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THE hp-VERSION OF THE BOUNDARY ELEMENT METHOD WITH QUASI-UNIFORM MESHES IN THREE DIMENSIONS *

Alexei Bespalov[1] and Norbert Heuer[1]

Abstract. We prove an *a priori* error estimate for the hp-version of the boundary element method with hypersingular operators on piecewise plane open or closed surfaces. The underlying meshes are supposed to be quasi-uniform. The solutions of problems on polyhedral or piecewise plane open surfaces exhibit typical singularities which limit the convergence rate of the boundary element method. On closed surfaces, and for sufficiently smooth given data, the solution is H^1-regular whereas, on open surfaces, edge singularities are strong enough to prevent the solution from being in H^1. In this paper we cover both cases and, in particular, prove an *a priori* error estimate for the h-version with quasi-uniform meshes. For open surfaces we prove a convergence like $O(h^{1/2}p^{-1})$, h being the mesh size and p denoting the polynomial degree. This result had been conjectured previously.

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Dedicated to Professor Ernst P. Stephan on the occasion of his 60th birthday.

1. Introduction

We study the hp-version of the boundary element Galerkin method (BEM) for hypersingular integral operators on piecewise plane surfaces. The particularly important case of open surfaces is included. We prove an *a priori* error estimate for the hp-version with quasi-uniform meshes which is, by heuristic arguments, optimal when singularities do not include logarithmic contributions. Whether our result is optimal for logarithmic singularities is unknown. Fixing polynomial degrees our error bounds yield error estimates for the h-version which (in the case of singularities) have been unknown.

In the finite element framework, many hp *a priori* error estimates have been proved, for quasi-uniform as well as graded meshes. For an overview see the book by Schwab [27] and the references given there. Recent error estimates, in particular for mixed methods, can be found in [2,3,16,25], see also the following discussion.

There are much fewer results in the case of the BEM (for an early overview see [30]). The first paper on the p-version of the BEM for problems in three dimensions appeared 1996 [28]. It covers only polyhedral domains (and hypersingular operators) where solutions are in H^1 (on the boundary). The second paper [23], which appeared

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[1] Department of Mathematical Sciences, Brunel University, Uxbridge, West London UB8 3PH, UK. albespalov@yahoo.com; norbert.heuer@gmail.com

1999, analyses the hp-version of the BEM with geometrically graded meshes on open surfaces, for hypersingular and weakly singular operators. This method uses appropriate combinations of graded meshes and highly non-uniform polynomial degrees to achieve a convergence that is faster than algebraic, even in the presence of strong singularities that are inherent to problems on open surfaces. From those results one cannot, however, deduce *a priori* error estimates for the p-version or hp-version with quasi-uniform meshes. In the latter cases polynomial degrees are large also on elements close to the singularities, whereas the hp-version with geometrically graded meshes uses lowest order polynomials at the singularities. The hp-version with geometrically graded meshes is numerically convincing and well analysed. However, the analysis of high order approximations of singularities is challenging and with this paper we fill one of the gaps in the existing literature.

In our previous paper [11] we studied the p-version of the BEM for hypersingular operators on open surfaces. The strongest singularities of typical solutions are edge singularities which behave like $y^{1/2}$ where y denotes the distance to an edge of the surface. Let us denote this surface by Γ. Then this edge singularity is in the Sobolev space $H^{1-\varepsilon}(\Gamma)$ for any $\varepsilon > 0$, but it is not an element of $H^1(\Gamma)$. The energy space of hypersingular operators is $\tilde{H}^{1/2}(\Gamma)$, sometimes denoted by $H^{1/2}_{00}(\Gamma)$. (For a definition of the Sobolev spaces see Sect. 3 below.) Therefore, in order to find an optimal *a priori* error estimate, one has to analyse the approximation in $\tilde{H}^{1/2}(\Gamma)$ of a function which is not in $H^1(\Gamma)$. One possibility to deal with this is to introduce weighted Sobolev spaces. In particular, Jacobi-weighted Sobolev and Besov spaces are appropriate to prove optimal error estimates for the p-version, see [20,21] for the BEM in two dimensions and [5,19] for the FEM in two and three dimensions. In order to obtain error estimates in the energy norm, a key ingredient is to prove that the interpolation between appropriate weighted spaces reproduces the energy space. For the space $\tilde{H}^{1/2}$ on open curves or surfaces this result is not immediate. In two dimensions it can be proved by using arguments of complex analysis (see [8], Lem. 3.1) and in three dimensions this is open. In this paper we follow the strategy of [11] and avoid the use of weighted spaces by performing the approximation analysis in fractional order Sobolev spaces. We note that in [12] we studied the p-version for weakly singular operators. These operators are inherently connected with negative order fractional Sobolev spaces and we reduced their analysis to the case of positive order fractional Sobolev spaces.

Considering the particular edge singularity $y^{1/2}$ for hypersingular operators, we proved a convergence like $O(p^{-1})$ for the p-version [11]. Here, p denotes the polynomial degree of the approximating functions. In this paper, we extend the analysis to the hp-version and the corresponding error estimate for the edge singularity gives an upper bound that behaves like $O(h^{1/2}p^{-1})$. Here, h refers to the maximum diameter of the elements. This result is in agreement with conjectures stated in [23].

Fixing polynomial degrees, our results on the hp-version in particular prove *a priori* error estimates for the h-version of the BEM with quasi-uniform meshes. In fact only very little has been proved for the h-version of the BEM in three dimensions. For problems with singularities we only know of [33] where von Petersdorff and Stephan present a sub-optimal error estimate (for quasi-uniform and graded meshes). In the case of an open surface their result states an error bound like $O(h^{1/2-\varepsilon})$ for piecewise polynomial approximations of lowest order on quasi-uniform meshes. Here, $\varepsilon > 0$ and the leading error term contains a factor $C(\varepsilon)$ whose behaviour for $\varepsilon \to 0$ is unknown. Fixing p in this paper we prove an error bound like $O(h^{1/2})$ for any polynomial degree. Applying arguments from [17] (where the weakly singular operator is studied) one can show that this result is optimal in the case $p = 1$ and for rectangular meshes in a neighbourhood of the edge singularities.

To prove results for the hp-version with quasi-uniform meshes one usually tries to make use of p-version results by scaling arguments. For the finite element method in two dimensions see [6] and for the BEM in two dimensions we refer to [31]. There are, however, two principal difficulties. First, p-version analysis employs different polynomial degrees in different parts of the approximation. When only p-asymptotic estimates are wanted one approximates, for instance, polynomial jumps of degree p over element interfaces by polynomial extensions of degree $2p + 1$ (*cf*. Lem. 3.4 below). This is not possible when aiming at h-version results where polynomial degrees are fixed (*e.g.* uniformly at p). In that sense hp-estimates do not directly follow from corresponding p-estimates by scaling arguments. Second, in this paper we are considering three-dimensional problems where different types of singularities appear. This fact, together with the need to directly work in fractional order Sobolev spaces, makes the use of scaling arguments non-trivial.

For our analysis we assume that the surface under consideration is open (which is the most singular case) and piecewise plane such that it can be discretised by meshes consisting of triangles and parallelograms. Since Sobolev spaces are invariant under sufficiently smooth mappings our results generalise to piecewise smooth surfaces and elements with curved boundaries. For ease of presentation we assume that $\Gamma \subset \mathbb{R}^3$ is a plane open surface with polygonal boundary. Our model problem reads: *Find* $u \in \tilde{H}^{1/2}(\Gamma)$ *such that*

$$\langle Wu, v\rangle = \langle f, v\rangle \quad \forall v \in \tilde{H}^{1/2}(\Gamma). \tag{1.1}$$

Here, $f \in H^{-1/2}(\Gamma)$ is a given functional and W is the hypersingular operator

$$Wu(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,\mathrm{d}S_y.$$

The operator $W:\ \tilde{H}^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is continuous, symmetric and positive definite such that any finite element method for (1.1) (then called boundary element method) converges quasi-optimally, see [14,29]. Here, $H^{-1/2}(\Gamma)$ is the dual space of $\tilde{H}^{1/2}(\Gamma)$ and the latter is defined in Section 3.

The rest of the paper is organised as follows. In the next section we define the *hp*-version of the BEM, recall a regularity result for the solution of (1.1), and formulate the main theorem stating an *a priori* error estimate for the *hp*-version of the BEM. In Section 3 we introduce the Sobolev spaces and collect several technical results. Of particular importance is Lemma 3.5 which bounds a fractional order norm by local contributions. This is needed to join local approximation results in fractional spaces to form a global estimate. Sections 4–6 are focused on the approximation analysis of particular singularities. In Section 7 we prove a general approximation theorem and the main result given in Section 2.

2. *hp*-BEM and *a priori* error estimate

For the approximate solution of (1.1) we apply the *hp*-version of the BEM on quasi-uniform meshes. In what follows, $h > 0$ and $p \geq 1$ will always specify the mesh parameter and a polynomial degree, respectively. For any $\Omega \subset \mathbb{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);\ B \text{ is a ball in } \Omega\}$. By $A \simeq B$ we mean that A is equivalent to B, *i.e.*, there exists a constant $C > 0$ such that $C\,B \leq A \leq C^{-1}B$ where B and A may depend on a parameter (usually h or p) but C does not.

Let $\mathcal{M} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1,\dots,J\}$ on Γ, where Γ_j are open triangles or parallelograms such that $\bar{\Gamma} = \cup_{j=1}^J \bar{\Gamma}_j$. For any $\Gamma_j \in \Delta_h$ we will denote $h_j = \mathrm{diam}(\Gamma_j)$ and $\rho_j = \rho_{\Gamma_j}$. Let $h = \max_j h_j$. In this paper we will consider a family $\mathcal{M}$ of quasi-uniform meshes Δ_h on Γ in the sense that there exist positive constants $\sigma_1,\ \sigma_2$ independent of h such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{M}$

$$h \leq \sigma_1 h_j, \qquad h_j \leq \sigma_2 \rho_j. \tag{2.1}$$

Let $Q = (-1,1)^2$ and $T = \{(x_1,x_2);\ 0 < x_1 < 1,\ 0 < x_2 < x_1\}$ be the reference square and triangle, respectively. Then for any $\Gamma_j \in \Delta_h$ one has $\Gamma_j = M_j(K)$, where M_j is an affine mapping with Jacobian $|J_j| \simeq h_j^2$ and $K = Q$ or T as appropriate.

Below we will refer to three different patches of elements. The union of the elements at a node v is denoted by A_v, *i.e.*, $\bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$, the union of the elements at one edge e by A_e (the endpoints of e are not included in e), $\bar{A}_e := \cup\{\bar{\Gamma}_j;\ \bar{\Gamma}_j \cap e \neq \emptyset\}$, and $A_{ev} := A_v \cap A_e$.

Further, $\mathcal{P}_p(I)$ denotes the set of polynomials of degree $\leq p$ on an interval $I \subset \mathbb{R}$. Moreover, $\mathcal{P}_p^1(T)$ is the set of polynomials on T of total degree $\leq p$, and $\mathcal{P}_p^2(Q)$ is the set of polynomials on Q of degree $\leq p$ in each variable. Let $K \subset \mathbb{R}^2$ be an arbitrary triangle or parallelogram, and let $K = M(T)$ or $K = M(Q)$ with an invertible affine mapping M. Then by $\mathcal{P}_p(K)$ we will denote the set of polynomials v on K such that $v \circ M \in \mathcal{P}_p^1(T)$ if K

is a triangle and $v \circ M \in \mathcal{P}_p^2(Q)$ if K is a parallelogram (in particular, we will use this notation for $K = Q$ and $K = T$). For given p, we then consider the space of continuous, piecewise polynomials on the mesh $\Delta_h \in \mathcal{M}$,

$$V_0^{h,p}(\Gamma) := \{v \in C^0(\Gamma);\ v|_{\partial\Gamma} = 0,\ v|_{\Gamma_j} \in \mathcal{P}_p(\Gamma_j),\ j = 1, \dots, J\}.$$

Note that $V_0^{h,p}(\Gamma) \subset \tilde{H}^{1/2}(\Gamma)$. Now, the hp-version of the BEM is: *Find* $u_{hp} \in V_0^{h,p}(\Gamma)$ *such that*

$$\langle W u_{hp}, v\rangle = \langle f, v\rangle \quad \forall v \in V_0^{h,p}(\Gamma). \tag{2.2}$$

Before giving our main result stating an *a priori* error estimate for (2.2) let us recall the typical structure of the solution of the model problem for a sufficiently smooth right-hand side function f.

Theorem 2.1 ([33])**.** *Let V and E denote the sets of vertices and edges of Γ, respectively. For $v \in V$, let $E(v)$ denote the set of edges with v as an end point. Then, for sufficiently smooth given f, the solution u of (1.1) has the form*

$$u = u_{\mathrm{reg}} + \sum_{e\in E} u^e + \sum_{v\in V} u^v + \sum_{v\in V}\sum_{e\in E(v)} u^{ev}, \tag{2.3}$$

where, using local coordinate systems (r_v, θ_v) and (x_{e1}, x_{e2}) with origin v, there hold the following representations:

(i) *The regular part $u_{\mathrm{reg}} \in H^k(\Gamma)$, $k > 1$.*

(ii) *The edge singularities u^e have the form*

$$u^e = \sum_{j=1}^{m_e}\left(\sum_{s=0}^{s_j^e} b_{js}^e(x_{e1})|\log x_{e2}|^s\right) x_{e2}^{\gamma_j^e}\, \chi_1^e(x_{e1})\chi_2^e(x_{e2}), \tag{2.4}$$

where $\gamma_{j+1}^e \ge \gamma_j^e \ge \frac{1}{2}$, and m_e, s_j^e are integers. Here, χ_1^e, χ_2^e are C^∞ cut-off functions with $\chi_1^e = 1$ in a certain distance to the end points of e and $\chi_1^e = 0$ in a neighbourhood of these vertices. Moreover, $\chi_2^e = 1$ for $0 \le x_{e2} \le \delta_e$ and $\chi_2^e = 0$ for $x_{e2} \ge 2\delta_e$ with some $\delta_e \in (0, \frac{1}{2})$. The functions $b_{js}^e\chi_1^e \in H^m(e)$ for m as large as required.

