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L^p-SOLUTIONS OF BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS WITH STOCHASTIC LIPSCHITZ CONDITION AND $p \in (1, 2)$

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Abstract. We study backward doubly stochastic differential equations where the coefficients satisfy stochastic Lipschitz condition. We prove the existence and uniqueness of the solution in L^p with $p \in (1, 2)$.

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1. Introduction

Backward doubly stochastic differential equations (BDSDEs in short) are equations driven by two independent Brownian motions, *i.e.*, equations which involve both a standard forward stochastic integral $\mathrm{d}W_t$ and a backward stochastic Kunita-Itô integral $\overleftarrow{\mathrm{d}B_t}$:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\mathrm{d}s + \int_t^T g(s, Y_s, Z_s)\overleftarrow{\mathrm{d}B_s} - \int_t^T Z_s \mathrm{d}W_s, \quad t \in [0, T], \tag{1.1}$$

where ξ is a random variable called the terminal condition, f and g are the coefficients (also called generators) and (Y, Z) are the unknown processes that we study the existence under certain conditions on the data (ξ, f, g). This kind of equations, in the nonlinear case, has been introduced by Pardoux and Peng [1]. They obtained the first result on the existence and uniqueness of solution in $L^p, p \geq 2$ with Lipschitz coefficients. Recently, Aman [2] replaced the Lipschitz condition on f in the variable y from [1] with a monotone one and provided the existence and uniqueness of the solution for BDSDEs (1.1) in $L^p, p \in (1, 2)$.

More recently, Owo [3] proved the existence and uniqueness of the solution for BDSDEs (1.1), when the coefficients f and g are stochastic Lipschitz continuous, *i.e.*, the constants of Lipschitz in [1, 2] are replaced with stochastic ones. However the solution in Owo [3] is taken in L^2 space. This limits the scope for several applications. For example, let $T = 1$ and suppose that the terminal condition is given by $\xi = \mathrm{e}^{\left(\frac{W_1^2}{2p} - W_1\right)} \mathbf{1}_{\{W_1 > p\}}$ for some $p \in (1, 2)$. A simple calculation of the expectation of $|\xi|^2$ and $|\xi|^p$ for $p \in (1, 2)$, yields that

$$\mathbb{E}(|\xi|^2) = +\infty \quad \text{and} \quad \mathbb{E}(|\xi|^p) = \frac{1}{\sqrt{2\pi}p}\mathrm{e}^{\left(-p^2\right)} < +\infty.$$

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So that the existence result in Owo [3] can not be applied to solve the above BDSDE with such a terminal condition ξ. To correct this shortcoming, we study in this paper, the L^p-solution with $p \in (1,2)$ for BDSDEs with stochastic Lipschitz coefficients. Our work provides an extension of result obtained in L^p, $p \in (1,2)$ by J. Wang *et al.* [4] for BSDEs with a stochastic Lipschitz coefficient, that is when $g \equiv 0$.

The paper is organized as follows. In Section 2, we introduce some preliminaries including some notations and some spaces. In Section 3, some useful *a priori* estimates are given. Section 4 is devoted to the main result, *i.e.*, the existence and uniqueness solution in L^p with $p \in (1,2)$.

2. Preliminaries

The standard inner product of $\mathbb{R}^k$ is denoted by $\langle .,. \rangle$ and the Euclidean norm by $|\ .\ |$.

A norm on $\mathbb{R}^{d\times k}$ is defined by $\sqrt{Tr(zz^\star)}$, where $z^\star$ is the the transpose of z. We will also denote this norm by $|\ .\ |$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and T be a fixed final time.

Throughout this paper $\{W_t : 0 \le t \le T\}$ and $\{B_t : 0 \le t \le T\}$ will denote two independent Brownian motions, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively.

Let $\mathcal{N}$ denote the class of $\mathbb{P}$-null sets of $\mathcal{F}$. For each $t \in [0,T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B,$$

where for any process $\{\eta_t : t \ge 0\}$; $\mathcal{F}_{s,t}^\eta = \sigma\{\eta_r - \eta_s; s \le r \le t\} \vee \mathcal{N}$ and $\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta$.

Note that $\{\mathcal{F}_{0,t}^W, t \in [0,T]\}$ is an increasing filtration and $\{\mathcal{F}_{t,T}^B, t \in [0,T]\}$ is a decreasing filtration, and the collection $\{\mathcal{F}_t, t \in [0,T]\}$ is neither increasing nor decreasing, so it does not constitute a filtration.

For every random process $(a(t))_{t\ge 0}$ with positive values, such that $a(t)$ is $\mathcal{F}_t^W$-measurable for a.e $t \ge 0$, we define an increasing process $(A(t))_{t\ge 0}$ by setting $A(t) = \displaystyle\int_0^t a^2(s)\mathrm{d}s$.

For $p > 1$ and $\beta > 0$, we denote by:

- $\mathcal{H}_\beta^p(a,T,\mathbb{R}^n)$ the set of jointly measurable processes φ: $\Omega \times [0,T] \to \mathbb{R}^n$, such that $\varphi(t)$ is $\mathcal{F}_t$-measurable, for a.e. $t \in [0,T]$, with $\|\varphi\|^p_{\mathcal{H}^p_\beta} = \mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(t)}|\varphi(t)|^2\mathrm{d}t\right)^{\frac{p}{2}}\right] < \infty$.
- $\mathcal{H}_\beta^{p,a}(a,T,\mathbb{R}^n)$ the set of jointly measurable processes φ: $\Omega \times [0,T] \to \mathbb{R}^n$, such that $\varphi(t)$ is $\mathcal{F}_t$-measurable, for a.e. $t \in [0,T]$, with $\|\varphi\|^p_{\mathcal{H}^{p,a}_\beta} = \mathbb{E}\left[\int_0^T a^2(t)\mathrm{e}^{\frac{p}{2}\beta A(t)}|\varphi(t)|^p\mathrm{d}t\right] < \infty$.
- $\mathcal{S}_\beta^p(a,T,\mathbb{R}^n)$ the set of jointly measurable continuous processes φ: $\Omega \times [0,T] \to \mathbb{R}^n$, such that $\varphi(t)$ is $\mathcal{F}_t$-measurable, for any $t \in [0,T]$, with $\|\varphi\|^p_{\mathcal{S}^p_\beta} = \mathbb{E}\left[\sup\limits_{0\le t\le T} \mathrm{e}^{\frac{p}{2}\beta A(t)} \mid \varphi(t) \mid^p\right] < \infty$.

Note that the space $\mathcal{H}_\beta^{p,a}(a,T,\mathbb{R}^k)$ (resp. $\mathcal{H}_\beta^p(a,T,\mathbb{R}^{k\times d})$) with the norm $\|.\|_{\mathcal{H}^{p,a}_\beta}$ (resp. $\|.\|_{\mathcal{H}^p_\beta}$) is a Banach space. So is the space

$$\mathcal{M}_\beta^p(a,T) = \mathcal{H}_\beta^{p,a}(a,T,\mathbb{R}^k) \times \mathcal{H}_\beta^p(a,T,\mathbb{R}^{k\times d}),$$

with the norm $\|(Y,Z)\|^p_{\mathcal{M}^p_\beta} = \|Y\|^p_{\mathcal{H}^{p,a}_\beta} + \|Z\|^p_{\mathcal{H}^p_\beta}$. Also is the space

$$\mathcal{M}_{\beta,c}^p(a,T) = \left(\mathcal{S}_\beta^p(a,T,\mathbb{R}^k) \cap \mathcal{H}_\beta^{p,a}(a,T,\mathbb{R}^k)\right) \times \mathcal{H}_\beta^p(a,T,\mathbb{R}^{k\times d}),$$

with the norm $\|(Y,Z)\|^p_{\mathcal{M}^p_{\beta,c}} = \|Y\|^p_{\mathcal{S}^p_\beta} + \|Y\|^p_{\mathcal{H}^{p,a}_\beta} + \|Z\|^p_{\mathcal{H}^p_\beta}$.

