t^T |Y_s|^{p-2} I_{\{Y_s \neq 0\}} |Z_s|^2 ds \leq |\xi|^p + p \int_t^T |Y_s|^{p-1} (\hat{\eta}_s, f(s, Y_s, Z_s)) ds
+ c(p) \int_t^T |Y_s|^{p-2} I_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 ds
+ p \int_t^T |Y_s|^{p-1} (\hat{\eta}_s, g(s, Y_s, Z_s) B_s) - p \int_t^T |Y_s|^{p-1} (\hat{\eta}_s, Z_s dW_s),
\]
where, $c(p) = \frac{p(p-1)^{\lambda_1}}{2}$ and $\hat{\eta} = \text{sign}(y) = |y|^{-1} y I_{\{y \neq 0\}}.$

As a consequence of lemma 2.4, we have the following result

**Corollary 2.5.** Let $(Y, Z)$ be an $(\alpha, \beta)$-solution of BDSDE (1.1). Then, for any $p \geq 1, \beta \geq 0$ and any $t \in [0, T],$
\[
e^{\frac{1}{2} \beta A(t)} |Y_t|^p + c(p) \int_t^T e^{\frac{1}{2} \beta A(s)} |Y_s|^{p-2} I_{\{Y_s \neq 0\}} |Z_s|^2 ds + \frac{p}{2} \beta \int_t^T a^2(s) e^{\frac{1}{2} \beta A(s)} |Y_s|^p ds
\leq e^{\frac{1}{2} \beta A(T)} |\xi|^p + p \int_t^T e^{\frac{1}{2} \beta A(s)} |Y_s|^{p-1} (\hat{\eta}_s, f(s, Y_s, Z_s)) ds
+ c(p) \int_t^T e^{\frac{1}{2} \beta A(s)} |Y_s|^{p-2} I_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 ds - p \int_t^T e^{\frac{1}{2} \beta A(s)} |Y_s|^{p-1} (\hat{\eta}_s, Z_s dW_s)
+ p \int_t^T e^{\frac{1}{2} \beta A(s)} |Y_s|^{p-1} (\hat{\eta}_s, g(s, Y_s, Z_s) B_s) - p \int_t^T |Y_s|^{p-1} (\hat{\eta}_s, Z_s dW_s),
\]
where, $c(p) = \frac{p(p-1)^{\lambda_1}}{2}$ and $\hat{\eta} = \text{sign}(y) = |y|^{-1} y I_{\{y \neq 0\}}.$
Proof. Firstly, we show that

\[
e^{\frac{1}{2} \beta A(t)} Y_t = e^{\frac{1}{2} \beta A(t)} \xi + \int_t^T \left[ e^{\frac{1}{2} \beta A(s)} f(s, Y_s, Z_s) - \frac{1}{2} \beta a^2(s) e^{\frac{1}{2} \beta A(s)} Y_s \right] ds
\]

\[
+ \int_t^T e^{\frac{1}{2} \beta A(s)} g(s, Y_s, Z_s) dB_s - \int_t^T e^{\frac{1}{2} \beta A(s)} Z_s dW_s, \quad t \in [0, T].
\]

(2.4)

Indeed, let \( X_t = e^{\frac{1}{2} \beta A(t)} \), for \( t \in [0, T] \) with \( A(t) = \int_0^t a^2(s) ds \). Thus, by assumption (H2), \( X \) is a continuous and finite variation process. And by Itô’s formula, \( X_t = 1 + \frac{1}{2} \beta \int_0^t a^2(s) e^{\frac{1}{2} \beta A(s)} ds \).

Let \( \pi = \{ t = t_0 < t_1 < \ldots < t_n = T \} \), for \( t \in [0, T] \). Then,

\[
X_{t_{i+1}} Y_{t_{i+1}} - X_{t_i} Y_{t_i} = X_{t_i} (Y_{t_{i+1}} - Y_{t_i}) + Y_{t_i} (X_{t_{i+1}} - X_{t_i}) (Y_{t_{i+1}} - Y_{t_i})
\]

\[=
- \int_{t_i}^{t_{i+1}} X_{t_i} f(s, Y_s, Z_s) ds - \int_{t_i}^{t_{i+1}} X_{t_{i+1}} g(s, Y_s, Z_s) dB_s + \int_{t_i}^{t_{i+1}} X_{t_i} Z_s dW_s
\]

\[+
Y_{t_i} (X_{t_{i+1}} - X_{t_i}) + (X_{t_{i+1}} - X_{t_i}) \int_{t_i}^{t_{i+1}} f(s, Y_s, Z_s) ds - (X_{t_{i+1}} - X_{t_i}) \int_{t_i}^{t_{i+1}} Z_s dW_s.
\]

Therefore, taking the sum from \( i = 0 \) to \( i = n - 1 \), we get

\[
e^{\frac{1}{2} \beta A(t)} Y_t = e^{\frac{1}{2} \beta A(T)} \xi + I_1^n + I_2^n + I_3^n + I_4^n + I_5^n + I_6^n,
\]

where,

\[
I_1^n = \sum_{i=0}^{n-1} X_{t_i} (C_{t_{i+1}}^f - C_{t_i}^f), \quad I_2^n = - \sum_{i=0}^{n-1} X_{t_{i+1}} (M_s^{g} - M_{t_i}^{g})
\]

\[
I_3^n = - \sum_{i=0}^{n-1} X_{t_i} (M_{t_{i+1}}^{z} - M_s^{z}), \quad I_4^n = - \sum_{i=0}^{n-1} Y_{t_i} (X_{t_{i+1}} - X_{t_i})
\]

\[
I_5^n = - \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i}) (C_{t_{i+1}}^f - C_{t_i}^f), \quad I_6^n = \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i}) (M_{t_{i+1}}^{z} - M_s^{z})
\]

where, \( (C_f, M^g, M^z) \) are defined by:

\[
C_t^f = \int_0^t f(s, Y_s, Z_s) ds, \quad M_t^g = \int_t^T g(s, Y_s, Z_s) dB_s, \quad M_t^z = \int_0^t Z_s dW_s, \quad \text{ for } t \in [0, T].
\]

Since \((Y, Z)\) is an \((a, \beta)\)-solution, \( C^f \) is a continuous and finite variation process and the process \( M^g \) (resp. \( M^z \)) is a backward (resp. a forward) continuous martingale.