(iii) *The vertex singularities u^v have the form*

$$u^v = \chi^v(r_v)\sum_{i=1}^{n_v}\sum_{t=0}^{q_i^v} B_{it}^v|\log r_v|^t r_v^{\lambda_i^v} w_{it}^v(\theta_v), \tag{2.5}$$

where $\lambda_{i+1}^v \ge \lambda_i^v > 0$, n_v, $q_i^v \ge 0$ are integers, and B_{it}^v are real numbers. Here, χ^v is a C^∞ cut-off function with $\chi^v = 1$ for $0 \le r_v \le \tau_v$ and $\chi^v = 0$ for $r_v \ge 2\tau_v$ with some $\tau_v \in (0, \frac{1}{2})$. The functions $w_{it}^v \in H^q(0, \omega_v)$ for q as large as required. Here, ω_v denotes the interior angle (on Γ) between the edges meeting at v.

(iv) *The edge-vertex singularities u^{ev} have the form*

$$u^{ev} = u_1^{ev} + u_2^{ev},$$

where

$$u_1^{ev} = \sum_{j=1}^{m_e}\sum_{i=1}^{n_v}\left(\sum_{s=0}^{s_j^e}\sum_{t=0}^{q_i^v}\sum_{l=0}^{s} B_{ijlts}^{ev}|\log x_{e1}|^{s+t-l}|\log x_{e2}|^l\right) x_{e1}^{\lambda_i^v-\gamma_j^e} x_{e2}^{\gamma_j^e}\, \chi^v(r_v)\chi^{ev}(\theta_v) \tag{2.6}$$

and

$$u_2^{ev} = \sum_{j=1}^{m_e}\sum_{s=0}^{s_j^e} B_{js}^{ev}(r_v)|\log x_{e2}|^s x_{e2}^{\gamma_j^e}\, \chi^v(r_v)\chi^{ev}(\theta_v) \tag{2.7}$$

with

$$B^{ev}_{js}(r_v) = \sum_{l=0}^{s} B^{ev}_{jsl}(r_v)|\log r_v|^l. \tag{2.8}$$

Here, q^v_i, s^e_j, λ^v_i, γ^e_j, χ^v *are as above,* B^{ev}_{ijlts} *are real numbers, and* χ^{ev} *is a* C^∞ *cut-off function with* $\chi^{ev} = 1$ *for* $0 \le \theta_v \le \beta_v$ *and* $\chi^{ev} = 0$ *for* $\frac{3}{2}\beta_v \le \theta_v \le \omega_v$ *for some* $\beta_v \in (0, \min\{\omega_v/2, \pi/8\}]$*. The functions* B^{ev}_{jsl} *may be chosen such that*

$$B^{ev}_{js}(r_v)\,\chi^v(r_v)\chi^{ev}(\theta_v) = \chi_{js}(x_{e1}, x_{e2})\,\chi^e_2(x_{e2}), \tag{2.9}$$

where the extension of χ_{js} *by zero onto* $\mathbb{R}^{2+} := \{(x_{e1}, x_{e2});\ x_{e2} > 0\}$ *lies in* $H^m(\mathbb{R}^{2+})$ *for* m *as large as required. Here,* χ^e_2 *is a* C^∞ *cut-off function as in* (ii)*.*

Remark 2.1. (i) For an open surface there holds $u_{\text{reg}} \in H^k(\Gamma) \cap H^1_0(\Gamma)$ and w^v_{it} in (2.5) satisfies $w^v_{it} \in H^q(0,\omega_v) \cap H^1_0(0,\omega_v)$. This will be needed in the proofs of Theorems 6.1 and 7.1.

(ii) The singularity structure of u is being obtained by analysing the Neumann problem for the Laplacian in the bounded domain Ω whose boundary is Γ (in the case of a closed surface Γ) or in $\Omega := \mathbb{R}^3 \setminus \bar\Gamma$ (in the case of an open surface Γ). In the former case u is the trace on Γ of the solution to the boundary value problem and in the latter case it is its jump across Γ. The required smoothness of f in (1.1) relates to the smoothness of the given Neumann datum g. Sufficient for the result above is that g is the normal derivative of a $C^\infty_{\text{loc}}(\bar\Omega)$-function. In particular there holds $f = (1/2 + K')g$ (with K' being the adjoint of the double layer potential operator) and, depending on the number of singularities and the wanted regularity for u_{reg}, the smoothness requirement on g can be relaxed to standard Sobolev regularity, for details see [33].

The following theorem is the main result of this paper.

Theorem 2.2. *Let* $u \in \tilde H^{1/2}(\Gamma)$ *be the solution of* (1.1) *with sufficiently smooth given function* f *such that the representation from Theorem* 2.1 *holds. Let* $v_0 \in V$, $e_0 \in E(v_0)$ *be such that* $\min\{\lambda^{v_0}_1 + 1/2, \gamma^{e_0}_1\} = \min_{v\in V, e\in E(v)} \min\{\lambda^v_1 + 1/2, \gamma^e_1\}$*, with* λ^v_1 *and* γ^e_1 *being as in* (2.4)–(2.7)*. Then, for any* $h > 0$ *and every* $p \ge \min\{\lambda^{v_0}_1, \gamma^{e_0}_1 - 1/2\}$*, the BE approximation* u_{hp} *defined by* (2.2) *satisfies*

$$\|u - u_{hp}\|_{\tilde H^{1/2}(\Gamma)} \le C\, h^{\min\{\lambda^{v_0}_1 + 1/2, \gamma^{e_0}_1\}}\, p^{-2\min\{\lambda^{v_0}_1 + 1/2, \gamma^{e_0}_1\}}\, (1 + \log(p/h))^{\beta+\nu}, \tag{2.10}$$

where

$$\beta = \begin{cases} q^{v_0}_1 + s^{e_0}_1 + \frac{1}{2} & \text{if } \lambda^{v_0}_1 = \gamma^{e_0}_1 - \frac{1}{2}, \\ q^{v_0}_1 + s^{e_0}_1 & \text{otherwise,} \end{cases} \tag{2.11}$$

for numbers $q^{v_0}_1$, $s^{e_0}_1$ *as given in* (2.6)*, and*

$$\nu = \begin{cases} \frac{1}{2} & \text{if } p = \min\{\lambda^{v_0}_1,\ \gamma^{e_0}_1 - \frac{1}{2}\}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

If $1 \le p < \min\{\lambda^{v_0}_1, \gamma^{e_0}_1 - 1/2\}$*, then for any* $h > 0$ *there holds*

$$\|u - u_{hp}\|_{\tilde H^{1/2}(\Gamma)} \le C\, h^{p+1/2}. \tag{2.13}$$

The positive constants C *in* (2.10) *and* (2.13) *are independent of* h *and* p*.*

Remark 2.2. For problems in two dimensions with singularities comprising poly-logarithmic terms it is known that the p-approximation results have poly-logarithmic contributions where the order is reduced by one if the exponents of the singularities are integers, see [5,20]. It is an open problem whether this is true also in three dimensions.

A proof of Theorem 2.2 is given in Section 7.

3. Preliminaries

We introduce the Sobolev spaces and prove several technical lemmas.

For details concerning Sobolev spaces we refer to [18,24]. For a domain $\Omega \subset \mathbb{R}^n$ and an integer s, let $H^s(\Omega)$ be the closure of $C^\infty(\Omega)$ with respect to the norm

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^{s-1}(\Omega)} + |u|^2_{H^s(\Omega)} \quad (s \geq 1).$$

Here,

$$|u|^2_{H^s(\Omega)} = \int_\Omega |D^s u(x)|^2 \, \mathrm{d}x, \quad \text{and} \quad H^0(\Omega) = L_2(\Omega),$$

where $|D^s u(x)|^2 = \sum_{|\alpha|=s} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For a positive non-integer s with $s = m + \sigma$ with integer $m \geq 0$ and $0 < \sigma < 1$, the norm in $H^s(\Omega)$ is

$$\|u\|^2_{H^s(\Omega)} = \|u\|^2_{H^m(\Omega)} + |u|^2_{H^s(\Omega)}$$

with semi-norm

$$|u|^2_{H^s(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}} \, \mathrm{d}x \, \mathrm{d}y.$$

For $0 < s \leq 1$, the closure of $C_0^\infty(\Omega)$ with respect to the above norms is denoted by $H_0^s(\Omega)$. For a domain Ω with Lipschitz boundary $\partial\Omega$, $\tilde{H}^{1/2}(\Omega)$ denotes the space of functions in $H^{1/2}(\Omega)$ whose extensions by zero are elements of $H^{1/2}(\mathbb{R}^n)$. A norm in this space is

$$\|u\|^2_{\tilde{H}^{1/2}(\Omega)} = \|u\|^2_{L_2(\Omega)} + |u|^2_{H^{1/2}(\Omega)} + \int_\Omega \frac{|u(x)|^2}{\operatorname{dist}(x, \partial\Omega)} \, \mathrm{d}x.$$

For non-integer s, we equivalently define the Sobolev spaces by real interpolation:

$$H^s(\Omega) = \Big(L_2(\Omega), H^1(\Omega)\Big)_{s,2} \quad (0 < s < 1)$$

and

$$\tilde{H}^{1/2}(\Omega) = \Big(L_2(\Omega), H_0^s(\Omega)\Big)_{\frac{1}{2s},2} \quad (1/2 < s \leq 1).$$

We will also need the Besov spaces $B^s_\infty(\Omega)$ defined *via* interpolation between the above Sobolev spaces (see [9,10]): let $s_1, s_2 \in \mathbb{R}$, $0 \leq s_1 < s_2$, and $s = (1-\theta)s_1 + \theta s_2$ for $0 < \theta < 1$, then

$$B^s_\infty(\Omega) = \Big(H^{s_1}(\Omega), H^{s_2}(\Omega)\Big)_{\theta,\infty}.$$

This space is equipped with the norm

$$\|u\|_{B^s_\infty(\Omega)} = \sup_{t>0} t^{-\theta} K(u,t),$$

where

$$K(u,t) = \inf_{u=v+w} \big(\|v\|_{H^{s_1}(\Omega)} + t\,\|w\|_{H^{s_2}(\Omega)}\big).$$

For integer $k \geq 0$ and $\mu \in [0,1]$ we consider the spaces of continuously differentiable functions $C^k(\bar{\Omega})$ and $C^{k,\mu}(\bar{\Omega})$ with norms

$$\|u\|_{C^k(\bar{\Omega})} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|$$

and

$$\|u\|_{C^{k,\mu}(\bar\Omega)} = \|u\|_{C^k(\bar\Omega)} + \sum_{|\alpha|=k} \sup_{x,y\in\Omega,\, x\neq y} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\mu}.$$

Now let us collect several technical lemmas. We will need the following scaling result.

Lemma 3.1. *Let K^h and K be two open subsets of $\mathbb{R}^n$ such that $K^h = M(K)$ under an invertible affine mapping M. Let $\operatorname{diam} K^h \simeq \rho_{K^h} \simeq h$ and $\operatorname{diam} K \simeq \rho_K \simeq 1$. If $u \in H^m(K^h)$ with integer $m \geq 0$, then $\hat u = u \circ M \in H^m(K)$ and there exists a positive constant C depending on m but not on h or u such that*

$$|\hat u|_{H^m(K)} \leq C h^{m-\frac{n}{2}} |u|_{H^m(K^h)}. \tag{3.1}$$

Analogously for any $\hat u \in H^m(K)$ there holds

$$|u|_{H^m(K^h)} \leq C h^{\frac{n}{2}-m} |\hat u|_{H^m(K)}. \tag{3.2}$$

Moreover, if $\hat u \in H^s(K)$ with real $s \in [0,m]$, then

$$C_1 h^{\frac{n}{2}} \|\hat u\|_{H^s(K)} \leq \|u\|_{H^s(K^h)} \leq C_2 h^{\frac{n}{2}-s} \|\hat u\|_{H^s(K)}. \tag{3.3}$$

For the proof of (3.1), (3.2) see [13], Theorem 3.1.2. Inequalities (3.3) then follow by interpolation (see [4], Lem. 4.3).

Remark 3.1. The notation introduced in Lemma 3.1 will be used frequently in this paper. If not specified otherwise, $K^h \subset \mathbb{R}^2$ is assumed to be a triangle or parallelogram (an element of the mesh Δ_h) such that $\operatorname{diam} K^h \simeq \rho_{K^h} \simeq h$ (see (2.1)) and $K^h = M(K)$, where $K \subset \mathbb{R}^2$ is a triangle or parallelogram with $\operatorname{diam} K \simeq \rho_K \simeq 1$ and M is an invertible affine mapping of K onto K^h. The functions u and $\hat u$ defined on K^h and K, respectively, satisfy the relations: $\hat u = u \circ M$ and $u = \hat u \circ M^{-1}$.

The following two lemmas are Theorem 3.8 and Lemma 5.5 of Chapter 2 in [26] (for the case of a triangle or parallelogram K).

Lemma 3.2. *Let $m > 1$ be real. Let $\mu = m-1$ if $m < 2$, $\mu < 1$ if $m = 2$, and $\mu = 1$ if $m > 2$. Then $H^m(K) \subset C^{0,\mu}(\bar K)$, and*

$$\|u\|_{C^{0,\mu}(\bar K)} \leq C \|u\|_{H^m(K)}.$$

Lemma 3.3. *Let $u \in H^s(K)$ for real $s \geq 0$, and $v \in C^{[s]'-1,1}(\bar K)$, where $[s]'$ denotes the minimal integer such that $s \leq [s]'$. Then $uv \in H^s(K)$, and*

$$\|uv\|_{H^s(K)} \leq C \|u\|_{H^s(K)} \|v\|_{C^{[s]'-1,1}(\bar K)}.$$

The next lemma is the scaled version of Lemma 9.2 in [28].

