# THICK OBSTACLE PROBLEMS WITH DYNAMIC ADHESIVE CONTACT 

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#### Abstract

In this work, we consider dynamic frictionless contact with adhesion between a viscoelastic body of the Kelvin-Voigt type and a stationary rigid obstacle, based on the Signorini's contact conditions. Including the adhesion processes modeled by the bonding field, a new version of energy function is defined. We use the energy function to derive a new form of energy balance which is supported by numerical results. Employing the time-discretization, we establish a numerical formulation and investigate the convergence of numerical trajectories. The fully discrete approximation which satisfies the complementarity conditions is computed by using the nonsmooth Newton's method with the Kanzow-Kleinmichel function. Numerical simulations of a viscoelastic beam clamped at two ends are presented.


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## 1. Introduction

The quasi-static or dynamic adhesive contact between a purely elastic (or viscoelastic) body and a perfectly rigid (or deformable foundation with normal compliance) has been considered in many papers (see, for example, $[6,7,10,11,18,19,25]$ and reference therein). Most papers $[6,10,11,18,19]$ regarding frictionless contact show only the existence of solution and uniqueness, using the variational inequalities and Banach fixed point arguments. In the paper [25], Raous et al. considered quasistatic contact with Coulomb friction and adhesion, based on Frémond's work [15].

In frictionless contact problems, one of the important issues is to investigate conservation of energy or energy balance, while the papers listed above do not deal with that issue. The study regarding conservation of energy or energy balance has been done in [1,5,22,23,26,29-31]; especially, Petrov and Schatzman [23] and Stewart [2931] have obtained results of energy conservation or energy balance via convolution complementarity problems. The most recent results on the energy balance can be found in [5,31], where it has been shown that energy loss can be regarded as only viscosity and external body forces (not included in [5]). However, those papers have considered the energy balance without the adhesive energy.

In this paper, by adding the adhesive energy a new form of energy balance is formulated and evidence of the energy balance is provided by numerical experiments. Unlike the numerical results presented in most of works (for example, the quasi-static problem [6,10] and even the dynamic problem [13]) we perform numerical

[^0]experiments describing the comparison of the magnitude of contact forces, in which we are able to see whether the effect of adhesion has influence on the contact forces or not. Consequently, the adhesion process makes contact forces $N_{\mathrm{C}}$ neither more nor less regular, while viscosity affects the contact forces less regularly as theoretically shown in [23] and numerically shown in [5].

The most similar system has been studied in the paper [6]. It requires a quasi-static second order partial differential equation without viscosity, while we consider dynamic adhesive contact problems with viscosity expressed by the dynamic fourth order partial differential equation. Readers who are interested in adhesive contact may refer to monographs $[27,28]$ which include several different adhesive contact problems.

Basically, the contact problems that we consider are included in a class of thick obstacle problems. The meaning of "thick" is that for time $t \geq 0$ and an open domain $\Omega \subset \mathbf{R}^{d}$ with some $d>0$ the obstacles (or constraints) are applied over a subset $\Omega_{c} \subset \overline{\Omega_{c}} \subset \Omega$, while the meaning of "thin" is that the obstacles are applied over a subset of the boundary $\partial \Omega$. The approach studied in [3] is applicable to some examples of thick obstacle problems without viscosity; particularly, one of the examples is an Euler-Bernoulli beam equation which has been studied in [2] numerically and in [4] by the penalty method, and another example is a vibrating string problem studied in [1] theoretically and numerically. Those works have focused on conservation of energy. One example of thin obstacle problems with viscosity is considered in [5].

In this work, the contact is adhesive. Following Frémond's original approach [14,15], the process of adhesion is assumed to be irreversible. In contrast to this work, reversible processes are assumed in the paper [7] and the papers $[13,19]$ deal with history dependent processes. Those adhesive processes will be taken into consideration in a future work.

This paper is organized as follows. In Section 2 our dynamic adhesive contact problem is described and formulated in detail. Section 3 provides a mathematical background and explains some notations. In Section 4, the results of existence are stated. In Section 5 energy balance is investigated, introducing the total energy function for the adhesive contact. In Section 6, we prove that the numerical trajectories are convergent to solutions of the thick obstacle problem with adhesive contact. In Section 7, we set up the fully discrete numerical formulation of a viscoelastic beam clamped at two ends, employing the time-discretization and the finite element method. We also introduce the nonsmooth Newton's method which solves the complementarity problems at each time step. In Section 8, numerical simulations are shown and discussed. The numerical results are presented that demonstrate the energy balance and compare the magnitude of contact forces for several cases. Finally we present the conclusion in Section 9.

## 2. Problem statement and formulation

In this section we state the thick obstacle problems for a clamped boundary with dynamic adhesive contact. The solution $u=u(t, \mathbf{x})$ for our problem represents the displacement of viscoelastic bodies at $(t, \mathbf{x}) \in[0, T] \times \Omega$, where $T>0$ is given and $\Omega \subset \mathbf{R}^{d}$ is an open bounded domain with Lipschitz boundary $\partial \Omega$. The continuous function $\varphi(\mathbf{x})$ represents a stationary rigid obstacle whose surface is glued. One of the examples is a thin viscoelastic plate which moves vertically and whose boundary is clamped horizontally, i.e., the plate is fixed and flat at its boundary. The simple model is illustrated in Figure 1.

Now we describe the dynamic adhesive contact. According to Frémond's works [14,15], we introduce the bonding field $\beta=\beta(t, \mathbf{x})$, called the adhesion intensity, which measures the active microscopic bonds between the viscoelastic body and the rigid obstacle. $\beta=1$ implies that all bonds are perfectly active and $\beta=0$ that adhesion does not apply to the body, and $0<\beta<1$ that there is a partial bonding. The adhesive restoring forces prevent the body from bouncing away from the rigid obstacle and thus its direction is downward. Since the adhesive forces are proportional to the gap between the viscoelastic body and the rigid obstacle and to $\beta^{2}$, they are represented by $-\kappa(u-\varphi) \beta^{2}$, where the constant $\kappa>0$ is the interface stiffness or the bonding coefficient and $\kappa \beta^{2}$ the spring constant of the bonding field. The more detailed explanation about the adhesion processes can be found in articles (e.g., $[6,18,25]$ ).


Figure 1. Viscoelastic plate with dynamic adhesive contact.

Since the viscoelastic body of the Kelvin-Voigt type is not allowed to penetrate a perfectly rigid obstacle, it is placed above the obstacle;

$$
u(t, \mathbf{x}) \geq \varphi(\mathbf{x}) \quad \text { in }(0, T] \times \Omega
$$

When the viscoelastic body touches the rigid obstacle, there are occurring contact forces; if $u(t, \mathbf{x})=\varphi(\mathbf{x})$, the magnitude of contact forces $N_{\mathrm{C}}(t, \mathbf{x}) \geq 0$ in $[0, T] \times \Omega$ and thus their direction is upward. Otherwise $(u(t, \mathbf{x}) \geq \varphi(\mathbf{x}))$, the contact forces do not take place, i.e., $N_{\mathrm{C}}(t, \mathbf{x})=0$. Those situations lead to the Signorini's contact conditions.

Let $v=\dot{u}$ be the velocity and $\alpha>0$ a viscosity constant and $f$ a body force. Throughout this paper the notation "( $\cdot$ )" is used for the derivative with respect to time $t$. If the frictionless Signorini's contact conditions with adhesion are imposed over the viscoelastic body, the dynamic motion of the body is expressed by:

$$
\begin{equation*}
\dot{v}=-\Delta^{2} u-\alpha \Delta^{2} v+f(t, \mathbf{x})+N_{\mathrm{C}}-\kappa(u-\varphi) \beta^{2} \quad \text { in }(0, T] \times \Omega \tag{2.1}
\end{equation*}
$$

where $N_{\mathrm{C}}(t, \mathbf{x})$ satisfies the complementarity condition

$$
\begin{equation*}
0 \leq u(t, \mathbf{x})-\varphi(\mathbf{x}) \quad \perp \quad N_{\mathrm{C}}(t, \mathbf{x}) \geq 0 \quad \text { in }(0, T] \times \Omega \tag{2.2}
\end{equation*}
$$

We notice that the Signorini's contact conditions may be interpreted as the complementarity conditions (2.2). In general, the complementarity condition $\mathbf{0} \leq \mathbf{a} \perp \mathbf{b} \geq \mathbf{0}$ means that $\mathbf{a}, \mathbf{b} \geq \mathbf{0}$ component-wise and $\mathbf{a}^{T} \cdot \mathbf{b}=0$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors. The vectors and matrices will be denoted by bold characters. In the scalar case, $0 \leq a \perp b \geq 0$ means that both are nonnegative and either $a$ or $b$ is zero.

Let $N=N_{\mathrm{C}}-\kappa(u-\varphi) \beta^{2}$ be the contact force and adhesive force of the body. Then it is easy to see from (2.2) that both forces do not happen simultaneously. Thus if $u-\varphi>0, N=-\kappa(u-\varphi) \beta^{2} \leq 0$, which implies that when the body does not hit the rigid obstacle, only adhesive forces are applied to the body. In addition, we can
see that (2.1) and (2.2) are equivalent to

$$
\begin{aligned}
\dot{v} & =-\Delta^{2} u-\alpha \Delta^{2} v+f(t, \mathbf{x})+N(t, \mathbf{x}) \quad \text { in }(0, T] \times \Omega \\
0 \leq u(t, \mathbf{x})-\varphi(\mathbf{x}) & \perp N+\kappa(u-\varphi) \beta^{2} \geq 0 \quad \text { in }(0, T] \times \Omega
\end{aligned}
$$

Since the adhesive processes are irreversible, we assume that the evolution of adhesion is formulated by the first order ordinary differential equation (see [16,17,25]):

$$
\begin{equation*}
\dot{\beta}=-\frac{\kappa}{a}(u-\varphi)^{2} \beta \tag{2.3}
\end{equation*}
$$

where the constant $a>0$ is the adhesion rate. We notice from (2.3) that once debonding occurs, there is no rebonding because $\dot{\beta} \leq 0$.

Thus we are led to the following PDE system:

$$
\begin{align*}
\dot{v} & =-\Delta^{2} u-\alpha \Delta^{2} v+f(t, \mathbf{x})+N(t, \mathbf{x}) \quad \text { in }(0, T] \times \Omega,  \tag{2.4}\\
0 \leq N_{C}(t, \mathbf{x}) & \perp u(t, \mathbf{x})-\varphi(\mathbf{x}) \geq 0 \quad \text { in }(0, T] \times \Omega,  \tag{2.5}\\
\dot{\beta} & =-\frac{\kappa}{a}(u-\varphi)^{2} \beta \quad \text { in }(0, T] \times \Omega  \tag{2.6}\\
u(t, \mathbf{x}) & =\nabla u \cdot \mathbf{n}=0 \quad \text { on }(0, T] \times \partial \Omega  \tag{2.7}\\
u(0, \mathbf{x}) & =u^{0} \quad \text { in } \Omega,  \tag{2.8}\\
v(0, \mathbf{x}) & =v^{0} \quad \text { in } \Omega,  \tag{2.9}\\
\beta(0, \mathbf{x}) & =\beta^{0} \quad \text { and } \quad 0<\beta^{0} \leq 1 \quad \text { in } \Omega, \tag{2.10}
\end{align*}
$$

where $u^{0}$ is the initial displacement and $v^{0}$ the initial velocity, and $\beta^{0}$ the initial bonding field. The boundary $\partial \Omega$ is fixed and flat and so we have the essential boundary conditions (2.7) in which $\mathbf{n}$ is the outer normal vector.

