EXTREMES AND LIMIT THEOREMS FOR DIFFERENCE OF CHI-TYPE PROCESSES *

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Abstract. Let $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}, \kappa > 0$ be random processes defined as the differences of two independent stationary chi-type processes with m and k degrees of freedom. In this paper we derive the asymptotics of $\mathbb{P}\left\{\sup_{t \in [0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\}, u \to \infty$ under some assumptions on the covariance structures of the underlying Gaussian processes. Further, we establish a Berman sojourn limit theorem and a Gumbel limit result.

1. Introduction

Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_{m+k}(t)), t \geq 0, m \geq 1, k \geq 0$ be a vector process with independent components which are centered stationary Gaussian processes with almost surely (a.s.) continuous sample paths, and unit variance. Set $r_i(t) = \mathbb{E}\{X_i(t)X_i(0)\}, t \geq 0$ and suppose that

$$r_i(t) = 1 - C_i |t|^\alpha + o(|t|^\alpha), \quad t \to 0 \quad \text{and} \quad r_i(t) < 1, \quad \forall t \neq 0, \ 1 \leq i \leq m+k, \tag{1.1}$$

where $\alpha \in (0,2]$ and $\boldsymbol{C} := (C_1, \ldots, C_{m+k}) \in (0,\infty)^{m+k}$. Define the random processes $\left\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\right\}, \kappa > 0$ by

$$\zeta_{m,k}^{(\kappa)}(t) := \left(\sum_{i=1}^{m} X_i^2(t)\right)^{\kappa/2} - \left(\sum_{i=m+1}^{m+k} X_i^2(t)\right)^{\kappa/2} =: |\boldsymbol{X}^{(1)}(t)|^\kappa - |\boldsymbol{X}^{(2)}(t)|^\kappa, \quad t \geq 0. \tag{1.2}$$

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In this paper we shall investigate the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,T]}\zeta_{m,k}^{(\kappa)}(t) > u\right\}, \quad u \to \infty \tag{1.3}$$

for given positive T. The study of the asymptotics of (1.3) is of interest in engineering applications concerned with the safety of structures; see, *e.g.*, [17–19] and the references therein. More precisely, of interest is the calculation of the probability that the Gaussian vector process exits a predefined safety region $\boldsymbol{S}_u \subset \mathbb{R}^{m+k}$ up to the time T, *i.e.*,

$$\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{S}_u, \text{ for some } t \in [0,T]\right\}.$$

In the aforementioned papers, various types of safety regions $\boldsymbol{S}_u$ have been considered for smooth Gaussian vector processes. Particularly, a safety region given by a ball centered at 0 with radius $u > 0$

$$\boldsymbol{B}_u = \left\{(x_1,\ldots,x_{m+k}) \in \mathbb{R}^{m+k} : \left(\sum_{i=1}^{m+k} x_i^2\right)^{1/2} \leq u\right\}$$

has been extensively studied; see, *e.g.*, [2, 5, 14, 22]. Referring to [1, 2], we know that for $k = 0$

$$\mathbb{P}\left\{\boldsymbol{X}(t) \notin \boldsymbol{B}_u, \text{ for some } t \in [0,T]\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]}|\boldsymbol{X}(t)| > u\right\}$$
$$= T H_{\alpha,1}^{m,0}(\boldsymbol{C}) u^{\frac{2}{\alpha}} \mathbb{P}\left\{|\boldsymbol{X}(0)| > u\right\}(1 + o(1)), \quad u \to \infty,$$

where $H_{\alpha,1}^{m,0}(\boldsymbol{C})$ is a positive constant (see (2.3) below for a precise definition). Very recently [23] obtained the tail asymptotics of the product of two Gaussian processes which has the same tail asymptotic behavior as $\sup_{t\in[0,T]}\zeta_{1,1}^{(2)}(t)$.

Our first result extends the findings of [2, 23] and suggests an asymptotic approximation for the exit probability of $\boldsymbol{X}$ from the safety regions $\boldsymbol{S}_u^{(\kappa)}$ defined by

$$\boldsymbol{S}_u^{(\kappa)} = \left\{(x_1,\ldots,x_{m+k}) \in \mathbb{R}^{m+k} : |\boldsymbol{x}^{(1)}|^\kappa - |\boldsymbol{x}^{(2)}|^\kappa \leq u\right\}.$$

Since chi-type processes appear naturally as limiting processes (see, *e.g.*, [3, 4, 20]), when one considers two independent asymptotic models, the study of the supremum of the difference of the two chi-type processes is also of some interest in mathematical statistics and its applications.

Although for $k \geq 1$ the random process $\zeta_{m,k}^{(\kappa)}$ is not Gaussian and the analysis of the supremum cannot be directly transformed into the study of the supremum of a related Gaussian random field (which is the case for chi-type processes; see, *e.g.*, [10, 17–19, 21, 22, 25]), it turns out that it is possible to apply the techniques for dealing with extremes of stationary processes developed mainly in [2, 5, 6]. In the second part of Section 2 we derive a sojourn limit theorem for $\zeta_{m,k}^{(\kappa)}$. Further, we show a Gumbel limit theorem for the supremum of $\zeta_{m,k}^{(\kappa)}$ over an increasing infinite interval. We refer to [2–4, 15, 22, 26] for results on the Gumbel limit theorem for Gaussian processes and chi-type processes.

Brief outline of the paper: our main results are stated in Section 2. In Section 3 we present proofs of Theorem 2.1, Theorems 2.2 and 2.3 followed then by an appendix.

2. MAIN RESULTS

We first introduce some notation. Let $\{Z(t), t \geq 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $\alpha/2 \in (0,1]$, *i.e.*, it is a centered Gaussian process with a.s. continuous sample paths and covariance function

$$\text{Cov}(Z(s), Z(t)) = \frac{1}{2}\left(s^\alpha + t^\alpha - |s-t|^\alpha\right), \quad s, t \geq 0.$$

In the following, let $\{Z_i(t), t \geq 0\}, 1 \leq i \leq m + k$ be independent copies of Z and define $\mathcal{W}_\kappa$ to be a Gamma distributed random variable with parameter $(k/\kappa, 1)$. Further let $\boldsymbol{O}_1 = (O_1, \ldots, O_m), \boldsymbol{O}_2 = (O_{m+1}, \ldots, O_{m+k})$ denote two random vectors uniformly distributed on the unit sphere of $\mathbb{R}^m$ and $\mathbb{R}^k$, respectively. Hereafter we shall suppose that $\boldsymbol{O}_1, \boldsymbol{O}_2, \mathcal{W}_\kappa$ and Z_i's are mutually independent. Define for $m \geq 1, \; k \geq 0, \; \kappa > 0$

$$\eta_{m,k}^{(\kappa)}(t) = \widetilde{Z}_{m,k}^{(\kappa)}(t) + E, \quad t \geq 0, \tag{2.1}$$

where E is a unit mean exponential random variable being independent of all the other random elements involved, and (recall $\boldsymbol{C} = (C_1, \ldots, C_{m+k})$ given in (1.1))

$$\widetilde{Z}_{m,k}^{(\kappa)}(t) = \begin{cases} L_1(t), & \kappa > 1, \\ L_1(t) + L_2(t), & \kappa = 1, \\ L_2(t), & \kappa < 1, \end{cases} \text{ with } \begin{cases} L_1(t) = \sum_{i=1}^m \sqrt{2C_i} O_i Z_i(t) - \left(\sum_{i=1}^m C_i O_i^2\right) t^\alpha, \\ L_2(t) = \mathcal{W}_\kappa - \left(\mathcal{W}_\kappa^{2/\kappa} + 2(\mathcal{W}_\kappa/\kappa)^{1/\kappa} \sum_{i=m+1}^{m+k} \sqrt{2C_i} O_i Z_i(t) \right. \\ \left. + 2\kappa^{-2/\kappa} \sum_{i=m+1}^{m+k} C_i Z_i^2(t) \right)^{\kappa/2}, \end{cases} \tag{2.2}$$

Here we set $\sum_{i=m+1}^m c_i =: 0$.

We state next our main result.

Theorem 2.1. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$ is given by (1.2) with the involved Gaussian processes X_i's satisfying (1.1), then, for any $T > 0$*

$$\mathbb{P}\left\{\sup_{t \in [0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\} = T H_{\alpha,\kappa}^{m,k}(\boldsymbol{C}) u^{\frac{2\tau}{\alpha\kappa}} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}(1 + o(1))$$

holds as $u \to \infty$, where $\tau = 2/\kappa - 1$ for $\kappa \in (0, 1)$, and 1 otherwise, and with $\eta_{m,k}^{(\kappa)}$ given by (2.1),

$$H_{\alpha,\kappa}^{m,k}(\boldsymbol{C}) = \lim_{a \downarrow 0} \frac{1}{a} \mathbb{P}\left\{\sup_{j \geq 1} \eta_{m,k}^{(\kappa)}(aj) \leq 0\right\} \in (0, \infty). \tag{2.3}$$

Remarks:

(a) The tail asymptotics of the Gaussian chaos $\zeta_{m,k}^{(\kappa)}(0)$ is discussed in Lemma 3.1 below.