Throughout the paper, the coefficients $f : \Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^k$ and g: $\Omega\times[0,T]\times\mathbb{R}^k\times\mathbb{R}^{k\times d}\to\mathbb{R}^{k\times l}$, and the terminal value ξ: $\Omega\to\mathbb{R}^k$ satisfy the following assumptions, for $\beta>0$:

(H1) f and g are jointly measurable, and there exist three nonnegative processes $\{r(t):t\in[0,T]\}$, $\{\theta(t):t\in[0,T]\}$, $\{v(t):t\in[0,T]\}$ and a constant $0<\alpha<1$, such that:
(i) for a.e. $t\in[0,T]$, $r(t)$, $\theta(t)$ and $v(t)$ are $\mathcal{F}_t^W$-measurable;
(ii) for all $t\in[0,T]$ and all $(y,z),(y',z')\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$\begin{cases} |f(t,y,z)-f(t,y',z')|\le r(t)\ |y-y'|+\theta(t)|z-z'| \\ \\ |g(t,y,z)-g(t,y',z')|^2\le v(t)|y-y'|^2+\alpha|z-z'|^2. \end{cases}$$

(H2) For all $t\in[0,T]$, $\quad a^2(t)=r(t)+\theta^2(t)+v(t)>0$, with $A(T)<L$, $\mathbb{P}-$a.s., where L is a positive constant.

(H3) (i) ξ is a $\mathcal{F}_T$-measurable random variable, such that $\mathbb{E}\left[\mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p\right]<+\infty$;
(ii) for a.e. $t\in[0,T]$ and any $(y,z)\in\mathbb{R}^k\times\mathbb{R}^{k\times d}$, $f(t,y,z)$ and $g(t,y,z)$ are $\mathcal{F}_t$-measurable, such that $\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(s)}\frac{|f_s^0|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]<+\infty$, where $f_s^0=f(s,0,0)$ and $g_s^0=g(s,0,0)$.

Definition 2.1. A solution of BDSDE (1.1) is a pair of progressively measurable processes $(Y,Z):\Omega\times[0,T]\to\mathbb{R}^k\times\mathbb{R}^{k\times d}$ such that $\mathbb{P}-$a.s., $t\mapsto f(t,Y_t,Z_t)$ belongs to $L^1(0,T)$, $t\mapsto g(t,Y_t,Z_t)$ and $t\mapsto Z_t$ belong to $L^2(0,T)$ and satisfy equation (1.1).

Moreover, let $\beta>0$ and let a be an $\mathcal{F}^W$-adapted process, a solution (Y,Z) is said to be an (a,β)-solution of the BDSDE (1.1) if $\mathbb{P}-$a.s., $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}f(t,Y_t,Z_t)$ and $t\mapsto a^2(t)\mathrm{e}^{\frac{1}{2}\beta A(t)}Y_t$ belong to $L^1(0,T)$, $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}g(t,Y_t,Z_t)$ and $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}Z_t$ belong to $L^2(0,T)$.

For $p>1$, a solution is said to be an L^p-solution if we have, moreover $(Y,Z)\in\mathcal{M}^p_{\beta,c}(a,T)$.

Remark 2.2. Because of assumption **(H2)**, the space $\mathcal{M}^p_{\beta,c}(a,T)$ does not depend anymore on β.

Under assumptions **(H1)**–**(H3)**, as we can see in the following Lemma, for $p>1$, any L^p-solution in the sense of definition 2.1, is an (a,β)-solution.

Lemma 2.3. *For $p>1$, if $(Y,Z)\in\mathcal{M}^p_{\beta,c}(a,T)$ and* **(H1)**–**(H3)** *hold, then $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}f(t,Y_t,Z_t)$ and $t\mapsto a^2(t)\mathrm{e}^{\frac{1}{2}\beta A(t)}Y_t$ belong to $L^1(0,T)$, $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}g(t,Y_t,Z_t)$ and $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}Z_t$ belong to $L^2(0,T)$, $\mathbb{P}-$a.s.*

Proof. It is obvious that $t\mapsto\mathrm{e}^{\frac{1}{2}\beta A(t)}Z_t$ belongs to $L^2(0,T)$.
First, for $p\in(1,2)$, we have

$$\begin{aligned}\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s &= \int_0^T\left(\mathrm{e}^{(1-\frac{p}{2})\beta A(s)}|Y_s|^{2-p}\right)\left(a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\right)\mathrm{d}s\\ &\le\left(\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right)^{\frac{2-p}{p}}\left(\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\right). \end{aligned}\tag{2.1}$$

Next, for $p\ge2$, we have

$$\begin{aligned}\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s &= \int_0^T\left(a^{\frac{2(p-2)}{p}}(s)\right)\left(a^{\frac{4}{p}}(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\right)\mathrm{d}s\\ &\le\left(\int_0^T a^2(s)\mathrm{d}s\right)^{\frac{(p-2)}{p}}\left(\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\right)^{\frac{2}{p}}.\end{aligned}$$

Then, for $p>1$ and since $(Y,Z)\in\mathcal{M}^p_{\beta,c}(a,T)$, we get that

$$\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s<+\infty. \tag{2.2}$$

Therefore,

$$\int_0^T a^2(s)\mathrm{e}^{\frac{1}{2}\beta A(s)}|Y_s|\mathrm{d}s\le\left(\int_0^T a^2(s)\mathrm{d}s\right)^{\frac12}\left(\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s\right)^{\frac12}<+\infty. \tag{2.3}$$

On the other hand, from the assumptions on (f,g) and noting that $a^2(t)=r(t)+\theta^2(t)+v(t)$ together with (2.2) and (2.3), we get that

$$\begin{aligned}\int_0^T \mathrm{e}^{\frac12\beta A(s)}\left|f(s,Y_s,Z_s)\right|\mathrm{d}s&\le\int_0^T \mathrm{e}^{\frac12\beta A(s)}\left(\left|f_s^0\right|+a^2(s)|Y_s|+a(s)|Z_s|\right)\mathrm{d}s\\&\le\left(\int_0^T a^2(s)\mathrm{d}s\right)^{\frac12}\left(\int_0^T \mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s\right)^{\frac12}+\int_0^T a^2(s)\mathrm{e}^{\frac12\beta A(s)}|Y_s|\mathrm{d}s\\&\quad+\left(\int_0^T a^2(s)\mathrm{d}s\right)^{\frac12}\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac12}<+\infty,\end{aligned}$$

and

$$\int_0^T \mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\le2\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s+2\alpha\int_0^T \mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s+2\int_0^T \mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s<+\infty.\quad\square$$

In order to establish *a priori* estimates of L^p-solution of our BDSDE (1.1), we recall the Corollary 2.1 in Aman [2].