## 3. Notations and PRELIMINARIES

The spaces that we are mostly dealing with are based on Gelfand triple (see [34], Sect. 17.1):

$$
V \subset H \subset V^{\prime}
$$

where all spaces are separable Hilbert spaces and $\subset$ means compactly and densely embedding. Note that " 1 " is denoted for a dual space. In this work, those spaces shall be Sobolev spaces of functions defined on the domain $\Omega$. Let $X$ be a Banach space. Then the duality pairing between $X^{\prime}$ and $X$ is denoted by $\langle\cdot, \cdot\rangle_{X^{\prime} \times X}$. When a duality pairing is defined on the well-known space, $\langle\cdot, \cdot\rangle$ is used. Similarly, inner product $(\cdot, \cdot)_{H}$ is employed instead of $(\cdot, \cdot)_{H \times H}$. We note that for any $x \in H, y \in V\langle x, y\rangle_{V^{\prime} \times V}=(x, y)_{H \times V}=(x, y)_{H}$ in Gelfand triple.

In our problems, let the pivot space $H=L^{2}(\Omega)$ and $V=H_{0}^{2}(\Omega)$, where $H_{0}^{2}=\left\{u \in H^{2}(\Omega) \mid u=\nabla u \cdot \mathbf{n}=0\right.$ on $\partial \Omega\}$. From the essential boundary conditions, $A u=\Delta^{2} u$ and $B v=\alpha \Delta^{2} v$ are elliptic self-adjoint operators from $V$ to $V^{\prime}$. In addition, for the fixed $\beta \in L^{\infty}(\Omega)$ a self-adjoint operator $\mathcal{B}(\beta, \cdot)$ from $V$ to $H$ is needed to show the existence of solutions. Then the operators $A, B$ and $\mathcal{B}(\beta, \cdot)$ are defined as follows:

$$
\begin{align*}
\langle A u, w\rangle & =\int_{\Omega} \Delta u \Delta w \mathrm{~d} x  \tag{3.1}\\
\langle B v, w\rangle & =\alpha \int_{\Omega} \Delta v \Delta w \mathrm{~d} x  \tag{3.2}\\
(\mathcal{B}(\beta, u), w)_{H} & =\int_{\Omega} \beta^{2} u w \mathrm{~d} x . \tag{3.3}
\end{align*}
$$

We notice that $\langle\mathcal{B}(\beta, u), w\rangle_{V^{\prime} \times V}=(\mathcal{B}(\beta, u), w)_{H \times V}=(\mathcal{B}(\beta, u), w)_{H}$ by the continuous extension in Gelfand triple.

In order to show convergence of our numerical trajectories, we need scales of interpolation spaces $V_{\theta}$ for any $\theta \in \mathbb{R}$. See, for example, Bramble and Zhang [8], Appendix B, or Taylor [32], Section 5.12, for the interpolation theory. Taking $A=B=\Delta^{2}$, those spaces $V_{\theta}$ can be defined as

$$
V_{\theta}=\left\{u \in H \mid\left\langle A^{\theta} u, u\right\rangle<\infty\right\}
$$

which is a Hilbert space with $(u, w)_{V_{\theta}}=\left\langle A^{\theta} u, w\right\rangle$. In our problem, $V_{\theta}$ is taken as $H_{0}^{2 \theta}(\Omega)$ and thus $V_{\theta}=H_{0}^{2 \theta}(\Omega)$ with $0 \leq \theta \leq 1$. In particular, $V=V_{1}, H=V_{0}, V^{\prime}=V_{-1}$, and $\left(V_{\theta}\right)^{\prime}=V_{-\theta}$. We also note that $V_{\theta}$ is compactly embedded in $V_{\theta-\epsilon}$ for $\epsilon>0$.

The space $C^{p}\left(0, T ; V_{\theta}\right)$ consists of Hölder continuous functions from $[0, T]$ to $V_{\theta}$ with exponent $0<p \leq 1$. If $u \in C^{p}\left(0, T ; V_{\theta}\right)$, its norm is defined as

$$
\|u\|_{C^{p}\left(0, T ; V_{\theta}\right)}=\|u\|_{C\left(0, T ; V_{\theta}\right)}+\sup _{s \neq t} \frac{\|u(t)-u(s)\|_{V_{\theta}}}{|t-s|^{p}}
$$

Moreover, the following inequality will be used to prove that the solution $u$ is in the Hölder space $C^{p}\left(0, T ; V_{\theta}\right)$ : for $u \in V$

$$
\begin{equation*}
\|u\|_{V_{\theta}} \leq C_{\theta}\|u\|_{H}^{1-\theta}\|u\|_{V}^{\theta} \tag{3.4}
\end{equation*}
$$

where $0 \leq \theta \leq 1$. This inequality can be found in Bramble and Xu [8], Appendix A, Theorems A. 1 and A.2, and Kuttler [21], Section 22.6, equation (62), and Triebel [33], Theorem 1.3.3(g).

Let $K \subset X$ be a closed convex cone and $K^{\prime} \subset X^{\prime}$ its dual cone, where the dual cone $K^{\prime}$ is defined by

$$
\begin{equation*}
K^{\prime}=\{\mu \mid\langle\mu, y\rangle \geq 0 \text { for all } y \in K\} \tag{3.5}
\end{equation*}
$$

Extending the definition (3.5) to our dynamic problem, the following Lemma 3.1 (see [30], Lem. 4.3, for the proof) is presented which is useful to derive the energy balance formula. The application of Lemma 3.1 will be seen in Section 5.

Lemma 3.1. Suppose that $X$ and $X^{\prime}$ both have the Radon-Nikodym Property (see [9]). (This is true, for example, if $X$ is reflexive). Then suppose that $K$ is a closed convex cone and that $X \supset K \ni y(t) \perp \mu(t) \in$ $K^{\prime} \subset X^{\prime}$ for Lebesgue almost all $t$ with $y \in W^{1, q}(0, T ; X)$ and $\mu \in L^{p}\left(0, T ; X^{\prime}\right)$ with $1 / p+1 / q=1$. Then we have $\langle\mu(t), \dot{y}(t)\rangle=0$ for almost all $t$.

## 4. Existence Results

We want to find a solution $u:[0, T] \rightarrow V$ and $\beta:[0, T] \rightarrow L^{\infty}(\Omega)$ such that

$$
\begin{align*}
\dot{v} & =-A u-B v+f(t)+N(t) \quad \text { in }(0, T] \times \Omega,  \tag{4.1}\\
0 \leq N_{C}(t) & \perp u(t)-\varphi \geq 0 \quad \text { in }(0, T] \times \Omega,  \tag{4.2}\\
\dot{\beta} & =-\frac{\kappa}{a}(u-\varphi)^{2} \beta \quad \text { in }(0, T] \times \Omega,  \tag{4.3}\\
u(0, \mathbf{x}) & =u^{0} \quad \text { in } \Omega,  \tag{4.4}\\
v(0, \mathbf{x}) & =v^{0} \text { in } \Omega,  \tag{4.5}\\
\beta(0, \mathbf{x}) & =\beta^{0} \text { and } 0<\beta^{0} \leq 1 \quad \text { in } \Omega, \tag{4.6}
\end{align*}
$$

where the equation (4.1) needs to be considered in the sense of distributions.
Now we impose an important condition which is necessary to prove the boundedness of the contact forces in a suitable space. Our physical situation requires that contact forces do not occur near the boundary,
as the body is moving up and down. So we assume that for $\varepsilon>0$ there is a subdomain $\Omega_{\delta} \subset \overline{\Omega_{\delta}} \subset \Omega \subset \mathbb{R}^{d}$ with $\delta>0$ such that $u(t, \mathbf{x})-\varphi(\mathbf{x})>\varepsilon$ for all $(t, \mathbf{x}) \in[0, T] \times \Omega \backslash \Omega_{\delta}$, where $\Omega_{\delta}=\{\mathbf{x} \in \Omega \mid \operatorname{dist}(\mathbf{x}, \partial \Omega)>\delta\}$. This implies that there is always a gap between the body and the rigid obstacle near the boundary $\partial \Omega$. Indeed, this condition causes the motivation to define the strong pointedness from the abstract point of view. The definition of the strong pointedness is presented in [3].

In order to prove that the solutions satisfy the complementarity conditions in the weak sense, we require that the contact forces $N_{\mathrm{C}}$ are nonnegative Borel measures on $[0, T] \times \bar{\Omega}$ and $u(t, \mathbf{x})-\varphi(\mathbf{x}) \geq 0$ for almost all $(t, \mathbf{x}) \in[0, T] \times \bar{\Omega}$.

Finally our main result is presented in the following theorem.
Theorem 4.1. Assume that the initial datum $u^{0} \in V, v^{0} \in H, \beta^{0} \in L^{\infty}(\Omega), f \in L^{\infty}(0, T ; H)$, and $\varphi \in C(\bar{\Omega})$. Then there exist the solutions $u$ to (4.1)-(4.6) such that $u$ is in $L^{\infty}\left(0, T ; V_{\theta}\right) \cap C([0, T] \times \bar{\Omega}) \cap C^{1 / 2}(0, T ; V)$, where $d / 4<\theta<1$. Moreover, $v$ is in $L^{\infty}(0, T ; H) \cap L^{2}(0, T ; V)$ and $\beta$ is in $\mathcal{W}$, where $\mathcal{W}=\{\zeta \mid 0 \leq \zeta \leq 1$ and $\left.\zeta \in W^{1, \infty}(0, T ; H)\right\}$.

We notice from Theorem 4.1 that the thick obstacle problems that we consider work for only $d=1,2,3$. The proof of this result will be shown in Section 6.

## 5. Energy balance

Before we prove the existence of solutions, we investigate the energy balance in this section. The total energy function $E$ for the adhesive contact is defined to be

$$
\begin{equation*}
E(t):=E[u(t), v(t), \beta(t)]=\frac{1}{2}\left(\|v\|_{H}^{2}+\langle A u, u\rangle+\kappa\|\beta(u-\varphi)\|_{H}^{2}\right), \tag{5.1}
\end{equation*}
$$

where the first term is called the kinetic energy and the second term the elastic energy and the last term the adhesion energy which is obtained from the free surface energy with the Dupré adhesion energy (see [17]). As we shall see in the next section, the energy function plays an important role in showing the boundedness of numerical trajectories.