Lemma 3.4. *Let K^h be a triangle (respectively, a parallelogram) satisfying the assumptions of Lemma 3.1, and let l^h be a side of K^h with vertices v_1, v_2. Let $w_{hp} \in \mathcal{P}_p(l^h)$ be such that $w_{hp}(v_1) = w_{hp}(v_2) = 0$, and $\|w_{hp}\|_{L_2(l^h)} \leq f(h,p)$. Then there exists $u_{hp} \in \mathcal{P}_{2p+1}(K^h)$ (respectively, $u_{hp} \in \mathcal{P}_p(K^h)$) such that $u_{hp} = w_{hp}$ on l^h, $u_{hp} = 0$ on $\partial K^h \backslash l^h$, and for $0 \leq s \leq 1$*

$$\|u_{hp}\|_{H^s(K^h)} \leq C\, h^{1/2-s}\, p^{-1+2s}\, f(h,p).$$

Proof. One has (see Rem. 3.1) $K^h = M(K)$ with $K = T$ (respectively, $K = Q$). Let l be a side of K such that $l^h = M(l)$. Then $\hat w_{hp} = w_{hp} \circ M \in \mathcal{P}_p(l)$ and by Lemma 3.1 there holds

$$\|\hat w_{hp}\|_{L_2(l)} \leq C h^{-1/2} \|w_{hp}\|_{L_2(l^h)} \leq C h^{-1/2} f(h,p).$$

Applying now Lemma 9.2 of [28] to the function $\hat{w}_{hp}$ we find a polynomial $\hat{u}_{hp} \in \mathcal{P}^1_{2p+1}(K),\ K = T$ (respectively, $\hat{u}_{hp} \in \mathcal{P}^2_p(K),\ K = Q$) such that $\hat{u}_{hp} = \hat{w}_{hp}$ on l, $\hat{u}_{hp} = 0$ on $\partial K \backslash l$, and for $0 \le s \le 1$

$$\|\hat{u}_{hp}\|_{H^s(K)} \le C\, h^{-1/2}\, p^{-1+2s}\, f(h,p).$$

Setting $u_{hp} = \hat{u}_{hp} \circ M^{-1}$ and using again Lemma 3.1 it is easy to see that u_{hp} satisfies all conditions of the lemma. □

The next lemma is to split the norm in a fractional order Sobolev space onto sub-domains and is critical to prove global approximation results by using local approximation results on sub-domains. Since this result is of wider interest we present it in a more general form than needed in this paper.

Let $\Gamma \subset \mathbb{R}^n$ $(n = 2, 3)$ be a polygon $(n = 2)$ or a polyhedron $(n = 3)$, and let $\Delta = \{\Gamma_j\}$ be a regular mesh on Γ consisting of shape regular elements (being affine mappings of a bounded number of reference elements). For each $\Gamma_j \in \Delta$ we denote $h_j = \mathrm{diam}(\Gamma_j)$. In the lemma below we will consider a locally quasi-uniform mesh Δ on Γ in the sense that there exists a positive constant σ_1 independent of the mesh such that for any patch $\delta = \{\Gamma_i\} \subset \Delta$ of neighbouring elements there holds

$$\max_{j:\,\Gamma_j \in \delta} h_j \le \sigma_1 h_i \qquad \text{for each } \Gamma_i \in \delta.$$

Lemma 3.5. *Let $\Gamma \subset \mathbb{R}^n$ $(n = 2, 3)$ be a polygon $(n = 2)$ or a polyhedron $(n = 3)$, and let $\Delta = \{\Gamma_j\}$ be a locally quasi-uniform mesh on Γ. Then for $0 < s < 1$*

$$\|u\|^2_{H^s(\Gamma)} \ge \sum_j \|u\|^2_{H^s(\Gamma_j)} \quad \forall u \in H^s(\Gamma), \tag{3.4}$$

and for $1/2 < s < 1$ there holds

$$\|u\|^2_{H^s(\Gamma)} \le C \sum_j \left(h_j^{-2s} \|u\|^2_{L_2(\Gamma_j)} + |u|^2_{H^s(\Gamma_j)} \right) \quad \forall u \in H^s(\Gamma). \tag{3.5}$$

The positive constant C in (3.5) is independent of u and the mesh Δ.

Proof. Since $\|u\|^2_{L_2(\Gamma)} = \sum_j \|u\|^2_{L_2(\Gamma_j)}$, it is enough to consider the semi-norm in $H^s(\Gamma)$. For $s \in (0,1)$ one has

$$\begin{aligned}
|u|^2_{H^s(\Gamma)} &= \sum_{i,j} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y \\
&= \left(\sum_{i,j:\,\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset} + \sum_{i,j:\,\bar{\Gamma}_i \cap \bar{\Gamma}_j \ne \emptyset,\ i \ne j} + \sum_{i=j} \right) \int_{\Gamma_i} \int_{\Gamma_j} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, \mathrm{d}x\, \mathrm{d}y \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{3.6}$$

This immediately leads to (3.4), because $I_1,\ I_2 \ge 0$ and

$$I_3 = \sum_{j:\,\Gamma_j \in \Delta} |u|^2_{H^s(\Gamma_j)}. \tag{3.7}$$

Let $\frac{1}{2} < s < 1$. We will estimate the terms I_1 and I_2 in (3.6) separately. Let $\Gamma_i,\ \Gamma_j \in \Delta$ be such that $\bar{\Gamma}_i \cap \bar{\Gamma}_j = \emptyset$.

Denoting $d_{ij} = \mathrm{dist}\,(\Gamma_i, \Gamma_j)$ we have

$$\begin{aligned}\int_{\Gamma_i}\int_{\Gamma_j} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,\mathrm{d}x\,\mathrm{d}y &\le \frac{2}{d_{ij}^{n+2s}}\left(\int_{\Gamma_j}|u(x)|^2\,\mathrm{d}x\int_{\Gamma_i}\mathrm{d}y + \int_{\Gamma_j}\mathrm{d}x\int_{\Gamma_i}|u(y)|^2\,\mathrm{d}y\right)\\ &\le \frac{C}{d_{ij}^{n+2s}}\left(h_i^n\|u\|^2_{L_2(\Gamma_j)} + h_j^n\|u\|^2_{L_2(\Gamma_i)}\right).\end{aligned}$$

Hence

$$\begin{aligned} I_1 &\le C\sum_{i,j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset} \frac{h_i^n\|u\|^2_{L_2(\Gamma_j)} + h_j^n\|u\|^2_{L_2(\Gamma_i)}}{d_{ij}^{n+2s}} = C\sum_{i,j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset}\frac{2h_j^n\|u\|^2_{L_2(\Gamma_i)}}{d_{ij}^{n+2s}}\\ &= C\sum_i \|u\|^2_{L_2(\Gamma_i)}\sum_{j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset}\frac{2h_j^n}{d_{ij}^{n+2s}}. \end{aligned} \tag{3.8}$$

Let us fix an arbitrary $\Gamma_i \in \Delta$. We introduce polar coordinates with the origin at some point $x^i \in \Gamma_i$ and denote by $r_i = r_i(x) = |x - x^i|$ the polar radius. Then there exists a positive constant C independent of i and the mesh Δ such that

$$d_{ij} = \mathrm{dist}\,(\Gamma_i,\Gamma_j) \ge Cr_i(x) \qquad \forall x\in\Gamma_j, \quad \forall\,\Gamma_j\in\{\Gamma_j;\ \bar\Gamma_j\cap\bar\Gamma_i=\varnothing\}.$$

Moreover,

$$\cup\{\bar\Gamma_j;\ \bar\Gamma_j\cap\bar\Gamma_i=\varnothing\}\subset\{x\in\bar\Gamma;\ \kappa h_i\le|x-x_i|\le R\}$$

with some constants κ and R independent of the mesh. Therefore we estimate for fixed Γ_i

$$\begin{aligned}\sum_{j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset}\frac{h_j^n}{d_{ij}^{n+2s}} &\le C\sum_{j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset}\int_{\Gamma_j}\frac{\mathrm{d}x}{d_{ij}^{n+2s}} \le C\sum_{j:\,\bar\Gamma_i\cap\bar\Gamma_j=\emptyset}\int_{\Gamma_j}\frac{\mathrm{d}x}{r_i^{n+2s}(x)}\\ &\le C\int_{\kappa h_i}^{R} r_i^{-n-2s}\,r_i^{n-1}\,\mathrm{d}r_i \le C\,h_i^{-2s}.\end{aligned}$$

Then we obtain by (3.8)

$$I_1 \le C\sum_{i:\,\Gamma_i\in\Delta} h_i^{-2s}\,\|u\|^2_{L_2(\Gamma_i)}. \tag{3.9}$$

In order to estimate I_2 we again fix an arbitrary $\Gamma_i \in \Delta$ and denote by K^{h_i} the patch of neighbouring elements touching Γ_i, *i.e.*, $\bar K^{h_i} = \cup\{\bar\Gamma_j;\ \bar\Gamma_j\cap\bar\Gamma_i\ne\varnothing\}$. Observe that the number of elements in any patch K^{h_i} is bounded by a constant independent of i and Δ. Let K be an open subset in $\mathbb{R}^n$ such that $K^{h_i} = M(K)$, where M is the affine mapping (scaling) satisfying $M : x_k = h_i\hat x_k,\ k = 1,\dots,n,\ x\in K^{h_i},\ \hat x\in K$. Then $\bar K = \cup_j \bar K_j$, where $K_j = M^{-1}(\Gamma_j)$ for each $\Gamma_j \subset K^{h_i}$. Moreover, due to the local quasi-uniformity of the mesh, $\mathrm{diam}\,K \simeq \mathrm{diam}\,K_j \simeq 1$ for each $K_j \subset K$. Therefore

$$|u|^2_{H^s(K^{h_i})}\simeq h_i^{n-2s}|\hat u|^2_{H^s(K)},\quad \|u\|^2_{L_2(\Gamma_j)}\simeq h_i^n\|\hat u\|^2_{L_2(K_j)},\quad |u|^2_{H^s(\Gamma_j)}\simeq h_i^{n-2s}|\hat u|^2_{H^s(K_j)}$$

with $\hat{u} = u \circ M$, and applying Lemma 3.1 of [11] we obtain

$$\begin{aligned} |u|^2_{H^s(K^{h_i})} &\simeq h_i^{n-2s} |\hat{u}|^2_{H^s(K)} \le C h_i^{n-2s} \sum_{j: K_j \subset K} \Big(\|\hat{u}\|^2_{L_2(K_j)} + |\hat{u}|^2_{H^s(K_j)} \Big) \\ &\le C h_i^{n-2s} \sum_{j:\Gamma_j \subset K^{h_i}} \Big(h_i^{-n} \|u\|^2_{L_2(\Gamma_j)} + h_i^{-n+2s} |u|^2_{H^s(\Gamma_j)} \Big) \\ &\le C \sum_{j:\Gamma_j \subset K^{h_i}} \Big(h_i^{-2s} \|u\|^2_{L_2(\Gamma_j)} + |u|^2_{H^s(\Gamma_j)} \Big). \end{aligned} \tag{3.10}$$

Since $h_j \simeq h_i$ for every $\Gamma_j \subset K^{h_i}$ and each patch K^{h_i} has a bounded number of elements, we estimate by (3.10)

$$\begin{aligned} I_2 &= \sum_i \sum_{j:\, \bar{\Gamma}_j \cap \bar{\Gamma}_i \ne \emptyset,\, j \ne i} \int_{\Gamma_i} \int_{\Gamma_j} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \,\mathrm{d}x \,\mathrm{d}y \le \sum_i |u|^2_{H^s(K^{h_i})} \\ &\le C \sum_{j:\, \Gamma_j \in \Delta} \Big(h_j^{-2s} \|u\|^2_{L_2(\Gamma_j)} + |u|^2_{H^s(\Gamma_j)} \Big). \end{aligned} \tag{3.11}$$

Now inequality (3.5) follows from (3.7), (3.9), and (3.11) making use of decomposition (3.6). □

Remark 3.2. Inequality (3.4) was given in [32], Lemma 3.2, for the case when the norm in H^s is defined by the method of complex interpolation, and was proved in [4] in the case of real interpolation.

4. Auxiliary approximation results

In this section we formulate several results regarding the approximation of smooth and singular functions. For the approximation of smooth functions we will need the following lemma. For a more elaborate discussion of polynomial interpolation operators see [15].

Lemma 4.1. *Let K^h and K be two triangles (parallelograms) satisfying the assumptions of Lemma 3.1, and let l be a side of K. Suppose that $u \in H^m(K^h)$. Then $\hat{u} = u \circ M \in H^m(K)$ and there exists a family of operators $\{\hat{\pi}_p\}$, $p = 1, 2, \ldots$, $\hat{\pi}_p : H^m(K) \to \mathcal{P}_p(K)$ such that*

$$\|\hat{u} - \hat{\pi}_p \hat{u}\|_{H^q(K)} \le C h^{\mu-1} p^{-(m-q)} \|u\|_{H^m(K^h)}, \quad m \ge 0, \quad 0 \le q \le m, \tag{4.1}$$

$$|(\hat{u} - \hat{\pi}_p \hat{u})(\hat{x})| \le C h^{\mu-1} p^{-(m-1)} \|u\|_{H^m(K^h)}, \quad m > 1, \quad \hat{x} \in K, \tag{4.2}$$

$$\|\hat{u} - \hat{\pi}_p \hat{u}\|_{H^s(l)} \le C h^{\mu-1} p^{-(m-s-1/2)} \|u\|_{H^m(K^h)}, \quad m > 3/2, \quad s = 0, 1, \tag{4.3}$$

where $\mu = \min\{m, p+1\}$, and the positive constants C in (4.1)–(4.3) are independent of u, p, and h but depend on m.