Lemma 2.4. *Let (Y,Z) be a solution of BDSDE (1.1). Then, for any $p\ge1$ and any $t\in[0,T]$,*

$$\begin{aligned}|Y_t|^p+c(p)\int_t^T|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|Z_s|^2\mathrm{d}s&\le|\xi|^p+p\int_t^T|Y_s|^{p-1}\langle\hat{Y}_s,f(s,Y_s,Z_s)\rangle\mathrm{d}s\\&\quad+c(p)\int_t^T|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|g(s,Y_s,Z_s)|^2\mathrm{d}s\\&\quad+p\int_t^T|Y_s|^{p-1}\langle\hat{Y}_s,g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle-p\int_t^T|Y_s|^{p-1}\langle\hat{Y}_s,Z_s\mathrm{d}W_s\rangle,\end{aligned}$$

where, $c(p)=\frac{p[(p-1)\wedge1]}{2}$ and $\hat{y}=\mathrm{sign}(y)=|y|^{-1}y\mathbf{1}_{\{y\neq0\}}$.

As a consequence of lemma 2.4, we have the following result

Corollary 2.5. *Let (Y,Z) be an (a,β)-solution of BDSDE (1.1). Then, for any $p\ge1$, $\beta\ge0$ and any $t\in[0,T]$,*

$$\begin{aligned}&\mathrm{e}^{\frac p2\beta A(t)}|Y_t|^p+c(p)\int_t^T\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|Z_s|^2\mathrm{d}s+\frac p2\beta\int_t^T a^2(s)\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^p\mathrm{d}s\\&\le\mathrm{e}^{\frac p2\beta A(T)}|\xi|^p+p\int_t^T\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^{p-1}\langle\hat{Y}_s,f(s,Y_s,Z_s)\rangle\mathrm{d}s\\&\quad+c(p)\int_t^T\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|g(s,Y_s,Z_s)|^2\mathrm{d}s-p\int_t^T\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^{p-1}\langle\hat{Y}_s,Z_s\mathrm{d}W_s\rangle\\&\quad+p\int_t^T\mathrm{e}^{\frac p2\beta A(s)}|Y_s|^{p-1}\langle\hat{Y}_s,g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle,\end{aligned}$$

where, $c(p)=\frac{p[(p-1)\wedge1]}{2}$ and $\hat{y}=sign(y)=|y|^{-1}y\mathbf{1}_{\{y\neq0\}}$.

Proof. Firstly, we show that

$$e^{\frac{1}{2}\beta A(t)}Y_t = e^{\frac{1}{2}\beta A(t)}\xi + \int_t^T \left[e^{\frac{1}{2}\beta A(s)} f(s, Y_s, Z_s) - \frac{1}{2}\beta a^2(s) e^{\frac{1}{2}\beta A(s)} Y_s\right] ds + \int_t^T e^{\frac{1}{2}\beta A(s)} g(s, Y_s, Z_s) \overleftarrow{dB_s} - \int_t^T e^{\frac{1}{2}\beta A(s)} Z_s dW_s, \quad t \in [0, T]. \tag{2.4}$$

Indeed, let $X_t = e^{\frac{1}{2}\beta A(t)}$, for $t \in [0, T]$ with $A(t) = \int_0^t a^2(s) ds$. Thus, by assumption **(H2)**, X is a continuous and finite variation process. And by Itô's formula, $X_t = 1 + \frac{1}{2}\beta \int_0^t a^2(s) e^{\frac{1}{2}\beta A(s)} ds$.

Let $\pi = \{t = t_0 < t_1 < \ldots < t_n = T\}$, for $t \in [0, T]$. Then,

$$\begin{aligned} X_{t_{i+1}}Y_{t_{i+1}} - X_{t_i}Y_{t_i} &= X_{t_i}(Y_{t_{i+1}} - Y_{t_i}) + Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \\ &= -\int_{t_i}^{t_{i+1}} X_{t_i} f(s, Y_s, Z_s) ds - \int_{t_i}^{t_{i+1}} X_{t_{i+1}} g(s, Y_s, Z_s) \overleftarrow{dB_s} + \int_{t_i}^{t_{i+1}} X_{t_i} Z_s dW_s \\ &\quad + Y_{t_i}(X_{t_{i+1}} - X_{t_i}) + (X_{t_{i+1}} - X_{t_i}) \int_{t_i}^{t_{i+1}} f(s, Y_s, Z_s) ds - (X_{t_{i+1}} - X_{t_i}) \int_{t_i}^{t_{i+1}} Z_s dW_s. \end{aligned}$$

Therefore, taking the sum from $i = 0$ to $i = n - 1$, we get

$$e^{\frac{1}{2}\beta A(t)}Y_t = e^{\frac{1}{2}\beta A(T)}\xi + I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5 + I_n^6.$$

where,

$$\begin{aligned} I_n^1 &= \sum_{i=0}^{n-1} X_{t_i}(C_{t_{i+1}}^f - C_{t_i}^f), \quad & I_n^2 &= -\sum_{i=0}^{n-1} X_{t_{i+1}}(M_{t_{i+1}}^g - M_{t_i}^g) \\ I_n^3 &= -\sum_{i=0}^{n-1} X_{t_i}(M_{t_{i+1}}^z - M_{t_i}^z), \quad & I_n^4 &= -\sum_{i=0}^{n-1} Y_{t_i}(X_{t_{i+1}} - X_{t_i}) \\ I_n^5 &= -\sum_{i=0}^{n-1}(X_{t_{i+1}} - X_{t_i})(C_{t_{i+1}}^f - C_{t_i}^f), \quad & I_n^6 &= \sum_{i=0}^{n-1}(X_{t_{i+1}} - X_{t_i})(M_{t_{i+1}}^z - M_{t_i}^z) \end{aligned}$$

where, (C^f, M^g, M^z) are defined by:

$$C_t^f = \int_0^t f(s, Y_s, Z_s) ds, \quad M_t^g = \int_t^T g(s, Y_s, Z_s) \overleftarrow{dB_s}, \quad M_t^z = \int_0^t Z_s dW_s, \quad \text{for } t \in [0, T].$$

Since (Y, Z) is an (a, β)-solution, C^f is a continuous and finite variation process and the process M^g (resp. M^z) is a backward (resp. a forward) continuous martingale.

By continuity of X and Y, and the definition of Stieltjes integrals, together with the fact that (Y, Z) is an (a, β)-solution, it follows that

$$\begin{aligned} I_n^1 &\longrightarrow \int_t^T X_s dC_s^f = \int_t^T e^{\frac{1}{2}\beta A(s)} f(s, Y_s, Z_s) ds \quad \text{a.s.}, \\ I_n^4 &\longrightarrow -\int_t^T Y_s dX_s = -\frac{1}{2}\beta \int_t^T a^2(s) e^{\frac{1}{2}\beta A(s)} Y_s ds \quad \text{a.s.} \end{aligned}$$

Moreover, by the definition of backward-forward stochastic integrals with respect to martingales

$$I_n^2 \longrightarrow -\int_t^T X_s \mathrm{d}M_s^g = \int_t^T \mathrm{e}^{\frac{1}{2}\beta A(s)} g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B_s} \quad \text{in probability,}$$
$$I_n^3 \longrightarrow -\int_t^T X_s \mathrm{d}M_s^z = -\int_t^T \mathrm{e}^{\frac{1}{2}\beta A(s)} Z_s \mathrm{d}W_s \quad \text{in probability.}$$

On the other hand, we have,

$$|I_n^5| \leq \sup_{0\leq i\leq n-1}\Big(|C^f_{t_{i+1}} - C^f_{t_i}|\Big)\,\mathrm{e}^{\frac{1}{2}\beta A(T)} \;\longrightarrow\; 0 \quad \text{in probability,}$$

due to the fact that the first term converges to zero almost surely by the continuity of C^f, and the second is finite $\mathbb{P}-$a.s. by assumption **(H2)**.