(b) The most obvious choice of κ is 1, which corresponds to the difference of L_2-norm of two independent multivariate Gaussian processes. For the case $\kappa = 2$ and $m = k = 1$ the problem was (implicitly) investigated by considering the product of two independent Gaussian processes in the recent contribution [23].

(c) Since $\boldsymbol{O}_1$ is uniformly distributed on the unit sphere of $\mathbb{R}^m$, we have, for $\kappa > 1$ and $\boldsymbol{C} = \boldsymbol{1}$, that $\eta_{m,k}^{(\kappa)}(t) \stackrel{d}{=} \sqrt{2}Z(t) - t^\alpha + E$. In such a case, the constant $H_{\alpha,\kappa}^{m,k}(\boldsymbol{1})$ coincides with the classical Pickands constant H_α; see, *e.g.*, [22]. Approximation of Pickands constant H_α has been considered by a number of authors; see the recent contribution [8] which gives some simulation algorithms. Precise estimation of the general Pickands constant $H_{\alpha,\kappa}^{m,k}(\boldsymbol{C})$ seems to be hard to find, due to the complexity of the process $\eta_{m,k}^{(\kappa)}$.

(d) We see from Theorem 2.1 and Lemma 3.1 that, if $\kappa > 2$, then, for any $m, k \geq 1$

$$\mathbb{P}\left\{\sup_{t \in [0,T]} \zeta_{m,k}^{(\kappa)}(t) > u\right\} = \mathbb{P}\left\{\sup_{t \in [0,T]} \zeta_{m,0}^{(\kappa)}(t) > u\right\}(1 + o(1))$$

holds as $u \to \infty$, which means that $X_{m+1}, \ldots, X_{m+k}$ do not influence the tail asymptotic of $\sup_{t \in [0,T]} \zeta_{m,k}^{(\kappa)}(t)$. This is not so surprising as the tail asymptotic behavior of $\zeta_{m,k}^{(\kappa)}(0)$ is subexponential.

Next, we consider the sojourn time of $\zeta_{m,k}^{(\kappa)}$ above a threshold $u > 0$ in the time interval $[0, t]$ defined by

$$L_{m,k,t}^{(\kappa)}(u) = \int_0^t \mathbb{I}\{\zeta_{m,k}^{(\kappa)}(s) > u\} \, \mathrm{d}s, \quad t > 0,$$

where $\mathbb{I}\{\cdot\}$ stands for the indicator function. Our second result below establishes a Berman sojourn limit theorem for $\zeta_{m,k}^{(\kappa)}$. See [5] for related discussions on sojourn times of Gaussian processes and related processes.

Theorem 2.2. *Under the assumptions and notation of Theorem* 2.1, *we have, for any* $x > 0$

$$\int_x^\infty \mathbb{P}\left\{u^{\frac{2\tau}{\alpha\kappa}}L_{m,k,t}^{(\kappa)}(u) > y\right\}\mathrm{d}y = u^{\frac{2\tau}{\alpha\kappa}}\mathbb{E}\left\{L_{m,k,t}^{(\kappa)}(u)\right\}\Upsilon_\kappa(x)(1 + o(1))$$

holds as $u \to \infty$ *for all continuity point* $x > 0$ *of* $\Upsilon_\kappa(x) := \mathbb{P}\left\{\int_0^\infty \mathbb{I}\{\eta_{m,k}^{(\kappa)}(s) > 0\}\,\mathrm{d}s > x\right\}.$

In the following, we derive a Gumbel limit theorem for $\sup_{t\in[0,T]}\zeta_{m,k}^{(\kappa)}(t)$ under a linear normalization, which is also of interest in extreme value analysis and statistical tests. We refer to [4, 6, 13, 15] for its applications in deriving approximations of the critical values of the proposed test statistics.

Theorem 2.3. *Under the assumptions and notation of Theorem* 2.1, *if further the following Berman-type condition*

$$\lim_{t\to\infty}\max_{1\le l\le m+k}|r_l(t)|(\ln t)^c = 0, \quad \text{with } c = \begin{cases} 2/\kappa - 1, & 0 < \kappa < 1, \\ 1, & 1 \le \kappa \le 2, \\ k + 1 - 2k/\kappa, & \kappa > 2 \end{cases} \tag{2.4}$$

holds, then

$$\lim_{T\to\infty}\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left\{a_T^{(\kappa)}\left(\sup_{t\in[0,T]}\zeta_{m,k}^{(\kappa)}(t) - b_T^{(\kappa)}\right) \le x\right\} - \exp\left(-\mathrm{e}^{-x}\right)\right| = 0,$$

where, for all T *large*

$$a_T^{(\kappa)} = \frac{(2\ln T)^{1-\kappa/2}}{\kappa}, \quad b_T^{(\kappa)} = (2\ln T)^{\kappa/2} + \frac{\kappa}{2(2\ln T)^{1-\kappa/2}}\left(K_0\ln\ln T + \ln D_0\right), \tag{2.5}$$

with (*below* $\Gamma(\cdot)$ *denotes the Euler Gamma function*)

$$D_0 = \left(\frac{H_{\alpha,\kappa}^{m,k}(\boldsymbol{C})}{\Gamma(m/2)\Gamma(k/2)}\right)^2 \times \begin{cases} 2^{\frac{2}{\alpha}\left(\frac{2}{\kappa}-1\right)+2\left(1-\frac{k}{\kappa}\right)}\left(\Gamma\left(\frac{k}{\kappa}\right)\kappa^{(k/\kappa-1)}\right)^2, & 0 < \kappa \le 1, \\ 2^{\frac{2}{\alpha}+2\left(1-\frac{k}{\kappa}\right)}\left(\Gamma\left(\frac{k}{\kappa}\right)\kappa^{(k/\kappa-1)}\right)^2, & 1 < \kappa < 2, \\ 2^{\frac{2}{\alpha}-2}\left(\Gamma\left(\frac{k}{2}\right)\right)^2, & \kappa = 2, \\ 2^{\frac{2}{\alpha}}\left(\Gamma\left(\frac{k}{2}\right)\right)^2, & \kappa > 2 \end{cases}$$

$$K_0 = \begin{cases} m - 2 + (2/\alpha)(2/\kappa - 1) + k(1 - 2/\kappa), & 0 < \kappa \le 1, \\ m - 2 + 2/\alpha + k(1 - 2/\kappa), & 1 < \kappa < 2, \\ m - 2 + 2/\alpha, & \kappa \ge 2. \end{cases}$$

Under the assumptions of Theorem 2.3, we have the following convergence in probability (denoted by $\xrightarrow{p}$)

$$\frac{\sup_{t\in[0,T]}\zeta_{m,k}^{(\kappa)}(t)}{(2\ln T)^{\kappa/2}} \xrightarrow{p} 1, \quad T \to \infty,$$

which follows from the fact that $\lim_{T\to\infty}b_T^{(\kappa)}/(2\ln T)^{\kappa/2} = 1$ and that $a_T^{(\kappa)}$ is bounded away from zero, together with elementary considerations. In several cases such a convergence in probability can be strengthened to the *p*th mean convergence which is referred to as the Seleznjev *p*th mean convergence since the idea was first suggested by Seleznjev in [24], see also [12]. In order to show the Seleznjev *p*th mean convergence of crucial importance

is the Piterbarg inequality (see [22], Thm. 8.1). Since the Piterbarg inequality holds also for chi-square processes (see [25], Prop. 3.2), using further the fact that

$$\zeta_{m,k}^{(\kappa)}(t) \leq |\boldsymbol{X}^{(1)}(t)|^{\kappa}, \quad t \geq 0,$$

we immediately get the Piterbarg inequality for the difference of chi-type processes by simply applying the aforementioned proposition. Specifically, under the assumptions of Theorem 2.3 for any $T > 0$ and all large u

$$\mathbb{P}\left\{ \sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t) > u \right\} \leq KTu^{\beta} \exp\left(-\frac{1}{2}u^{2/\kappa} \right),$$

where K and β are two positive constants not depending on T and u. Note that the above result also follows immediately from Theorem 2.1 combined with Lemma 3.1 below. Hence utilizing Lemma 4.5 in [25] we arrive at our last result.