By continuity of \( X \) and \( Y \), and the definition of Stieltjes integrals, together with the fact that \((Y, Z)\) is an \((a, \beta)\)-solution, it follows that

\[
I_1^n \longrightarrow \int_t^T X_s dC_s^f = \int_t^T e^{\frac{1}{2} \beta A(s)} f(s, Y_s, Z_s) ds \quad \text{a.s.,}
\]

\[
I_4^n \longrightarrow - \int_t^T Y_s dX_s = - \frac{1}{2} \beta \int_t^T a^2(s) e^{\frac{1}{2} \beta A(s)} Y_s ds \quad \text{a.s.}
\]
Moreover, by the definition of backward-forward stochastic integrals with respect to martingales
\[ I_n^2 \rightarrow - \int_t^T X_s \, dM_s^2 = \int_t^T e^{\frac{1}{2} \beta A(s)} g(s, Y_s, Z_s) \, dB_s \text{ in probability}, \]
\[ I_n^3 \rightarrow - \int_t^T X_s \, dM_s^3 = - \int_t^T e^{\frac{1}{2} \beta A(s)} Z_s \, dW_s \text{ in probability}. \]
On the other hand, we have,
\[ |I_n^5| \leq \sup_{0 \leq t \leq n-1} \left( |C_n^I - C_{n+1}^I| \right) e^{\frac{1}{2} \beta A(T)} \rightarrow 0 \text{ in probability}, \]
due to the fact that the first term converges to zero almost surely by the continuity of \( C^I \), and the second is finite \( \mathbb{P} \) – a.s. by assumption \( (H2) \).

Also, by the continuity of \( M^Z \), we have
\[ |I_n^6| \leq \sup_{0 \leq i \leq n-1} \left( |M_n^I - M_{n+1}^I| \right) e^{\frac{1}{2} \beta A(T)} \rightarrow 0 \text{ in probability}, \]
so that we obtain (2.4).

Now letting \( \bar{Y}_t = e^{\frac{1}{2} \beta A(t)} Y_t \), \( \bar{Z}_t = e^{\frac{1}{2} \beta A(t)} Z_t \) and \( \bar{\xi} = e^{\frac{1}{2} \beta A(T)} \xi \), we get
\[ \bar{Y}_t = \bar{\xi} + \int_t^T \bar{f}(s, \bar{Y}_s, \bar{Z}_s) \, ds + \int_t^T \bar{g}(s, \bar{Y}_s, \bar{Z}_s) \, dB_s - \int_t^T \bar{Z}_s \, dW_s, \quad t \in [0, T], \]
where, \( \bar{f} \) and \( \bar{g} \) are defined by:
\[ \bar{f}(t, y, z) = e^{\frac{1}{2} \beta A(t)} f(t, e^{-\frac{1}{2} \beta A(t)} y, e^{-\frac{1}{2} \beta A(t)} z) - \frac{1}{2} \beta a^2(t) y, \]
\[ \bar{g}(t, y, z) = e^{\frac{1}{2} \beta A(t)} g(t, e^{-\frac{1}{2} \beta A(t)} y, e^{-\frac{1}{2} \beta A(t)} z). \]
Thus, by Definition 2.1 and Lemma 2.4, we deduce the result. \( \square \)

3. A PRIORI ESTIMATES

**Lemma 3.1.** Let \( \beta \geq 0, \ p \in [1, 2] \) and assume that \( (H1) \)–\( (H3) \) hold. Let \( (Y, Z) \) be an \( (a, \beta) \)-solution of BDSDE (1.1). If \( Y \in \mathcal{S}_p^\alpha(a, T, \mathbb{R}) \cap \mathcal{H}_p^\beta(a, T, \mathbb{R}^{k \times d}) \), then \( Z \in \mathcal{H}_p^\beta(a, T, \mathbb{R}^{k \times d}) \) and there exists a constant \( C_p \) depending on \( p, \alpha \) such that for some \( \beta > 0 \),
\[ |||Z|||_p \leq C_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{1}{2} \beta A(t)} |Y_t|^p + \left( \int_0^T e^{\beta A(s)} \left| \frac{\partial f}{\partial a^2(s)} \right|^2 \, ds \right)^{\frac{p}{2}} + \left( \int_0^T e^{\beta A(s)} \left| \frac{\partial g}{\partial a^2(s)} \right|^2 \, ds \right)^{\frac{p}{2}} \right]. \]