To see the energy balance, we assume that solutions are sufficiently smooth and there is no body force, i.e., $f(t, \mathbf{x})=0$. By the extension of $(\cdot, \cdot)_{H}$ on $V^{\prime} \times V$ we compute

$$
\begin{aligned}
\frac{\mathrm{d} E(t)}{\mathrm{d} t} & =(\dot{v}, v)+\langle A u, v\rangle+\kappa \int_{\Omega}\left(\beta \dot{\beta}(u-\varphi)^{2}+\beta^{2}(u-\varphi) v\right) \mathrm{d} x \\
& =\left\langle N_{\mathrm{C}}-\kappa(u-\varphi) \beta^{2}, v\right\rangle-\langle B v, v\rangle+\kappa \int_{\Omega}\left(\beta \dot{\beta}(u-\varphi)^{2}+\beta^{2}(u-\varphi) v\right) \mathrm{d} x \\
& =\left\langle N_{\mathrm{C}}, v\right\rangle-\langle B v, v\rangle+\kappa \int_{\Omega} \beta \dot{\beta}(u-\varphi)^{2} \mathrm{~d} x
\end{aligned}
$$

Then using (2.3), we have

$$
\begin{equation*}
\frac{\mathrm{d} E(t)}{\mathrm{d} t}=\left\langle N_{\mathrm{C}}, v\right\rangle-\langle B v, v\rangle-a\|\dot{\beta}\|_{H}^{2} . \tag{5.2}
\end{equation*}
$$

In fact, if $\left\langle N_{\mathrm{C}}, v\right\rangle$ is interpreted in the ordinary sense, it is not difficult to see that $\left\langle N_{\mathrm{C}}, v\right\rangle=0$. So we are able to derive the energy balance (5.3) directly. However, in general the velocity $v$ cannot be differentiable in the ordinary sense. Therefore we need more sophisticated analysis.

Lemma 5.1. Suppose that for all $t \in[0, T] u(t)-\varphi \in K \subset V$ and $N_{\mathrm{C}}(t) \in K^{\prime} \subset V^{\prime}$ and $N_{\mathrm{C}} \in L^{1}\left(0, T ; V^{\prime}\right)$ and $u \in W^{1, \infty}(0, T ; V)$. Then we have the following form of energy balance

$$
\begin{equation*}
E(T)=E(0)-\int_{0}^{T}\langle B v, v\rangle \mathrm{d} t-a \int_{0}^{T}\|\dot{\beta}\|_{H}^{2} \mathrm{~d} t \tag{5.3}
\end{equation*}
$$

Proof. Since the Radon-Nikodym Property (RNP) holds if $X$ is reflexive (see [30]), we can apply Lemma 3.1 into our dynamic problem. Then we see that $\left\langle N_{\mathrm{C}}, \mathrm{d}(u-\varphi) / \mathrm{d} t\right\rangle_{V^{\prime} \times V}=\left\langle N_{\mathrm{C}}, v\right\rangle_{V^{\prime} \times V}=0$. Therefore it follows from (5.2) that

$$
\begin{equation*}
\frac{\mathrm{d} E(t)}{\mathrm{d} t}=-\langle B v, v\rangle-a\|\dot{\beta}\|_{H}^{2} . \tag{5.4}
\end{equation*}
$$

Thus integrating both sides of the equation (5.4) on $[0, T]$, we are led to the energy balance (5.3).
As we can see (5.3), energy dissipates due to viscosity and the evolution of adhesion under the assumption of the sufficient regularity. The regularity $\dot{\beta} \in L^{\infty}(0, T ; H)$ will be shown in the next Section 6 . Therefore possibility to conserve energy depends on viscosity coefficient $\alpha>0$ and adhesion rate $a>0$ : as $\alpha \downarrow 0$ and $a \downarrow 0$, energy conserves. This argument will be supported by numerical results in Section 8.

## 6. Time-Discretization and its convergence

In this section we set up a numerical formulation of (2.4)-(2.10), using a hybrid of two numerical schemes in time space:

- Elasticity $\left(\Delta^{2} u\right)$ and viscosity $\left(\Delta^{2} v\right)$ - Midpoint rule is used.
- Complementarity condition and bonding field $\beta$ - Implicit Euler method is used.

The starting point is that the time interval $[0, T]$ will be partitioned with the time step size $\left(h_{t}\right)_{l}=t_{l+1}-t_{l}$ :

$$
0=t_{0}<t_{1}<t_{2}<t_{3}<\ldots<t_{l-1}<t_{l}<t_{l+1}<\ldots<t_{n}=T
$$

For simplicity we use a uniform spacing $h_{t}=t_{l+1}-t_{l}$ and $T / h_{t}$ is assumed to be a positive integer $n$. Thus each time step $t_{l}$ becomes $t_{l}=l \cdot h_{t}$ and the end time $T=t_{n}$. Then the numerical solution of displacement $u\left(t_{l}, \mathbf{x}\right)$ is denoted by $u^{l}$ and the numerical solution of velocity $v\left(t_{l}, \mathbf{x}\right)$ by $v^{l}$ and the numerical solution of contact force $N_{\mathrm{C}}\left(t_{l}, \mathbf{x}\right)$ by $N_{\mathrm{C}}^{l}$, and the numerical solution of bonding field $\beta\left(t_{l}, \mathbf{x}\right)$ by $\beta^{l}$. For a given body force $f \in L^{\infty}(0, T ; H)$ the discrete-time formulation is presented below:

$$
\begin{align*}
\frac{v^{l+1}-v^{l}}{h_{t}}= & -\frac{A\left(u^{l+1}+u^{l}\right)}{2}-\frac{B\left(v^{l+1}+v^{l}\right)}{2}+f^{l} \\
& +N_{\mathrm{C}}^{l}-\kappa \mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right),  \tag{6.1}\\
\frac{v^{l+1}+v^{l}}{2}= & \frac{u^{l+1}-u^{l}}{h_{t}},  \tag{6.2}\\
0 \leq u^{l+1}-\varphi \perp & N_{\mathrm{C}}^{l} \geq 0,  \tag{6.3}\\
\frac{\beta^{l+1}-\beta^{l}}{h_{t}}= & -\frac{\kappa}{a}\left(u^{l}-\varphi\right)^{2}\left(\beta^{l+1}\right), \tag{6.4}
\end{align*}
$$

where (6.1) is defined in the distributional sense. We note that the body force $f:[0, T] \rightarrow H$ is assumed to be approximated by a step function: $f(t)=\sum_{l=0}^{n-1} f^{l}$ with a constant vector valued function $f^{l}:\left[t_{l}, t_{l+1}\right) \rightarrow H$ for each $0 \leq l \leq n-1$.

Now we begin constructing the approximate solutions. We denote by $u_{h_{t}}(t, \cdot)$ the numerical trajectory of displacement and by $v_{h_{t}}(t, \cdot)$ the numerical trajectory of velocity and by $\beta_{h_{t}}(t, \cdot)$ the numerical trajectory of the bonding field and $\beta_{h_{t}}(t, \cdot)$ the numerical trajectory of velocity of the bonding field, respectively. In our approximation, $u_{h_{t}}$ is a piecewise linear continuous interpolant with $u_{h_{t}}\left(t_{l}, \cdot\right)=u^{l}$ and $u_{h_{t}}\left(t_{l+1}, \cdot\right)=u^{l+1}$ for $t \in\left[t_{l}, t_{l+1}\right]$ and also $\beta_{h_{t}}$ is a piecewise linear continuous interpolant with $\beta_{h_{t}}\left(t_{l}, \cdot\right)=\beta^{l}$ and $\beta_{h_{t}}\left(t_{l+1}, \cdot\right)=\beta^{l+1}$ for $t \in\left[t_{l}, t_{l+1}\right]$, whereas $v_{h_{t}}(t, \cdot)$ is a constant interpolant of $v_{h_{t}}(t, \cdot)=v^{l+1}$ for $t \in\left(t_{l}, t_{l+1}\right]$ and $\dot{\beta_{h_{t}}}(t, \cdot)$ is a constant interpolant of $\dot{\beta_{h_{t}}}(t, \cdot)=(\dot{\beta})^{l+1}$ for $t \in\left(t_{l}, t_{l+1}\right]$. Then from (6.2) we can obtain $u_{h_{t}}(t, \cdot)=u^{l}+$
$\frac{1}{2} \int_{t_{l}}^{t} v_{h_{t}}\left(\tau-h_{t}, \cdot\right)+v_{h_{t}}(\tau, \cdot) \mathrm{d} \tau$ for $t \in\left(t_{l}, t_{l+1}\right]$. We also set up a step function $\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \cdot)$ as $\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \cdot)=N_{\mathrm{C}}^{l}$ for $t \in\left[t_{l}, t_{l+1}\right)$. Then the approximate contact force $\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \mathbf{x})$ can be defined to be

$$
\begin{equation*}
\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \mathbf{x})=h_{t} \sum_{l=0}^{n-1} \delta\left(t-(l+1) h_{t}\right) N_{\mathrm{C}}^{l}(\mathbf{x}), \tag{6.5}
\end{equation*}
$$

where $\delta$ is the Dirac delta function.
In the semi-discrete case, the energy function $E^{l}$ is defined as

$$
\begin{equation*}
E\left(t_{l}\right):=E^{l}=\frac{1}{2}\left(\left\|v^{l}\right\|_{H}^{2}+\left\langle A u^{l}, u^{l}\right\rangle+\kappa\left\|\beta^{l}\left(u^{l}-\varphi\right)\right\|_{H}^{2}\right) . \tag{6.6}
\end{equation*}
$$

Based on the energy function $E^{l}$, we want to prove that the solutions $\left(u^{l}, v^{l}, \beta^{l}\right)$ are in suitable spaces for each discretized time $t_{l}$, independent of time step $h_{t}>0$. Moreover, we can observe in the next Lemma 6.1 that energy dissipates, i.e., $E^{l} \geq E^{l+1}$. Taking $\kappa=1$ for the sake of convenience, we consider the following lemmas.

Lemma 6.1. If the discretized solutions $\left(u^{l}, v^{l}\right)$ satisfy the numerical formulation (6.1)-(6.4), then $\left(u^{l}, v^{l}\right) \in$ $V \times H$ for any $l \geq 1$, independent of $h_{t}>0$.