Corollary 2.4 (Seleznjev pth mean theorem)**.** *Under the assumptions of Theorem 2.3, we have, for any $p > 0$*

$$\lim_{T\to\infty} \mathbb{E}\left\{ \left(\frac{\sup_{t\in[0,T]} \zeta_{m,k}^{(\kappa)}(t)}{(2\ln T)^{\kappa/2}} \right)^{p} \right\} = 1.$$

3. FURTHER RESULTS AND PROOFS

Before presenting the proof of Theorem 2.1 we first give some preliminary lemmas. Hereafter we use the same notation and assumptions as in Section 1. By $\xrightarrow{d}$ and $\overset{d}{=}$ we shall denote the convergence in distribution (or the convergence of finite dimensional distributions if both sides of it are random processes) and equality in distribution function, respectively. Further, we write $f_{\xi}(\cdot)$ for the pdf of a random variable ξ and write $h_1 \sim h_2$ if two functions $h_i(\cdot), i = 1, 2$ are such that h_1/h_2 goes to 1 as the argument tends to some limit. For simplicity we shall denote, with $\kappa > 0$ and $\tau = \max(2/\kappa - 1, 1)$,

$$q_{\kappa} = q_{\kappa}(u) = u^{-2\tau/(\alpha\kappa)}, \quad w_{\kappa}(u) = \frac{1}{\kappa}u^{2/\kappa-1}, \quad u > 0.$$

In the proofs of Lemmas 3.1–3.3, we denote $u_{\kappa,x} = u + x/w_{\kappa}(u)$ for all $u, x > 0$.

Lemma 3.1. *Let $\{\zeta_{m,k}^{(\kappa)}(t), \ t \geq 0\}$ be given by (1.2). For all integers $m \geq 1, k \geq 0$ we have as $u \to \infty$*

$$\mathbb{P}\left\{ \zeta_{m,k}^{(\kappa)}(0) > u \right\} \sim \frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}{w_{\kappa}(u)} \sim \frac{2^{2-(m+k)/2}}{\kappa^2 \Gamma(k/2)\Gamma(m/2)} \frac{u^{m/\kappa-1}}{w_{\kappa}(u)} \exp\left(-\frac{1}{2}u^{2/\kappa} \right) \times \begin{cases} \frac{\Gamma(k/\kappa)}{(w_{\kappa}(u))^{k/\kappa}}, & \kappa < 2, \\ \Gamma(k/2), & \kappa = 2, \\ \kappa 2^{k/2-1}\Gamma(k/2), & \kappa > 2, \end{cases}$$

where $\Gamma(k/\kappa)/\Gamma(k/2) := 1$ for $k = 0$ and all $\kappa > 0$.

Proof. The claim follows from Theorem 1 in [11]. □

Lemma 3.2. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$ is as in Theorem 2.1, then*

$$\left\{ w_{\kappa}(u)(\zeta_{m,k}^{(\kappa)}(q_{\kappa}t) - u)|\{\zeta_{m,k}^{(\kappa)}(0) > u\}, \ t \geq 0 \right\} \xrightarrow{d} \left\{ \eta_{m,k}^{(\kappa)}(t), \ t \geq 0 \right\}, \quad u \to \infty,$$

with $\eta_{m,k}^{(\kappa)}$ given by (2.1). Recall that $\xrightarrow{d}$ stands for the convergence of finite dimensional distributions.

Proof. We henceforth adopt the notation introduced in Section 2. By Lemma 3.1, we have

$$w_\kappa(u)\left(\zeta_{m,k}^{(\kappa)}(0) - u\right)\Big|\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} \xrightarrow{d} E, \quad u \to \infty.$$

Thus, in view of Theorem 5.1 in [5], it suffices to show that, for any $0 < t_1 < \cdots < t_n < \infty, n \in \mathbb{N}$

$$
\begin{aligned}
p_k(u) &:= \mathbb{P}\left\{\cap_{j=1}^n \left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j) \leq u_{\kappa,z_j}\right\} \Big| \zeta_{m,k}^{(\kappa)}(0) = u_{\kappa,x}\right\} \\
&\to \mathbb{P}\left\{\cap_{j=1}^n \left\{\widetilde{Z}_{m,k}^{(\kappa)}(t_j) + x \leq z_j\right\}\right\}, \quad u \to \infty
\end{aligned}
\tag{3.1}
$$

holds for all $x > 0$ and $z_j \in \mathbb{R}, 1 \leq j \leq n$. Define below

$$\Delta_{iu}(t_j) = X_i(q_\kappa t_j) - r_i(q_\kappa t_j)X_i(0), \quad 1 \leq i \leq m+k, \ 1 \leq j \leq n.$$

By (1.1) we have

$$
\begin{aligned}
u^{2\tau/\kappa}\mathrm{Cov}(\Delta_{iu}(s), \Delta_{iu}(t)) &\to C_i(s^\alpha + t^\alpha - |s-t|^\alpha) \\
&= 2C_i\mathrm{Cov}(Z_i(s), Z_i(t)), \quad u \to \infty, \ s,t > 0, \ 1 \leq i \leq m+k.
\end{aligned}
$$

Therefore,

$$\{u^{\tau/\kappa}\Delta_{iu}(t), t \geq 0\} \xrightarrow{d} \{\sqrt{2C_i}Z_i(t), t \geq 0\}, \quad u \to \infty, \ 1 \leq i \leq m+k.$$

Furthermore, by the independence of $\Delta_{iu}(t)$'s and $X_i(0)$'s, the random processes Z_i's can be chosen such that they are independent of $\zeta_{m,k}^{(\kappa)}(0)$. Note that $\boldsymbol{X}^{(1)}(0) \overset{d}{=} R_1\boldsymbol{O}_1$ holds for some $R_1 > 0$ which is independent of $\boldsymbol{O}_1$. Then, using the Taylor's expansion of $(1+x)^{\kappa/2} = 1 + \kappa x/2 + o(x), x \to 0$ for any $z_j \in \mathbb{R}, 1 \leq j \leq n$ we have

$$
\begin{aligned}
p_0(u) &= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa \leq u_{\kappa,z_j}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x}\right\} \\
&= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{w_\kappa(u)\left(R_1^\kappa\left(1 + \frac{1}{R_1^2}V_u(t_j)\right)^{\kappa/2} - R_1^\kappa\right) \leq z_j - x\right\} \Big| R_1^\kappa = u_{\kappa,x}\right\} \\
&= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\frac{\kappa}{2}w_\kappa(u)R_1^{\kappa-2}V_u(t_j)(1 + o_p(1)) \leq z_j - x\right\} \Big| R_1^\kappa = u_{\kappa,x}\right\} \\
&= \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\sum_{i=1}^m \frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1 + o_p(1)) - \left(\sum_{i=1}^m \frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1 + o_p(1)) + x \leq z_j\right\}\right\}, \quad u \to \infty,
\end{aligned}
\tag{3.2}
$$

where $V_u(t_j) := \sum_{i=1}^m \Delta_{iu}^2(t_j) + 2\sum_{i=1}^m \Delta_{iu}(t_j)r_i(q_\kappa t_j)X_i(0) - \sum_{i=1}^m (1 - r_i^2(q_\kappa t_j))X_i^2(0)$. Consequently, the claim for $k = 0$ follows. Next, for $k \geq 1$, we rewrite $p_k(u)$ as (recall that $u_{\kappa,x} = u + x/w_\kappa(u)$)

$$
\begin{aligned}
p_k(u) &= \int_0^\infty \mathbb{P}\left\{\bigcap_{j=1}^n \left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t_j) \leq u_{\kappa,z_j}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\} \\
&\quad \times \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u)f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})} \, \mathrm{d}y \\
&= \int_0^\infty \mathbb{P}\left\{\bigcap_{j=1}^n \left\{|\boldsymbol{X}^{(1)}(q_\kappa t_j)|^\kappa \leq u_{\kappa,z_j+w_\kappa(u)\cdot|\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa}\right\} \Big| |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\} \\
&\quad \times h_{\kappa,u}(y) \, \mathrm{d}y,
\end{aligned}
\tag{3.3}
$$

where

$$h_{\kappa,u}(y) := \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u)f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})}$$

$$= \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y})}{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})}\frac{f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}{f_{\zeta_{m,k}^{(\kappa)}(0)}(u_{\kappa,x})}\frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{w_\kappa(u)f_{\zeta_{m,k}^{(\kappa)}(0)}(u)}$$

$$\sim \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))}{\int_0^\infty f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,y})f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}(y/w_\kappa(u))\,\mathrm{d}y}, \quad u \to \infty. \tag{3.4}$$

Here the last step follows by Lemma 3.1.