Proof. Making use of Lemma 4.4 in [6], estimates (4.1)–(4.3) follow from the corresponding results of [7], Lemma 3.1 (for details, see [6], Lem. 4.5, in particular, estimates (4.14), (4.16) therein). □

Now we can prove the result on the approximation of smooth functions. It gives estimates for the error of this approximation in the norms of the spaces $\tilde{H}^{1/2}(\Gamma)$ and $H^s(\Gamma)$, $s \in [0,1]$. For the space $H^1(\Gamma)$ this result has been proved before in [6], Theorem 4.6.

Proposition 4.1. *Let $m > 1$. Then for $u \in H^m(\Gamma) \cap H^1_0(\Gamma)$ there exists $u_{hp} \in V_0^{h,p}(\Gamma)$ such that for $s \in [0,1]$*

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C h^{\mu-s} p^{-(m-s)} \|u\|_{H^m(\Gamma)}, \quad \mu = \min\{m, p+1\} \tag{4.4}$$

if the mesh Δ_h *on* Γ *does not contain triangles, and*

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le Ch^{\mu-s}p^{-(m-\tilde{s})}\|u\|_{H^m(\Gamma)} \tag{4.5}$$

if Δ_h *contains triangles; here* μ *is the same as in* (4.4) *and*

$$\tilde{s} = \begin{cases} 1/2 & \textit{if} \;\; s \in [0, 1/2), \\ 1/2 + \varepsilon, \;\; \varepsilon > 0 & \textit{if} \;\; s = 1/2, \\ s & \textit{if} \; s \in (1/2, 1]. \end{cases} \tag{4.6}$$

Moreover,

$$\|u - u_{hp}\|_{\tilde{H}^{1/2}(\Gamma)} \le C\, h^{\min\{m,p+1\}-1/2}\, p^{-(m-1/2-\varepsilon)}\, \|u\|_{H^m(\Gamma)}, \tag{4.7}$$

where $\varepsilon = 0$ *if* Δ_h *does not contain triangles, and* $\varepsilon > 0$ *if* Δ_h *contains triangles.*

Proof. First, let us assume that $m > 3/2$. Let $K^h = \Gamma_j \in \Delta_h$ and $K = Q$ (or $K = T$) so that $K^h = M_j(K)$. Thus K^h and K satisfy the assumptions of Lemma 3.1 and, due to Lemma 4.1, there exists $\hat{v}_j = \hat{\pi}_p(u \circ M_j) \in \mathcal{P}_p(K)$ such that for $s = 0, 1$

$$\|\hat{u} - \hat{v}_j\|_{H^s(K)} \le Ch^{\mu-1}p^{-(m-s)}\|u\|_{H^m(\Gamma_j)}, \tag{4.8}$$

$$\|\hat{u} - \hat{v}_j\|_{H^s(l)} \le Ch^{\mu-1}p^{-(m-s-1/2)}\|u\|_{H^m(\Gamma_j)}, \tag{4.9}$$

where $l \subset \partial K$ denotes a side of K, $\mu = \min\{m, p+1\}$. Since $m > 3/2$, we can modify $\hat{v}_j$ as in Theorem 4.1 of [7] to obtain $\hat{v}_j = \hat{u}$ at the vertices of K.

Let $v_j = \hat{v}_j \circ M_j^{-1}$. Then $v_j \in \mathcal{P}_p(\Gamma_j)$ and we obtain by Lemma 3.1 and (4.8)

$$\|u - v_j\|_{H^s(\Gamma_j)} \le Ch^{\mu-s}p^{-(m-s)}\|u\|_{H^m(\Gamma_j)}, \quad \mu = \min\{m, p+1\}, \quad s = 0, 1. \tag{4.10}$$

Further we consider two elements Γ_i, $\Gamma_j \in \Delta_h$ having the common edge $l^h = \bar{\Gamma}_i \cap \bar{\Gamma}_j$. Let $v_i \in \mathcal{P}_p(\Gamma_i)$ and $v_j \in \mathcal{P}_p(\Gamma_j)$ be the polynomials constructed above. Then the jump $w = (v_j - v_i)|_{l^h} \in \mathcal{P}_p(l^h)$ vanishes at the end points of l^h. Furthermore, using (4.9) and standard interpolation arguments, we find

$$\|\hat{w}\|_{H^s(l)} \le \|\hat{u} - \hat{v}_i\|_{H^s(l)} + \|\hat{u} - \hat{v}_j\|_{H^s(l)} \le Ch^{\mu-1}p^{-(m-s-1/2)}\|u\|_{H^m(\Gamma_i \cup \Gamma_j)}, \;\; s = 0, 1, \tag{4.11}$$

$$\|\hat{w}\|_{\tilde{H}^{1/2}(l)} \le Ch^{\mu-1}p^{-(m-1)}\|u\|_{H^m(\Gamma_i \cup \Gamma_j)}, \tag{4.12}$$

where $l = M_i^{-1}(l^h)$, $M_i : K \to \Gamma_i$, $\mu = \min\{m, p+1\}$.

We will adjust the function v_i on Γ_i to obtain the continuity of the approximation on the inter-element edge. If Γ_i is a parallelogram, we use the constructions from the proof of Theorem 4.1 in [7]. In this case $K = Q = I \times I$, $I = (-1, 1)$ and without loss of generality we can assume that $l = \{(\hat{x}_1, \hat{x}_2);\; \hat{x}_1 \in I, \hat{x}_2 = -1\}$. Then there exists a polynomial $\hat{\psi}_p(\hat{x}_2) \in \mathcal{P}_p(I)$ such that (see [7], pp. 759–760)

$$\hat{\psi}_p(-1) = 1, \qquad \hat{\psi}_p(1) = 0,$$

and

$$\|\hat{\psi}_p\|_{H^s(I)} \le Cp^{s-1/2}, \quad s = 0, 1. \tag{4.13}$$

Let us define $\hat{z} := \hat{w}\hat{\psi}_p(\hat{x}_2)$. Then $\hat{z} \in \mathcal{P}_p^2(Q)$, $\hat{z} = \hat{w}$ on l, $\hat{z} = 0$ on $\partial Q\backslash l$, and making use of (4.11), (4.13) we prove

$$\begin{aligned}\|\hat{z}\|_{H^1(Q)} &\le C\left(\|\hat{w}\|_{H^1(l)}\|\hat{\psi}_p\|_{H^0(I)} + \|\hat{w}\|_{H^0(l)}\|\hat{\psi}_p\|_{H^1(I)}\right)\\ &\le Ch^{\mu-1}p^{-(m-1)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)},\end{aligned}$$

$$\|\hat{z}\|_{H^0(Q)} = \|\hat{w}\|_{H^0(l)}\|\hat{\psi}_p\|_{H^0(I)} \le Ch^{\mu-1}p^{-m}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}.$$

If Γ_i is a triangle, then we use the result of [1], Theorem 1, giving stable, polynomial preserving trace liftings on Γ_i: there exists $\hat{z} \in \mathcal{P}_p^1(T)$ such that $\hat{z} = \hat{w}$ on l, $\hat{z} = 0$ on $\partial T\backslash l$,

$$\|\hat{z}\|_{H^1(T)} \le C\|\hat{w}\|_{\tilde{H}^{1/2}(l)}, \qquad \|\hat{z}\|_{H^{1/2}(T)} \le C\|\hat{w}\|_{L_2(l)}.$$

Then using (4.11), (4.12), and interpolation arguments we obtain

$$\begin{aligned}\|\hat{z}\|_{H^s(T)} &\le Ch^{\mu-1}p^{-(m-s)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}, \qquad s\in[1/2,1],\\ \|\hat{z}\|_{H^s(T)} &\le \|\hat{z}\|_{H^{1/2}(T)} \le Ch^{\mu-1}p^{-(m-1/2)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}, \qquad s\in[0,1/2).\end{aligned}$$

Now for both cases considered above we define $z := \hat{z}\circ M_i^{-1} \in \mathcal{P}_p(\Gamma_i)$. Then setting $\tilde{v} = v_i + z$ on Γ_i and $\tilde{v} = v_j$ on Γ_j, we find a continuous piecewise polynomial on $\Gamma_i \cup \Gamma_j \cup l^h$ such that $\|u-\tilde{v}\|_{H^s(\Gamma_j)}$ is bounded as in (4.10). On Γ_i we use Lemma 3.1 and corresponding estimates for $\|\hat{z}\|_{H^s(K)}$ with $K = Q$ or T:

$$\|u-\tilde{v}\|_{H^s(\Gamma_i)} \le \|u-v_i\|_{H^s(\Gamma_i)} + Ch^{1-s}\|\hat{z}\|_{H^s(Q)} \le Ch^{\mu-s}p^{-(m-s)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}, \quad s=0,1$$

if Γ_i is a parallelogram, and

$$\|u-\tilde{v}\|_{H^s(\Gamma_i)} \le Ch^{\mu-s}p^{-(m-s)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}, \quad s\in[1/2,1], \tag{4.14}$$

$$\|u-\tilde{v}\|_{H^s(\Gamma_i)} \le Ch^{\mu-s}p^{-(m-1/2)}\|u\|_{H^m(\Gamma_i\cup\Gamma_j)}, \quad s\in[0,1/2) \tag{4.15}$$

if Γ_i is a triangle.

Repeating these procedures for each pair of adjacent elements as well as for the elements Γ_i having the side $l^h \subset \partial\Gamma$ we construct the function $u_{hp} \in V_0^{h,p}(\Gamma)$. If the mesh Δ_h on Γ consists only of parallelograms, then u_{hp} satisfies (4.4) for $s = 0, 1$. For real $s \in (0,1)$ this result then follows by interpolation.

If the mesh Δ_h on Γ contains triangular elements, then we deduce (4.5) from (4.14), (4.15). In fact, for $s \in [0, \frac{1}{2})$, (4.5) immediately follows from (4.15), because $\tilde{H}^s(\Gamma) = H^s(\Gamma) = H_0^s(\Gamma)$ for these values of s (see [18]). If $s \in (\frac{1}{2}, 1)$, then we use Lemma 3.5 and estimates (4.14), (4.15):

$$\begin{aligned}\|u-u_{hp}\|^2_{H^s(\Gamma)} &\le C\left(h^{-2s}\|u-u_{hp}\|^2_{L_2(\Gamma)} + \sum_{j:\Gamma_j\subset\Gamma}|u-u_{hp}|^2_{H^s(\Gamma_j)}\right)\\ &\le C\left(h^{-2s}h^{2\mu}p^{-2(m-1/2)}\|u\|^2_{H^m(\Gamma)} + h^{2(\mu-s)}p^{-2(m-s)}\|u\|^2_{H^m(\Gamma)}\right)\\ &\le Ch^{2(\mu-s)}p^{-2(m-s)}\|u\|^2_{H^m(\Gamma)}.\end{aligned}$$

For $s = \frac{1}{2}$, estimate (4.5) then follows *via* interpolation between $H^{s'}(\Gamma)$ and $H^{s''}(\Gamma)$, where $s' = \frac{1}{2} - 2\varepsilon$, $s'' = \frac{1}{2} + 2\varepsilon$, $0 < \varepsilon < \frac{1}{4}$.

Since $(u - u_{hp}) \in H_0^s(\Gamma)$ for any $s \in (\frac{1}{2}, 1]$, we prove (4.7) (for the meshes of both types) by interpolation between $H_0^{s'}(\Gamma)$ and $H_0^{s''}(\Gamma)$ with the same s', s'' as above.

So far we assumed that $m > \frac{3}{2}$. Now we prove the assertion for $1 < m \le \frac{3}{2}$. We will consider only the case of meshes containing triangles. The arguments for meshes without triangles are analogous.