Proof. Multiplying both sides of (6.1) by each side of (6.2), it follows from extension of $(\cdot, \cdot)_{H}$ that

$$
\begin{aligned}
\frac{1}{2 h_{t}}\left(\left\|v^{l+1}\right\|_{H}^{2}-\left\|v^{l}\right\|_{H}^{2}\right)= & -\frac{1}{2 h_{t}}\left(\left\langle A u^{l+1}, u^{l+1}\right\rangle-\left\langle A u^{l}, u^{l}\right\rangle\right)-\frac{1}{4}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle \\
& +\frac{1}{h_{t}}\left(f^{l}, u^{l+1}-u^{l}\right)_{H}+\frac{1}{h_{t}}\left\langle N_{\mathrm{C}}^{l}, u^{l+1}-u^{l}\right\rangle \\
& -\frac{1}{h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-u^{l}\right)_{H} \\
= & -\frac{1}{2 h_{t}}\left(\left\langle A u^{l+1}, u^{l+1}\right\rangle-\left\langle A u^{l}, u^{l}\right\rangle\right)-\frac{1}{4}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle \\
& +\frac{1}{h_{t}}\left(f^{l}, u^{l+1}-u^{l}\right)_{H}+\frac{1}{h_{t}}\left\langle N_{\mathrm{C}}^{l}, u^{l+1}-\varphi\right\rangle-\frac{1}{h_{t}}\left\langle N_{\mathrm{C}}^{l}, u^{l}-\varphi\right\rangle \\
& -\frac{1}{h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-u^{l}\right)_{H} .
\end{aligned}
$$

From the complementarity condition (6.3), we obtain

$$
\begin{align*}
\frac{1}{2 h_{t}}\left(\left\|v^{l+1}\right\|_{H}^{2}-\left\|v^{l}\right\|_{H}^{2}\right) \leq-\frac{1}{2 h_{t}}\left(\left\langle A u^{l+1}, u^{l+1}\right\rangle-\left\langle A u^{l}, u^{l}\right\rangle\right)- & \frac{1}{4}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle \\
& -\frac{1}{h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-u^{l}\right)_{H} \tag{6.7}
\end{align*}
$$

We now modify $\left(u^{l+1}-\varphi\right)$ on the last term (6.7):

$$
\begin{equation*}
\left(u^{l+1}-\varphi\right)=\frac{1}{2}\left(u^{l+1}-\varphi+u^{l}-\varphi+u^{l+1}-u^{l}\right) \tag{6.8}
\end{equation*}
$$

Since the visco operator $B$ is elliptic, applying the equation (6.8) to (6.7), we obtain

$$
\begin{align*}
\frac{1}{2 h_{t}} & \left(\left\|v^{l+1}\right\|_{H}^{2}+\left\langle A u^{l+1}, u^{l+1}\right\rangle\right) \leq \frac{1}{2 h_{t}}\left(\left\|v^{l}\right\|_{H}^{2}+\left\langle A u^{l+1}, u^{l}\right\rangle\right)+\frac{1}{h_{t}}\left(f^{l}, u^{l+1}-u^{l}\right)_{H} \\
& -\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi+u^{l}-\varphi+u^{l+1}-u^{l}\right), u^{l+1}-u^{l}\right)_{H} \\
\leq & \frac{1}{2 h_{t}}\left(\left\|v^{l}\right\|_{H}^{2}+\left\langle A u^{l}, u^{l}\right\rangle\right)+\frac{1}{h_{t}}\left(f^{l}, u^{l+1}-u^{l}\right)_{H} \\
& -\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-u^{l}\right)_{H}-\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l}-\varphi\right), u^{l+1}-u^{l}\right)_{H} \tag{6.9}
\end{align*}
$$

Now we change (6.9) as follows:

$$
\begin{aligned}
- & \frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-u^{l}\right)_{H}-\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l}-\varphi\right), u^{l+1}-u^{l}\right)_{H} \\
= & -\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-\varphi\right)_{H}+\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l}-\varphi\right)_{H} \\
& -\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l}-\varphi\right), u^{l+1}-\varphi\right)_{H}+\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l}-\varphi\right), u^{l}-\varphi\right)_{H} \\
= & -\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), u^{l+1}-\varphi\right)_{H}+\frac{1}{2 h_{t}}\left(\mathcal{B}\left(\beta^{l+1}-\beta^{l}+\beta^{l}, u^{l}-\varphi\right), u^{l}-\varphi\right)_{H} .
\end{aligned}
$$

Thus using (3.3), (6.4), (6.7), and (6.9), we have

$$
E^{l+1}+\frac{h_{t}}{4}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle \leq E^{l}+\left(f^{l}, u^{l+1}-u^{l}\right)_{H} .
$$

Using the equation (6.2), by the telescoping sum from $i=0$ to $i=l-1$ we can see that

$$
\begin{equation*}
E^{l}+\frac{h_{t}}{4} \sum_{i=0}^{l-1}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle \leq E^{0}+\frac{h_{t}}{2} \sum_{i=0}^{l-1}\left\|f^{i}\right\|_{H}\left(\left\|v^{i+1}\right\|_{H}+\left\|v^{i}\right\|_{H}\right) . \tag{6.10}
\end{equation*}
$$

Now applying the trapezoidal rule and the Cauchy inequality, from (6.10) we obtain

$$
\begin{align*}
E^{l}+\frac{h_{t}}{4} \sum_{i=0}^{l-1}\left\langle B\left(v^{l+1}+v^{l}\right), v^{l+1}+v^{l}\right\rangle & \leq E^{0}+\|f\|_{L^{\infty}(0, T ; H)} \sum_{i=0}^{l-1} \int_{\left(t_{i}, t_{i+1}\right]}\left\|v_{h_{t}}(\tau)\right\|_{H} \mathrm{~d} \tau \\
& =E^{0}+\|f\|_{L^{\infty}(0, T ; H)} \int_{0}^{t_{l}}\left\|v_{h_{t}}(\tau)\right\|_{H} \mathrm{~d} \tau \\
& \leq E^{0}+\frac{1}{2}\left(T\|f\|_{L^{\infty}(0, T ; H)}^{2}+\int_{0}^{t_{l}}\left\|v_{h_{t}}(\tau)\right\|_{H}^{2} \mathrm{~d} \tau\right) \tag{6.11}
\end{align*}
$$

Thus employing Gronwall inequality and recalling the construction of $v_{h_{t}}$, from (6.6) and (6.11) we obtain

$$
\begin{equation*}
\left\|v^{l}\right\|_{H}^{2}=\left\|v_{h_{t}}\left(t_{l}\right)\right\|_{H}^{2} \leq \int_{0}^{t_{l}}\left\|v_{h_{t}}(\tau)\right\|_{H}^{2} \mathrm{~d} \tau+2 E^{0}+T\|f\|_{L^{\infty}(0, T ; H)}^{2} \leq C_{1}\left(1+T e^{T}\right) \tag{6.12}
\end{equation*}
$$

where $C_{1}=2 E^{0}+T\|f\|_{L^{\infty}(0, T ; H)}^{2}$. So $v^{l}$ is bounded in $H$ for any $l \geq 1$. To see that $u^{l}$ is bounded in $V$,
we use (6.10) to obtain

$$
\left\langle A u^{l}, u^{l}\right\rangle \leq 2 E^{0}+T\|f\|_{L^{\infty}(0, T ; H)} \max _{0 \leq l \leq n}\left\|v^{l}\right\|_{H} \leq 2 E^{0}+C_{2} \sqrt{C_{1}\left(1+T e^{T}\right)}
$$

where $C_{2}=T\|f\|_{L^{\infty}(0, T ; H)}$. Thus the proof is complete.
From the previous Lemma 6.1 we can see easily that $u_{h_{t}} \in C(0, T ; V)$ and $v_{h_{t}} \in L^{\infty}(0, T ; H)$. We note that throughout this paper, $C$ may be different in each occurrence and does not depend on $h_{t}$.

Lemma 6.2. The numerical solutions $v_{h_{t}}$ are uniformly bounded in $L^{2}(0, T ; V)$ and $u_{h_{t}}$ are uniformly Hölder continuous from $[0, T] \rightarrow V$ with exponent $p=1 / 2$, independent of sufficiently small $h_{t}>0$.

The previous Lemma 6.2 has been proved in [5]. Applying Alaoglu's theorem, it can be shown from Lemmas 6.1 and 6.2 that there is a subsequence denoted by $v_{h_{t}}$ such that $v_{h_{t}} \rightharpoonup^{*} v$ in $L^{\infty}(0, T ; H) \cap L^{2}(0, T ; V)$ as $h_{t} \downarrow 0$.

Now we start showing the boundedness of contact force in the measure sense. Recalling (6.5), we can see easily that

$$
\int_{0}^{T} \int_{\Omega}\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \mathbf{x}) \mathrm{d} x \mathrm{~d} t=h_{t} \sum_{l=0}^{n-1} \int_{\Omega} N_{\mathrm{C}}^{l}(\mathbf{x}) \mathrm{d} x .
$$

We also identify the numerical trajectory of the contact force $\left(N_{\mathrm{C}}\right)_{h_{t}}$ as the Borel measure on $[0, T] \times \bar{\Omega}$ :

$$
\left(N_{C}\right)_{h_{t}}(B)=\int_{B}\left(N_{C}\right)_{h_{t}}(t, \mathbf{x}) \mathrm{d} x \mathrm{~d} t
$$

where $B \subseteq[0, T] \times \bar{\Omega}$ is a Borel set.
Lemma 6.3. There is a constant $C>0$ independent of $h_{t}$ such that

$$
\int_{0}^{T} \int_{\Omega}\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \mathbf{x}) \mathrm{d} x \mathrm{~d} t \leq C
$$

Proof. We claim that

$$
\int_{0}^{T} \int_{\Omega} N_{\mathrm{C}}(t) \mathrm{d} x \mathrm{~d} t=h_{t} \sum_{l=0}^{n-1} \int_{\Omega} N_{\mathrm{C}}^{l} \mathrm{~d} x \leq C
$$

Recalling (6.1), for any $h_{t}>0$ we have

$$
\begin{align*}
h_{t} N_{\mathrm{C}}^{l}= & \left(v^{l+1}-v^{l}\right)+\frac{h_{t}}{2}\left(A u^{l+1}+A u^{l}\right)+\frac{h_{t}}{2}\left(B v^{l+1}+B v^{l}\right)-h_{t} f^{l} \\
& +h_{t} \mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right) . \tag{6.13}
\end{align*}
$$

We construct a function $w \in V$ satisfying the essential boundary conditions as follows:

$$
w(\mathbf{x})=\left\{\begin{array}{c}
1 \quad \text { in } \Omega_{\delta} \\
-\frac{2}{\delta^{3}}|\mathbf{x}-\mathbf{y}|^{3}+\frac{3}{\delta^{2}}|\mathbf{x}-\mathbf{y}|^{2} \text { in } \Omega \backslash \Omega_{\delta} \text { for } \mathbf{y} \in \partial \Omega
\end{array}\right.
$$

where $\mathbf{y}$ is a projection of $\mathbf{x}$ onto $\partial \Omega$. Notice that $w \in C^{1}(\bar{\Omega})$. For all $w \in V$ we define $\left\langle N_{\mathrm{C}}^{l}, w\right\rangle$ at each time step $t_{l}$ as

$$
\left\langle N_{\mathrm{C}}^{l}, w\right\rangle=\int_{\Omega} N_{\mathrm{C}}^{l} w \mathrm{~d} x
$$

Since $N_{\mathrm{C}}^{l}=0$ on $\Omega \backslash \Omega_{\delta}$ for any $l \geq 0$, it follows that

$$
\int_{\Omega} N_{\mathrm{C}}^{l} w \mathrm{~d} x=\int_{\Omega_{\delta}} N_{\mathrm{C}}^{l} w \mathrm{~d} x+\int_{\Omega_{\backslash \Omega_{\delta}}} N_{\mathrm{C}}^{l} w \mathrm{~d} x=\int_{\Omega_{\delta}} N_{\mathrm{C}}^{l} \mathrm{~d} x .
$$