**Proof.** Let \( p \in [1, 2] \). For each integer \( n > 0 \), let us introduce the stopping time
\[ \tau_n = \inf \left\{ t \in [0, T], \int_0^t e^{\beta A(s)} |Z_s|^2 \, ds \geq n \right\} \wedge T. \]
Applying Itô’s formula to \( e^{\beta A(t)} |Y_t|^2 \), we have
\[ e^{\beta A(t)} |Y_t|^2 + \beta \int_t^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 \, ds + \int_t^{\tau_n} e^{\beta A(s)} |Z_s|^2 \, ds \]
\[ = e^{\beta A(\tau_n)} |Y_{\tau_n}|^2 + 2 \int_t^{\tau_n} e^{\beta A(s)} \langle Y_s, f(s, Y_s, Z_s) \rangle \, ds + \int_t^{\tau_n} e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \, ds \]
\[ + 2 \int_t^{\tau_n} e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) \rangle \, dB_s - 2 \int_t^{\tau_n} e^{\beta A(s)} \langle Y_s, Z_s \rangle \, dW_s. \]
From (H1) and Young’s inequality for every $\sigma > 0$ such that $\sigma + \alpha < 1$, we have
\[
2 \langle Y_s, f(s, Y_s, Z_s) \rangle \leq 2r(s) |Y_s|^2 + 2\theta(s) |Y_s| |Z_s| + 2 |Y_s| f_0^s \\
\leq \left( 3 + \frac{1}{\sigma} \right) a^2(s) |Y_s|^2 + \sigma |Z_s|^2 + \frac{|f_0|^2}{a^2(s)}
\]
and for every $\gamma > 0$,
\[
|g(s, Y_s, Z_s)|^2 \leq (1 + \gamma) a^2(s) |Y_s|^2 + (1 + \gamma) \alpha |Z_s|^2 + \left( 1 + \frac{1}{\gamma} \right) |g_0|^2. \quad (3.2)
\]
Finally, it follows that
\[
e^{\beta A(t)} |Y_t|^2 + D_1 \int_t^T a^2(s) e^{\beta A(s)} |Y_s|^2 ds + D_2 \int_t^T e^{\beta A(s)} |Z_s|^2 ds \\
\leq e^{\beta A(t)} |Y_t|^2 + \int_t^T e^{\beta A(s)} \frac{|f_0|^2}{a^2(s)} ds + \left( 1 + \frac{1}{\gamma} \right) \int_t^T e^{\beta A(s)} |g_0|^2 ds \\
- 2 \int_t^T e^{\beta A(s)} \langle Y_s, Z_s dW_s \rangle + 2 \int_t^T e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) dB_s \rangle,
\]
where, $D_1 = \beta - 4 - \gamma - \frac{1}{\gamma}$ and $D_2 = 1 - \sigma - (1 + \gamma) \alpha$.

Choosing $\gamma > 0$, $\beta > 0$ such that $\gamma < \frac{1 - (\sigma + \alpha)}{\alpha}$ and $\beta > 4 + \gamma + \frac{1}{\sigma}$, we get $D_1 > 0$ and $D_2 > 0$.

Therefore, since $\tau_n \leq T$, putting $t = 0$, we have
\[
D_1 \int_0^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 ds + D_2 \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 ds \\
\leq \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^2 + \int_0^T e^{\beta A(s)} \frac{|f_0|^2}{a^2(s)} ds + \left( 1 + \frac{1}{\gamma} \right) \int_0^T e^{\beta A(s)} |g_0|^2 ds \\
- 2 \int_0^{\tau_n} e^{\beta A(s)} \langle Y_s, Z_s dW_s \rangle + 2 \int_0^{\tau_n} e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) dB_s \rangle
\]
and thus, raising both sides to the power $\frac{\beta}{2} < 1$, and taking expectation, we derive
\[
\mathbb{E} \left[ \left( \int_0^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 ds \right)^{\frac{\beta}{2}} + \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 ds \right)^{\frac{\beta}{2}} \right] \\
\leq \lambda_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^p + \left( \int_0^T e^{\beta A(s)} \frac{|f_0|^2}{a^2(s)} ds \right)^{\frac{\beta}{p}} + \left( \int_0^T e^{\beta A(s)} |g_0|^2 ds \right)^{\frac{\beta}{p}} \right] \\
+ \left( \int_0^{\tau_n} e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) dB_s \rangle \right)^{\frac{\beta}{2}} + \left( \int_0^{\tau_n} e^{\beta A(s)} \langle Y_s, Z_s dW_s \rangle \right)^{\frac{\beta}{2}}. \quad (3.4)
\]

But by the BDG and Young’s inequalities, we get for a given constant $d_p > 0$ and any $\gamma_1 > 0$,
\[
\lambda_p \mathbb{E} \left[ \left( \int_0^{\tau_n} e^{\beta A(s)} \langle Y_s, Z_s dW_s \rangle \right)^{\frac{\beta}{2}} \right] \\
\leq \lambda_p d_p \mathbb{E} \left[ \left( \int_0^{\tau_n} e^{\beta A(s)} |Y_s|^2 e^{\beta A(s)} |Z_s|^2 ds \right)^{\frac{\beta}{2}} \right] \\
\leq \lambda_p d_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^p \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 ds \right)^{\frac{\beta}{p}} \right] \\
\leq \mathbb{E} \left[ \frac{\lambda_p^2 d_p^2}{\gamma_1} \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^p + \gamma_1 \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 ds \right)^{\frac{\beta}{p}} \right]
\]
and
\[
\lambda_p \mathbb{E} \left[ \int_0^{\tau_n} e^{\beta A(s)} (Y_s, g(s, Y_s, Z_s) \mathrm{d} B_s) \right]^\frac{\gamma}{p} \leq \lambda_p d_p \mathbb{E} \left[ \int_0^{\tau_n} e^{\beta A(s)} |Y_s|^2 e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \mathrm{d}s \right]^\frac{\gamma}{p}
\]
\[
\leq \lambda_p d_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{e^{\beta A(t)}}{\beta A(t)}} |Y_t|^2 \left( \int_0^{\tau_n} e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right]
\]
\[
\leq \mathbb{E} \left[ \frac{\lambda_p^2 d_p^2}{\gamma_1} \sup_{0 \leq t \leq T} e^{\frac{e^{\beta A(t)}}{\beta A(t)}} |Y_t|^p + \gamma_1 \left( \int_0^{\tau_n} e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right].
\]