As it is shown in [9],

$$H^m(\Gamma) \cap H_0^1(\Gamma) = (H^1(\Gamma), H^2(\Gamma))_{m-1,2} \cap H_0^1(\Gamma) = (H_0^1(\Gamma), H^2(\Gamma) \cap H_0^1(\Gamma))_{m-1,2}$$

and there holds

$$H^m(\Gamma) \cap H_0^1(\Gamma) \subset B_\infty^m(\Gamma) \cap H_0^1(\Gamma).$$

On the other hand, by the reiteration theorem we find that

$$(H_0^1(\Gamma), H^2(\Gamma) \cap H_0^1(\Gamma))_{m-1,2} = (H_0^s(\Gamma), H^k(\Gamma) \cap H_0^1(\Gamma))_{\theta,2} \quad \text{and} \quad B_\infty^m(\Gamma) = (H^s(\Gamma), H^k(\Gamma))_{\theta,\infty}$$

with $s \in [0,1]$, $\frac{3}{2} < k \le 2$, $\theta = \frac{m-s}{k-s} \in (0,1)$. Therefore,

$$H^m(\Gamma) \cap H_0^1(\Gamma) = (H_0^s(\Gamma), H^k(\Gamma) \cap H_0^1(\Gamma))_{\theta,2} \subset B_\infty^m(\Gamma) \cap H_0^1(\Gamma) = (H^s(\Gamma), H^k(\Gamma))_{\theta,\infty} \cap H_0^1(\Gamma).$$

Decomposing $u \in H^m(\Gamma) \cap H_0^1(\Gamma)$ as

$$u = v + w, \quad v \in H_0^s(\Gamma), \quad w \in H^k(\Gamma) \cap H_0^1(\Gamma)$$

and applying (4.5) with $m := k$ to the function w we find $w_{hp} \in V_0^{h,p}(\Gamma)$ satisfying

$$\|u - w_{hp}\|_{H^s(\Gamma)} \le C\left(\|v\|_{H^s(\Gamma)} + t\|w\|_{H^k(\Gamma)}\right).$$

Here $t = h^{k-s}p^{-(k-\tilde{s})}$ and $\tilde{s}$ is defined by (4.6). Hence,

$$\|u - w_{hp}\|_{H^s(\Gamma)} \le C \inf_{u=v+w}\left(\|v\|_{H^s(\Gamma)} + t\|w\|_{H^k(\Gamma)}\right),$$

and recalling the definition of the Besov space $B_\infty^m(\Gamma)$ we obtain

$$\|u - w_{hp}\|_{H^s(\Gamma)} \le Ct^\theta \|u\|_{B_\infty^m(\Gamma)} \le Ch^{m-s}p^{-\frac{(k-\tilde{s})(m-s)}{k-s}}\|u\|_{H^m(\Gamma)},$$

which gives (4.5) for $s \in [1/2, 1]$. If $s \in [0, 1/2)$, then $\tilde{s} = 1/2$ and (4.5) also holds:

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le Ch^{m-s}p^{-\frac{(k-1/2)(m-s)}{k-s}}\|u\|_{H^m(\Gamma)} \le Ch^{m-s}p^{-(m-1/2)}\|u\|_{H^m(\Gamma)}.$$

This finishes the proof of (4.5) for $1 < m \le 3/2$. As before, estimate (4.7) is proved by interpolation. $\square$

Remark 4.1. The given error estimate is of the expected order in h. In p the order is the expected one when no triangular elements are present or when the error is measured in $H^s(\Gamma)$ for $s > 1/2$. Otherwise the p-result appears to be non-optimal. The difficulty lies in finding an hp-interpolation operator that works equally well in $L_2(\Gamma)$ and $H^1(\Gamma)$ such that results in fractional order Sobolev spaces can be obtained by interpolation. The continuity of interpolating functions across element boundaries is generally incorporated by using appropriate extension operators, *cf.* Lemma 3.4. For triangles, such an extension operator which works equally well with respect to h and p in L_2 and H^1 is unknown. Note, however, that there is a quasi-optimal result for an extension operator that could be used to improve the result of Proposition 4.1 for the case $s = 1/2$, see [22]. In this paper we consider singular functions and the slight perturbation of the p-estimate for smooth functions (on triangles when estimating in $\tilde{H}^{1/2}(\Gamma)$) does not affect the main result Theorem 2.2.

Let us recall some known results regarding the approximation of singularities by polynomials of arbitrary degree in fractional order Sobolev spaces on triangles (parallelograms) of fixed size. In the propositions below $K \subset \mathbb{R}^2$ will always denote a triangle or parallelogram satisfying the assumptions of Lemma 3.1. The particular location of K in $\mathbb{R}^2$ will be additionally specified in each proposition. We will consider three types of singular functions on K which correspond to the vertex singularity (see (2.5)) and to the edge-vertex singularities of both types (see (2.6)–(2.9)):

$$u_1(x) = r^\lambda |\log r|^\beta \chi(r) w(\theta), \tag{4.16}$$

$$u_2(x) = x_1^{\lambda-\gamma} x_2^\gamma |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi(r) \tilde{\chi}(\theta), \tag{4.17}$$

$$u_3(x) = x_2^\gamma |\log x_2|^\beta \chi_1(x_1, x_2) \chi_2(x_2), \tag{4.18}$$

where λ and γ are real parameters to be specified, β, β_1, $\beta_2 \geq 0$ are integers, (r, θ) are polar coordinates in $\mathbb{R}^2$, χ, $\tilde{\chi}$, χ_2 are C^∞ cut-off functions satisfying

$$\operatorname{supp} \chi \subset [0, \tau_0], \quad \operatorname{supp} \tilde{\chi} \subset [0, \beta_0], \quad \operatorname{supp} \chi_2 \subset [0, \delta_0]$$

for some $\tau_0, \beta_0, \delta_0 > 0$, and the functions w, χ_1 are sufficiently smooth.

Proposition 4.2 ([28], Thm. 8.2)**.** *Let $K \subset \mathbb{R}^2$ and suppose that the origin O is a vertex of K. Let u_1 be given by (4.16) with $\lambda > 0$ and $\operatorname{supp} \chi \subset [0, \tau_0]$ for $0 < \tau_0 < \rho_K$. Then there exists a sequence $u_{1,p} \in \mathcal{P}_p(K)$, $p = 1, 2, \ldots,$ such that for $0 \leq s \leq 1$*

$$\|u_1 - u_{1,p}\|_{H^s(K)} \leq C\, p^{-2(\lambda+1-s)} (1 + \log p)^\beta. \tag{4.19}$$

Moreover, $u_{1,p}(0, 0) = 0$, $u_{1,p} = 0$ on the sides $l_i \subset \partial K$, $\bar{l}_i \not\ni O$, and

$$\|u_1 - u_{1,p}\|_{L_2(l_k)} \leq C\, p^{-2(\lambda+1/2)} (1 + \log p)^\beta \quad \text{for each side } l_k \subset \partial K,\ O \in \bar{l}_k. \tag{4.20}$$

Proposition 4.3. *Let $K \subset \mathbb{R}^{2+}$. Suppose that the origin O is a vertex of K and one of the other vertices of K lies on the right semi-axis Ox_1. Let u_2 be given by (4.17) with $\lambda > -1/2$, $\gamma > 0$, and assume that $\operatorname{supp} u_2 \subset \bar{S}_0 = \{(r, \theta);\ 0 \leq r \leq \tau_0,\ 0 \leq \theta \leq \beta_0 < \frac{\pi}{4}\} \subset \bar{K}$. Then there exists a sequence $u_{2,p} \in \mathcal{P}_p(K)$, $p = 1, 2, \ldots,$ such that $u_{2,p} = 0$ on ∂K and for $0 \leq s < \min\{1, \lambda + 1, \gamma + 1/2\}$*

$$\|u_2 - u_{2,p}\|_{H^s(K)} \leq C\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)} (1 + \log p)^{\beta_1+\beta_2+\sigma}, \tag{4.21}$$

where $\sigma = \frac{1}{2}$ if $\lambda = \gamma - \frac{1}{2}$, and $\sigma = 0$ otherwise.

This statement was first proved in [28], Theorem 7.2, under the assumptions that $\lambda > 0$, $\gamma > \frac{1}{2}$. Later, in [11], Theorem 3.5, we generalised that result to λ and γ with $\frac{1}{2} < \min\{\lambda + 1, \gamma + \frac{1}{2}\} \leq 1$.

Proposition 4.4. *Let $K \subset \mathbb{R}^{2+}$ and suppose that at least one vertex of K lies on the axis Ox_1. Let $l_k \subset \partial K$ ($k = \overline{1,3}$ or $k = \overline{1,4}$) denote the sides of K, $\tau = \{l_k \subset \partial K;\ \bar{l}_k \cap Ox_1 = \emptyset\}$, and $\mathcal{A} = \{l_k \subset \partial K;\ \bar{l}_k \cap Ox_1$ contains only a single point$\}$. Let u_3 be given by (4.18) with $\gamma > 0$, $\chi_1 \in H^m(K)$, $m > 2\gamma + 2$, and assume that $(\operatorname{supp} u_3) \cap \bar{l}_k = \emptyset$ for each $l_k \in \tau$. Then there exists a sequence $u_{3,p} \in \mathcal{P}_p(K)$, $p = 0, 1, 2, \ldots,$ such that for $0 \leq s < \min\{1, \gamma + 1/2\}$*

$$\|u_3 - u_{3,p}\|_{H^s(K)} \leq C\, (p + 1)^{-2(\gamma+1/2-s)} (1 + \log(p + 1))^\beta. \tag{4.22}$$

Moreover, $u_{3,p}$ vanishes at the vertices of K, $u_{3,p} = 0$ on $(\partial K \cap Ox_1) \cup \tau$, and for every side $l_k \in \mathcal{A}$,

$$\|u_3 - u_{3,p}\|_{L_2(l_k)} \leq C\, (p + 1)^{-2(\gamma+1/2)} (1 + \log(p + 1))^\beta. \tag{4.23}$$

Proof. If $p = 0$, then we set $u_{3,p} = 0$ on K, and (4.22), (4.23) are valid. Let $p \ge 1$. Then for $\gamma > \frac{1}{2}$ the assertion is proved in [28], Theorem 6.2. For $0 < \gamma \le \frac{1}{2}$ see Theorem 3.2 and estimates (3.20), (3.21) in [11]. □

Now we will study the approximation of a certain singular function with small support. For this function we prove an approximation result which plays an essential role in our further analysis.

Let $e \in E$ be an edge of Γ with vertices v, w. Recalling that A_e denotes the union of elements at the edge e, we consider the function

$$u(x_{e1}, x_{e2}) = x_{e2}^{\gamma} |\log x_{e2}|^{\beta} \chi_1(x_{e1}, x_{e2}) \chi_2(x_{e2}/h_0), \quad (x_{e1}, x_{e2}) \in A_e, \tag{4.24}$$

where $\gamma > 0$, $\beta \ge 0$ is integer, $h_0 = (\sigma_1\sigma_2)^{-1}h$ with σ_1, σ_2 being the same as in (2.1), χ_2 is a C^∞ cut-off function with support in $[0, \delta]$ for some $0 < \delta < 1$, $\chi_1 \in H^m(A_e)$ with integer $m > 2\gamma + 2$, and χ_1 vanishes on the edges l_v, $l_w \subset \partial A_e$ with $\bar{l}_v \cap \bar{e} = \{v\}$ and $\bar{l}_w \cap \bar{e} = \{w\}$.

Observe that $u \in H^s(A_e)$ for any $s \in [0, 1/2 + \gamma)$. Due to (2.1), $h_0 \le \rho_j$ for any $\Gamma_j \subset A_e$, and hence $\operatorname{supp} u \subset \bar{A}_e$.

Lemma 4.2. *Let u be given by (4.24). Then for every $p = 1, 2, \ldots$ there exists a continuous function u_{hp} defined on A_e such that $u_{hp} \in \mathcal{P}_p(\Gamma_j)$ for each $\Gamma_j \subset A_e$, $u_{hp} = 0$ on ∂A_e, and for $0 \le s < \min\{1, \gamma + 1/2\}$*

$$\|u - u_{hp}\|_{H^s(A_e)} \le C\, h^{\gamma+1-s}\, p^{-2(\gamma+1/2-s)}\, (1 + \log(p/h))^{\beta} \sum_{t=0}^{m} h^{t-1}\, |\chi_1|_{H^t(A_e)}. \tag{4.25}$$

Proof. For simplicity of notation, and when not leading to ambiguity, we will omit e in the subscripts of the coordinates x_{e1}, x_{e2}. Let $K^h = \Gamma_j \subset A_e$, and let $K \subset \mathbb{R}^{2+}$ be a triangle or parallelogram such that $K^h = M(K)$, where M is the affine mapping

$$M : x_i = h\hat{x}_i, \quad i = 1, 2, \quad x \in K^h, \quad \hat{x} \in K.$$

Then at least one vertex of K lies on the axis $O\hat{x}_1$ and

$$\begin{aligned} \hat{u}(\hat{x}) &= u(h\hat{x}_1, h\hat{x}_2) = h^{\gamma}\hat{x}_2^{\gamma} |\log(h\hat{x}_2)|^{\beta} \chi_1(h\hat{x}_1, h\hat{x}_2)\chi_2(\sigma_1\sigma_2\hat{x}_2), \\ &= h^{\gamma}\hat{x}_2^{\gamma} \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k |\log \hat{x}_2|^{\beta-k} \hat{\chi}_1(\hat{x}_1, \hat{x}_2)\chi_2(\sigma_1\sigma_2\hat{x}_2) = \hat{\varphi}(\hat{x})\hat{\chi}_1(\hat{x}), \end{aligned}$$

where

$$\hat{\varphi}(\hat{x}) = h^{\gamma} \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k \hat{\varphi}_{\beta-k}(\hat{x}_2) \quad \text{with} \quad \hat{\varphi}_i(\hat{x}_2) = \hat{x}_2^{\gamma} |\log \hat{x}_2|^i \chi_2(\sigma_1\sigma_2\hat{x}_2), \; i = 0, \ldots, \beta.$$

Using Proposition 4.4 for each function $\hat{\varphi}_i$, $i = 0, \ldots, \beta$, we find polynomials $\hat{\varphi}_{i,p} \in \mathcal{P}_p(K)$ such that $\hat{\varphi}_{i,p} = 0$ at the vertices of K and on $(\partial K \cap O\hat{x}_1) \cup \tau$,

$$\begin{aligned} \|\hat{\varphi}_i - \hat{\varphi}_{i,p}\|_{H^s(K)} &\le C\, p^{-2(\gamma+1/2-s)}\, (1 + \log p)^i, \quad 0 \le s < \min\{1, \gamma + 1/2\}, \\ \|\hat{\varphi}_i - \hat{\varphi}_{i,p}\|_{L_2(l)} &\le C\, p^{-2(\gamma+1/2)}\, (1 + \log p)^i \quad \text{for every } l \in \mathcal{A}. \end{aligned}$$

Hence, setting

$$\hat{\varphi}_p(\hat{x}) := h^{\gamma} \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k \hat{\varphi}_{\beta-k,p}(\hat{x})$$

we obtain the estimates

$$\begin{aligned}\|\hat\varphi-\hat\varphi_p\|_{H^s(K)} &\le h^\gamma\sum_{k=0}^{\beta}\binom{\beta}{k}|\log h|^k\|\hat\varphi_{\beta-k}-\hat\varphi_{\beta-k,p}\|_{H^s(K)}\\ &\le h^\gamma p^{-2(\gamma+1/2-s)}\sum_{k=0}^{\beta}\binom{\beta}{k}C(k)\log^k(1/h)(1+\log p)^{\beta-k}\\ &\le C(\beta)h^\gamma p^{-2(\gamma+1/2-s)}(1+\log(p/h))^\beta,\quad 0\le s<\min\{1,\gamma+1/2\}, \qquad (4.26)\\ \|\hat\varphi-\hat\varphi_p\|_{L_2(l)} &\le C(\beta)\,h^\gamma\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta\quad \text{for every } l\in\mathcal{A}; \qquad (4.27)\end{aligned}$$

moreover, $\hat\varphi_p=0$ at the vertices of K and on $(\partial K\cap O\hat x_1)\cup\tau$.