Thus taking any $w \in V$ and using (6.2), from (6.13) we have

$$
\begin{aligned}
& h_{t} \sum_{l=0}^{n-1} \int_{\Omega} N_{\mathrm{C}}^{l} \mathrm{~d} x=h_{t} \sum_{l=0}^{n-1} \int_{\Omega} N_{\mathrm{C}}^{l} w \mathrm{~d} x=\sum_{l=0}^{n-1}\left(v^{l+1}-v^{l}, w\right)+\frac{h_{t}}{2} \sum_{l=0}^{n-1}\left\langle A\left(u^{l+1}+u^{l}\right), w\right\rangle+h_{t} \sum_{l=0}^{n-1}\left(f^{l}, w\right) \\
& \quad+\frac{h_{t}}{2} \sum_{l=0}^{n-1}\left\langle B\left(v^{l+1}+v^{l}\right), w\right\rangle+\sum_{l=0}^{n-1}\left(\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right), w\right) \\
& \leq\left(\left\|v^{l}\right\|_{H}+\left\|v^{0}\right\|_{H}\right)\|w\|_{V}+C T \max _{1 \leq l \leq n}\left\|u^{l}\right\|_{V}\|w\|_{V}+T\|f\|_{L^{\infty}(0, T ; H)}\|w\|_{V} \\
& \quad+T\left(\left\|u^{l}\right\|_{H}+\left\|u^{0}\right\|_{H}\right)\|w\|_{V}+T \max _{1 \leq l \leq n}\left\|\mathcal{B}\left(\beta^{l+1}, u^{l+1}-\varphi\right)\right\|_{H}\|w\|_{V} .
\end{aligned}
$$

Therefore it follows from Lemma 6.1 that

$$
\int_{0}^{T} \int_{\Omega}\left(N_{\mathrm{C}}\right)_{h_{t}}(t, \mathbf{x}) \mathrm{d} x \mathrm{~d} t=h_{t} \sum_{l=0}^{n-1} \int_{\Omega} N_{\mathrm{C}}^{l} \mathrm{~d} x \leq C,
$$

where $C$ does not depend on $h_{t}>0$. The proof is complete.
Applying the Alouglu theorem and Riesz representation theorem into Lemma 6.3, there is a subsequence, denoted by $\left(N_{\mathrm{C}}\right)_{h_{t}}$ such that $\left(N_{\mathrm{C}}\right)_{h_{t}} \rightharpoonup^{*} N_{\mathrm{C}}$ in the sense of measures as $h_{t} \downarrow 0$.

Using the inequality (3.4) and Lemma 6.1, we can easily show that the numerical trajectories $u_{h_{t}} \in C^{p}\left(0, T ; V_{\theta}\right)$, where $p=1-\theta$ with $0 \leq \theta<1$. Thus by the Sobolev embedding theorem and the Arzela-Ascoli theorem $C^{p}\left(0, T ; V_{\theta}\right)$ is compactly embedded in $C(0, T ; C(\bar{\Omega}))=C([0, T] \times(\bar{\Omega}))$, where $0<p \leq 1$ and $d / 4<\theta<1$. Therefore there is a subsequence, denoted by $u_{h_{t}}$ such that $u_{h_{t}} \rightarrow u$ in $C(0, T ; C(\bar{\Omega}))$ as $h_{t} \downarrow 0$. Let $u_{h_{t}}$ be the suitable subsequence which is corresponding to $\left(N_{\mathrm{C}}\right)_{h_{t}}$.

Now we claim that the solutions $\left(u, N_{\mathrm{C}}\right)$ converging by such subsequences $\left(u_{h_{t}},\left(N_{\mathrm{C}}\right)_{h_{t}}\right)$ satisfy the complementarity condition (2.2) in the weak sense. Using (6.5), from the complementarity conditions (6.3) we obtain

$$
\begin{aligned}
\int_{0}^{T} \int_{\Omega}\left(N_{\mathrm{C}}\right)_{h_{t}}\left(u_{h_{t}}-\varphi\right) \mathrm{d} x \mathrm{~d} t & =\int_{\Omega} \int_{0}^{T} h_{t} \sum_{l=0}^{n-1} \delta\left(t-(l+1) h_{t}\right) N_{\mathrm{C}}^{l}\left(u_{h_{t}}-\varphi\right) \mathrm{d} t \mathrm{~d} x \\
& =\int_{\Omega} h_{t} \sum_{l=0}^{n-1} N_{\mathrm{C}}^{l}\left(u^{l+1}-\varphi\right) \mathrm{d} x=0
\end{aligned}
$$

Therefore we can see that

$$
0=\int_{0}^{T} \int_{\Omega}\left(N_{\mathrm{C}}\right)_{h_{t}}\left(u_{h_{t}}-\varphi\right) \mathrm{d} x \mathrm{~d} t \rightarrow \int_{0}^{T} \int_{\Omega} N_{\mathrm{C}}(u-\varphi) \mathrm{d} x \mathrm{~d} t \quad \text { as } h_{t} \downarrow 0
$$

which implies that the solutions converging by our numerical trajectories hold the complementarity conditions in the weak sense.

Let $\Omega_{c}=\left\{\mathbf{x} \in \Omega_{\delta} \mid u(\mathbf{x})-\varphi(\mathbf{x})=0\right\}$, which is a subset of $\Omega_{\delta}$. Then putting $\kappa=a=1$ for the sake of simplicity, we consider the following Lemma 6.4.

Lemma 6.4. Suppose that $\beta^{l} \in L^{2}\left(\Omega_{c}\right)$ for some $l>0$. Then $\beta_{h_{t}}$ is uniformly bounded in $\mathcal{W}$, independent of $h_{t}>0$.
Proof. First we claim that for any $l \geq 0 \beta^{l}$ is uniformly bounded in $H$. For sufficiently small $\varepsilon>0$

$$
\begin{align*}
\varepsilon^{2}\left\|\beta^{l}\right\|_{H}^{2} & =\varepsilon^{2}\left(\int_{\Omega \backslash \Omega_{\delta}}\left(\beta^{l}\right)^{2} \mathrm{~d} x+\int_{\Omega_{\delta}}\left(\beta^{l}\right)^{2} \mathrm{~d} x\right) \\
& \leq \int_{\Omega \backslash \Omega_{\delta}}\left(u^{l}-\varphi\right)^{2}\left(\beta^{l}\right)^{2} \mathrm{~d} x+\int_{\Omega_{\delta}} \varepsilon^{2}\left(\beta^{l}\right)^{2} \mathrm{~d} x \tag{6.14}
\end{align*}
$$

Now we consider two cases: the first case is $u^{l}(\mathbf{x})-\varphi(\mathbf{x})>0$ on $\Omega_{\delta}$ for some $l \geq 0$ and the second case is that for $l \neq m_{c}>0$ there are subsets $\Omega_{c} \subset \Omega_{\delta}$ such that $\Omega_{c}=\left\{\mathbf{x} \mid u^{m_{c}}(\mathbf{x})-\varphi(\mathbf{x})=0\right\}$. For the first case, we can choose an $\eta_{1}>\varepsilon>0$ such that $u^{l}(\mathbf{x})-\varphi(\mathbf{x})>\eta_{1}$ on $\Omega_{\delta}$. Then recalling the energy function (6.6) and applying (6.11) and (6.12), from (6.14) we have

$$
\begin{aligned}
\varepsilon^{2}\left\|\beta^{l}\right\|_{H}^{2} & \leq \int_{\Omega \backslash \Omega_{\delta}}\left(u^{l}-\varphi\right)^{2}\left(\beta^{l}\right)^{2} \mathrm{~d} x+\int_{\Omega_{\delta}}\left(u^{l}-\varphi\right)^{2}\left(\beta^{l}\right)^{2} \mathrm{~d} x \\
& =\left\|\left(u^{l}-\varphi\right) \beta^{l}\right\|_{H}^{2} \leq 2 E^{0}+C_{2} \sqrt{C_{1}\left(1+T e^{T}\right)}
\end{aligned}
$$

For the second case, we similarly choose an $\eta_{2}>\varepsilon>0$ such that $u^{l}(\mathbf{x})-\varphi(\mathbf{x})>\eta_{2}$ on $\Omega_{\delta} \backslash \Omega_{c}$. Thus we obtain

$$
\begin{aligned}
\varepsilon^{2}\left\|\beta^{m_{c}}\right\|_{H}^{2} & =\varepsilon^{2}\left(\int_{\Omega \backslash \Omega_{\delta}}\left(\beta^{m_{c}}\right)^{2} \mathrm{~d} x+\int_{\Omega_{c}}\left(\beta^{m_{c}}\right)^{2} \mathrm{~d} x+\int_{\Omega_{\delta} \backslash \Omega_{c}}\left(\beta^{m_{c}}\right)^{2} \mathrm{~d} x\right) \\
& \leq \int_{\Omega_{\backslash \Omega_{\delta}}}\left(u^{m_{c}}-\varphi\right)^{2}\left(\beta^{m_{c}}\right)^{2} \mathrm{~d} x+\int_{\Omega_{\delta} \backslash \Omega_{c}}\left(u^{m_{c}}-\varphi\right)^{2}\left(\beta^{m_{c}}\right)^{2} \mathrm{~d} x+C \\
& \leq\left\|\left(u^{m_{c}}-\varphi\right) \beta^{m_{c}}\right\|_{H}^{2}+C \leq 2 E^{0}+C_{2} \sqrt{C_{1}\left(1+T e^{T}\right)}+C
\end{aligned}
$$

Thus from both cases $\beta^{l}$ is bounded in $H$ for any $l \geq 0$. Since $\beta_{h_{t}}$ is a continuous linear interpolant on time, $\beta_{h_{t}}$ is bounded in $L^{\infty}(0, T ; H)$. Furthermore it follows from (6.4) that for $0 \leq l \leq n-1$

$$
\left\|(\dot{\beta})^{l+1}\right\|_{H} \leq \max _{0 \leq l \leq n}\left\|\left(u^{l}-\varphi\right)^{2}\right\|_{L^{\infty}(\Omega)} \max _{0 \leq l \leq n-1}\left\|\beta^{l+1}\right\|_{H} \leq C
$$

which implies that $\dot{\beta_{h_{t}}}$ is bounded in $L^{\infty}(0, T ; H)$.
To see that $0 \leq \beta^{l} \leq 1$ for all $l \geq 0$, employing (6.4) we compute

$$
\begin{equation*}
\beta^{l+1}=\frac{\beta^{l}}{1+\left(u^{l}-\varphi\right)^{2} h_{t}} \tag{6.15}
\end{equation*}
$$

If $0 \leq \beta^{l} \leq 1$ for any $l \geq 0$, it is obvious from (6.15) that $0 \leq \beta^{l+1} \leq 1$, independent of $h_{t}>0$. Inductively, it is easy to show that the interpolant $\beta_{h_{t}}$ satisfies $0 \leq \beta_{h_{t}} \leq 1$ on $[0, T] \times \Omega$. The proof is complete.