Noting that $\boldsymbol{X}^{(2)}(0) \stackrel{d}{=} R_2\boldsymbol{O}_2$ holds for some $R_2 > 0$ which is independent of $\boldsymbol{O}_2$, we have by similar arguments as in (3.2) that, for any $t \geq 0$

$$\left(w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t)|^\kappa\right)^{2/\kappa}\Big|\{w_\kappa(u)|\boldsymbol{X}^{(2)}(0)|^\kappa = y\}$$

$$= (w_\kappa(u))^{2/\kappa}\Bigg(\sum_{i=m+1}^{m+k}X_i^2(0) + 2\sum_{i=m+1}^{m+k}r_i(q_\kappa t)X_i(0)\Delta_{iu}(t) + \sum_{i=m+1}^{m+k}\Delta_{iu}^2(t)$$

$$- \sum_{i=m+1}^{m+k}(1-r_i(q_\kappa t)^2)X_i^2(0)\Bigg)\Bigg|\left\{R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}}\right\}$$

$$= (w_\kappa(u))^{2/\kappa}\Bigg(R_2^2 + 2\frac{R_2}{u^{\tau/\kappa}}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t)(1+o_p(1)) + \frac{2}{u^{2\tau/\kappa}}\sum_{i=m+1}^{m+k}C_iZ_i^2(t)(1+o_p(1))$$

$$- 2\left(\frac{R_2}{u^{\tau/\kappa}}\right)^2\sum_{i=m+1}^{m+k}C_iO_i^2t^\alpha(1+o_p(1))\Bigg)\Bigg|\left\{R_2^\kappa = \frac{y}{(w_\kappa(u))^{1/\kappa}}\right\}$$

$$= y^{2/\kappa} + 2y^{1/\kappa}\left(\frac{w_\kappa(u)}{u^\tau}\right)^{1/\kappa}\sum_{i=m+1}^{m+k}\sqrt{2C_i}O_iZ_i(t)(1+o_p(1)) + 2\left(\frac{w_\kappa(u)}{u^\tau}\right)^{2/\kappa}\sum_{i=m+1}^{m+k}C_iZ_i^2(t)(1+o_p(1))$$

$$=: \theta_{\kappa,u}(y,t). \tag{3.5}$$

This together with (3.2) and (3.3) implies that

$$p_k(u) = \int_0^\infty \mathbb{P}\Bigg\{\bigcap_{j=1}^n\Bigg\{\sum_{i=1}^m\frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1)) - \left(\sum_{i=1}^m\frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1)) + x + y$$

$$\leq z_j + w_\kappa(u)|\boldsymbol{X}^{(2)}(q_\kappa t_j)|^\kappa\Bigg\}\Bigg||\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\Bigg\}h_{\kappa,u}(y)\,\mathrm{d}y$$

$$= \int_0^\infty \mathbb{P}\Bigg\{\bigcap_{j=1}^n\Bigg\{\sum_{i=1}^m\frac{\sqrt{2C_i}O_iZ_i(t_j)}{u^{(\tau-1)/\kappa}}(1+o_p(1)) - \left(\sum_{i=1}^m\frac{C_iO_i^2}{u^{2(\tau-1)/\kappa}}\right)t_j^\alpha(1+o_p(1)) + x + y$$

$$\leq z_j + (\theta_{\kappa,u}(y,t_j))^{\frac{\kappa}{2}}\Bigg\}\Bigg\}h_{\kappa,u}(y)\,\mathrm{d}y. \tag{3.6}$$

Recalling that $\tau = \max(2/\kappa - 1, 1)$ and $w_\kappa(u) = (1/\kappa)u^{2/\kappa-1}$, we have by (3.5) that $(\theta_{\kappa,u}(y,t_j))^{\kappa/2} = y + o_p(1)$ for $\kappa > 1$. While for $\kappa \in (0,1]$, it follows by (3.4) and Lemma 3.1 that,

$$h_{\kappa,\infty}(y) := \lim_{u\to\infty}h_{\kappa,u}(y) = \frac{1}{\Gamma(k/\kappa)}y^{k/\kappa-1}\mathrm{e}^{-y}, \quad y > 0, \tag{3.7}$$

which is the pdf of a Gamma distributed random variable with parameter $(k/\kappa, 1)$. Hence, combining (3.4)–(3.7) and (2.2) for the definition of $\widetilde{Z}^{(\kappa)}_{m,k}(t)$, the claim in (3.1) follows. Consequently, the proof of Lemma 3.2 is complete.   □

The next lemma corresponds to Condition B in [2]; see also [1,3]. As shown in Chapter 5 in [1] this condition is crucial in ensuring that the double sum part is asymptotically negligible with respect to the principal sum. Denote in the following by $[x]$ the integer part of $x \in \mathbb{R}$.

Lemma 3.3. *If* $\{\zeta^{(\kappa)}_{m,k}(t), t \geq 0\}$ *is as in Theorem* 2.1, *then for any* $T, a > 0$

$$\limsup_{u \to \infty} \sum_{j=N}^{[T/(aq_\kappa)]} \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(aq_\kappa j) > u \,\Big|\, \zeta^{(\kappa)}_{m,k}(0) > u\right\} \to 0, \quad N \to \infty.$$

Proof. Note first that the case $k = 0$ is treated in [2], page 119. Using the fact that the standard bivariate Gaussian distribution is exchangeable for $u > 0$ we have

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t) > u \,\Big|\, \zeta^{(\kappa)}_{m,k}(0) > u\right\} = 2\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t) > u, |\boldsymbol{X}^{(1)}(q_\kappa t)| > |\boldsymbol{X}^{(1)}(0)| \,\Big|\, \zeta^{(\kappa)}_{m,k}(0) > u\right\} =: 2\Theta(u).$$

Further, it follows from Lemma 3.1 that, for any $k \geq 1$

$$\Theta(u) = \int_0^\infty \int_0^\infty \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t) > u, |\boldsymbol{X}^{(1)}(q_\kappa t)| > |\boldsymbol{X}^{(1)}(0)| \,\Big|\, |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}, |\boldsymbol{X}^{(2)}(0)|^\kappa = \frac{y}{w_\kappa(u)}\right\}$$

$$\times \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}} \, dx dy$$

$$\leq \int_0^\infty \int_0^\infty \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \,\Big|\, |\boldsymbol{X}^{(1)}(0)|^\kappa = u_{\kappa,x+y}\right\} \frac{f_{|\boldsymbol{X}^{(1)}(0)|^\kappa}(u_{\kappa,x+y}) f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right)}{w_\kappa^2(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}} \, dx dy$$

$$= \int_0^\infty \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \,\Big|\, |\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\} \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}} f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right) dy.$$

Moreover, in view of the treatment of the case $k = 0$ in [2], page 119 we readily see that, for any $p \geq 1$, with $R(t) := \max_{1 \leq i \leq m} r_i(t), r(t) := \min_{1 \leq i \leq m} r_i(t)$ and $\Phi(\cdot)$ denoting the $N(0,1)$ distribution function

$$\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u_{\kappa,y} \,\Big|\, |\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\} \leq 4m\left(1 - \Phi\left(\frac{(1 - R(q_\kappa t))u^{1/\kappa}}{\sqrt{m(1 - r^2(q_\kappa t))}}\right)\right)$$

$$\leq K_p t^{-\alpha p/2}, \quad \forall q_\kappa t \in (0, T]$$

holds for some $K_p > 0$ not depending on u, t and y. Consequently,

$$\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(q_\kappa t) > u \,\Big|\, \zeta^{(\kappa)}_{m,k}(0) > u\right\} \leq 2K_p t^{-\alpha p/2} \int_0^\infty \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)|^\kappa > u_{\kappa,y}\right\}}{w_\kappa(u)\mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(0) > u\right\}} f_{|\boldsymbol{X}^{(2)}(0)|^\kappa}\left(\frac{y}{w_\kappa(u)}\right) dy$$

$$= 2K_p t^{-\alpha p/2}, \quad \forall q_\kappa t \in (0, T]. \tag{3.8}$$

Therefore, with $p = 4/\alpha$

$$\limsup_{u \to \infty} \sum_{j=N}^{[T/(aq_\kappa)]} \mathbb{P}\left\{\zeta^{(\kappa)}_{m,k}(aq_\kappa j) > u \,\Big|\, \zeta^{(\kappa)}_{m,k}(0) > u\right\} \leq 2K_p \int_{aN}^\infty x^{-2} \, dx = \frac{2K_p}{aN} \to 0, \quad N \to \infty$$

establishing the proof.   □

The lemma below concerns the accuracy of the discrete approximation to the continuous process, which is related to Condition C in [2]. As shown in [3] (see Eq. (7) therein), in order to verify Condition C the following lemma is sufficient. Its proof is relegated to the appendix.

Lemma 3.4. *If $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$ is as in Theorem* 2.1*, then there exist some constants $C, p > 0, d > 1$ and $\lambda_0, u_0 > 0$ such that*

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \leq u\right\} \leq Ct^d \lambda^{-p} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}$$

for $0 < t^\varpi < \lambda < \lambda_0$ and $u > u_0$. Here ϖ is $\alpha/2$ for $\kappa \geq 1$, and $(\alpha/2)\min(\kappa/(4(1-\kappa)), 1)$ otherwise.

Proof of Theorem 2.1. It follows from Lemmas 3.1–3.4 that all the assumptions of Theorem 1 in [2] are satisfied by the process $\zeta_{m,k}^{(\kappa)}$, which immediately establishes the proof. $\qquad\square$

Proof of Theorem 2.2. In view of (3.8) with $p = 4/\alpha$ and letting $v_\kappa = v_\kappa(u) = 1/q_\kappa(u) = u^{2\tau/(\alpha\kappa)}$, we obtain

$$v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \mathrm{d}s = \int_{N}^{v_\kappa T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s/v_\kappa) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \mathrm{d}s$$

$$\leq K_{4/\alpha} \int_{N}^{v_\kappa T} s^{-2}\, \mathrm{d}s \leq \frac{K_{4/\alpha}}{N}, \quad u \to \infty.$$

Hence,

$$\lim_{N\to\infty} \limsup_{u\to\infty} v_\kappa \int_{N/v_\kappa}^{T} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(s) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \mathrm{d}s = 0.$$

Since further Lemma 3.2 holds, the claim follows by Theorem 3.1 in [5]. $\qquad\square$

As shown by Theorem 10 in [2], in order to derive the Gumbel limit theorem for the random process $\zeta_{m,k}^{(\kappa)}$ two additional conditions, which were first addressed by the seminal contributions [15, 16], need to be checked, namely the mixing Condition D and the Condition D' therein. These two conditions will follow from Lemma 3.5 and Lemma 3.6 below; their proofs are displayed in the appendix.