Also, by the continuity of M^z, we have

$$|I_n^5| \leq \sup_{0\leq i\leq n-1}\Big(|M^z_{t_{i+1}} - M^z_{t_i}|\Big)\,\mathrm{e}^{\frac{1}{2}\beta A(T)} \;\longrightarrow\; 0 \quad \text{in probability,}$$

so that we obtain (2.4).

Now letting $\overline{Y}_t = \mathrm{e}^{\frac{1}{2}\beta A(t)} Y_t$, $\overline{Z}_t = \mathrm{e}^{\frac{1}{2}\beta A(t)} Z_t$ and $\overline{\xi} = \mathrm{e}^{\frac{1}{2}\beta A(T)}\xi$, we get

$$\overline{Y}_t = \overline{\xi} + \int_t^T \overline{f}(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}s + \int_t^T \overline{g}(s,\overline{Y}_s,\overline{Z}_s)\overleftarrow{\mathrm{d}B_s} - \int_t^T \overline{Z}_s \mathrm{d}W_s, \quad t\in[0,T], \tag{2.5}$$

where, $\overline{f}$ and $\overline{g}$ are defined by:

$$\overline{f}\,(t,y,z) = \mathrm{e}^{\frac{1}{2}\beta A(t)} f\big(t, \mathrm{e}^{-\frac{1}{2}\beta A(t)}y, \mathrm{e}^{-\frac{1}{2}\beta A(t)}z\big) - \frac{1}{2}\beta a^2(t) y,$$
$$\overline{g}\,(t,y,z) = \mathrm{e}^{\frac{1}{2}\beta A(t)} g\big(t, \mathrm{e}^{-\frac{1}{2}\beta A(t)}y, \mathrm{e}^{-\frac{1}{2}\beta A(t)}z\big).$$

Thus, by Definition 2.1 and Lemma 2.4, we deduce the result. □

3. *A priori* estimates

Lemma 3.1. *Let $\beta \geq 0$, $p\in]1,2[$ and assume that* **(H1)**–**(H3)** *hold. Let (Y,Z) be an (a,β)-solution of BDSDE (1.1). If $Y\in\mathcal{S}^p_\beta(a,T,\mathbb{R}^k)\cap\mathcal{H}^{p,a}_\beta(a,T,\mathbb{R}^{k\times d})$, then $Z\in\mathcal{H}^p_\beta(a,T,\mathbb{R}^{k\times d})$ and there exists a constant C_p depending on p,α such that for some $\beta>0$,*

$$||Z||^p_{\mathcal{H}^p_\beta} \leq C_p\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + \left(\int_0^T \mathrm{e}^{\beta A(s)}\frac{|f_s^0|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_0^T \mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]. \tag{3.1}$$

Proof. Let $p\in]1,2[$. For each integer $n>0$, let us introduce the stopping time

$$\tau_n = \inf\left\{t\in[0,T],\ \int_0^t \mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s \geq n\right\}\wedge T.$$

Applying Itô's formula to $\mathrm{e}^{\beta A(t)}|Y_t|^2$, we have

$$\begin{aligned}
&\mathrm{e}^{\beta A(t)}|Y_t|^2 + \beta\int_t^{\tau_n} a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s + \int_t^{\tau_n}\mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s\\
&= \mathrm{e}^{\beta A(\tau_n)}|Y_{\tau_n}|^2 + 2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, f(s,Y_s,Z_s)\rangle\mathrm{d}s + \int_t^{\tau_n}\mathrm{e}^{\beta A(s)}|g(s,Y_s,Z_s)|^2\mathrm{d}s\\
&\quad + 2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B_s}\rangle - 2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle.
\end{aligned}$$

From **(H1)** and Young's inequality for every $\sigma > 0$ such that $\sigma + \alpha < 1$, we have

$$\begin{aligned}2\langle Y_s, f(s, Y_s, Z_s)\rangle &\le 2r(s)\left|Y_s\right|^2 + 2\theta(s)\left|Y_s\right|\left|Z_s\right| + 2\left|Y_s\right|\left|f_s^0\right| \\ &\le \left(3 + \frac{1}{\sigma}\right)a^2(s)\left|Y_s\right|^2 + \sigma\left|Z_s\right|^2 + \frac{\left|f_s^0\right|^2}{a^2(s)}\end{aligned}$$

and for every $\gamma > 0$,

$$\left|g(s, Y_s, Z_s)\right|^2 \le (1+\gamma)\, a^2(s)\left|Y_s\right|^2 + (1+\gamma)\,\alpha\left|Z_s\right|^2 + \left(1 + \frac{1}{\gamma}\right)\left|g_s^0\right|^2. \tag{3.2}$$

Finally, it follows that

$$\begin{aligned}&\mathrm{e}^{\beta A(t)}\left|Y_t\right|^2 + D_1\int_t^{\tau_n} a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s + D_2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s \\ &\le \mathrm{e}^{\beta A(\tau_n)}\left|Y_{\tau_n}\right|^2 + \int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s + \left(1+\frac{1}{\gamma}\right)\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s \\ &\quad - 2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle + 2\int_t^{\tau_n}\mathrm{e}^{\beta A(s)}\left\langle Y_s, g(s, Y_s, Z_s)\overleftarrow{\mathrm{d}B_s}\right\rangle,\end{aligned} \tag{3.3}$$

where, $D_1 = \beta - 4 - \gamma - \frac{1}{\sigma}$ and $D_2 = 1 - \sigma - (1+\gamma)\,\alpha$.

Choosing $\gamma > 0$, $\beta > 0$ such that $\gamma < \frac{1-(\sigma+\alpha)}{\alpha}$ and $\beta > 4 + \gamma + \frac{1}{\sigma}$, we get $D_1 > 0$ and $D_2 > 0$.

Therefore, since $\tau_n \le T$, putting $t = 0$, we have

$$\begin{aligned}&D_1\int_0^{\tau_n} a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s + D_2\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s \\ &\le \sup_{0\le t\le T}\mathrm{e}^{\beta A(t)}|Y_t|^2 + \int_0^{T}\mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s + \left(1+\frac{1}{\gamma}\right)\int_0^{T}\mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s \\ &\quad - 2\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle + 2\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left\langle Y_s, g(s, Y_s, Z_s)\overleftarrow{\mathrm{d}B_s}\right\rangle,\end{aligned}$$

and thus, raising both sides to the power $\frac{p}{2} < 1$, and taking expectation, we derive

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^{\tau_n} a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right] \\ &\le \lambda_p\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + \left(\int_0^{T}\mathrm{e}^{\beta A(s)}\frac{|f_s^0|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}} + \left(\int_0^{T}\mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s\right)^{\frac{p}{2}}\right. \\ &\quad \left. + \left|\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, g(s, Y_s, Z_s)\overleftarrow{\mathrm{d}B_s}\rangle\right|^{\frac{p}{2}} + \left|\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle\right|^{\frac{p}{2}}\right].\end{aligned} \tag{3.4}$$