Therefore we can conclude from the previous Lemma 6.4 that $\beta_{h_{t}} \rightharpoonup^{*} \beta$ in $\mathcal{H}$ and $\dot{\beta_{h_{t}}} \rightharpoonup^{*} \dot{\beta}$ in $L^{\infty}(0, T ; H)$ as $h_{t} \downarrow 0$, where $\mathcal{H}=\left\{\varsigma \mid 0 \leq \varsigma \leq 1\right.$ and $\left.\varsigma \in L^{\infty}(0, T ; H)\right\}$.
Remark 6.5. We recall the ordinary differential equation (2.3). The uniqueness of the bonding field $\beta$ may be shown, using the Banach fixed point theorem. However, the hard part is to prove the uniqueness of the solution $u$ and $N_{\mathrm{C}}$, which is still an open question.

## 7. Fully discrete scheme

### 7.1. The time discretization and the Galerkin approximation

A dynamic adhesive contact problem that we implement is a clamped viscoelastic beam equation with a rigid obstacle. So we consider the fully discrete scheme about the one-dimensional problem, using the time discretization in time space $[0, T]$ and the finite element method with B-splines in space $[0, L]$. Here $L$ is a length of a beam. Assuming that there is no body force, i.e., $f=0$, we set up the following one-dimensional problem equivalent to (2.4)-(2.5):

$$
\begin{align*}
\dot{v} & =-u^{(4)}-\alpha v^{(4)}+N_{\mathrm{C}}(t, x)-\kappa(u-\varphi) \beta^{2} \text { in }(0, T] \times(0, L),  \tag{7.1}\\
0 \leq N_{\mathrm{C}}(t, x) & \perp u(t, x)-\varphi(x) \geq 0 \text { in }(0, T] \times(0, L) .
\end{align*}
$$

Since the clamped beam is fixed and flat on its two ends $x=0$ and $x=L$, we have the essential boundary conditions for the clamped beam:

$$
u(t, x)=u_{x}(t, x)=0 \quad \text { on }[0, T] \times(0, L) .
$$

The notation $u^{(4)}$ (or $v^{(4)}$ ) implies the fourth derivative of $u$ (or $v$ ) with respect to $x$ and a subscript denotes derivative with respect to a subscripted variable. In order to approximate the spatial variables we introduce two finite dimensional space $V_{h_{x}} \subset V$ and $H_{h_{x}} \subset H$, where $V$ is

$$
V=H_{0}^{2}(0, L)=\left\{w \in H_{0}^{2}(0, L) \mid w(0)=w_{x}(0)=w(L)=w_{x}(L)=0\right\}
$$

and $H$ is $L^{2}(0, L)$. Now we partition the space $[0, L]$ uniformly:

$$
0=x_{0}<x_{1}<x_{2}<\ldots<x_{m}=L
$$

where the points $x_{i}$ with $0 \leq i \leq m$ are nodes. Then the mesh size denoted by $h_{x}$ becomes $h_{x}=x_{i+1}-x_{i}$. We note that for a piecewisely smooth functions $w_{h_{x}} \in H^{2}(0, L)$ if and only if $w_{h_{x}} \in C^{1}[0, L]$. Thus for an approximation of solutions $(u, v, N)$, we choose a finite dimensional space

$$
\begin{aligned}
V_{h_{x}}= & \left\{w_{h_{x}} \in C^{1}[0, L]\left|w_{h_{x}}\right|_{\left[x_{i}, x_{i+4}\right]} \in \mathcal{P}_{3}\left[x_{i}, x_{i+4}\right],\right. \\
& \left.w_{h_{x}}(0)=w_{h_{x}}^{\prime}(0)=w_{h_{x}}(L)=w_{h_{x}}^{\prime}(L)=0\right\},
\end{aligned}
$$

where $\mathcal{P}_{3}$ is the set of piecewise cubic functions and $\left({ }^{\prime}\right)$ denotes the derivative with respect to $x$. Then our basis functions $\psi_{i} \in V_{h_{x}}$ will be cubic B-splines which need to satisfy the essential boundary conditions. According to the cubic B-splines property, we are able to construct the standard B-spline function $S(z)$ on the reference interval $[-2,2]$ :

$$
S(z)=\frac{2}{3}\left\{\begin{array}{cl}
1+\frac{3}{4}|z|^{3}-\frac{3}{2}|z|^{2} & \text { if }|z| \leq 1 \\
\frac{1}{4}(2-|z|)^{3} & \text { if } 1 \leq|z| \leq 2 \\
0 & \text { if }|z| \geq 2
\end{array}\right.
$$

For $2 \leq i \leq m-2$ most of basis function $\psi_{i}$ is defined by the shifted B-splines:

$$
\psi_{i}(x)=S\left(\frac{x-x_{i}}{h_{x}}\right)=S\left(\frac{x}{h_{x}}-i\right)
$$

where $x_{i}=i \cdot h_{x}$. Since the first and last basis functions $\psi_{1}, \psi_{m-1}$ have to satisfy the essential boundary conditions, those basis functions is modified as follows:

$$
\psi_{1}(x)=\psi_{m-1}(x)=\frac{17}{28}\left(2 \cdot\left(S\left(\frac{x}{h_{x}}+1\right)+S\left(\frac{x}{h_{x}}-1\right)\right)-S\left(\frac{x}{h_{x}}\right)\right) .
$$

Now we write the fully approximate solutions $\left(u_{h_{x}}^{l}, v_{h_{x}}^{l}, N_{h_{x}}^{l}\right)$ at each time $t_{l}$ as

$$
\begin{equation*}
u_{h_{x}}^{l}(x)=\sum_{j=1}^{m-1} u_{j}^{l} \psi_{j}(x), \quad v_{h_{x}}^{l}(x)=\sum_{j=1}^{m-1} v_{j}^{l} \psi_{j}(x), \quad N_{h_{x}}^{l}(x)=\sum_{j=1}^{m-1} N_{j}^{l} \psi_{j}(x) \tag{7.2}
\end{equation*}
$$

Note that $N_{h_{x}}^{l}$ is the fully approximate solution of the normal contact force $N_{\mathrm{C}}$. We also let the rigid obstacle $\varphi$ be $\varphi_{h_{x}}=\sum_{j=1}^{m-1} \varphi_{j} \psi_{j}(x)$.

On the other hand, we use another finite dimensional space $H_{h_{x}}$ which is different from $V_{h_{x}}$ to approximate bonding field $\beta$. Let

$$
H_{h_{x}}=\left\{\varpi_{h_{x}} \in L^{\infty}[0, L]\left|\varpi_{h_{x}}\right|_{\left[x_{i}, x_{i+1}\right)} \in \mathcal{P}_{0}\left[x_{i}, x_{i+1}\right)\right\}
$$

where $\mathcal{P}_{0}$ is the set of piecewise constant functions. Then we write the fully approximate solution $\beta_{h_{x}}^{l}$ at each time $t^{l}$ as

$$
\begin{equation*}
\beta_{h_{x}}^{l}(x)=\sum_{j=1}^{m-1} \beta_{j}^{l} \phi_{j}(x), \tag{7.3}
\end{equation*}
$$

where the basis function $\phi_{j} \in H_{h_{x}}$ is a piecewise constant basis function unlike we use piecewise cubic basis functions for other approximate solutions $u_{h_{x}}, v_{h_{x}}, N_{h_{x}}$.

Recalling (6.1)-(6.2), we set up the following fully numerical formulation:

$$
\begin{align*}
\frac{v_{h_{x}}^{l+1}-v_{h_{x}}^{l}}{h_{t}}= & -\frac{\left(u_{h_{x}}^{l+1}\right)^{(4)}+\left(u_{h_{x}}^{l}\right)^{(4)}}{2}-\alpha \frac{\left(v_{h_{x}}^{l+1}\right)^{(4)}+\left(v_{h_{x}}^{l}\right)^{(4)}}{2} \\
& +N_{h_{x}}^{l}-\kappa\left(u_{h_{x}}^{l+1}-\varphi_{h_{x}}\right)\left(\beta_{h_{x}}^{l+1}\right)^{2}  \tag{7.4}\\
0 \leq N_{h_{x}}^{l} \perp & u_{h_{x}}^{l+1}-\varphi_{h_{x}} \geq 0  \tag{7.5}\\
\frac{v_{h_{x}}^{l+1}+v_{h_{x}}^{l}}{2}= & \frac{u_{h_{x}}^{l+1}-u_{h_{x}}^{l}}{h_{t}},  \tag{7.6}\\
\frac{\beta_{h_{x}}^{l+1}-\beta_{h_{x}}^{l}}{h_{t}}= & -\frac{\kappa}{a}\left(u_{h_{x}}^{l}-\varphi_{h_{x}}\right)^{2}\left(\beta_{h_{x}}^{l+1}\right) . \tag{7.7}
\end{align*}
$$

Then using (7.6), from (7.4) we derive the discrete-time equation of motion:

$$
\begin{align*}
&\left(2+h_{t}^{2} \kappa\left(\beta_{h_{x}}^{l+1}\right)^{2}\right) u_{h_{x}}^{l+1}+\left(\frac{h_{t}^{2}}{2}+\alpha h_{t}\right)\left(u_{h_{x}}^{l+1}\right)^{(4)}= \\
&\left(\alpha h_{t}-\frac{h_{t}^{2}}{2}\right)\left(u_{h_{x}}^{l}\right)^{(4)}+2 u_{h_{x}}^{l}+2 h_{t} v_{h_{x}}^{l}+h_{t}^{2} \kappa\left(\beta_{h_{x}}^{l+1}\right)^{2} \varphi_{h_{x}}+h_{t}^{2} N_{h_{x}}^{l} \tag{7.8}
\end{align*}
$$

Note that $\beta_{h_{x}}^{l+1}$ can be computed from (6.15), using previous solution $u_{h_{x}}^{l}$ and $\beta_{h_{x}}^{l}$. Now multiplying the basis function $\psi_{i}$ by both sides of (7.8) and using integration by parts, we have the Galerkin approximation for one time step:

$$
\begin{equation*}
\left(2 \mathbf{M}+h_{t}^{2} \kappa \mathbf{M}_{\beta}^{l+1}+\left(\frac{h_{t}^{2}}{2}+\alpha h_{t}\right) \mathbf{K}\right) \mathbf{u}^{l+1}=\left(\left(\alpha h_{t}-\frac{h_{t}^{2}}{2}\right) \mathbf{K}+2 \mathbf{M}\right) \mathbf{u}^{l}+2 h_{t} \mathbf{M} \mathbf{v}^{l}+h_{t}^{2} \kappa \mathbf{M}_{\beta}^{l+1} \boldsymbol{\varphi}+h_{t}^{2} \mathbf{M} \mathbf{N}^{l} \tag{7.9}
\end{equation*}
$$

and from (7.6)

$$
\begin{equation*}
\mathbf{v}^{l+1}=\frac{2}{h_{t}}\left(\mathbf{u}^{l+1}-\mathbf{u}^{l}\right)-\mathbf{v}^{l} . \tag{7.10}
\end{equation*}
$$