Lemma 3.5. *Let T and a be any given positive constants and $M \in (0, T)$. If $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$ is as in Theorem* 2.1*, then for any $0 \leq s_1 < \cdots < s_p < t_1 < \cdots < t_{p'}$ in $\{aq_\kappa j : j \in \mathbb{Z}, 0 \leq aq_\kappa j \leq T\}$ such that $t_1 - s_p \geq M$*

$$\left| \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) \leq u\}, \cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) \leq u\}\right\} - \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) \leq u\}\right\} \mathbb{P}\left\{\cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) \leq u\}\right\} \right|$$

$$\leq Ku^\varsigma \sum_{1\leq i\leq p, 1\leq j\leq p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \tag{3.9}$$

and

$$\left| \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) > u\}, \cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) > u\}\right\} - \mathbb{P}\left\{\cap_{i=1}^{p}\{\zeta_{m,k}^{(\kappa)}(s_i) > u\}\right\} \mathbb{P}\left\{\cap_{j=1}^{p'}\{\zeta_{m,k}^{(\kappa)}(t_j) > u\}\right\} \right|$$

$$\leq Ku^\varsigma \sum_{1\leq i\leq p, 1\leq j\leq p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)}\right) \tag{3.10}$$

hold for all $u > 0$ and some $K > 0$ not depending on u. Here $\varsigma = 2/\kappa\big(m - k(2/\kappa - 1) - 1 + \max(0, 2(1/\kappa - 1))\big)$ and $\widetilde{r}(t) := \max_{1\leq l\leq m+k} |r_l(t)|, t > 0$.

Lemma 3.6. *Under the assumptions of Theorem* 2.3, *for* ς, $\widetilde{r}(\cdot)$ *as in Lemma* 3.5 *and* T_κ *given by*

$$T_\kappa = T_\kappa(u) = \frac{1}{H_{\alpha,\kappa}^{m,k}(\boldsymbol{C})} \frac{q_\kappa(u)}{\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}}, \tag{3.11}$$

we have, for any given constant $\varepsilon \in (0, T_\kappa)$

$$u^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{\varepsilon \leq aq_\kappa j \leq T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right) \to 0, \quad u \to \infty. \tag{3.12}$$

Proof of Theorem 2.3. To establish Conditions D and D' in [2], we shall make use of Lemma 3.5 with $T = T_\kappa$ given by (3.11) and $M = \varepsilon \in (0, T_\kappa)$, and Lemma 3.6. First note that the right-hand side of (3.9) is bounded from above by

$$Ku^\varsigma \frac{T_\kappa}{aq_\kappa} \sum_{\varepsilon \leq aq_\kappa j \leq T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right),$$

which by an application of (3.12) implies that the mixing Condition D in [2] holds for the random process $\zeta_{m,k}^{(\kappa)}$.

Next, we prove Condition D' in [2], *i.e.*, for any given positive constants a and $\widetilde{T}$

$$\limsup_{u \to \infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \to 0, \quad \varepsilon \downarrow 0. \tag{3.13}$$

Indeed, by (3.10) for some $\widetilde{M} > \widetilde{T}$ and a positive constant K

$$\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} \leq \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} + Ku^\varsigma \frac{T_\kappa}{q_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right)$$

holds for $u > 0$ and $aq_\kappa j > \widetilde{M}$. Consequently,

$$\limsup_{u \to \infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\}$$

$$\leq \limsup_{u \to \infty} \sum_{j=[\widetilde{T}/(aq_\kappa)]}^{[\widetilde{M}/(aq_\kappa)]} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(aq_\kappa j) > u \Big| \zeta_{m,k}^{(\kappa)}(0) > u\right\} + \varepsilon$$

$$+ \limsup_{u \to \infty} Ku^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{j=[\widetilde{M}/(aq_\kappa)]}^{\left[\varepsilon/\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0)>u\right\}\right]} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right),$$

which equals ϵ by an application of Lemma 3.3 and (3.12), respectively. It follows then that (3.13) holds. Consequently, in view of Theorem 10 in [2] we have, for T_κ given by (3.11)

$$\lim_{u \to \infty} \mathbb{P}\left\{\sup_{t \in [0, T_\kappa]} \zeta_{m,k}^{(\kappa)}(t) \leq u + \frac{x}{w_\kappa(u)}\right\} = \exp\left(-\mathrm{e}^{-x}\right), \quad x \in \mathbb{R}.$$

Expressing u in terms of T_κ using (3.11) (see also (A.24)) we obtain the required claim with $a_T^{(\kappa)}, b_T^{(\kappa)}$ given by (2.5) for any $x \in \mathbb{R}$; the uniform convergence in x follows since all functions (with respect to x) are continuous, bounded and increasing. $\square$

APPENDIX A.

Proof of Lemma 3.4. By (1.1), for any small $\epsilon \in (0,1)$ there exists some positive constant B such that

$$r_i(t) \geq \frac{1}{2} \quad \text{and} \quad 1 - r_i(t) \leq Bt^\alpha, \quad \forall t \in (0, \epsilon], \ 1 \leq i \leq m + k.$$

Furthermore, for any positive t satisfying (recall $\varpi = \alpha/2\mathbb{I}\{\kappa \geq 1\} + \alpha/2 \min(\kappa/(4(1-\kappa)), 1)\mathbb{I}\{0 < \kappa < 1\}$)

$$0 < t^\varpi < \lambda < \lambda_0 := \min\left(\frac{1}{2^{\kappa+4}B}, \frac{\kappa}{2^{\kappa+2}}, \epsilon^\varpi\right)$$

and any $u > 2$

$$u^{2\tau/\kappa}\theta_\kappa(t) \leq 2^\kappa \kappa Bt^\alpha \leq \frac{\kappa t^{\alpha/2}}{16} \quad \text{with} \quad \theta_\kappa(t) := \frac{1}{(r(q_\kappa t))^\kappa} - 1, \quad r(t) := \min_{1 \leq i \leq m+k} r_i(t). \tag{A.14}$$

Let $(\boldsymbol{X}_{1/r}^{(1)}(t), \boldsymbol{X}_{1/r}^{(2)}(t)) := \left(X_1(t) - r_1^{-1}(t)X_1(0), \ldots, X_{m+k}(t) - r_{m+k}^{-1}(t)X_{m+k}(0)\right)$ which by definition is independent of $\{\zeta_{m,k}^{(\kappa)}(t), t \geq 0\}$. For $j = 1, 2$

$$\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(j)}(q_\kappa t)| > x\right\} \leq \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{x}{2\sqrt{2B}u^{-2\tau/\kappa t^\alpha}}\right\}, \quad u\theta_\kappa(t) \leq \frac{\lambda}{2w_\kappa(u)}. \tag{A.15}$$

In the following, the cases $\kappa = 1, \kappa \in (1, \infty)$ and $\kappa \in (0, 1)$ will be considered in turn.

Case $\kappa = 1$. Note by the triangular inequality

$$\zeta_{m,k}^{(1)}(q_1 t) \leq |\boldsymbol{X}_{1/r}^{(1)}(q_1 t)| + |\boldsymbol{X}_{1/r}^{(2)}(q_1 t)| + \frac{1}{r(q_1 t)}\zeta_{m,k}^{(1)}(0) + \theta_1(t)|\boldsymbol{X}^{(2)}(0)|.$$

Consequently, from (A.15) we get

$$\mathbb{P}\left\{\zeta_{m,k}^{(1)}(q_1 t) > u + \frac{\lambda}{u}, \zeta_{m,k}^{(1)}(0) \leq u\right\}$$

$$\leq \mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(1)}(q_1 t)| + |\boldsymbol{X}_{1/r}^{(2)}(q_1 t)| + \theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{2u}, \zeta_{m,k}^{(1)}(q_1 t) > u\right\}$$

$$\leq \mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(1)}(q_1 t)| + |\boldsymbol{X}_{1/r}^{(2)}(q_1 t)| > \frac{\lambda}{3u}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(1)}(q_1 t) > u\right\} + \mathbb{P}\left\{\theta_1(t)|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda}{6u}\right\}$$

$$=: I_{1u} + I_{2u}.$$

By (A.14) and (A.15), we have, for any $p > 1$

$$\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(1)}(q_1 t)| > \frac{\lambda}{6u}\right\} \leq \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda}{12\sqrt{2B}t^{\alpha/2}}\right\} \leq K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}$$

holds with some $K > 0$ (the values of p and K might change from line to line below). Similarly,

$$\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(2)}(q_1 t)| > \frac{\lambda}{6u}\right\} \leq K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}$$

and hence

$$I_{1u} \leq K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta_{m,k}^{(1)}(0) > u\right\}. \tag{A.16}$$

Moreover, in view of Lemma 3.1 and (A.14) we have for sufficiently large u that

$$I_{2u} \leq \frac{\mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{2\lambda u}{t^{\alpha/2}}\right\}}{\mathbb{P}\left\{\zeta_{m,k}^{(1)}(0) > u\right\}} \mathbb{P}\left\{\zeta_{m,k}^{(1)}(0) > u\right\} \leq K\left(\frac{\lambda}{t^{\alpha/2}}\right)^{-(p-k+2)} u^{-(p+m-2k)} \mathbb{P}\left\{\zeta_{m,k}^{(1)}(0) > u\right\}. \quad \text{(A.17)}$$

Hence, the claim for $\kappa = 1$ follows from (A.16) and (A.17) by choosing $p > \max(4/\alpha + k, 2k)$.