But by the BDG and Young's inequalities, we get for a given constant $d_p > 0$ and any $\gamma_1 > 0$,

$$\begin{aligned}\lambda_p\mathbb{E}\left[\left|\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle\right|^{\frac{p}{2}}\right] &\le \lambda_p d_p\mathbb{E}\left[\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{4}}\right] \\ &\le \lambda_p d_p\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{4}\beta A(t)}|Y_t|^{\frac{p}{2}}\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{4}}\right] \\ &\le \mathbb{E}\left[\frac{\lambda_p^2 d_p^2}{\gamma_1}\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + \gamma_1\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\end{aligned}$$

and

$$\lambda_p\mathbb{E}\left[\left|\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\langle Y_s,g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle\right|^{\frac{p}{2}}\right]\leq\lambda_pd_p\mathbb{E}\left[\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{p}{4}}\right]$$
$$\leq\lambda_pd_p\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{4}\beta A(t)}|Y_t|^{\frac{p}{2}}\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{p}{4}}\right]$$
$$\leq\mathbb{E}\left[\frac{\lambda_p^2d_p^2}{\gamma_1}\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p+\gamma_1\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Now, from (3.2), we have for any $\gamma_2>0$

$$\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\leq\left(1+\frac{1}{\gamma_2}\right)\int_0^T\mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s+(1+\gamma_2)\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left[a^2(s)|Y_s|^2+\alpha|Z_s|^2\right]\mathrm{d}s.$$

Thus, rising to power $\frac{p}{2}<1$, we get

$$\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\leq\left(1+\frac{1}{\gamma_2}\right)\left(\int_0^T\mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s\right)^{\frac{p}{2}}+(1+\gamma_2)\left(\int_0^{\tau_n}a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}$$
$$+(1+\gamma_2)\alpha^{\frac{p}{2}}\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}. \tag{3.5}$$

Therefore, coming back to (3.4), we have

$$\mathbb{E}\left[\left(\int_0^{\tau_n}a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$\leq\lambda(p)\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p+\left(\int_0^T\mathrm{e}^{\beta A(s)}\frac{|f_s^0|^2\mathrm{d}s}{a^2(s)}\right)^{\frac{p}{2}}+\left(\int_0^T\mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$+\left[\gamma_1+(1+\gamma_2)\gamma_1\alpha^{\frac{p}{2}}\right]\mathbb{E}\left[\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]+(1+\gamma_2)\gamma_1\mathbb{E}\left[\left(\int_0^{\tau_n}a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Consequently, choosing $\gamma_1,\gamma_2>0$ such that $\gamma_1+(1+\gamma_2)\gamma_1\alpha^{\frac{p}{2}}<1$ and $(1+\gamma_2)\gamma_1<1$, we derive, for any $n\geq1$

$$\mathbb{E}\left[\left(\int_0^{\tau_n}\mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]\leq C_p\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p+\left(\int_0^T\mathrm{e}^{\beta A(s)}\frac{|f_s^0|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^T\mathrm{e}^{\beta A(s)}|g_s^0|^2\mathrm{d}s\right)^{\frac{p}{2}}\right],$$

witch by Fatou's lemma yields the desired result. □

Proposition 3.2. *Let $\beta\geq0$, $p\in]1,2[$. Let (Y,Z) be an (a,β)-solution of BDSDE (1.1) with terms (ξ,f,g) satisfying* **(H1)**–**(H3)**, *where $Y\in\mathcal{S}_\beta^p(a,T,\mathbb{R}^k)\cap\mathcal{H}_\beta^{p,a}(a,T,\mathbb{R}^{k\times d})$. Then, there exists a constant $C_p=C_p(\beta,\alpha,T,L)$ satisfying the* a priori *estimate*

$$||Y||_{\mathcal{S}_\beta^p}^p+||Y||_{\mathcal{H}_\beta^{p,a}}^p+||Z||_{\mathcal{H}_\beta^p}^p\leq C_p\mathbb{E}\left[\mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p+\left(\int_0^T\mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}}\right.$$
$$\left.+\left(\int_0^T\mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s\right)^{\frac{p}{2}}+\int_0^T\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq0\}}|g_s^0|^2\mathrm{d}s\right]. \tag{3.6}$$

Proof. Let $p \in]1,2[$. From corollary 2.5, we have for any $\beta \geq 0$ and any $t \in [0,T]$,

$$\begin{aligned}
&\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + c(p)\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s + \frac{p}{2}\beta\int_t^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^p\mathrm{d}s\\
&\leq \mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p + p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, f(s,Y_s,Z_s)\rangle\mathrm{d}s\\
&\quad + c(p)\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g(s,Y_s,Z_s)|^2\mathrm{d}s - p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, Z_s\mathrm{d}W_s\rangle\\
&\quad + p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle.
\end{aligned}$$

From **(H1)**, we have

$$\left\langle \hat{Y}_s, f(s,Y_s,Z_s)\right\rangle \leq r(s)\,|Y_s| + \theta(s)\,|Z_s| + \left|f_s^0\right|,$$

which, together with (3.2), yields for every $\gamma > 0$,

$$\begin{aligned}
&\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + c(p)\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s + \frac{p}{2}\beta\int_t^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^p\mathrm{d}s\\
&\leq \mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p + p\int_t^T r(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s + p\int_t^T \theta(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}|Z_s|\mathrm{d}s\\
&\quad + p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}|f_s^0|\mathrm{d}s + c(p)\,(1+\gamma)\int_t^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\\
&\quad + c(p)\,(1+\gamma)\,\alpha\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s + p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle\\
&\quad - p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, Z_s\mathrm{d}W_s\rangle + c(p)\left(1+\frac{1}{\gamma}\right)\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2\mathrm{d}s.
\end{aligned}$$

By virtue of Young's inequality, we have for any $\varepsilon > 0$

$$\begin{aligned}
p\theta(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}|Z_s| &= \left(p\theta(s)\mathrm{e}^{\frac{p}{4}\beta A(s)}|Y_s|^{\frac{p}{2}}\right)\left(\mathrm{e}^{\frac{p}{4}\beta A(s)}|Y_s|^{\frac{p}{2}-1}|Z_s|\right)\\
&\leq \frac{2p}{\varepsilon[(p-1)\wedge 1]}\theta^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p + \varepsilon c(p)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2.
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
&\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + \delta_1\int_t^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s + \delta_2\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s\\
&\leq X + p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle - p\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, Z_s\mathrm{d}W_s\rangle,
\end{aligned}\tag{3.7}$$

where $\delta_1 = \frac{p}{2}\beta - p - c(p)\,(1+\gamma) - \frac{2p}{\varepsilon[(p-1)\wedge 1]}$, $\delta_2 = c(p)[1-(1+\gamma)\,\alpha - \varepsilon]$ and

$$X = \mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p + p\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}|f_s^0|\mathrm{d}s + c(p)\left(1+\frac{1}{\gamma}\right)\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2\mathrm{d}s.$$

From BDG inequality, one can show that

$$M = \left\{\int_0^t \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, Z_s\mathrm{d}W_s\rangle\right\}_{0\leq t\leq T} \text{ and } N = \left\{\int_t^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}\langle \hat{Y}_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle\right\}_{0\leq t\leq T}$$

are respectively uniformly integrable martingale. Indeed, we have, by Young's inequality

$$\mathbb{E}\langle M,M\rangle_T^{1/2} \le \mathbb{E}\left[\sup_{0\le t\le T} \mathrm{e}^{\frac{p-1}{2}\beta A(t)}|Y_t|^{p-1}\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{1}{2}}\right]$$
$$\le \frac{p-1}{p}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right]+\frac{1}{p}\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Also, in view of (3.2) and since $\frac{p}{2}<1$, we get

$$\mathbb{E}\langle N,N\rangle_T^{1/2} \le \frac{p-1}{p}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right]+\frac{1}{p}\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$\le \frac{p-1}{p}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right]+(1+\gamma)\,\mathbb{E}\left[\left(\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$+(1+\gamma)\,\alpha^{\frac{p}{2}}\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right]+\left(1+\frac{1}{\gamma}\right)\mathbb{E}\left[\left(\int_0^T \mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right].$$

Now, from (2.1) for $p\in(1,2)$, we derive by Young's inequality

$$\left(\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}\left|Y_s\right|^2\mathrm{d}s\right)^{\frac{p}{2}} \le \left(\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right)^{\frac{2-p}{2}}\left(\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\right)^{\frac{p}{2}}$$
$$\le \frac{2-p}{2}\left(\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right)+\frac{p}{2}\left(\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\right).$$

Since $Y\in\mathcal{S}^p_\beta(a,T,\mathbb{R}^k)\cap\mathcal{H}^{p,a}_\beta(a,T,\mathbb{R}^{k\times d})$, it follows from Lemma 3.1, that $Z\in\mathcal{H}^p_\beta(a,T,\mathbb{R}^{k\times d})$, which together with assumption (H3)(ii), yields that

$$\mathbb{E}\langle M,M\rangle_T^{1/2}<+\infty \quad\text{and}\quad \mathbb{E}\langle N,N\rangle_T^{1/2}<+\infty,$$

which implies that M and N are uniformly integrable martingale.