Here the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{K}$ have the the following form:

$$
\mathbf{M}=(M)_{i j}=\int_{0}^{L} \psi_{i}(x) \psi_{j}(x) \mathrm{d} x, \quad \mathbf{K}=(K)_{i j}=\int_{0}^{L} \psi_{i}^{\prime \prime}(x) \psi_{j}^{\prime \prime}(x) \mathrm{d} x
$$

and another mass matrix $\mathbf{M}_{\beta}^{l+1}$ containing the next approximation of bonding field $\beta^{l+1}$ is computed as follows:

$$
\begin{align*}
\mathbf{M}_{\beta}^{l+1}=\left(M_{\beta}^{l+1}\right)_{i j} & =\sum_{i, j=1}^{m-1} \int_{0}^{L}\left(\beta_{h_{x}}^{l+1}\right)^{2} \psi_{i}(x) \psi_{j}(x) \mathrm{d} x \\
& =\sum_{i, j=1}^{m-1} \sum_{k=1}^{m-1}\left(\beta_{k}^{l+1}\right)^{2} \int_{0}^{L} \phi_{k}(x) \psi_{i}(x) \psi_{j}(x) \mathrm{d} x \\
& =\sum_{i, j=1}^{m-1} \sum_{k=1}^{m-1}\left(\beta_{k}^{l+1}\right)^{2} \int_{x_{k}}^{x_{k+1}} \psi_{i}(x) \psi_{j}(x) \mathrm{d} x . \tag{7.11}
\end{align*}
$$

Note that the symmetric matrices $\mathbf{M}$ and $\mathbf{K}$ are positive definite and $\mathbf{M}_{\beta}^{l+1}$ is a semi-positive symmetric definite matrix. From the construction of basis function $\psi_{i}$ those matrices are banded matrices with three sub-diagonals and three super-diagonals, i.e., if $0 \leq|i-j| \leq 3$ those matrices contain nonzero entries and otherwise those zero entries. At each time step $t_{l}$, the discretized solutions $\mathbf{u}^{l}, \mathbf{v}^{l}, \mathbf{N}^{l}, \boldsymbol{\beta}^{l} \in \mathbb{R}^{m-1}$ have the form $\mathbf{u}^{l}=$ $\left(u_{1}^{l}, u_{2}^{l}, \ldots, u_{m-1}^{l}\right)^{T}, \mathbf{v}^{l}=\left(v_{1}^{l}, v_{2}^{l}, \ldots, v_{m-1}^{l}\right)^{T}, \mathbf{N}^{l}=\left(N_{1}^{l}, N_{2}^{l}, \ldots, N_{m-1}^{l}\right)^{T}$ and $\boldsymbol{\beta}^{l}=\left(\beta_{1}^{l}, \beta_{2}^{l}, \ldots, \beta_{m-1}^{l}\right)^{T}$ respectively. Also let $\boldsymbol{\varphi}=\left(\varphi_{1}, \varphi_{2}, \ldots, \varphi_{m-1}\right)$ be the approximation of the rigid obstacle.

In order to define energy function for the fully discrete case, we express the gap function $g_{h_{x}}^{l}$ at each time step $t_{l}$ between the viscoelastic clamped beam and the rigid obstacle by

$$
\begin{equation*}
g_{h_{x}}^{l}(x)=\sum_{i=1}^{m-1} g_{i}^{l} \psi_{i}(x), \tag{7.12}
\end{equation*}
$$

where $g_{i}^{l}=u_{i}^{l}-\varphi_{i}$. Let the approximate gap function $\mathbf{g}^{l}$ be $\mathbf{g}^{l}=\left(g_{1}^{l}, g_{2}^{l}, \ldots, g_{m-1}^{l}\right)$ at time step $t_{l}$. Now recalling (6.6), we define the energy function in the fully discrete case:

$$
\begin{equation*}
E\left(t_{l}\right):=E\left[\mathbf{u}^{l}, \mathbf{v}^{l}, \boldsymbol{\beta}^{l}\right]=\frac{1}{2}\left(\left(\mathbf{v}^{l}\right)^{T} \mathbf{M} \mathbf{v}^{l}+\left(\mathbf{u}^{l}\right)^{T} \mathbf{K} \mathbf{u}^{l}+\kappa\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l} \mathbf{g}^{l}\right) \tag{7.13}
\end{equation*}
$$

Assuming that there is no body force, it will be seen from the next Lemma 7.1 that energy does not increase in the fully discrete case. This result is the same as energy not including adhesive energy (see [1,2,5]) does not increase.

Lemma 7.1. Suppose that the fully discrete approximations obtained from (7.9)-(7.10) satisfy the complementarity conditions:

$$
\begin{equation*}
\mathbf{0} \leq \mathbf{u}^{l+1}-\boldsymbol{\varphi} \quad \perp \quad \mathbf{M} \mathbf{N}^{l} \geq \mathbf{0} \tag{7.14}
\end{equation*}
$$

Then energy does not increase, i.e., $E\left[\mathbf{u}^{l+1}, \mathbf{v}^{l+1}, \boldsymbol{\beta}^{l+1}\right] \leq E\left[\mathbf{u}^{l}, \mathbf{v}^{l}, \boldsymbol{\beta}^{l}\right]$ for any $l \geq 0$.
Proof. Employing integration by parts and using the mass $\mathbf{M}$ and stiffness matrix $\mathbf{K}$, it follows from the fully approximate formulation (7.4)-(7.6) and (7.12) that

$$
\begin{aligned}
\frac{1}{2 h_{t}}\left(\left(\mathbf{v}^{l+1}\right)^{T} \mathbf{M} \mathbf{v}^{l+1}-\left(\mathbf{v}^{l}\right)^{T} \mathbf{M} \mathbf{v}^{l}\right)=\frac{1}{2 h_{t}} & \left(\left(\mathbf{u}^{l+1}\right)^{T} \mathbf{K} \mathbf{u}^{l+1}-\left(\mathbf{u}^{l}\right)^{T} \mathbf{K} \mathbf{u}^{l}\right)+\frac{1}{h_{t}^{2}}\left(\mathbf{u}^{l+1}-\mathbf{u}^{l}\right)^{T} \mathbf{K}\left(\mathbf{u}^{l+1}-\mathbf{u}^{l}\right) \\
& +\frac{1}{h_{t}}\left(\mathbf{N}^{l}\right)^{T} \mathbf{M}\left(\mathbf{u}^{l+1}-\boldsymbol{\varphi}-\mathbf{u}^{l}+\varphi\right)-\frac{\kappa}{h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)
\end{aligned}
$$

Thus it follows from the complementarity conditions (7.14) that

$$
\begin{equation*}
\frac{1}{2 h_{t}}\left(\left(\mathbf{v}^{l+1}\right)^{T} \mathbf{M} \mathbf{v}^{l+1}-\left(\mathbf{v}^{l}\right)^{T} \mathbf{M} \mathbf{v}^{l}\right) \leq-\frac{1}{2 h_{t}}\left(\left(\mathbf{u}^{l+1}\right)^{T} \mathbf{K} \mathbf{u}^{l+1}-\left(\mathbf{u}^{l}\right)^{T} \mathbf{K} \mathbf{u}^{l}\right)-\frac{\kappa}{h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) \tag{7.15}
\end{equation*}
$$

Then we want to modify the last term on the right side of (7.15):

$$
\frac{\kappa}{h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)=\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}+\mathbf{g}^{l}+\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) .
$$

Thus since $\mathbf{M}_{\beta}^{l}$ is semi-positive symmetric definite for each $l \geq 0$, we have

$$
\begin{aligned}
\frac{1}{2 h_{t}} & \left(\left(\mathbf{v}^{l+1}\right)^{T} \mathbf{M} \mathbf{v}^{l+1}-\left(\mathbf{v}^{l}\right)^{T} \mathbf{M} \mathbf{v}^{l}\right)-\frac{1}{2 h_{t}}\left(\left(\mathbf{u}^{l+1}\right)^{T} \mathbf{K} \mathbf{u}^{l+1}-\left(\mathbf{u}^{l}\right)^{T} \mathbf{K} \mathbf{u}^{l}\right) \\
\leq & -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}+\mathbf{g}^{l}+\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) \\
= & -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)-\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) \\
& -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) \\
\leq & -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right)-\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1}\left(\mathbf{g}^{l+1}-\mathbf{g}^{l}\right) \\
= & -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l+1}+\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l} \\
& -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l+1}+\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l} \\
= & -\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l+1}+\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T}\left(\mathbf{M}_{\beta}^{l+1}-\mathbf{M}_{\beta}^{l}\right) \mathbf{g}^{l}+\frac{\kappa}{2 h_{t}}\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l} \mathbf{g}^{l} .
\end{aligned}
$$

Now we can observe from (7.7) and (7.11) that $\left(\mathbf{g}^{l}\right)^{T}\left(\mathbf{M}_{\beta}^{l+1}-\mathbf{M}_{\beta}^{l}\right)\left(\mathbf{g}^{l}\right) \leq 0$. Therefore we obtain

$$
\begin{aligned}
& \frac{1}{2}\left(\left(\mathbf{v}^{l+1}\right)^{T} \mathbf{M} \mathbf{v}^{l+1}+\left(\mathbf{u}^{l+1}\right)^{T} \mathbf{K} \mathbf{u}^{l+1}+\kappa\left(\mathbf{g}^{l+1}\right)^{T} \mathbf{M}_{\beta}^{l+1} \mathbf{g}^{l+1}\right) \\
& \quad \leq \frac{1}{2}\left(\left(\mathbf{v}^{l}\right)^{T} \mathbf{M} \mathbf{v}^{l}+\left(\mathbf{u}^{l}\right)^{T} \mathbf{K} \mathbf{u}^{l}+\kappa\left(\mathbf{g}^{l}\right)^{T} \mathbf{M}_{\beta}^{l} \mathbf{g}^{l}\right)
\end{aligned}
$$

as required.
We note that the previous Lemma 7.1 will be a good indication of stability for our numerical results. Indeed, in the next Section 8 it will be seen that the numerical solutions computed from our numerical scheme support that energy does not increase.