Now, from (3.2), we have for any \( \gamma_2 > 0 \)
\[
\int_0^{\tau_n} e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \mathrm{d}s \leq \left( 1 + \frac{1}{\gamma_2} \right) \int_0^T e^{\beta A(s)} |g_0|^2 \mathrm{d}s + (1 + \gamma_2) \int_0^{\tau_n} e^{\beta A(s)} \left[ a^2(s)|Y_s|^2 + \alpha |Z_s|^2 \right] \mathrm{d}s.
\]
Thus, rising to power \( \frac{p}{2} < 1 \), we get
\[
\left( \int_0^{\tau_n} e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \leq \left( 1 + \frac{1}{\gamma_2} \right) \left( \int_0^T e^{\beta A(s)} |g_0|^2 \mathrm{d}s \right)^\frac{\gamma}{p} + (1 + \gamma_2) \left( \int_0^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} + (1 + \gamma_2) \alpha^2 \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p}.
\] (3.5)

Therefore, coming back to (3.4), we have
\[
\mathbb{E} \left[ \left( \int_0^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} + \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right]
\]
\[
\leq \lambda(p) \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{e^{\beta A(t)}}{\beta A(t)}} |Y_t|^p + \left( \int_0^T e^{\beta A(s)} \frac{|f_0|^2 \mathrm{d}s}{a^2(s)} \right)^\frac{\gamma}{p} + \left( \int_0^T e^{\beta A(s)} |g_0|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right]
\]
\[
+ \left[ \gamma_1 + (1 + \gamma_2) \gamma_1 \alpha^2 \right] \mathbb{E} \left[ \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right] + (1 + \gamma_2) \gamma_1 \mathbb{E} \left[ \left( \int_0^{\tau_n} a^2(s) e^{\beta A(s)} |Y_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right].
\]

Consequently, choosing \( \gamma_1, \gamma_2 > 0 \) such that \( \gamma_1 + (1 + \gamma_2) \gamma_1 \alpha^2 < 1 \) and \( (1 + \gamma_2) \gamma_1 < 1 \), we derive, for any \( n \geq 1 \)
\[
\mathbb{E} \left[ \left( \int_0^{\tau_n} e^{\beta A(s)} |Z_s|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right] \leq C_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{e^{\beta A(t)}}{\beta A(t)}} |Y_t|^p + \left( \int_0^T e^{\beta A(s)} \frac{|f_0|^2 \mathrm{d}s}{a^2(s)} \right)^\frac{\gamma}{p} + \left( \int_0^T e^{\beta A(s)} |g_0|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right],
\]
with by Fatou’s lemma yields the desired result.

\( \square \)

**Proposition 3.2.** Let \( \beta \geq 0, \ p \in [1, 2] \). Let \((Y, Z)\) be an \((a, \beta)\)-solution of BDSDE (1.1) with terms \((\xi, f, g)\) satisfying (H1)-(H3), where \(Y \in S^p_{\beta}(a, T, \mathbb{R}^k) \cap \mathcal{H}^p_{\beta}(a, T, \mathbb{R}^{k \times d})\). Then, there exists a constant \(C_p = C_p(\beta, a, T, L)\) satisfying the a priori estimate
\[
\|Y\|_{S^p_{\beta}}^p + \|Y\|_{\mathcal{H}^p_{\beta}}^p + \|Z\|_{\mathcal{H}^p_{\beta}}^p \leq C_p \mathbb{E} \left[ e^{\frac{e^{\beta A(T)}}{\beta A(T)}} \|\xi\|^p + \left( \int_0^T e^{\beta A(s)} \frac{|f_0|^2 \mathrm{d}s}{a^2(s)} \right)^\frac{\gamma}{p} + \left( \int_0^T e^{\beta A(s)} |g_0|^2 \mathrm{d}s \right)^\frac{\gamma}{p} \right] \geq 0 \right] \|g_0|^2 \mathrm{d}s.\] (3.6)
Proof. Let \( p \in [1, 2] \). From corollary 2.5, we have for any \( \beta \geq 0 \) and any \( t \in [0, T] \),
\[
e^{\frac{\beta}{2}A(t)}|Y_t|^p + c(p) \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^p ds + \frac{p}{2} \beta \int_t^T a^2(s)e^{\beta A(s)}|Y_s|^p ds
\leq e^{\frac{\beta}{2}A(T)}|\xi|^p + p \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} \hat{f}(s, f(s, Y_s, Z_s)) ds
\]

From BDG inequality, one can show that
\[
\int_0^T e^{\frac{\beta}{2}A(s)}|Y_s|^p ds \leq \int_0^T e^{\frac{\beta}{2}A(s)}|Y_s|^p ds + p \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} \hat{f}(s, f(s, Y_s, Z_s)) ds.
\]

From (H1), we have
\[
\left< \hat{Y}_s, f(s, Y_s, Z_s) \right> \leq r(s) |Y_s| + \theta(s) |Z_s| + |f_1^0|,
\]
which, together with (3.2), yields for every \( \gamma > 0 \),
\[
e^{\frac{\beta}{2}A(t)}|Y_t|^p + c(p) \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^p ds + \frac{p}{2} \beta \int_t^T a^2(s)e^{\beta A(s)}|Y_s|^p ds
\leq e^{\frac{\beta}{2}A(T)}|\xi|^p + p \int_t^T r(s)e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} |Z_s| ds
\]

By virtue of Young’s inequality, we have for any \( \varepsilon > 0 \),
\[
p\theta(s)e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} |Z_s| = \left( p\theta(s)e^{\frac{\beta}{2}A(s)}|Y_s|^{\frac{p}{2}} \right) \left( e^{\frac{\beta}{2}A(s)}|Y_s|^{\frac{p}{2}} |Z_s|^{\frac{p}{2}} \right)
\]
\[
\leq \frac{2p}{[(p-1)\wedge 1]} \theta^2(s)e^{\frac{\beta}{2}A(s)}|Y_s|^p + \varepsilon c(p)e^{\frac{\beta}{2}A(s)}|Y_s|^{p-2} |Y_s|^{p-1} |Z_s|^2.
\]