Thus, taking expectation in (3.7) with $t=0$, we have

$$\mathbb{E}\left[\delta_1\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s+\delta_2\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s\right]\le\mathbb{E}(X). \tag{3.8}$$

Now, by choosing $\gamma,\varepsilon>0$ such that $(1+\gamma)\,\alpha+\varepsilon<1$ and $\beta>2+\frac{2c(p)}{p}\,(1+\gamma)+\frac{4}{\varepsilon[(p-1)\wedge 1]}$, it follows that $\delta_1,\ \delta_2>0$ and so taking the sup(.) and then the expectation in (3.7), we derive by Burkhölder–Davis–Gundy's inequality that

$$\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right]\le\mathbb{E}(X)+k_p\mathbb{E}\langle M,M\rangle_T^{1/2}+h_p\mathbb{E}\langle N,N\rangle_T^{1/2}. \tag{3.9}$$

But from Young's inequality and (3.8), we get

$$\begin{aligned} k_p\mathbb{E}\langle M,M\rangle_T^{1/2} &\le k_p\mathbb{E}\left[\sup_{0\le t\le T} \mathrm{e}^{\frac{p}{4}\beta A(t)}|Y_t|^{\frac{p}{2}}\left(\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\left|Z_s\right|^2\mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] + 4k_p^2\mathbb{E}\left[\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\left|Z_s\right|^2\mathrm{d}s\right] \\ &\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] + k_p'\mathbb{E}\left(X\right). \end{aligned} \tag{3.10}$$

Likewise

$$\begin{aligned} h_p\mathbb{E}\langle N,N\rangle_T^{1/2} &\le h_p\mathbb{E}\left[\sup_{0\le t\le T} \mathrm{e}^{\frac{p}{4}\beta A(t)}|Y_t|^{\frac{p}{2}}\left(\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right)^{\frac{1}{2}}\right] \\ &\le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] + 4h_p^2\mathbb{E}\left[\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s\right]. \end{aligned}$$

Now, in view of (3.2), it follows that

$$\begin{aligned} &\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}\left|g(s,Y_s,Z_s)\right|^2\mathrm{d}s \\ &\le (1+\gamma)\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s + (1+\gamma)\,\alpha\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|Z_s|^2\mathrm{d}s \\ &\quad + \left(1+\frac{1}{\gamma}\right)\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2\mathrm{d}s. \end{aligned}$$

Then, from (3.8) together with the definition of X, we have

$$h_p\mathbb{E}\langle N,N\rangle_T^{1/2} \le \frac{1}{4}\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] + h_p'\mathbb{E}\left(X\right). \tag{3.11}$$

Therefore, putting the estimates (3.10) and (3.11) into (3.9), we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] \le 2(1+k_p'+h_p')\mathbb{E}\left(X\right),$$

which together with (3.8), implies that

$$\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p + \delta_1\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s\right] \le C_p\mathbb{E}\left(X\right).$$

Applying Holder and Young's inequalities, we have, by (H2)

$$\begin{aligned} &p\int_0^T \mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-1}|f_s^0|\mathrm{d}s = p\int_0^T\left(a^{\frac{2(p-1)}{p}}(s)\mathrm{e}^{\frac{p-1}{2}\beta A(s)}|Y_s|^{p-1}\right)\left(a^{\frac{2-p}{p}}(s)\mathrm{e}^{\frac{1}{2}\beta A(s)}\frac{\left|f_s^0\right|}{a(s)}\right)\mathrm{d}s \\ &\le \frac{\delta_1}{2C_p}\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s + \left(\frac{2(p-1)C_p}{\delta_1}\right)^{p-1}\int_0^T a^{2-p}\left(s\right)\left(\mathrm{e}^{\frac{p}{2}\beta A(s)}\frac{\left|f_s^0\right|^p}{a^p(s)}\right)\mathrm{d}s \\ &\le \frac{\delta_1}{2C_p}\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\mathrm{d}s + \left(\frac{2(p-1)C_p}{\delta_1}\right)^{p-1}L^{1-\frac{p}{2}}\left(\int_0^T \mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}}. \end{aligned}$$

Finally, coming back to the definition of X, we obtain

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p+\int_t^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^p\mathrm{d}s\right]\\&\le C_p'\mathbb{E}\left[\mathrm{e}^{\frac{p}{2}\beta A(T)}|\xi|^p+\left(\int_0^T\mathrm{e}^{\beta A(s)}\frac{\left|f_s^0\right|^2}{a^2(s)}\mathrm{d}s\right)^{\frac{p}{2}}+\left(\int_0^T\mathrm{e}^{\beta A(s)}\left|g_s^0\right|^2\mathrm{d}s\right)^{\frac{p}{2}}\right.\\&\quad\left.+\int_0^T\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^{p-2}\mathbf{1}_{\{Y_s\neq 0\}}|g_s^0|^2\mathrm{d}s\right].\end{aligned}$$

The result follows from Lemma 3.1. □

4. Existence and uniqueness of a solution

In order to obtain the existence and uniqueness result for BDSDEs associated to data (ξ, f, g) in L^p, we make the following supplementary assumption:

(H4) $g(t,0,0)=0, \forall\, t\in[0,T]$.

Moreover, we recall the following result due to Owo ([3], Thm. 3.3).

Theorem 4.1. *For $p=2$ and any β, assume that* **(H1)–(H3)** *hold. Then, the BDSDE* (1.1) *has a unique solution $(Y,Z)\in\mathcal{M}^2_{\beta,c}(a,T)$.*

From Lemma 2.3, the unique solution $(Y,Z)\in\mathcal{M}^2_{\beta,c}(a,T)$ in Theorem 4.1 is an (a,β)-solution of BDSDE (1.1). Now we give a basic estimate concerning the solution.

Lemma 4.2. *For $p\in]1,2[$ and any β, assume that* **(H1)–(H4)** *hold. Let $(Y,Z)\in\mathcal{M}^2_{\beta,c}(a,T)$ be a solution of BDSDE* (1.1) *and assume that $\mathbb{P}-a.s.$,*

$$\sup_{0\le t\le T}\mathrm{e}^{\frac{1}{2}\beta A(t)}\frac{|f_t^0|}{a(t)}\le n,\qquad \mathrm{e}^{\frac{1}{2}\beta A(T)}\xi\le n, \tag{4.1}$$

then $Y\in\mathcal{S}^p_\beta(a,T,\mathbb{R}^k)\cap\mathcal{H}^{p,a}_\beta(a,T,\mathbb{R}^{k\times d})$.