### 7.2. Nonsmooth Newton's method

Employing nonsmooth Newton's method (see [24]), we implement the fully numerical scheme which has been discussed in the previous Section 7.1. We recall the linear system (7.9), in which the next step solution $\mathbf{u}^{l+1}$ can be computed. However, $\mathbf{u}^{l+1}$ has to satisfy the complementarity condition (7.14). We can see easily from the linear system (7.9) that $\mathbf{M} \mathbf{N}^{l}$ can be expressed in terms of $\mathbf{u}^{l+1}$

$$
\begin{aligned}
\mathbf{M N}^{l}=\frac{1}{h_{t}^{2}} & {\left[\left(2 \mathbf{M}+h_{t}^{2} \kappa \mathbf{M}_{\beta}^{l+1}+\left(\frac{h_{t}^{2}}{2}+\alpha h_{t}\right) \mathbf{K}\right) \mathbf{u}^{l+1}\right.} \\
& \left.-\left(\left(\alpha h_{t}-\frac{h_{t}^{2}}{2}\right) \mathbf{K}+2 \mathbf{M}\right) \mathbf{u}^{l}-2 h_{t} \mathbf{M} \mathbf{v}^{l}-h_{t}^{2} \kappa \mathbf{M}_{\beta}^{l+1} \boldsymbol{\varphi}\right]
\end{aligned}
$$

where the previous solutions $\mathbf{u}^{l}, \mathbf{v}^{l}$ are already known and $\boldsymbol{\beta}^{l+1}$ contained in matrix $\mathbf{M}_{\beta}^{l+1}$ can be computed from (6.15). Bringing the new smooth mapping $\mathbf{F}: \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}$, we define $\mathbf{M} \mathbf{N}^{l}$ as $\mathbf{M N}{ }^{l}:=\mathbf{F}\left(\mathbf{u}^{l+1}\right)$. Thus we are led to the NCP (nonlinear complementarity problem):

$$
\begin{equation*}
\mathbf{0} \leq \mathbf{u}^{l+1}-\boldsymbol{\varphi} \quad \perp \quad \mathbf{F}\left(\mathbf{u}^{l+1}\right) \geq \mathbf{0} \tag{7.16}
\end{equation*}
$$

Many methods for solving NCP are presented in the book [12] by Facchinei and Pang. Those methods are based on a $C$-function called the (nonlinear) complementarity function. We note that $C$-function is called a NCP-function in other papers (e.g., [20]). $C$-function $\xi: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is a mapping having the following property

$$
\xi(x, y)=0 \Leftrightarrow 0 \leq x \perp y \geq 0
$$

The basic example of $C$-function is a minimum function, i.e., $\xi(x, y)=\min (x, y)$. Many $C$-functions are listed in [12]. The advantage of $C$-function is that it converts the complementarity condition to a single equation and then can be applied to the nonsmooth Newton's method. While the Fischer-Burmeister function is used as a $C$-function in [1], the Kanzow-Kleinmichel function will be done in this work as in [2]. We note that the Kanzow-Kleinmichel function is studied in the paper [20] and defined as

$$
\xi(x, y)=\frac{\sqrt{(x-y)^{2}+2 q x y}-(x+y)}{2-q} \quad \text { for } 0 \leq q<2
$$

Applying the Kanzow-Kleinmichel $C$-function with $q=1$ into (7.16), we establish the following equation:

$$
\mathbf{F}_{\mathrm{kk}}\left(\mathbf{u}^{l+1}\right):=\left(\begin{array}{c}
\xi\left(u_{1}^{l+1}, F_{1}\left(u_{1}^{l+1}\right)\right) \\
\cdots \\
\xi\left(u_{m-1}^{l+1}, F_{m-1}\left(u_{m-1}^{l+1}\right)\right)
\end{array}\right)=\mathbf{0}
$$

where $\mathbf{F}_{\mathrm{kk}}: \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}$ is a semismooth function. Practically, putting a smoothing parameter into the function $\mathbf{F}_{\mathrm{kk}}$, we can implement the smoothed guarded Newton's method, where an Armijo/Goldstine line search strategy is used. Since our detailed strategy for computing $\mathbf{u}^{l+1}$ has been already described in [1,2], we omit it here. When our numerical solutions are computed with the guarded Newton's method, the average number of linear systems solved per time step is mostly in between 19 and 31 . This indicates that the average number seems not to increase as $h_{t}$ and $h_{x}$ go to zero. Consequently our Newton's method performs fairly efficiently as done in the previous papers $[1,2]$.

## 8. Numerical experiments and discussion

In this section, we present numerical simulations (using the size of subinterval $h_{x}=0.02$ ) and discuss them. In the computation, the following datum are used: the length of a clamped beam is $L=20$ and its initial displacement is expressed by the polynomial $u^{0}(x)=\left(x^{4}-40 x^{3}+400 x^{2}\right) / 1000$ and the initial velocity $v^{0}(x)=-10$ and the final time $T=4$ and the body force $f=0$, and the rigid foundation is a piecewise quadratic function

$$
\varphi(x)=\left\{\begin{array}{c}
-4(x-5)^{2} \quad \text { on }[0,8], \\
-17(x-14)^{2} / 18-2 \quad \text { on }[8,20] .
\end{array}\right.
$$

Note that we do not consider the units of measurement for any datum. In addition, coefficients $a=0.01$ and $\kappa=1$ and $\alpha=0.05$ (in the visco case) are used in most of numerical experiments. In the equation of motion (7.1) we take the coefficient of $u^{(4)}$ as 50 rather than 1 , in order to obtain more realistic simulation. We notice that $u^{0}$ holds the essential boundary conditions and the sign of $v^{0}$ is negative, because our clamped beam is assumed to move initially downward.

Recalling the energy function (7.13), we observe three pictures for the total energy in Figure 2. As one can see, every graph drawn in (a) (including different time step $h_{t}$ ), (b), and (c) is going down, which supports dissipation of energy shown in Lemma 7.1. One more important thing is that in some parts where graphs are decreasing, it means that the beam is hitting the rigid foundation, i.e., contact forces are taking place. Those numerical results are extremely significant to justify that our numerical results are stable. There is no error analysis, because the uniqueness of solution is not still proved. Probably error analysis will not be able to be considered unless the uniqueness is shown. (a) makes a conjecture on energy conservation with adhesion energy in the limit, i.e., as $h_{t} \downarrow 0$ and $h_{x} \downarrow 0$. While all graphs drawn in (a) show the possibility of energy conservation for small fixed $a=0.01$, three graphs ( $h_{t}=0.00125$ used) in (b) support the energy balance. We notice that the different scale is used in (b) to have a close look at the energy balance. Furthermore, we need to note that the reason that after around $t=3.5$ both energy with $a=0.01$ and energy with $a=100$ are bigger than the one ( $a=0.000001$ ) is that there is already occurring the second contact in the case ( $a=0.000001$ ). (c) shows the larger rate of energy dissipation is due to viscosity and even adhesion. This picture (c) supports the energy balance (5.3). Note that the time step size used in (b) and (c) is $h_{t}=0.005$.

Figure 3 presents the flow of the bonding field $\beta$. From (a) and (b) we can see that the bonding field is decreasing very rapidly before the beam reaches the obstacle, because the small adhesion rate $a=0.01$ is used. Moreover we can observe that $\beta$ in the viscoelastic case drops down more slowly than in the purely elastic case. However we can guess that the bonding field $\beta$ for these two cases will eventually be zero. On the other hand, on some parts of the beams the bonding field keeps the same value, which means that contact forces are occurring, i.e., there is no gap between the body and obstacle. Those descriptions can be confirmed with the differential equation (2.3).

In Figure 4, there are four pictures regarding contact forces. One of important things is that the viscosity makes contact forces more singular. This has been already confirmed in both the theoretical approach of [23] and the numerical experiments of [5]. When comparing pictures (a), (b), (c), and (d), we can recognize that the bonding field $\beta$ seems not to give an effect of the magnitude of contact forces $N_{\mathrm{C}}$. In fact, we have already seen from the Signorini's conditions (the complementarity conditions) that $\beta$ does not have any influence on $N_{\mathrm{C}}$.

Figures 5 and 6 show the motion of the purely elastic clamped beam. The remarkable difference between the two cases (with adhesion and without adhesion) is that the deformation of the beam without adhesion is bigger than its deformation with adhesion, because the bonding field works downward and prevents the beam from extending upward.

Figures 7 and 8 show small deformations of the viscoelastic beam, compared with the purely elastic beam.
On the other hand, the velocity of the purely elastic beam without adhesion is presented in Figure 9 and with adhesion in Figure 10, while the velocity of the viscoelastic beam without adhesion is shown in Figure 11 and with adhesion in Figure 12. When the beam touches the rigid foundation, the elastic beam has higher frequency than the viscoelastic beam does. According to this physical phenomenon, we are able to guess that the rate of the deformation of the purely elastic beam is much faster than the viscoelastic beam. Moreover, we can observe that viscosity causes the regularity of velocity. As one can see in those Figures 9-12, the vibration of elastic and viscoelastic beam looks a little bit different depending on whether adhesion processes are applied or not, which means that the effect of adhesion does not have a big influence on the vibration of the beams.

Finally, we present the velocity of the bonding field in Figures 13 and 14. We notice that the last pictures for each beam are drawn in different scale, to see how the velocity $\dot{\beta}$ of each beam performs around the time when $\dot{\beta}$ become zero. The rate that $\dot{\beta}$ is going to zero is slower in the viscoelastic case than in the purely elastic case. However, it seems that the velocity of the bonding field will become zero after all. This is also confirmed from the evolution (2.3) of the bonding field.

## 9. Conclusions

In this work, dynamic frictionless contact problems of viscoelastic bodies with the adhesion processes are considered. It is shown that there is a subsequence of the numerical approximations that converges to a solution of the adhesive contact problem in the weak sense. Unlike energy balance only caused by viscoelasticity, we derive


Figure 2. Energy functions with a hybrid of two numerical schemes.


Figure 3. Flow of bonding field $\beta$.


Figure 4. Contact forces.


Figure 5. The motion of the purely elastic clamped beam without adhesion.


Figure 6. The motion of the purely elastic clamped beam with adhesion.


Figure 7. The motion of the viscoelastic clamped beam without adhesion.
a new form of energy balance. Our numerical scheme is implemented using time-discretization and the FEM with B-splines. The complementarity problem that set up at each time step is solved, employing nonsmooth Newton's method with the Kanzow-Kleinmichel function. Our numerical methods are performed reasonably well and provide evidence of energy balance.


Figure 8. The motion of the viscoelastic clamped beam with adhesion.

(a) The velocity of the beam (b) The velocity of the beam (c) The velocity of the beam from 1st to 78 th time step. from 79 th to 432 nd time step. from 433 rd to 800 th time step.

Figure 9. The velocity of each point on the purely elastic beam without adhesion.

(a) The velocity of the beam (b) The velocity of the beam (c) The velocity of the beam from 1st to 78 th time step. from 79 th to 432 nd time step. from 433 rd to 800 th time step.

Figure 10. The velocity of each point on the purely elastic beam with adhesion.

(a) The velocity of the beam (b) The velocity of the beam (c) The velocity of the beam from 1st to 78th time step. from 79th to 432nd time step. from 433rd to 800th time step.

Figure 11. The velocity of each point on the viscoelastic beam without adhesion.

(a) The velocity of the beam from 1st to 78th time step.


(b) The velocity of the beam
(c) The velocity of the beam from 79 th to 432 nd time step. from 433 rd to 800 th time step.

Figure 12. The velocity of each point on the viscoelastic beam with adhesion.

(a) The velocity of the bonding (b) The velocity of the bonding (c) The velocity of the bonding field from 1st to 78 th time step. field from 79 th to 432 nd time field from 433 rd to 800 th time step. step.

Figure 13. Velocity $\dot{\beta}$ of bonding field in the purely elastic case.


Figure 14. Velocity $\dot{\beta}$ of bonding field in the viscoelastic case.

One of our main concerns is to investigate the relation between the contact forces and adhesion. Indeed, adhesion processes do not make contact force more regular and numerical results also seem to support this, whereas viscosity tends to make the contact forces less regular. Investigating an effect which makes the normal contact force more regular and proving the uniqueness of the solution $u$ and the contact force $N_{\mathrm{C}}$ will be future research.

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