Since $\hat\varphi\in H^s(K)$ and $\|\hat\varphi\|_{H^s(K)}\le Ch^\gamma\log^\beta(1/h)$, we estimate by (4.26)

$$\begin{aligned}\|\hat\varphi_p\|_{H^s(K)} &\le \|\hat\varphi-\hat\varphi_p\|_{H^s(K)}+\|\hat\varphi\|_{H^s(K)}\\ &\le Ch^\gamma(1+\log(p/h))^\beta,\quad 0\le s<\min\{1,\gamma+1/2\}, \qquad (4.28)\end{aligned}$$

and similarly by (4.27)

$$\|\hat\varphi_p\|_{L_2(l)}\le Ch^\gamma(1+\log(p/h))^\beta\quad\text{for every } l\in\mathcal{A}. \qquad (4.29)$$

Now let us approximate the smooth function $\hat\chi_1\in H^m(K)$. Using [6], Lemma 4.1, we find a polynomial $\hat\chi_{1,p}=\hat\pi_p\hat\chi_1\in\mathcal{P}_p(K)$ satisfying

$$\begin{aligned}\|\hat\chi_1-\hat\chi_{1,p}\|_{H^q(K)} &\le Cp^{-(m-q)}\|\hat\chi_1\|_{H^m(K)},\quad 0\le q\le m, \qquad (4.30)\\ |(\hat\chi_1-\hat\chi_{1,p})(\hat x)| &\le Cp^{-(m-1)}\|\hat\chi_1\|_{H^m(K)},\quad m>1,\ \hat x\in K. \qquad (4.31)\end{aligned}$$

We define $\hat\psi(\hat x):=\hat\varphi_p(\hat x)\,\hat\chi_{1,p}(\hat x)$. Then $\hat\psi\in\mathcal{P}_{2p}(K)$, $\hat\psi=0$ at the vertices of K and on $(\partial K\cap O\hat x_1)\cup\tau$, and for $0\le s<\min\{1,\gamma+1/2\}$

$$\|\hat u-\hat\psi\|_{H^s(K)}\le\|\hat\chi_1(\hat\varphi-\hat\varphi_p)\|_{H^s(K)}+\|(\hat\chi_1-\hat\chi_{1,p})\hat\varphi_p\|_{H^s(K)}. \qquad (4.32)$$

First, let us consider the case when $1/2<s<\min\{1,\gamma+1/2\}$. Applying Lemmas 3.2 and 3.3 we have for any $\varepsilon>0$

$$\|\hat\chi_1(\hat\varphi-\hat\varphi_p)\|_{H^s(K)}\le C\|\hat\chi_1\|_{C^{0,1}(\bar K)}\|\hat\varphi-\hat\varphi_p\|_{H^s(K)}\le C\|\hat\chi_1\|_{H^{2+\varepsilon}(K)}\|\hat\varphi-\hat\varphi_p\|_{H^s(K)}.$$

Hence, taking ε sufficiently small $(2+\varepsilon<m)$ and using estimate (4.26) we find

$$\|\hat\chi_1(\hat\varphi-\hat\varphi_p)\|_{H^s(K)}\le C\,h^\gamma\,p^{-2(\gamma+1/2-s)}\,(1+\log(p/h))^\beta\|\hat\chi_1\|_{H^m(K)}. \qquad (4.33)$$

For the second term on the right-hand side of (4.32) we again use Lemmas 3.2, 3.3, and then estimates (4.28), (4.30):

$$\begin{aligned}\|(\hat\chi_1-\hat\chi_{1,p})\hat\varphi_p\|_{H^s(K)} &\le C\|\hat\chi_1-\hat\chi_{1,p}\|_{H^{2+\varepsilon}(K)}\|\hat\varphi_p\|_{H^s(K)}\\ &\le Ch^\gamma p^{-(m-2-\varepsilon)}(1+\log(p/h))^\beta\|\hat\chi_1\|_{H^m(K)},\ 2+\varepsilon<m. \qquad (4.34)\end{aligned}$$

Now we deduce from (4.32)–(4.34) for $s \in (1/2, \min\{1, \gamma+1/2\})$

$$\begin{aligned}\|\hat{u}-\hat{\psi}\|_{H^s(K)} &\le C\,h^\gamma \max\left\{p^{-2(\gamma+1/2-s)}, p^{-(m-2-\varepsilon)}\right\}(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^m(K)}\\ &\le C\,h^\gamma\,p^{-2(\gamma+1/2-s)}\,(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^m(K)}. \qquad (4.35)\end{aligned}$$

Here we have chosen ε small enough such that $1+\varepsilon < 2s$, since then one can estimate $p^{-m+2+\varepsilon} \le p^{-2\gamma-1+2s}$ for $m > 2\gamma+2$.

To treat the case $s=0$ we use similar arguments relying on the inequality

$$\|u\,v\|_{H^0(K)} \le C\|u\|_{C^0(\bar{K})}\,\|v\|_{H^0(K)},$$

the embedding $H^{1+\varepsilon}(K) \subset C^0(\bar{K})$ $(\varepsilon > 0)$, and estimates (4.26), (4.28), (4.31), (4.32):

$$\begin{aligned}\|\hat{u}-\hat{\psi}\|_{H^0(K)} &\le C\,h^\gamma\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^{1+\varepsilon}(K)}\\ &\quad + C\,h^\gamma\,p^{-(m-1)}\,(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^m(K)}\\ &\le C\,h^\gamma\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^m(K)}. \qquad (4.36)\end{aligned}$$

Analogously, using (4.27), (4.29), (4.31) we obtain for every side $l \in \mathcal{A}$

$$\|\hat{u}-\hat{\psi}\|_{L_2(l)} \le C\,h^\gamma\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta\,\|\hat{\chi}_1\|_{H^m(K)}. \qquad (4.37)$$

Observe that adjusting the constants C in (4.35)–(4.37) we can obtain these estimates for a polynomial $\hat{\psi} \in \mathcal{P}_p(K)$ for every $p = 1, 2, \ldots$ Therefore, recalling the notation $K^h = \Gamma_j$ and setting $\psi_j := \hat{\psi} \circ M^{-1}$ we find a polynomial $\psi_j \in \mathcal{P}_p(\Gamma_j)$, $p = 1, 2, \ldots$ such that $\psi_j = 0$ at the vertices of Γ_j, on $(\partial\Gamma_j \cap \bar{e})$, and on $\tau^j = M(\tau) = \{l_k \subset \partial\Gamma_j;\ \bar{l}_k \cap \bar{e} = \varnothing\}$. Moreover, making use of Lemma 3.1 we deduce from (4.35)–(4.37)

$$\|u-\psi_j\|_{H^s(\Gamma_j)} \le C\,h^{\gamma+1-s}\,p^{-2(\gamma+1/2-s)}\,(1+\log(p/h))^\beta \sum_{t=0}^{m} h^{t-1}\,|\chi_1|_{H^t(\Gamma_j)} \qquad (4.38)$$

for $s \in \{0\} \cup (1/2, \min\{1, \gamma+1/2\})$, and

$$\|u-\psi_j\|_{L_2(l^h)} \le C\,h^{\gamma+1/2}\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta \sum_{t=0}^{m} h^{t-1}\,|\chi_1|_{H^t(\Gamma_j)} \qquad (4.39)$$

for every $l^h \in \mathcal{A}^j = M(\mathcal{A})$.

Suppose that $\Gamma_i, \Gamma_j \subset A_e$ are two elements having the common edge $l^h = \bar{\Gamma}_i \cap \bar{\Gamma}_j$. Let $\psi_i \in \mathcal{P}_p(\Gamma_i)$ and $\psi_j \in \mathcal{P}_p(\Gamma_j)$ be the approximations to u constructed above and satisfying estimates (4.38), (4.39). Then the jump $w = (\psi_j - \psi_i)|_{l^h}$ vanishes at the end points of l^h and, because of (4.39),

$$\begin{aligned}\|w\|_{L_2(l^h)} &\le \|u-\psi_i\|_{L_2(l^h)} + \|u-\psi_j\|_{L_2(l^h)}\\ &\le C\,h^{\gamma+1/2}\,p^{-2(\gamma+1/2)}\,(1+\log(p/h))^\beta \sum_{t=0}^{m} h^{t-1}\,|\chi_1|_{H^t(\Gamma_i\cup\Gamma_j)}.\end{aligned}$$

In the case that Γ_i is a parallelogram, we use Lemma 3.4 to find a polynomial $z \in \mathcal{P}_p(\Gamma_i)$ such that

$$z = w \;\text{ on } l^h, \qquad z = 0 \;\text{ on } \partial\Gamma_i \backslash l^h, \qquad (4.40)$$

and for $0 \le s \le 1$

$$\|z\|_{H^s(\Gamma_i)} \le C\, h^{\gamma+1-s}\, p^{-2(\gamma+1-s)}\, (1+\log(p/h))^\beta \sum_{t=0}^{m} h^{t-1}\, |\chi_1|_{H^t(\Gamma_i\cup\Gamma_j)}. \tag{4.41}$$

In the case that Γ_i is a triangle, we note that (4.38) and (4.39) also hold for a polynomial ψ_j of degree $\left[\frac{p-1}{2}\right]$ (with different constants C for the upper bounds in (4.38) and (4.39)). Then Lemma 3.4 yields a polynomial $z \in P_p(\Gamma_i)$ which satisfies (4.40), (4.41) for Γ_i being a triangle.

Further we set

$$\tilde\psi = \psi_i + z \quad \text{on} \quad \Gamma_i, \qquad \tilde\psi = \psi_j \quad \text{on} \quad \Gamma_j.$$

Then $\tilde\psi$ is continuous on $\Gamma_i \cup \Gamma_j \cup l^h$, the norms $\|u-\tilde\psi\|_{H^s(\Gamma_j)}$, $\|u-\tilde\psi\|_{L_2(l^h)}$ are bounded as in (4.38), (4.39), and on the element Γ_i there holds

$$\begin{aligned}\|u-\tilde\psi\|_{H^s(\Gamma_i)} &\le \|u-\psi_i\|_{H^s(\Gamma_i)} + \|z\|_{H^s(\Gamma_i)}\\ &\le C\, h^{\gamma+1-s}\, p^{-2(\gamma+1/2-s)}\, (1+\log(p/h))^\beta \sum_{t=0}^{m} h^{t-1}\, |\chi_1|_{H^t(\Gamma_i\cup\Gamma_j)}.\end{aligned}$$

Using the same arguments as above we can adjust also the polynomial ψ_i on each element $\Gamma_i \subset A_e \cap (A_v \cup A_w)$. We construct the function $\tilde\psi$ satisfying estimates (4.38), (4.39) and vanishing on the side $l^h \subset \partial\Gamma_i$ such that $l^h \cap \bar e = \{v\}$ or $l^h \cap \bar e = \{w\}$ (*i.e.*, l^h is l_v or l_w). In this case the jump is $w = (-\psi_i)|_{l^h}$ and we set $\tilde\psi = \psi_i + z$ on Γ_i, where $z \in \mathcal{P}_p(\Gamma_i)$ is constructed using Lemma 3.4. Obviously $\tilde\psi = 0$ on l^h, and estimates (4.38), (4.39) remain valid because $u|_{l^h} = 0$.

Repeating this procedure, we obtain a continuous function u_{hp} defined on A_e such that $u_{hp} \in \mathcal{P}_p(\Gamma_j)$ for $\Gamma_j \subset A_e$, $u_{hp} = 0$ on ∂A_e, and for $s \in \{0\} \cup (1/2, \min\{1, \gamma+1/2\})$

$$\sum_{j:\ \Gamma_j\subset A_e} \|u-u_{hp}\|^2_{H^s(\Gamma_j)} \le C\, h^{2(\gamma+1-s)}\, p^{-4(\gamma+1/2-s)}\, (1+\log(p/h))^{2\beta} \sum_{t=0}^{m} h^{2(t-1)}\, |\chi_1|^2_{H^t(A_e)}. \tag{4.42}$$

For $s=0$ this immediately leads to (4.25). If $1/2 < s < \min\{1,\gamma+1/2\}$, then we also obtain (4.25) from (4.42) by using Lemma 3.5. Estimate (4.25) for any $s \in (0, 1/2]$ then follows by interpolation between $H^0(A_e)$ and $H^{s'}(A_e)$ with $1/2 < s' < \min\{1, \gamma+1/2\}$. □

5. Approximation of edge-vertex singularities

Let $e \in E$ be the edge of Γ with vertices v, w. As before, we denote by l_v and l_w the edges of ∂A_e such that $\bar l_v \cap \bar e = \{v\}$ and $\bar l_w \cap \bar e = \{w\}$.

Let us consider the cut-off functions χ^v and χ^{ev} which appear in the expressions for the edge-vertex singularities u_1^{ev} and u_2^{ev} (see (2.6), (2.7)). We adjust the supports of these cut-off functions as follows:

$$\operatorname{supp}\chi^v \subset [0, 2\tau_v] \quad \text{with} \quad 0 < \tau_v < \min\{\tfrac14 \operatorname{dist}\{v,w\}, \tfrac12\},$$

$$\operatorname{supp}\chi^{ev} \subset [0, \tfrac32\beta_v] \quad \text{with} \quad 0 < \beta_v \le \min\{\tfrac12\theta_0, \tfrac12\omega_v, \tfrac{\pi}{8}\},$$

where θ_0 is the minimal angle of the elements in the mesh Δ_h. Then u_1^{ev} and u_2^{ev} vanish outside the sector $S = \{(r_v,\theta_v);\ 0 < r_v < 2\tau_v,\ 0 < \theta_v < \frac32\beta_v\}$, in particular, $u_1^{ev} = u_2^{ev} = 0$ on $l_v \cup l_w$.