Proof. Applying Itô's formula to $\mathrm{e}^{\beta A(t)}|Y_t|^2$, we have for any $t\in[0,T]$,

$$\begin{aligned}&\mathrm{e}^{\beta A(t)}|Y_t|^2+\beta\int_t^T\mathrm{e}^{\beta A(s)}|Y_s|^2\mathrm{d}s+\int_t^T\mathrm{e}^{\beta A(s)}|Z_s|^2\mathrm{d}s\\&=\mathrm{e}^{\beta A(T)}|\xi|^2+2\int_t^T\mathrm{e}^{\beta A(s)}\langle Y_s,f(s,Y_s,Z_s)\rangle\mathrm{d}s+\int_t^T\mathrm{e}^{\beta A(s)}|g(s,Y_s,Z_s)|^2\mathrm{d}s\\&\quad+2\int_t^T\mathrm{e}^{\beta A(s)}\langle Y_s,g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B}_s\rangle-2\int_t^T\mathrm{e}^{\beta A(s)}\langle Y_s,Z_s\mathrm{d}W_s\rangle.\end{aligned}$$

From **(H1)** and Young's inequality, we have

$$\begin{aligned}2\langle Y_s,f(s,Y_s,Z_s)\rangle&\le 2r(s)\left|Y_s\right|^2+2\theta(s)\left|Y_s\right|\left|Z_s\right|+2\left|Y_s\right|\left|f_s^0\right|\\&\le\left(3+\frac{2}{1-\alpha}\right)a^2(s)\left|Y_s\right|^2+\frac{1-\alpha}{2}\left|Z_s\right|^2+\frac{\left|f_s^0\right|^2}{a^2(s)}\end{aligned}$$

and from **(H1)** and **(H4)**

$$|g(s,Y_s,Z_s)|^2 \leq a^2(s)\,|Y_s|^2 + \alpha\,|Z_s|^2\,.$$

Finally, in view of (4.1), it follows that

$$\begin{aligned}
&\mathrm{e}^{\beta A(t)}\,|Y_t|^2 + \left(\beta - 4 - \frac{2}{1-\alpha}\right)\int_t^T a^2(s)\mathrm{e}^{\beta A(s)}\,|Y_s|^2\,\mathrm{d}s + \left(\frac{1-\alpha}{2}\right)\int_t^T \mathrm{e}^{\beta A(s)}\,|Z_s|^2\,\mathrm{d}s \\
&\leq n^2 + n^2T - 2\int_t^T \mathrm{e}^{\beta A(s)}\,\langle Y_s, Z_s\mathrm{d}W_s\rangle + 2\int_t^T \mathrm{e}^{\beta A(s)}\left\langle Y_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B_s}\right\rangle. \qquad (4.2)
\end{aligned}$$

By the same argument as in the previous proof on the uniform integrability of M and N, we prove that $\left\{\int_0^t \mathrm{e}^{\beta A(s)}\langle Y_s, Z_s\mathrm{d}W_s\rangle\right\}_{0\leq t\leq T}$ and $\left\{\int_t^T \mathrm{e}^{\beta A(s)}\langle Y_s, g(s,Y_s,Z_s)\overleftarrow{\mathrm{d}B_s}\rangle\right\}_{0\leq t\leq T}$ are respectively uniformly integrable martingale. Therefore, taking expectation in (4.2), we have

$$\mathbb{E}\left[\left(\beta - 4 - \frac{2}{1-\alpha}\right)\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}\,|Y_s|^2\,\mathrm{d}s + \left(\frac{1-\alpha}{2}\right)\int_t^T \mathrm{e}^{\beta A(s)}\,|Z_s|^2\,\mathrm{d}s\right] \leq n^2 + n^2T. \qquad (4.3)$$

Now, choosing $\beta > 4 + \frac{2}{1-\alpha}$, and taking $\sup_{0\leq t\leq T}(.)$ in (4.2) and applying Burkhölder–Davis–Gundy's inequality and Young's inequality $2ab \leq \delta a^2 + \frac{1}{\delta}b^2$, for every $\delta > 0$, we deduce that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\beta A(t)}\,|Y_t|^2\right] &\leq n^2 + n^2T + 2c\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{1}{2}\beta A(t)}|Y_t|\left(\int_0^T \mathrm{e}^{\beta A(s)}\,|Z_s|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\quad + 2c\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{1}{2}\beta A(t)}|Y_t|\left(\int_0^T \mathrm{e}^{\beta A(s)}\,|g(s,Y_s,Z_s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}\right] \\
&\leq n^2 + n^2T + 2\delta\mathbb{E}\left(\sup_{0\leq t\leq T}\mathrm{e}^{\beta A(t)}|Y_t|^2\right) \qquad (4.4)\\
&\quad + (1+\alpha)\frac{c^2}{\delta}\mathbb{E}\left(\int_0^T \mathrm{e}^{\beta A(s)}\,|Z_s|^2\,\mathrm{d}s\right) + \frac{c^2}{\delta}\mathbb{E}\left(\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}\,|Y_s|^2\,\mathrm{d}s\right).
\end{aligned}$$

Therefore, combining (4.3) and (4.4), and choosing $\delta < \frac{1}{2}$, we derive

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\beta A(t)}\,|Y_t|^2 + \int_0^T \mathrm{e}^{\beta A(s)}a^2(s)\,|Y_s|^2\,\mathrm{d}s + \int_0^T \mathrm{e}^{\beta A(s)}\,|Z_s|^2\,\mathrm{d}s\right] \leq c'(n^2 + n^2T), \qquad (4.5)$$

which since $p \in]1,2[$ and together with Hölder's inequality yields

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\frac{p}{2}\beta A(t)}|Y_t|^p\right] \leq \left(\mathbb{E}\left[\sup_{0\leq t\leq T}\mathrm{e}^{\beta A(t)}|Y_t|^2\right]\right)^{\frac{p}{2}} < \infty$$

and

$$\mathbb{E}\left[\int_0^T a^2(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\,\mathrm{d}s\right]=\mathbb{E}\left[\int_0^T \left(a^{2-p}(s)\right)a^p(s)\mathrm{e}^{\frac{p}{2}\beta A(s)}|Y_s|^p\,\mathrm{d}s\right]$$
$$\leq \mathbb{E}\left[\left(\int_0^T a^2(s)\mathrm{d}s\right)^{1-\frac{p}{2}}\left(\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\,\mathrm{d}s\right)^{\frac{p}{2}}\right]$$
$$\leq \left(\mathbb{E}\left[\int_0^T a^2(s)\mathrm{d}s\right]\right)^{1-\frac{p}{2}}\left(\mathbb{E}\left[\int_0^T a^2(s)\mathrm{e}^{\beta A(s)}|Y_s|^2\,\mathrm{d}s\right]\right)^{\frac{p}{2}}<\infty.$$

□

We now state and prove our main result.

Theorem 4.3. *For $p\in]1,2[$, let assume* **(H1)–(H4)**. *Then, for β sufficiently large, the BDSDE (1.1) has a unique solution $(Y,Z)\in\mathcal{M}^p_{\beta,c}(a,T)$.*

Proof. (Uniqueness). Let (Y,Z), $(Y',Z')\in\mathcal{M}^p_{\beta,c}(a,T)$ be two solutions of BDSDE (1.1).