Therefore, we get
\[
e^{\frac{\beta}{2}A(t)}|Y_t|^p + \delta_1 \int_t^T a^2(s)e^{\frac{\beta}{2}A(s)}|Y_s|^p ds + \delta_2 \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} \hat{f}(s, f(s, Y_s, Z_s)) ds
\]

where \( \delta_1 = \frac{\beta}{2} - p - c(p)(1 + \gamma) - \frac{2p}{[(p-1)\wedge 1]} \delta_2 = c(p)[1 - (1 + \gamma) \alpha - \varepsilon] \) and

\[
X = e^{\frac{\beta}{2}A(T)}|\xi|^p + p \int_0^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} |f_0^s|^{p-1} ds + c(p) \left( 1 + \frac{1}{\gamma} \right) \int_0^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-2} |Y_s|^{p-1} |f_0^s|^2 ds.
\]

From BDG inequality, one can show that
\[
M = \left\{ \int_0^t e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} \hat{f}(s, Y_s, Z_s) ds \right\}_{0 \leq t \leq T} \quad \text{and} \quad N = \left\{ \int_t^T e^{\frac{\beta}{2}A(s)}|Y_s|^{p-1} \hat{f}(s, Y_s, Z_s) ds \right\}_{0 \leq t \leq T}.
\]
are respectively uniformly integrable martingale. Indeed, we have, by Young’s inequality

$$\mathbb{E} \langle M, M \rangle_T^{1/2} \leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{p-1}{2} \alpha^{2}(t)} |Y_t|^{p-1} \left( \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right)^{\frac{p}{2}} \right]$$

$$\leq \frac{p-1}{p} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{p}{2} \alpha^{2}(t)} |Y_t|^p \right] + \frac{1}{p} \mathbb{E} \left[ \left( \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right)^{\frac{p}{2}} \right].$$

Also, in view of (3.2) and since $\frac{p}{2} < 1$, we get

$$\mathbb{E} \langle N, N \rangle_T^{1/2} \leq \frac{p-1}{p} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{p}{2} \alpha^{2}(t)} |Y_t|^p \right] + (1 + \gamma) \mathbb{E} \left[ \left( \int_0^T a^2(s) e^{\beta A(s)} |Y_s|^2 \, ds \right)^{\frac{p}{2}} \right]$$

$$+ (1 + \gamma) \alpha^2 \mathbb{E} \left[ \left( \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right)^{\frac{p}{2}} \right] + \left( 1 + \frac{1}{\gamma} \right) \mathbb{E} \left[ \left( \int_0^T e^{\beta A(s)} |g_0|^2 \, ds \right)^{\frac{p}{2}} \right].$$

Now, from (2.1) for $p \in (1, 2)$, we derive by Young’s inequality

$$\left( \int_0^T a^2(s) e^{\beta A(s)} |Y_s|^2 \, ds \right)^{\frac{p}{2}} \leq \left( \sup_{0 \leq t \leq T} e^{\frac{p}{2} \alpha^{2}(t)} |Y_t|^p \right) \left( \int_0^T a^2(s) e^{\frac{p}{2} \alpha^{2}(s)} |Y_s|^p \, ds \right)^{\frac{p}{2}}$$

$$\leq 2 \frac{p-1}{2} \left( \sup_{0 \leq t \leq T} e^{\frac{p}{2} \alpha^{2}(t)} |Y_t|^p \right) + \frac{p}{2} \left( \int_0^T a^2(s) e^{\frac{p}{2} \alpha^{2}(s)} |Y_s|^p \, ds \right).$$

Since $Y \in \mathcal{S}_\beta^{p}(a, T, \mathbb{R}^k) \cap \mathcal{H}_\beta^{p\cdot a}(a, T, \mathbb{R}^{k \times d})$, it follows from Lemma 3.1, that $Z \in \mathcal{H}_\beta^{p\cdot a}(a, T, \mathbb{R}^{k \times d})$, which together with assumption (H3)(ii), yields that

$$\mathbb{E} \langle M, M \rangle_T^{1/2} < +\infty \quad \text{and} \quad \mathbb{E} \langle N, N \rangle_T^{1/2} < +\infty,$$

which implies that $M$ and $N$ are uniformly integrable martingale.

Thus, taking expectation in (3.7) with $t = 0$, we have

$$\mathbb{E} \left[ \delta_1 \int_0^T a^2(s) e^{\frac{p}{2} \alpha^{2}(s)} |Y_s|^p \, ds + \delta_2 \int_0^T e^{\frac{p}{2} \alpha^{2}(s)} |Y_s|^p \, ds \right] \leq \mathbb{E} (X). \quad (3.8)$$

Now, by choosing $\gamma, \varepsilon > 0$ such that $(1 + \gamma) \alpha + \varepsilon < 1$ and $\beta > 2 + \frac{2\varepsilon(p)}{p} (1 + \gamma) + \frac{4}{(p-1)p} \gamma$, it follows that $\delta_1, \delta_2 > 0$ and so taking the sup(,) and then the expectation in (3.7), we derive by Burkholder–Davis–Gundy’s inequality that

$$\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{p}{2} \alpha^{2}(t)} |Y_t|^p \right] \leq \mathbb{E} (X) + k_p \mathbb{E} \langle M, M \rangle_T^{1/2} + h_p \mathbb{E} (N, N)_T^{1/2}. \quad (3.9)$$
Likewise, but from Young's inequality and (3.8), we get
\[
 k_p \mathbb{E}(M, M)^{1/2} \leq k_p \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} \left| Y_t \right|^p \right] \left( \int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-2} 1_{\{Y_s \neq 0\}} |Z_s|^2 \, ds \right)^{\frac{1}{p}}
\]
\[
 \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t|^p \right] + 4k_p^2 \mathbb{E} \left[ \int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-2} 1_{\{Y_s \neq 0\}} |Z_s|^2 \, ds \right]
\]
\[
 \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t|^p \right] + k_p \mathbb{E} (X). \tag{3.10}
\]