In the two sub-sections below we will study the approximation of the singular functions u_1^{ev} and u_2^{ev}.

5.1. Approximation of the function u_1^{ev}

Theorem 5.1. *Let* $u = u_1^{ev}$ *be given by* (2.6)*. Then there exists* $u_{hp} \in V_0^{h,p}(\Gamma)$ *with* $p \geq \min\{\lambda, \gamma - \frac{1}{2}\}$ *such that for* $s \in [0, \min\{1, \lambda+1, \gamma+1/2\})$,

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq C\, h^{\min\{\lambda+1,\gamma+1/2\}-s}\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\,(1+\log(p/h))^{\beta+\nu}, \tag{5.1}$$

where $\lambda = \lambda_1^v > -1/2$*,* $\gamma = \gamma_1^e > 0$*,*

$$\beta = \begin{cases} q_1^v + s_1^e + \frac{1}{2} & \text{if } \lambda_1^v = \gamma_1^e - \frac{1}{2}, \\ q_1^v + s_1^e & \text{otherwise,} \end{cases}$$

and

$$\nu = \begin{cases} \frac{1}{2} & \text{if } p = \min\{\lambda, \gamma - \frac{1}{2}\}, \\ 0 & \text{otherwise.} \end{cases}$$

If $1 \leq p < \min\{\lambda, \gamma - \frac{1}{2}\}$*, then there exists* $u_{hp} \in V_0^{h,p}(\Gamma)$ *satisfying for* $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq C\, h^{p+1-s}. \tag{5.2}$$

Proof. For simplicity we consider the singular function

$$u(x_1, x_2) = x_1^{\lambda-\gamma} x_2^{\gamma} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi^v(r) \chi^{ev}(\theta), \tag{5.3}$$

where $\lambda = \lambda_1^v > -1/2$, $\gamma = \gamma_1^e > 0$, and β_1, $\beta_2 \geq 0$ are integers.

Let us introduce an auxiliary cut-off function $\chi_2 \in C^\infty(\mathbb{R}^+)$ such that for some $\delta \in (0,1)$

$$\chi_2(t) = 1 \;\; \text{for } 0 \leq t \leq \delta/2 \quad \text{and} \quad \chi_2(t) = 0 \;\; \text{for } t \geq \delta.$$

Denoting $h_0 = (\sigma_1\sigma_2)^{-1} h$ we decompose the function u in (5.3) as

$$\begin{aligned} u &= u\chi^v(r/h_0) + u(1-\chi^v(r/h_0))\chi_2(x_2/h_0) + u(1-\chi^v(r/h_0))(1-\chi_2(x_2/h_0)) \\ &=: \varphi_1 + \varphi_2 + \varphi_3. \end{aligned} \tag{5.4}$$

We will approximate the functions φ_i $(i = 1,2,3)$ in (5.4) separately.

Approximation of φ_1. Due to the adjustment of the supports of the cut-off functions χ^v and χ^{ev}, the singular function φ_1 has small support, $\operatorname{supp}\varphi_1 \subset \bar{K}^h$, where $K^h = \Gamma_1 \subset A_{ev}$ is the element touching simultaneously the edge e and the vertex v. Let $K \subset \mathbb{R}^{2+}$ be a triangle or parallelogram such that $K^h = M(K)$, where M is the affine mapping

$$M : x_i = h\hat{x}_i, \quad i = 1,2, \quad x \in K^h, \quad \hat{x} \in K.$$

Then K satisfies the assumptions of Proposition 4.3, and for $h < \frac{1}{2}$ we have

$$\begin{aligned} \hat{\varphi}_1(\hat{x}) &= \varphi_1(h\hat{x}_1, h\hat{x}_2) \\ &= h^\lambda \hat{x}_1^{\lambda-\gamma} \hat{x}_2^{\gamma} \sum_{k_1=0}^{\beta_1} \sum_{k_2=0}^{\beta_2} \binom{\beta_1}{k_1} \binom{\beta_2}{k_2} |\log h|^{k_1+k_2} |\log \hat{x}_1|^{\beta_1-k_1} |\log \hat{x}_2|^{\beta_2-k_2} \chi^v(\sigma_1\sigma_2\hat{r}) \chi^{ev}(\hat{\theta}), \end{aligned}$$

where $\hat{r} = (\hat{x}_1^2 + \hat{x}_2^2)^{1/2}$, $\hat{\theta} = \arctan(\hat{x}_2/\hat{x}_1)$.

By Proposition 4.3, for each pair (k_1, k_2) with $k_i = 0, \dots, \beta_i$ $(i = 1, 2)$ there exists a polynomial $\hat\psi_{k_1,k_2} \in \mathcal{P}_p(K)$ vanishing on ∂K and satisfying for $0 \le s < \min\{1, \lambda+1, \gamma+1/2\}$

$$\Big\|\hat x_1^{\lambda-\gamma}\hat x_2^{\gamma}|\log\hat x_1|^{k_1}|\log\hat x_2|^{k_2}\chi^v(\sigma_1\sigma_2\hat r)\chi^{ev}(\hat\theta) - \hat\psi_{k_1,k_2}\Big\|_{H^s(K)} \le \\ C\,p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}\,(1+\log p)^{k_1+k_2+\sigma}.$$

Setting

$$\hat\psi_1(\hat x) := h^\lambda \sum_{k_1=0}^{\beta_1}\sum_{k_2=0}^{\beta_2}\binom{\beta_1}{k_1}\binom{\beta_2}{k_2}|\log h|^{k_1+k_2}\hat\psi_{\beta_1-k_1,\beta_2-k_2}(\hat x),$$

we estimate

$$\begin{aligned}
&\|\hat\varphi_1 - \hat\psi_1\|_{H^s(K)} \le \\
&\quad h^\lambda \sum_{k_1,k_2=0}^{\beta_1,\beta_2}\binom{\beta_1}{k_1}\binom{\beta_2}{k_2}|\log h|^{k_1+k_2}C(k_1,k_2)p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}(1+\log p)^{\beta_1+\beta_2-k_1-k_2+\sigma}\\
&\le\ C(\beta_1,\beta_2)h^\lambda p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}(1+\log(p/h))^{\beta_1+\beta_2}(1+\log p)^\sigma. \qquad (5.5)
\end{aligned}$$

Let $\psi_1 := \hat\psi_1\circ M^{-1}$ on $K^h = \Gamma_1$. Then $\psi_1 \in \mathcal{P}_p(\Gamma_1)$, $\psi_1 = 0$ on $\partial\Gamma_1$, and making use of Lemma 3.1 we deduce from (5.5)

$$\begin{aligned}
\|\varphi_1 - \psi_1\|_{H^s(\Gamma_1)} &\le Ch^{1-s}\|\hat\varphi_1 - \hat\psi_1\|_{H^s(K)}\\
&\le Ch^{\lambda+1-s}p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}(1+\log(p/h))^{\beta_1+\beta_2}(1+\log p)^\sigma, \qquad (5.6)
\end{aligned}$$

where $0 \le s < \min\{1, \lambda+1, \gamma+1/2\}$, $\sigma = 1/2$ if $\lambda = \gamma - 1/2$, and $\sigma = 0$ otherwise.

Approximation of φ_2. The function φ_2 in (5.4) has a singular behaviour only with respect to x_2 and has small support, $\operatorname{supp}\varphi_2 \subset (\bar A_e \cap \bar R_1^h)$, where $R_1^h = \{(r,\theta);\ \tau_v h_0 < r < 2\tau_v,\ 0 < \theta < \frac{3}{2}\beta_v\}$. Thus we can write φ_2 in the form given by (4.24):

$$\begin{aligned}
\varphi_2(x_1,x_2) &= x_1^{\lambda-\gamma}x_2^\gamma|\log x_1|^{\beta_1}|\log x_2|^{\beta_2}\chi^v(r)\chi^{ev}(\theta)(1-\chi^v(r/h_0))\chi_2(x_2/h_0)\\
&= x_2^\gamma|\log x_2|^{\beta_2}\chi_1(x_1,x_2)\chi_2(x_2/h_0),
\end{aligned}$$

where

$$\chi_1(x_1,x_2) := x_1^{\lambda-\gamma}|\log x_1|^{\beta_1}\chi^v(r)\chi^{ev}(\theta)(1-\chi^v(r/h_0)). \qquad (5.7)$$

Note that $\chi_1 \in C^\infty(A_e)$, $\operatorname{supp}\chi_1 \subset \bar R_1^h$, in particular, $\chi_1 = 0$ on the edges $l_v, l_w \subset \partial A_e$.

Now we can apply Lemma 4.2 to find a piecewise polynomial approximation of φ_2 on A_e: there exists a function ψ_2 such that $\psi_2 \in \mathcal{P}_p(\Gamma_j)$ for each $\Gamma_j \subset A_e$, $\psi_2 = 0$ on ∂A_e, and for $0 \le s < \min\{1, \gamma+1/2\}$

$$\|\varphi_2 - \psi_2\|_{H^s(A_e)} \le C\,h^{\gamma+1-s}\,p^{-2(\gamma+1/2-s)}\,(1+\log(p/h))^{\beta_2}\sum_{t=0}^m h^{t-1}\,|\chi_1|_{H^t(A_e)} \qquad (5.8)$$

for some integer $m > 2\gamma + 2$.

To evaluate semi-norms of the function χ_1 given by (5.7) we use the following inequalities:

$$\left|\frac{\partial r}{\partial x_1}\right| = |\cos\theta| \le 1, \qquad \left|\frac{\partial r}{\partial x_2}\right| = |\sin\theta| \le 1,$$
$$\left|\frac{\partial \theta}{\partial x_1}\right| = \left|\frac{\sin\theta}{r}\right| \le \frac{1}{r}, \qquad \left|\frac{\partial \theta}{\partial x_2}\right| = \left|\frac{\cos\theta}{r}\right| \le \frac{1}{r}.$$

Hence it follows by induction that for any integer $k, l \ge 0$

$$\left|\frac{\partial^{k+l} r}{\partial x_1^k \partial x_2^l}\right| \le C r^{1-k-l}, \quad \left|\frac{\partial^{k+l} \theta}{\partial x_1^k \partial x_2^l}\right| \le C r^{-k-l}. \tag{5.9}$$

Furthermore, for any integer $k \ge 1$ one has

$$\begin{aligned} \left|\frac{\partial^k}{\partial r^k}\Big(1-\chi^v(r/h_0)\Big)\right| &= \begin{cases} 0 & \text{for } 0<r<\tau_v h_0 \text{ and } r>2\tau_v h_0, \\ |(\chi^v)^{(k)}| h_0^{-k} & \text{for } \ \tau_v h_0 \le r \le 2\tau_v h_0 \end{cases} \\ &\le \ C\, r^{-k} \qquad \text{for } \ r>0. \end{aligned} \tag{5.10}$$

Since $\operatorname{supp}\chi_1 \subset \bar{R}_1^h$, $x_1 \simeq r$ on R_1^h, and $\chi^v, \chi^{ev} \in C^\infty(R^+)$, we estimate by (5.7), (5.9), (5.10) for $t = 0, \ldots, m$

$$|\chi_1|^2_{H^t(A_e)} \le C(\log(1/h))^{2\beta_1} \int\limits_{A_e \cap R_1^h} x_1^{2(\lambda-\gamma-t)} \mathrm{d}x \le C(\log(1/h))^{2\beta_1} \int\limits_0^h \int\limits_{\kappa h}^{2\tau_v} x_1^{2(\lambda-\gamma-t)} \mathrm{d}x_1 \mathrm{d}x_2$$

for a positive constant κ independent of h. Hence

$$|\chi_1|_{H^t(A_e)} \le C \log^{\beta_1}(1/h) h^{1/2-t} \begin{cases} h^{\lambda-\gamma+1/2} & \text{if } \lambda < \gamma - 1/2, \\ \log^{1/2}(1/h) & \text{if } \lambda = \gamma - 1/2, \\ 1 & \text{if } \lambda > \gamma - 1/2, \end{cases}$$

and we obtain by (5.8)

$$\|\varphi_2 - \psi_2\|_{H^s(A_e)} \ \le \ C\, h^{\min\{\lambda+1, \gamma+1/2\}-s}\, p^{-2(\gamma+1/2-s)}\, (\log(1/h))^{\beta_1+\sigma}\, (1+\log(p/h))^{\beta_2}, \tag{5.11}$$

where $0 \le s < \min\{1, \gamma+1/2\}$ and σ is the same as in (5.6).

Approximation of φ_1 and φ_2 on Γ. Let us extend ψ_i $(i = 1, 2)$ by zero onto the remaining parts of Γ. Then $\psi_i \in V_0^{h,p}(\Gamma)$, $i = 1, 2$ and there hold the following estimates

$$\|\varphi_1 - \psi_1\|_{H^s(\Gamma)} \le C h^{\lambda+1-s} p^{-2(\min\{\lambda+1, \gamma+1/2\}-s)} (1+\log(p/h))^{\beta_1+\beta_2} (1+\log p)^{\sigma} \tag{5.12}$$

for $0 \le s < \min\{1, \lambda+1, \gamma+1/2\}$, and

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \ \le \ C\, h^{\min\{\lambda+1, \gamma+1/2\}-s}\, p^{-2(\gamma+1/2-s)}\, (\log(1/h))^{\beta_1+\sigma}\, (1+\log(p/h))^{\beta_2} \tag{5.13}$$

for $0 \le s < \min\{1, \gamma+1/2\}$.