Let denote by $\left(\overline{Y},\overline{Z}\right)$ the process $(Y-Y',Z-Z')$. Then, it is obvious that $\left(\overline{Y},\overline{Z}\right)$ is a solution in $\mathcal{M}^p_{\beta,c}(a,T)$ to the following BDSDE:

$$\overline{Y}_t=\int_t^T F(s,\overline{Y}_s,\overline{Z}_s)\mathrm{d}s+\int_t^T G(s,\overline{Y}_s,\overline{Z}_s)\overleftarrow{\mathrm{d}B_s}-\int_t^T \overline{Z}_s\mathrm{d}W_s, \tag{4.6}$$

where F,G stand for the random functions

$$F(t,y,z)=f(t,y+Y'_t,z+Z'_t)-f(t,Y'_t,Z'_t)$$
$$G(t,y,z)=g(t,y+Y'_t,z+Z'_t)-g(t,Y'_t,Z'_t).$$

It is easy to verify that BDSDE (4.6) satisfies assumptions **(H1)–(H3)**. Noting that $F^0_t=0$ and $G^0_t=0$, by Proposition 3.2, we get immediately that $\left(\overline{Y},\overline{Z}\right)=(0,0)$.

Existence. For each $n\geq 1$, let $q_n(x)=x\frac{n}{|x|\vee n}$ and define $\xi_n=\mathrm{e}^{-\frac{1}{2}\beta A(T)}q_n\left(\mathrm{e}^{\frac{1}{2}\beta A(T)}\xi\right)$ and

$$f_n(t,y,z)=f(t,y,z)-f^0_t+a(t)\mathrm{e}^{-\frac{1}{2}\beta A(t)}q_n\left(\mathrm{e}^{\frac{1}{2}\beta A(t)}\frac{f^0_t}{a(t)}\right).$$

By definition, $q_n(x)\leq n$, for any $n\geq 1$. So we have

$$\sup_{0\leq t\leq T}\mathrm{e}^{\frac{1}{2}\beta A(t)}\frac{|f_n(t,0,0)|}{a(t)}\leq n\quad\text{and}\quad \mathrm{e}^{\frac{1}{2}\beta A(T)}\xi_n\leq n.$$

Then, it follows that ξ_n,f_n satisfy the assumptions **(H1)–(H3)** for $p=2$. Thus, from Theorem 4.1, for each $n\geq 1$, there exists a unique solution $(Y^n,Z^n)\in\mathcal{M}^2_{\beta,c}(a,T)$ for the following BDSDE:

$$Y^n_t=\xi_n+\int_t^T f_n(s,Y^n_s,Z^n_s)\mathrm{d}s+\int_t^T g(s,Y^n_s,Z^n_s)\overleftarrow{\mathrm{d}B_s}-\int_t^T Z^n_s\mathrm{d}W_s.$$

Moreover, according to Lemma 4.2, $Y^n\in\mathcal{S}^p_\beta(a,T,\mathbb{R}^k)\cap\mathcal{H}^{p,a}_\beta(a,T,\mathbb{R}^{k\times d})$, so that from Lemma 3.1, $Z^n\in\mathcal{H}^p_\beta(a,T,\mathbb{R}^{k\times d})$. Hence, $(Y^n,Z^n)\in\mathcal{M}^p_{\beta,c}(a,T)$.

Now, for $(i,n)\in\mathbb{N}\times\mathbb{N}^*$, let $Y^{i,n}=Y^{n+i}-Y^n$, $Z^{i,n}=Z^{n+i}-Z^n$.
Then, it is obvious that $\left(Y^{i,n},Z^{i,n}\right)\in\mathcal{M}^p_{\beta,c}(a,T)$ and verifies the following BDSDE:

$$Y^{i,n}_t=\xi_{i,n}+\int_t^T f_{i,n}(s,Y^{i,n}_s,Z^{i,n}_s)\mathrm{d}s+\int_t^T g_{i,n}(s,Y^{i,n}_s,Z^{i,n}_s)\overleftarrow{\mathrm{d}B_s}-\int_t^T Z^{i,n}_s\mathrm{d}W_s, \tag{4.7}$$

where $\xi_{i,n}=\xi_{n+i}-\xi_n$ and, $f_{i,n}$ and $g_{i,n}$ stand for the random functions

$$\begin{aligned} f_{i,n}(t,y,z)&=f_{n+i}(t,y+Y^n_t,z+Z^n_t)-f_n(t,Y^n_t,Z^n_t)\\ g_{i,n}(t,y,z)&=g(t,y+Y^n_t,z+Z^n_t)-g(t,Y^n_t,Z^n_t). \end{aligned}$$

From assumptions on (ξ,f,g) and the fact that $|q_n(x)|\le|x|$, for any $n\ge1$, it is easy to check that $(\xi_{i,n},f_{i,n},g_{i,n})$ satisfy **(H1)**–**(H4)** with

$$\begin{aligned} \xi_{i,n}&=\mathrm{e}^{-\frac12\beta A(t)}\left[q_{n+i}\left(\mathrm{e}^{\frac12\beta A(T)}\xi\right)-q_n\left(\mathrm{e}^{\frac12\beta A(T)}\xi\right)\right]\\ f_{i,n}(t,0,0)&=a(t)\mathrm{e}^{-\frac12\beta A(t)}\left[q_{n+i}\left(\mathrm{e}^{\frac12\beta A(t)}\frac{f^0_t}{a(t)}\right)-q_n\left(\mathrm{e}^{\frac12\beta A(t)}\frac{f^0_t}{a(t)}\right)\right]\ \text{ and}\\ g_{i,n}(t,0,0)&=0. \end{aligned}$$

Therefore, since $Y^{i,n}\in\mathcal{S}^p_\beta(a,T,\mathbb{R}^k)\cap\mathcal{H}^{p,a}_\beta(a,T,\mathbb{R}^{k\times d})$ and $g_{i,n}(t,0,0)=0$, we obtain thanks to Proposition 3.2 that, for $(i,n)\in\mathbb{N}\times\mathbb{N}^*$,

$$||Y^{i,n}||^p_{\mathcal{S}^p_\beta}+||Y^{i,n}||^p_{\mathcal{H}^{p,a}_\beta}+||Z^{i,n}||^p_{\mathcal{H}^p_\beta}\le C_p\mathbb{E}\left[\mathrm{e}^{\frac p2\beta A(T)}|\xi_{i,n}|^p+\left(\int_0^T\mathrm{e}^{\beta A(t)}\frac{|f_{i,n}(t,0,0)|^2}{a^2(t)}\mathrm{d}t\right)^{\frac p2}\right].$$

Hence,

$$\begin{aligned} &||Y^{n+i}-Y^n||^p_{\mathcal{S}^p_\beta}+||Y^{n+i}-Y^n||^p_{\mathcal{H}^{p,a}_\beta}+||Z^{n+i}-Z^n||^p_{\mathcal{H}^p_\beta}\\ &\le C_p\mathbb{E}\left[\left|q_{n+i}\left(\mathrm{e}^{\frac12\beta A(T)}\xi\right)-q_n\left(\mathrm{e}^{\frac12\beta A(T)}\xi\right)\right|^p\right.\\ &\qquad\left.+\left(\int_0^T\left|q_{n+i}\left(\mathrm{e}^{\frac12\beta A(t)}\frac{f^0_t}{a(t)}\right)-q_n\left(\mathrm{e}^{\frac12\beta A(t)}\frac{f^0_t}{a(t)}\right)\right|^2\mathrm{d}t\right)^{\frac p2}\right]. \end{aligned}$$

From **(H3)**, it follows by the dominated convergence theorem that the right-hand side of the above inequality tends to 0, as $n\to\infty$, uniformly in i, so (Y^n,Z^n) is a Cauchy sequence in $\mathcal{M}^p_{\beta,c}(a,T)$ and the limit is a solution of BDSDE (ξ,f,g) (1.1). □

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