Now, in view of (3.2), it follows that
\[
\int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-2} 1_{\{Y_s \neq 0\}} |g(s, Y_s, Z_s)|^2 \, ds
\]
\[
\leq (1 + \gamma) \int_0^T a^2(s) e^{\frac{\beta A(s)}{2}} |Y_s|^p \, ds + (1 + \gamma) \alpha \int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-2} 1_{\{Y_s \neq 0\}} |Z_s|^2 \, ds
\]
\[
+ \left( 1 + \frac{1}{\gamma} \right) \int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-2} 1_{\{Y_s \neq 0\}} |g_s|_0^2 \, ds.
\]

Then, from (3.8) together with the definition of $X$, we have
\[
 h_p \mathbb{E}(N, N)^{1/2} \leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t|^p \right] + h_p \mathbb{E} (X). \tag{3.11}
\]

Therefore, putting the estimates (3.10) and (3.11) into (3.9), we obtain
\[
 \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t|^p \right] \leq 2(1 + k_p + h_p) \mathbb{E} (X),
\]

which together with (3.8), implies that
\[
 \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t|^p + \frac{\delta_1}{2} \int_0^T a^2(s) e^{\frac{\beta A(s)}{2}} |Y_s|^p \, ds \right] \leq C_p \mathbb{E} (X).
\]

Applying Holder and Young's inequalities, we have, by (H2)
\[
p \int_0^T e^{\frac{\beta A(s)}{2}} |Y_s|^{p-1} |g_s|^p \, ds = \int_0^T \left( a^{2(p-1)}(s) e^{\frac{\beta A(s)}{2}} |Y_s|^{p-1} \right) \left( a^2(s) e^{\frac{\beta A(s)}{2}} |f_s|^p \right) \, ds
\]
\[
\leq \frac{\delta_1}{2C_p} \int_0^T a^2(s) e^{\frac{\beta A(s)}{2}} |Y_s|^p \, ds + \left( \frac{2(p-1)C_p}{\delta_1} \right)^{p-1} \int_0^T a^2(s) e^{\frac{\beta A(s)}{2}} \frac{|f_s|^p}{a^p(s)} \, ds
\]
\[
\leq \frac{\delta_1}{2C_p} \int_0^T a^2(s) e^{\frac{\beta A(s)}{2}} |Y_s|^p \, ds + \left( \frac{2(p-1)C_p}{\delta_1} \right)^{p-1} \left( \int_0^T e^{\beta A(s)} \frac{|f_s|^2}{a^2(s)} \, ds \right)^{\frac{p}{2}}.
\]
Finally, coming back to the definition of $X$, we obtain

\[
E \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta}{2} A(t)} |Y_t|^p + \int_t^T a^2(s) e^{\beta A(s)} |Y_s|^p ds \right] 
\leq C_p' E \left[ e^{\frac{\beta}{2} A(T)} |\xi|^p + \left( \int_0^T e^{\beta A(s)} |f_0|^2 a^2(s) ds \right)^\frac{p}{2} + \left( \int_0^T e^{\beta A(s)} |g_s|^2 ds \right)^\frac{p}{2} 
+ \int_0^T e^{\frac{\beta}{2} A(s)} |Y_s|^p \cdot 21_{\{Y_s \neq 0\}} |g_s|^2 ds \right].
\]

The result follows from Lemma 3.1. □

4. Existence and uniqueness of a solution

In order to obtain the existence and uniqueness result for BDSDEs associated to data $(\xi, f, g)$ in $L^p$, we make the following supplementary assumption:

\[ (H4) \quad g(t, 0, 0) = 0, \forall \ t \in [0, T]. \]

Moreover, we recall the following result due to Owo ([3], Thm. 3.3).

Theorem 4.1. For $p = 2$ and any $\beta$, assume that (H1)–(H3) hold. Then, the BDSDE (1.1) has a unique solution $(Y, Z) \in M_{\beta, c}^2 (a, T)$.

From Lemma 2.3, the unique solution $(Y, Z) \in M_{\beta, c}^2 (a, T)$ in Theorem 4.1 is an $(a, \beta)$-solution of BDSDE (1.1). Now we give a basic estimate concerning the solution.

Lemma 4.2. For $p \in [1, 2]$ and any $\beta$, assume that (H1)–(H4) hold. Let $(Y, Z) \in M_{\beta, c}^2 (a, T)$ be a solution of BDSDE (1.1) and assume that $\mathbb{P}$-a.s.,

\[
\sup_{0 \leq t \leq T} e^{\frac{\beta}{2} A(t)} \frac{|f_0|}{a(t)} \leq n, \quad e^{\frac{\beta}{2} A(T)} \xi \leq n, \tag{4.1}
\]

then $Y \in S^p_{\beta}(a, T, \mathbb{R}^k) \cap H^{p,a}_{\beta}(a, T, \mathbb{R}^{k \times d})$.

Proof. Applying Itô’s formula to $e^{\beta A(t)} |Y_t|^2$, we have for any $t \in [0, T]$,

\[
e^{\beta A(t)} |Y_t|^2 + \beta \int_t^T e^{\beta A(s)} |Y_s|^2 ds + \int_t^T e^{\beta A(s)} |Z_s|^2 ds 
= e^{\beta A(T)} |\xi|^2 + 2 \int_t^T e^{\beta A(s)} \langle Y_s, f(s, Y_s, Z_s) \rangle ds + \int_t^T e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 ds 
+ 2 \int_t^T e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) dB_s \rangle - 2 \int_t^T e^{\beta A(s)} \langle Y_s, Z_s dW_s \rangle.
\]