Case $\kappa \in (1, \infty)$. Denote below by $(\boldsymbol{Y}^{(1)}(t), \boldsymbol{Y}^{(2)}(t)) := (r_1^{-1}(t)X_1(0), \ldots, r_{m+k}^{-1}(t)X_{m+k}(0))$. Note that $|\boldsymbol{Y}^{(1)}(t)| \leq |\boldsymbol{X}^{(1)}(0)|/r(t)$ and $|\boldsymbol{X}^{(2)}(0)| \leq |\boldsymbol{Y}^{(2)}(t)| \leq |\boldsymbol{X}^{(2)}(0)|/r(t)$ for all $t < \varepsilon$, and for some constants K_1, $K_2 > 0$ whose values might change from line to line below

$$|1 + x|^\kappa \geq 1 + \kappa x, \quad x \in \mathbb{R} \quad \text{and} \quad (1 + x)^\kappa \leq 1 + K_1 x + K_2 x^\kappa, \quad x \geq 0.$$

We have further by the triangle inequality

$$\begin{aligned}
\zeta_{m,k}^{(\kappa)}(q_\kappa t) &\leq \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)| + |\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|\right)^\kappa - \left||\boldsymbol{Y}^{(2)}(q_\kappa t)| - |\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)|\right|^\kappa \\
&\leq |\boldsymbol{Y}^{(1)}(q_\kappa t)|^\kappa + K_1|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)||\boldsymbol{Y}^{(1)}(q_\kappa t)|^{\kappa-1} + K_2|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa \\
&\quad - |\boldsymbol{Y}^{(2)}(q_\kappa t)|^\kappa + \kappa|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1} \\
&\leq K_1|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} + K_2|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa \\
&\quad + K_3|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1} + \frac{\zeta_{m,k}^{(\kappa)}(0)}{(r(q_\kappa t))^\kappa} + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa
\end{aligned}$$

holds for $q_\kappa t \leq \epsilon$ and some constant $K_3 > 0$. Therefore, with $\mu = 1/(2(\kappa - 1))$ and $\varphi = \alpha/(4(\kappa - 1))$,

$$\begin{aligned}
&\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \leq u\right\} \\
&\leq \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\} + \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| > \frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right\} \\
&\quad + \mathrm{P}\left\{K_1|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1} + K_2|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|^\kappa + K_3|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)|\left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi}\right)^{\kappa-1}\right. \\
&\quad \left. + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa \geq \frac{\lambda}{2w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\} \\
&=: \tilde{I}_{1u} + \tilde{I}_{2u} + \tilde{I}_{3u}.
\end{aligned} \quad \text{(A.18)}$$

Note by (A.14) that $\lambda^\mu/t^\varphi > 1$. Similar arguments as in (A.17) yield that

$$\begin{aligned}
\tilde{I}_{1u} &\leq K\left(\tfrac{\lambda^\mu}{t^\varphi}\right)^{-(p-m+2)} u^{-(p-k(2/\kappa-1)\mathbb{I}\{\kappa \leq 2\})/\kappa} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\} \\
\tilde{I}_{2u} &\leq K\left(\tfrac{\lambda^\mu}{t^\varphi}\right)^{-(p-k+2)} u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa \leq 2\})/\kappa} \mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}
\end{aligned}$$

and

$$
\begin{aligned}
\tilde{I}_{3u} \leq & \left(\mathbb{P}\left\{ K_1 |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi} \right)^{\kappa-1} > \frac{\lambda}{8 w_\kappa(u)} \right\} + \mathbb{P}\left\{ K_2 |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)|^\kappa > \frac{\lambda}{8 w_\kappa(u)} \right\} \right. \\
& \left. + \mathbb{P}\left\{ K_3 |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)| \left(\frac{\lambda^\mu u^{1/\kappa}}{t^\varphi} \right)^{\kappa-1} > \frac{\lambda}{8 w_\kappa(u)} \right\} \right) \mathbb{P}\left\{ \zeta^{(\kappa)}_{m,k}(q_\kappa t) > u \right\} \\
& + \mathbb{P}\left\{ \theta_\kappa(t) |\boldsymbol{X}^{(2)}(0)|^\kappa > \frac{\lambda}{8 w_\kappa(u)} \right\} \\
=: & (II_{1u} + II_{2u} + II_{3u}) \mathbb{P}\left\{ \zeta^{(\kappa)}_{m,k}(0) > u \right\} + II_{4u}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
II_{1u} &\leq \mathbb{P}\left\{ |\boldsymbol{X}^{(1)}(0)| > K_1 \frac{\lambda^{1/2} u^{-1/\kappa}}{t^{-\alpha/4}(r^{-2}(q_\kappa t) - 1)^{1/2}} \right\} \\
&\leq \mathbb{P}\left\{ |\boldsymbol{X}^{(1)}(0)| > K_1 \frac{\lambda^{1/2}}{t^{\alpha/4}} \right\} \leq K \left(\frac{\lambda}{t^{\alpha/2}} \right)^{-p/2}.
\end{aligned}
$$

Similarly,

$$
II_{2u} \leq K \left(\frac{\lambda u^{2(1-1/\kappa)}}{t^{\alpha\kappa/2}} \right)^{-p/\kappa}, \quad II_{3u} \leq K \left(\frac{\lambda}{t^{\alpha/2}} \right)^{-p/2}.
$$

Next, we deal with II_{4u}. We have by (A.14) that $2^{\kappa+4} B t^{\alpha/2} \leq 1$. Hence as in the proof of (A.17), we have

$$
\begin{aligned}
II_{4u} &\leq \mathbb{P}\left\{ |\boldsymbol{X}^{(2)}(0)|^\kappa > \frac{2\lambda u}{t^{\alpha/2}} \frac{1}{2^{\kappa+4} B t^{\alpha/2}} \right\} \\
&\leq K \left(\frac{\lambda}{t^{\alpha/2}} \right)^{-(p-k+2)} u^{-(p-k+m-k(2/\kappa-1)\mathbb{I}\{\kappa\leq 2\})/\kappa} \mathbb{P}\left\{ \zeta^{(\kappa)}_{m,k}(0) > u \right\}. \quad (\mathrm{A.19})
\end{aligned}
$$

Therefore, the claim for $\kappa \in (1,\infty)$ follows from (A.18) and the inequalities for $\tilde{I}_{1u}, \tilde{I}_{2u}$ and II_{1u}– II_{4u} by choosing $p > \max(8(\kappa-1)/\alpha + k + m, 2k)$.

Case $\kappa \in (0,1)$. Note that

$$
(1+x)^\kappa \leq 1+x, \quad x \geq 0 \quad \text{and} \quad -|1-x|^\kappa \leq -(1-x), \quad x \in [0,\infty).
$$

We have further by the triangle inequality

$$
\begin{aligned}
\zeta^{(\kappa)}_{m,k}(q_\kappa t) &\leq \left(|\boldsymbol{Y}^{(1)}(q_\kappa t)| + |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)| \right)^\kappa - \left| |\boldsymbol{Y}^{(2)}(q_\kappa t)| - |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)| \right|^\kappa \\
&\leq |\boldsymbol{Y}^{(1)}(q_\kappa t)|^\kappa + |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} - |\boldsymbol{Y}^{(2)}(q_\kappa t)|^\kappa + |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{Y}^{(2)}(q_\kappa t)|^{\kappa-1} \\
&\leq \frac{|\boldsymbol{X}^{(1)}(0)|^\kappa}{(r(q_\kappa t))^\kappa} + |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} - |\boldsymbol{X}^{(2)}(0)|^\kappa + |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1} \\
&= \frac{\zeta^{(\kappa)}_{m,k}(0)}{(r(q_\kappa t))^\kappa} + \theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa + |\boldsymbol{X}^{(1)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(1)}(0)|^{\kappa-1} + |\boldsymbol{X}^{(2)}_{1/r}(q_\kappa t)||\boldsymbol{X}^{(2)}(0)|^{\kappa-1}.
\end{aligned}
$$

Therefore, we have by (A.15), with $\psi = \alpha/(4(1-\kappa))$

$$
\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(0) \le u\right\}
$$

$$
\le \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa + \frac{|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)|}{(u^{-\tau/\kappa}t^\psi)^{1-\kappa}} + \frac{|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)|}{(u^{-\tau/\kappa}t^\psi)^{1-\kappa}} > \frac{\lambda}{2w_\kappa(u)}, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}
$$

$$
+ \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| \le u^{-\frac{\tau}{\kappa}}t^\psi, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\} + \mathbb{P}\left\{|\boldsymbol{X}^{(2)}(0)| \le u^{-\frac{\tau}{\kappa}}t^\psi, \zeta_{m,k}^{(\kappa)}(0) \le u, \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u + \frac{\lambda}{w_\kappa(u)}\right\}
$$

$$
=: I_{1u}^* + I_{2u}^* + I_{3u}^*.
$$

Now we deal with the three terms one by one. Clearly, for any $u > 2$

$$
I_{1u}^* \le \mathbb{P}\left\{\theta_\kappa(t)|\boldsymbol{X}^{(2)}(0)|^\kappa > \frac{\lambda}{6w_\kappa(u)}\right\} + \mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(1)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}
$$

$$
+ \mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(2)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\},
$$

where the first term can be treated as for II_{4u}, see (A.19). For the rest two terms, we have, by using (A.15)

$$
\mathbb{P}\left\{|\boldsymbol{X}_{1/r}^{(j)}(q_\kappa t)| > \frac{\lambda\kappa t^{\alpha/4}}{6u^{\tau/\kappa}}\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{\kappa}{12\sqrt{2B}}\frac{\lambda}{t^{\alpha/4}}\right\} \le K\left(\frac{\lambda}{t^{\alpha/4}}\right)^{-p}, \quad j = 1,2. \tag{A.20}
$$

In order to deal with I_{2u}^* and I_{3u}^*, set below $(\boldsymbol{X}_r^{(1)}(t), \boldsymbol{X}_r^{(2)}(t)) := (X_1(0) - r_1(t)X_1(t), \dots, X_{m+k}(0) - r_{m+k}(t)X_{m+k}(t))$ which by definition is independent of $\{\zeta_{m,k}^{(\kappa)}(t), t \ge 0\}$. For $j = 1,2$

$$
\mathbb{P}\left\{|\boldsymbol{X}_r^{(j)}(q_\kappa t)| > x\right\} \le \mathbb{P}\left\{|\boldsymbol{X}^{(j)}(0)| > \frac{2\sqrt{\lambda}x}{\sqrt{u^{-2\tau/\kappa}t^\alpha}}\right\}. \tag{A.21}
$$

Using further the triangle inequality $|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa \ge (r(q_\kappa t))^\kappa|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa - |\boldsymbol{X}^{(1)}(0)|^\kappa$ and (A.14) (recalling $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa \ge \zeta_{m,k}^{(\kappa)}(q_\kappa t) > u$), we have

$$
I_{2u}^* \le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa - \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}
$$

$$
\le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > \frac{(1-2^{-\kappa})u}{2^\kappa}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}
$$

$$
\le \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > (1-2^{-\kappa})^{1/\kappa}\frac{\sqrt{\lambda}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}
$$

$$
\le K\left(\frac{\lambda}{t^\alpha}\right)^{-p/2}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}. \tag{A.22}
$$

For I_{3u}^*, using $|\boldsymbol{X}^{(1)}(q_\kappa t)|^\kappa > u + \lambda/w_\kappa(u)$ and

$$
|\boldsymbol{X}^{(1)}(0)|^\kappa = \zeta_{m,k}^{(\kappa)}(0) + |\boldsymbol{X}^{(2)}(0)|^\kappa \le u\left(1 + \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)
$$

we have

$$
I_{3u}^* \le \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u\left((r(q_\kappa t))^\kappa\left(1 + \frac{\lambda}{uw_\kappa(u)}\right) - \left(1 + \frac{t^{\psi\kappa}}{u^{1+\tau}}\right)\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(q_\kappa t) > u\right\}
$$

$$
= \mathbb{P}\left\{|\boldsymbol{X}_r^{(1)}(q_\kappa t)|^\kappa > u^{-\tau}\left(\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1 - (r(q_\kappa t))^\kappa) - t^{\psi\kappa}\right)\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\},
$$

where by (A.14)

$$\lambda\kappa(r(q_\kappa t))^\kappa - u^{1+\tau}(1 - (r(q_\kappa t))^\kappa) - t^{\psi\kappa} \geq \frac{\lambda\kappa}{2^{\kappa+1}} - t^{\psi\kappa} \geq \frac{\lambda\kappa}{2^{\kappa+2}}.$$

Consequently, it follows further by (A.21) that

$$I_{3u}^* \leq \mathbb{P}\left\{|\boldsymbol{X}^{(1)}(0)| > 2^{-2/\kappa}\kappa^{1/\kappa}\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right\}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\}$$

$$\leq K\left(\frac{\lambda^{1/\kappa+1/2}}{t^{\alpha/2}}\right)^{-p}\mathbb{P}\left\{\zeta_{m,k}^{(\kappa)}(0) > u\right\},$$

which together with (A.19), (A.20) and (A.22) completes the proof for $\kappa \in (0,1)$ by taking $p > 4/\alpha + k$. Consequently, the desired claim of Lemma 3.4 follows. This completes the proof. $\qquad\square$

Proof of Lemma 3.5. We give only the proof for (3.9) since (3.10) follows by similar arguments. Since the claims for $k = 0$ are already shown in [1], we only consider that $k \geq 1$ below. Define, for $j = 1, 2$, independent random vectors $\left(|\boldsymbol{Y}^{(j)}(s_1)|, \ldots, |\boldsymbol{Y}^{(j)}(s_p)|\right)$ and $\left(|\widetilde{\boldsymbol{Y}}^{(j)}(t_1)|, \ldots, |\widetilde{\boldsymbol{Y}}^{(j)}(t_{p'})|\right)$, which are independent of the process $\zeta_{m,k}^{(\kappa)}$ and have the same distributions as those of $\left(|\boldsymbol{X}^{(j)}(s_1)|, \ldots, |\boldsymbol{X}^{(j)}(s_p)|\right)$ and $\left(|\boldsymbol{X}^{(j)}(t_1)|, \ldots, |\boldsymbol{X}^{(j)}(t_{p'})|\right)$, respectively. Note that, for any $u > 0$, the left-hand side of (3.9) is clearly bounded from above by

$$\left|\mathbb{P}\left\{\bigcap_{i=1}^p \left\{|\boldsymbol{X}^{(2)}(s_i)|^\kappa \geq |\boldsymbol{X}^{(1)}(s_i)|^\kappa - u\right\}, \bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(2)}(t_j)|^\kappa \geq |\boldsymbol{X}^{(1)}(t_j)|^\kappa - u\right\}\right\}\right.$$

$$\left. - \mathbb{P}\left\{\bigcap_{i=1}^p\left\{|\boldsymbol{Y}^{(2)}(s_i)|^\kappa \geq |\boldsymbol{X}^{(1)}(s_i)|^\kappa - u\right\}, \bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa \geq |\boldsymbol{X}^{(1)}(t_j)|^\kappa - u\right\}\right\}\right|$$

$$+ \left|\mathbb{P}\left\{\bigcap_{i=1}^p\left\{|\boldsymbol{X}^{(1)}(s_i)|^\kappa \leq |\boldsymbol{Y}^{(2)}(s_i)|^\kappa + u\right\}, \bigcap_{j=1}^{p'}\left\{|\boldsymbol{X}^{(1)}(t_j)|^\kappa \leq |\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa + u\right\}\right\}\right.$$

$$\left. - \mathbb{P}\left\{\bigcap_{i=1}^p\left\{|\boldsymbol{Y}^{(1)}(s_i)|^\kappa \leq |\boldsymbol{Y}^{(2)}(s_i)|^\kappa + u\right\}, \bigcap_{j=1}^{p'}\left\{|\widetilde{\boldsymbol{Y}}^{(1)}(t_j)|^\kappa \leq |\widetilde{\boldsymbol{Y}}^{(2)}(t_j)|^\kappa + u\right\}\right\}\right|. \quad (A.23)$$