From (H1) and Young’s inequality, we have

\[
2 \langle Y_s, f(s, Y_s, Z_s) \rangle \leq 2r(s) |Y_t|^2 + 2\theta(s) |Y_s| |Z_s| + 2 |Y_s| |f_0|^2 
\leq \left( 3 + \frac{2}{1 - \alpha} \right) a^2(s) |Y_s|^2 + \frac{1 - \alpha}{2} |Z_s|^2 + \frac{|f_0|^2}{a^2(s)}
\]

\[
\leq \left( 3 + \frac{2}{1 - \alpha} \right) a^2(s) |Y_s|^2 + \frac{1 - \alpha}{2} |Z_s|^2 + \frac{|f_0|^2}{a^2(s)}
\]
and from (H1) and (H4)

\[ |g(s, Y_s, Z_s)|^2 \leq a^2(s) |Y_s|^2 + \alpha |Z_s|^2. \]

Finally, in view of (4.1), it follows that

\[
e^{\beta A(t)} |Y_t|^2 + \left( \beta - 4 - \frac{2}{1 - \alpha} \right) \int_t^T a^2(s)e^{\beta A(s)} |Y_s|^2 \, ds + \left( \frac{1 - \alpha}{2} \right) \int_t^T e^{\beta A(s)} |Z_s|^2 \, ds
\leq n^2 + n^2T - 2 \int_t^T e^{\beta A(s)} \langle Y_s, Z_s \rangle dW_s + 2 \int_t^T e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) \rangle dB_s. \tag{4.2}
\]

By the same argument as in the previous proof on the uniform integrability of \(M\) and \(N\), we prove that \(\left\{ \int_0^T e^{\beta A(s)} \langle Y_s, Z_s \rangle dW_s \right\}_{0 \leq t \leq T}\) and \(\left\{ \int_0^T e^{\beta A(s)} \langle Y_s, g(s, Y_s, Z_s) \rangle dB_s \right\}_{0 \leq t \leq T}\) are respectively uniformly integrable martingale. Therefore, taking expectation in (4.2), we have

\[
\mathbb{E} \left[ \left( \beta - 4 - \frac{2}{1 - \alpha} \right) \int_0^T a^2(s)e^{\beta A(s)} |Y_s|^2 \, ds + \left( \frac{1 - \alpha}{2} \right) \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right] \leq n^2 + n^2T. \tag{4.3}
\]

Now, choosing \(\beta > 4 + \frac{2}{1 - \alpha}\), and taking \(\sup_{0 \leq t \leq T} (\cdot)\) in (4.2) and applying Burkholder–Davis–Gundy’s inequality and Young’s inequality \(2ab \leq \delta a^2 + \frac{1}{\delta} b^2\), for every \(\delta > 0\), we deduce that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^2 \right] \leq n^2 + n^2T + 2e\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t| \left( \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right)^{\frac{1}{2}} \right]
+ 2e \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{\beta A(t)}{2}} |Y_t| \left( \int_0^T e^{\beta A(s)} |g(s, Y_s, Z_s)|^2 \, ds \right)^{\frac{1}{2}} \right]
\leq n^2 + n^2T + 2\delta \mathbb{E} \left( \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^2 \right)
+ (1 + \alpha) \frac{1}{\delta} \mathbb{E} \left( \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right) + \frac{2}{\delta} \mathbb{E} \left( \int_0^T a^2(s)e^{\beta A(s)} |Y_s|^2 \, ds \right). \tag{4.4}
\]

Therefore, combining (4.3) and (4.4), and choosing \(\delta < \frac{1}{2}\), we derive

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^2 + \int_0^T e^{\beta A(s)} a^2(s) |Y_s|^2 \, ds + \int_0^T e^{\beta A(s)} |Z_s|^2 \, ds \right] \leq c'(n^2 + n^2T), \tag{4.5}
\]

which since \(p \in [1, 2]\) and together with Hölder’s inequality yields

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\frac{p}{2}\beta A(t)} |Y_t|^p \right] \leq \left( \mathbb{E} \left[ \sup_{0 \leq t \leq T} e^{\beta A(t)} |Y_t|^2 \right] \right)^{\frac{p}{2}} < \infty.
\]
and

\[
\mathbb{E}\left[\int_0^T a^2(s)e^{\frac{s}{\beta}\lambda}\beta A(s) | Y_s|^p ds \right] = \mathbb{E}\left[\int_0^T (a^{2p}(s)) a^p(s)e^{\frac{s}{\beta}\lambda}\beta A(s) | Y_s|^p ds \right] \\
\leq \mathbb{E}\left[\left(\int_0^T a^2(s)ds\right)^{1-p} \left(\int_0^T a^2(s)e^{\beta A(s)} | Y_s|^2 ds\right)^{1-p}\right] \\
\leq \left(\mathbb{E}\left[\int_0^T a^2(s)ds\right]\right)^{1-p} \left(\mathbb{E}\left[\int_0^T a^2(s)e^{\beta A(s)} | Y_s|^2 ds\right]\right)^{1-p} < \infty.
\]

\[\square\]

We now state and prove our main result.

**Theorem 4.3.** For \( p \in [1, 2] \), let assume (H1)–(H4). Then, for \( \beta \) sufficiently large, the BDSDE (1.1) has a unique solution \((Y, Z) \in \mathcal{M}_{\beta,c}^p(a, T)\).

**Proof.** (Uniqueness). Let \((Y, Z), (Y', Z') \in \mathcal{M}_{\beta,c}^p(a, T)\) be two solutions of BDSDE (1.1).

Let denote by \((\overline{Y}, \overline{Z})\) the process \((Y - Y', Z - Z')\). Then, it is obvious that \((\overline{Y}, \overline{Z})\) is a solution in \(\mathcal{M}_{\beta,c}^p(a, T)\) to the following BDSDE:

\[
\overline{Y}_t = \int_t^T F(s, \overline{Y}_s, \overline{Z}_s)ds + \int_t^T G(s, \overline{Y}_s, \overline{Z}_s)d\overline{B}_s - \int_t^T \overline{Z}_s dW_s, \tag{4.6}
\]

where \(F, G\) stand for the random functions

\[
F(t, y, z) = f(t, y + Y'_t, z + Z'_t) - f(t, Y'_t, Z'_t) \\
G(t, y, z) = g(t, y + Y'_t, z + Z'_t) - g(t, Y'_t, Z'_t).
\]

It is easy to verify that BDSDE (4.6) satisfies assumptions (H1)–(H3). Noting that \(F^0 = 0\) and \(G^0 = 0\), by Proposition 3.2, we get immediately that \((\overline{Y}, \overline{Z}) = (0, 0)\).

**Existence.** For each \( n \geq 1 \), let \( q_n(x) = x^{\frac{n}{|x|}} \) and define \( \xi_n = e^{-\frac{1}{\beta}\lambda A(T)}q_n\left(e^{\frac{1}{\beta}\lambda A(T)}\xi\right) \) and

\[
f_n(t, y, z) = f(t, y, z) - f^0(t) + a(t)e^{-\frac{1}{\beta}\lambda A(t)} q_n\left(e^{\frac{1}{\beta}\lambda A(t)} f^0(t) a(t)\right).
\]

By definition, \( q_n(x) \leq n \), for any \( n \geq 1 \). So we have

\[
\sup_{0 \leq t \leq T} e^{\frac{1}{\beta}\lambda A(t)} \left|\frac{f_n(t, 0, 0)}{a(t)}\right| \leq n \quad \text{and} \quad e^{\frac{1}{\beta}\lambda A(T)} \xi_n \leq n.
\]

Then, it follows that \( \xi_n, f_n \) satisfy the assumptions (H1)–(H3) for \( p = 2 \). Thus, from Theorem 4.1, for each \( n \geq 1 \), there exists a unique solution \((Y^n, Z^n) \in \mathcal{M}_{\beta,c}^p(a, T)\) for the following BDSDE:

\[
Y^n_t = \xi_n + \int_t^T f_n(s, Y^n_s, Z^n_s)ds + \int_t^T g(s, Y^n_s, Z^n_s)d\overline{B}_s - \int_t^T Z^n_s dW_s.
\]

Moreover, according to Lemma 4.2, \( Y^n \in S^0_\beta(a, T, \mathbb{R}^k) \cap \mathcal{H}_{\beta,c}^p(a, T, \mathbb{R}^{k \times d}) \), so that from Lemma 3.1, \( Z^n \in \mathcal{H}_{\beta,c}^p(a, T, \mathbb{R}^{k \times d}) \). Hence, \((Y^n, Z^n) \in \mathcal{M}_{\beta,c}^p(a, T)\).
Now, for \((i, n) \in \mathbb{N} \times \mathbb{N}^*,\) let \(Y^{i,n} = Y^{n+i} - Y^n,\) \(Z^{i,n} = Z^{n+i} - Z^n.\) Then, it is obvious that \((Y^{i,n}, Z^{i,n}) \in \mathcal{M}_{\beta,c}^p(a, T)\) and verifies the following BDSDE:

\[
Y^{i,n}_t = \xi_{i,n} + \int_t^T f_{i,n}(s, Y^{i,n}_s, Z^{i,n}_s)ds + \int_t^T g_{i,n}(s, Y^{i,n}_s, Z^{i,n}_s)dB_s - \int_t^T Z^{i,n}_s dW_s, \tag{4.7}
\]

where \(\xi_{i,n} = \xi_{n+i} - \xi_n\) and, \(f_{i,n}\) and \(g_{i,n}\) stand for the random functions

\[
f_{i,n}(t, 0, 0) = a(t)e^{-\frac{1}{2}\beta A(t)}\left[q_{n+i}\left(e^{\frac{1}{2}\beta A(T)}\xi_{n+i}\right) - q_n\left(e^{\frac{1}{2}\beta A(T)}\xi_n\right)\right]
\]

\[
g_{i,n}(t, 0, 0) = 0.
\]

From assumptions on \((\xi, f, g)\) and the fact that \(|q_n(x)| \leq |x|,\) for any \(n \geq 1,\) it is easy to check that \((\xi_{i,n}, f_{i,n}, g_{i,n})\) satisfy \((H1)-(H4)\) with

\[
||Y^{i,n}||_{\mathbb{S}^p_{\beta}} + ||Y^{i,n}||_{\mathcal{H}^{P,a}_{\beta}} + ||Z^{i,n}||_{\mathcal{H}^p_{\beta}} \leq C_p\mathbb{E}\left[e^{\frac{1}{2}\beta A(T)}||\xi_{i,n}||_{\mathbb{S}^{P^{i,n}}_{\beta}} + \left(\int_0^T e^{\beta A(t)}\frac{|f_{i,n}(t, 0, 0)|^2}{a^2(t)}dt\right)\right].
\]

Hence,

\[
||Y^{n+i} - Y^n||_{\mathbb{S}^p_{\beta}} + ||Y^{n+i} - Y^n||_{\mathcal{H}^{P,a}_{\beta}} + ||Z^{n+i} - Z^n||_{\mathcal{H}^p_{\beta}} \leq C_p\mathbb{E}\left[\left|q_{n+i}\left(e^{\frac{1}{2}\beta A(T)}\xi_{n+i}\right) - q_n\left(e^{\frac{1}{2}\beta A(T)}\xi_n\right)\right|^p + \left(\int_0^T e^{\beta A(t)}\frac{|f_{i,n}(t, 0, 0)|^2}{a^2(t)}dt\right)\right].
\]

From \((H3),\) it follows by the dominated convergence theorem that the right-hand side of the above inequality tends to 0, as \(n \to \infty,\) uniformly in \(i,\) so \((Y^n, Z^n)\) is a Cauchy sequence in \(\mathcal{M}_{\beta,c}^p(a, T)\) and the limit is a solution of BDSDE \((\xi, f, g) \ (1.1).\)

\[\square\]

**References**