Next, note by Cauchy−Schwarz inequality that $u^2 + v^2 \leq (u^2 - 2\rho uv + v^2)/(1 - |\rho|)$ for all $\rho \in (-1, 1)$ and $u, v \in \mathbb{R}$. It follows that, $f_{ij}(\cdot, \cdot)$, the joint density function of $\left(|\boldsymbol{X}^{(1)}(s_i)|, |\boldsymbol{X}^{(1)}(t_j)|\right)$, satisfies

$$f_{i,j}(x, y) = \int_{|\boldsymbol{x}|=x, |\boldsymbol{y}|=y}\prod_{l=1}^m \frac{1}{2\pi\sqrt{1 - r_l^2(t_j - s_i)}}\exp\left(-\frac{x_l^2 - 2r_l(t_j - s_i)x_ly_l + y_l^2}{2(1 - r_l^2(t_j - s_i))}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

$$\leq \frac{1}{(2\pi)^m(1 - (\widetilde{r}(t_j - s_i))^2)^{m/2}}\int_{|\boldsymbol{x}|=x, |\boldsymbol{y}|=y}\prod_{l=1}^m\exp\left(-\frac{x_l^2 + y_l^2}{2(1 + |r_l(t_j - s_i)|)}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

$$\leq \frac{1}{(2\pi)^m(1 - (\widetilde{r}(t_j - s_i))^2)^{m/2}}\exp\left(-\frac{x^2 + y^2}{2(1 + \widetilde{r}(t_j - s_i))}\right)\int_{|\boldsymbol{x}|=x, |\boldsymbol{y}|=y}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

$$= \frac{(xy)^{m-1}}{2^{m-2}(\Gamma(m/2))^2(1 - (\widetilde{r}(t_j - s_i))^2)^{m/2}}\exp\left(-\frac{x^2 + y^2}{2(1 + \widetilde{r}(t_j - s_i))}\right), \quad x, y > 0.$$

 Therefore, in view of Lemma 2 in [1], with K a constant whose value might change from line to line, the first absolute value in (A.23) is bounded from above by

$$K \sum_{i=1}^{p} \sum_{j=1}^{p'} \int_{x^\kappa > u} \int_{y^\kappa > u} \widetilde{r}(t_j - s_i) \left((x^\kappa - u)(y^\kappa - u) \right)^{(k-1)/\kappa} \exp\left(-\frac{(x^\kappa - u)^{2/\kappa} + (y^\kappa - u)^{2/\kappa}}{2(1 + \widetilde{r}(t_j - s_i))} \right) f_{ij}(x, y) \, \mathrm{d}x\mathrm{d}y$$

$$\leq K \sum_{i=1}^{p} \sum_{j=1}^{p'} \widetilde{r}(t_j - s_i) \left(\int_u^\infty (x - u)^{(k-1)/\kappa} x^{m/\kappa - 1} \exp\left(-\frac{x^{2/\kappa}}{2(1 + \widetilde{r}(t_j - s_i))} \right) \mathrm{d}x \right)^2$$

$$\leq K u^{(2/\kappa)(m - (k-1)(2/\kappa - 1) - 2)} \sum_{i=1}^{p} \sum_{j=1}^{p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)} \right),$$

where in the first inequality, we use first the bound $\mathrm{e}^{-x} \leq 1, x \geq 0$ and then a change of variable $x' = x^\kappa$, while the second inequality follows by a change of variable $x' = u^{2/\kappa - 1}(x - u)$ and Taylor's expansion of $(u + x'/u^{2/\kappa - 1})^{2/\kappa} = u^{2/\kappa} + (2/\kappa)x' + O(u^{-2/\kappa})$ for large u and $x' \geq 0$. Similarly, denoting by $g(\cdot)$ the pdf of $|\boldsymbol{X}^{(2)}(0)|$, we obtain that the second absolute value in (A.23) is bounded from above by

$$K \sum_{i=1}^{p} \sum_{j=1}^{p'} \widetilde{r}(t_j - s_i) \int_0^\infty \int_0^\infty \left((x^\kappa + u)(y^\kappa + u) \right)^{(m-1)/\kappa} \exp\left(-\frac{(x^\kappa + u)^{2/\kappa} + (y^\kappa + u)^{2/\kappa}}{2(1 + \widetilde{r}(t_j - s_i))} \right) g(x)g(y) \, \mathrm{d}x\mathrm{d}y$$

$$\leq K \sum_{i=1}^{p} \sum_{j=1}^{p'} \widetilde{r}(t_j - s_i) \left(\int_0^\infty (x^\kappa + u)^{(m-1)/\kappa} x^{k-1} \exp\left(-\frac{(x^\kappa + u)^{2/\kappa}}{2(1 + \widetilde{r}(t_j - s_i))} \right) \mathrm{d}x \right)^2$$

$$\leq K u^{(2/\kappa)(m - k(2/\kappa - 1) - 1)} \sum_{i=1}^{p} \sum_{j=1}^{p'} \widetilde{r}(t_j - s_i) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(t_j - s_i)} \right),$$

where the last step follows by a change of variable $x' = u^{2/\kappa - 1} x^\kappa$. Hence the proof of (3.9) is established since

$$\left(m - k(2/\kappa - 1) - 1 \right) - \left(m - (k-1)(2/\kappa - 1) - 2 \right) = -2(1/\kappa - 1).$$

The desired result in Lemma 3.5 follows. $\square$

Proof of Lemma 3.6. The proof follows by the same arguments as for Lemma 12.3.1 in [15], using alternatively the following asymptotic relation (recall (3.11) and Lem. 3.1)

$$u^{2/\kappa} = 2\ln T_\kappa + K_0 \ln\ln T_\kappa + \ln D_0 (1 + o(1)), \quad T_\kappa \to \infty \tag{A.24}$$

with D_0, K_0 defined in Theorem 2.3. We split the sum in (3.12) at T_κ^β, where β is a constant such that $0 < \beta < (1 - \delta)/(1 + \delta)$ and $\delta = \sup\{\widetilde{r}(t) : t \geq \epsilon\} < 1$ (see, *e.g.*, Lem. 8.1.1 (i) in [15]). Below K is again a positive constant which value might change from line to line. From (A.24) we conclude that $\exp\left(-u^{2/\kappa}/2 \right) \leq K/T_\kappa$ and $u^{2/\kappa} = 2\ln T_\kappa (1 + o(1))$. Further,

$$u^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{\varepsilon \leq aq_\kappa j \leq T_\kappa^\beta} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)} \right) \leq u^{\varsigma + \frac{4\tau}{\alpha\kappa}} T_\kappa^{\beta+1} \exp\left(-\frac{u^{2/\kappa}}{1 + \delta} \right) \leq K (\ln T_\kappa)^{\frac{\kappa\varsigma}{2} + \frac{2\tau}{\alpha}} T_\kappa^{\beta + 1 - \frac{2}{1+\delta}},$$

which tends to 0 as $T_\kappa \to \infty$ since $\beta + 1 - 2/(1 + \delta) < 0$. For the remaining sum, denoting $\delta(t) = \sup\{|\widetilde{r}(s)\ln s| : s \geq t\}$, $t > 0$, we have $\widetilde{r}(t) \leq \delta(t)/\ln t$ as $t \to \infty$, and thus in view of (A.24) for $aq_\kappa j \geq T_\kappa^\beta$

$$\exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)} \right) \leq \exp\left(-u^{2/\kappa} \left(1 - \frac{\delta(T_\kappa^\beta)}{\ln T_\kappa^\beta} \right) \right) \leq K \exp(-u^{2/\kappa}) \leq K T_\kappa^{-2} (\ln T_\kappa)^{-K_0}. \tag{A.25}$$

Consequently, with c given by Theorem 2.3 (recall $\tau = 2\max(1/\kappa - 1, 0) + 1$),

$$u^\varsigma \frac{T_\kappa}{q_\kappa} \sum_{T_\kappa^\beta \leq aq_\kappa j \leq T_\kappa} \widetilde{r}(aq_\kappa j) \exp\left(-\frac{u^{2/\kappa}}{1 + \widetilde{r}(aq_\kappa j)}\right)$$

$$\leq K u^\varsigma \left(\frac{T_\kappa}{q_\kappa}\right)^2 T_\kappa^{-2} (\ln T_\kappa)^{-K_0} \frac{1}{(\ln T_\kappa^\beta)^c} \frac{1}{T_\kappa/q_\kappa} \sum_{T_\kappa^\beta \leq aq_\kappa j \leq T_\kappa} \widetilde{r}(aq_\kappa j) \big(\ln(aq_\kappa j)\big)^c$$

$$\leq K (\ln T_\kappa)^{\frac{\kappa\varsigma}{2} + \frac{2\tau}{\alpha} - K_0 - c} \frac{1}{T_\kappa/q_\kappa} \sum_{T_\kappa^\beta \leq aq_\kappa j \leq T_\kappa} \widetilde{r}(aq_\kappa j) \big(\ln(aq_\kappa j)\big)^c. \tag{A.26}$$

Since $K_0 = m - 2 + 2\tau/\alpha + k\min(1 - 2/\kappa, 0)$ and $\varsigma := 2/\kappa(m - k(2/\kappa - 1) - 1 + \max(0, 2(1/\kappa - 1)))$, we have $\kappa\varsigma/2 + 2\tau/\alpha - K_0 - c = 0$ for all $\kappa > 0$. Noting further that the Berman-type condition $\lim_{t\to\infty} \widetilde{r}(t)(\ln t)^c = 0$ holds and $\beta < 1$, the right-hand side of (A.26) tends to 0 as $u \to \infty$. Thus the proof is complete. □

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