In fact, for $s = 0$ estimates (5.12) and (5.13) immediately follow from inequalities (5.6) and (5.11), respectively. If $1/2 < s < 1$, then we use Lemma 3.5:

$$\begin{aligned}\|\varphi_2 - \psi_2\|^2_{H^s(\Gamma)} &\le C\Big(h^{-2s}\|\varphi_2 - \psi_2\|^2_{L_2(\Gamma)} + \sum_{j:\Gamma_j\subset\Gamma} |\varphi_2 - \psi_2|^2_{H^s(\Gamma_j)}\Big)\\ &\le C\Big(h^{-2s}\|\varphi_2 - \psi_2\|^2_{L_2(A_e)} + \sum_{j:\Gamma_j\subset A_e} \|\varphi_2 - \psi_2\|^2_{H^s(\Gamma_j)}\Big)\\ &\le C\Big(h^{-2s}\|\varphi_2 - \psi_2\|^2_{L_2(A_e)} + \|\varphi_2 - \psi_2\|^2_{H^s(A_e)}\Big)\end{aligned}$$

and (5.13) follows from (5.11). The estimate (5.12) for $1/2 < s < 1$ is proved analogously. Finally, for $0 < s \le 1/2$, estimates (5.12), (5.13) follow *via* interpolation between $H^0(\Gamma)$ and $H^{s'}(\Gamma)$ for some $s' \in (\frac{1}{2}, 1)$.

Approximation of φ_3. Now we approximate the function φ_3 in (5.4). Observe that $\varphi_3 \in C_0^\infty(\Gamma)$ and $\operatorname{supp}\varphi_3 \subset \bar\Gamma \cap \bar R_1^h \cap \bar R_2^h$, where R_1^h is defined above and $R_2^h = \{(x_1, x_2);\ x_2 > \delta h_0/2\}$ for some $\delta \in (0,1)$. We also note that the mesh contains triangles and/or parallelograms. Therefore, applying Proposition 4.1, we find $\psi_3 \in V_0^{h,p}(\Gamma)$ such that for $s \in [0,1]$

$$\|\varphi_3 - \psi_3\|_{H^s(\Gamma)} \le C h^{\mu-s} p^{-(m-\tilde s)} \|\varphi_3\|_{H^m(\Gamma)}, \tag{5.14}$$

where $m > 1,\ \mu = \min\{m, p+1\}$, and $\tilde s$ is defined by (4.6).

Let us estimate the norm $\|\varphi_3\|_{H^m(\Gamma)}$. Similarly to (5.9), (5.10) one has for $k, l \ge 0$

$$\left|\frac{\partial^{k+l} r}{\partial x_1^k \partial x_2^l}\right| \le C r x_1^{-k} x_2^{-l}, \quad \left|\frac{\partial^{k+l}\theta}{\partial x_1^k \partial x_2^l}\right| \le C x_1^{-k} x_2^{-l},$$

$$\left|\frac{\partial^{k+l}}{\partial x_1^k \partial x_2^l}(1 - \chi^v(r/h_0))\right| \le C x_1^{-k} x_2^{-l}, \quad \left|\frac{\mathrm{d}^l}{\mathrm{d}x_2^l}(1 - \chi_2(x_2/h_0))\right| \le C x_2^{-l}.$$

Hence, recalling that

$$\varphi_3(x_1, x_2) = x_1^{\lambda-\gamma} x_2^{\gamma} |\log x_1|^{\beta_1} |\log x_2|^{\beta_2} \chi^v(r)\chi^{ev}(\theta)(1 - \chi^v(r/h_0))(1 - \chi_2(x_2/h_0)),$$

$\operatorname{supp}\varphi_3 \subset \bar R_1^h \cap \bar R_2^h$, and $\chi_2, \chi^v, \chi^{ev} \in C^\infty(R^+)$, we can estimate derivatives of φ_3 as

$$\left|\frac{\partial^{k+l}\varphi_3(x)}{\partial x_1^k \partial x_2^l}\right| \le \begin{cases} C(k,l)(\log(1/h))^{\beta_1+\beta_2} x_1^{\lambda-\gamma-k} x_2^{\gamma-l} & \text{for } x \in R_1^h \cap R_2^h,\\ 0 & \text{for } x \in \Gamma\backslash(R_1^h \cap R_2^h).\end{cases}$$

Since $(R_1^h \cap R_2^h) \subset T^h = \{(x_1, x_2);\ \kappa h < x_1 < 1,\ \kappa h < x_2 < x_1\}$ for some $\kappa > 0$, the above estimates for derivatives of φ_3 yield

$$\begin{aligned}\|\varphi_3\|^2_{H^m(\Gamma)} &\le C(\log(1/h))^{2(\beta_1+\beta_2)} \sum_{\substack{0\le k+l\le m\\ k,l\ge 0}} C^2(k,l) \int\limits_{R_1^h\cap R_2^h} x_1^{2(\lambda-\gamma-k)} x_2^{2(\gamma-l)}\,\mathrm{d}x\\ &\le C(m)(\log(1/h))^{2(\beta_1+\beta_2)} \int\limits_{T^h} x_1^{2(\lambda-\gamma)} x_2^{2(\gamma-m)}\,\mathrm{d}x.\end{aligned}$$

For any integer $m \geq \min\{\lambda+1, \gamma+\frac{1}{2}\}$ this implies

$$\|\varphi_3\|^2_{H^m(\Gamma)} \leq C(\log(1/h))^{2(\beta_1+\beta_2)} \begin{cases} \int\limits_{\kappa h}^{1} x_1^{2(\lambda-\gamma)} \int\limits_{\kappa h}^{x_1} x_2^{2(\gamma-m)} \mathrm{d}x_2 \mathrm{d}x_1 & \text{if } \lambda \geq \gamma - 1/2, \\ \int\limits_{\kappa h}^{1} x_2^{2(\gamma-m)} \int\limits_{x_2}^{1} x_1^{2(\lambda-\gamma)} \mathrm{d}x_1 \mathrm{d}x_2 & \text{if } \lambda < \gamma - 1/2 \end{cases}$$

$$\leq Ch^{2(\min\{\lambda+1,\gamma+1/2\}-m)}(\log(1/h))^{2(\beta_1+\beta_2+\sigma+\nu)}, \tag{5.15}$$

where σ is the same as in (5.6), $\nu = \frac{1}{2}$ if $m = \min\{\lambda+1, \gamma+\frac{1}{2}\}$, and $\nu = 0$ if $m > \min\{\lambda+1, \gamma+\frac{1}{2}\}$. Therefore we obtain by (5.14)

$$\|\varphi_3 - \psi_3\|_{H^s(\Gamma)} \leq Ch^{\mu-s+\min\{\lambda+1,\gamma+1/2\}-m} p^{-(m-\tilde{s})}(\log(1/h))^{\beta_1+\beta_2+\sigma+\nu}, \quad s \in [0,1], \tag{5.16}$$

where $m \geq \min\{\lambda+1, \gamma+1/2\}$, $m > 1$, $\mu = \min\{m, p+1\}$, and $\tilde{s}$ is defined by (4.6).

If $p > 2\min\{\lambda+1, \gamma+\frac{1}{2}\} - \frac{1}{2}$, we select an integer m satisfying

$$2\min\{\lambda+1, \gamma+\tfrac{1}{2}\} + \tfrac{1}{2} < m \leq p+1.$$

Then $\mu = m > \frac{3}{2}$ and $p^{-(m-\tilde{s})} \leq p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}$ for any $s \in [0,1]$.

If $\min\{\lambda+1, \gamma+\frac{1}{2}\} - 1 < p \leq 2\min\{\lambda+1, \gamma+\frac{1}{2}\} - \frac{1}{2}$ (*i.e.*, p is bounded), we choose an integer m such that

$$\max\left\{1, \min\{\lambda+1, \gamma+\tfrac{1}{2}\}\right\} < m \leq p+1,$$

and if $p = \min\{\lambda+1, \gamma+1/2\} - 1$, then we take $m = \min\{\lambda+1, \gamma+\frac{1}{2}\} = p+1$. In both these cases $\mu = m > 1$ and $p^{-(m-\tilde{s})} \leq C(\lambda, \gamma)\, p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}$ for any $s \in [0,1]$.

Thus, for any $p \geq \min\{\lambda, \gamma - \frac{1}{2}\}$, selecting m as indicated above we find by (5.16)

$$\|\varphi_3 - \psi_3\|_{H^s(\Gamma)} \leq Ch^{\min\{\lambda+1,\gamma+1/2\}-s} p^{-2(\min\{\lambda+1,\gamma+1/2\}-s)}(\log(1/h))^{\beta_1+\beta_2+\sigma+\nu}, \quad s \in [0,1], \tag{5.17}$$

where σ is the same as in (5.6), $\nu = \frac{1}{2}$ if $p = \min\{\lambda, \gamma - \frac{1}{2}\}$, and $\nu = 0$ otherwise.

Approximation of $u = \varphi_1 + \varphi_2 + \varphi_3$. Let us define $u_{hp} := \psi_1 + \psi_2 + \psi_3 \in V_0^{h,p}(\Gamma)$. Then combining estimates (5.12), (5.13), and (5.17) we obtain (5.1).

It remains to consider the case $1 \leq p < \min\{\lambda, \gamma - \frac{1}{2}\}$. In this case one does not need decomposition (5.4). Since $u \in H^m(\Gamma) \cap H_0^1(\Gamma)$ with $1 < m < \min\{\lambda+1, \gamma+\frac{1}{2}\}$, we apply Proposition 4.1 to find $u_{hp} \in V_0^{h,p}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq Ch^{\mu-s}\|u\|_{H^m(\Gamma)}, \quad \mu = \min\{m, p+1\}.$$

Hence, selecting $m \in [p+1, \min\{\lambda+1, \gamma+\frac{1}{2}\})$ we prove (5.2). □

5.2. Approximation of the function u_2^{ev}

In this sub-section we study the approximation of the edge-vertex singularity u_2^{ev} given by (2.7), (2.9).

Theorem 5.2. *Let $u = u_2^{ev}$ be given by (2.7), (2.9). Then there exists $u_{hp} \in V_0^{h,p}(\Gamma)$ with $p \geq \gamma - \frac{1}{2}$ such that for $s \in [0, \min\{1, \gamma+1/2\})$,*

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq C\, h^{\gamma+1/2-s}\, p^{-2(\gamma+1/2-s)}\,(1+\log(p/h))^{\beta+\nu}, \tag{5.18}$$

where $\gamma = \gamma_1^e > 0$, $\beta = s_1^e \geq 0$ is integer, $\nu = \frac{1}{2}$ if $p = \gamma - \frac{1}{2}$, and $\nu = 0$ otherwise.

If $1 \le p < \gamma - \frac{1}{2}$*, then there exists* $u_{hp} \in V_0^{h,p}(\Gamma)$ *satisfying for* $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C\, h^{p+1-s}. \tag{5.19}$$

Proof. For simplicity we consider one component of the function u_2^{ev}. Let

$$u(x_1, x_2) = x_2^{\gamma} |\log x_2|^{\beta} \chi_1(x_1, x_2) \chi_2^e(x_2), \tag{5.20}$$

where $\gamma = \gamma_1^e > 0$, $\beta \ge 0$ is integer, $\chi_2^e \in C^\infty(\mathbb{R}^+)$ is the same as in (2.4), $\chi_1 \in H^m(\Gamma)$ with m as large as required. Recalling that the supports of the cut-off functions χ^v and χ^{ev} (see (2.9)) were adjusted so that $\operatorname{supp} u_2^{ev} \subset \bar{S} = \{(r,\theta);\ 0 \le r \le 2\tau_v,\ 0 \le \theta \le \frac{3}{2}\beta_v\}$ with $\tau_v < \frac{1}{4} \operatorname{dist}\{v, w\}$ and $\beta_v \le \frac{1}{2}\theta_0$, we can assume that the function χ_1 in (5.20) vanishes on the edges $l_v, l_w \subset \partial A_e$ (l_v and l_w have been defined at the beginning of this section). Suppose that $h < \frac{1}{2}$. Letting $h_0 = (\sigma_1\sigma_2)^{-1} h$ we decompose u as

$$u = u\chi_2^e(x_2/h_0) + u(1 - \chi_2^e(x_2/h_0)) =: \varphi_1 + \varphi_2. \tag{5.21}$$

The singular part φ_1 of this decomposition has the form given by (4.24), and $\varphi_1 = 0$ on ∂A_e. Therefore, applying Lemma 4.2 we find a function ψ_1 such that $\psi_1 \in \mathcal{P}_p(\Gamma_j)$ for $\Gamma_j \subset A_e$, $\psi_1 = 0$ on ∂A_e, and for $0 \le s < \min\{1, \gamma + 1/2\}$ there holds

$$\|\varphi_1 - \psi_1\|^2_{H^s(A_e)} \le C\, h^{2(\gamma+1-s)}\, p^{-4(\gamma+1/2-s)}\, (1 + \log(p/h))^{2\beta} \sum_{t=0}^{k} h^{2(t-1)}\, |\chi_1|^2_{H^t(A_e)} \tag{5.22}$$

for some integer $k > 2\gamma + 2$.

Since $\operatorname{meas}(A_e) \simeq h$ and $\chi_1 \in H^m(\Gamma)$ for sufficiently large m, we estimate

$$\sum_{t=0}^{k} h^{2(t-1)}\, |\chi_1|^2_{H^t(A_e)} \le Ch^{-2}\, \|\chi_1\|^2_{C^k(\bar{A}_e)} \operatorname{meas}(A_e) \le Ch^{-1}\, \|\chi_1\|^2_{C^k(\bar{\Gamma})} \le Ch^{-1}\, \|\chi_1\|^2_{H^m(\Gamma)}.$$

Then we obtain by (5.22)

$$\|\varphi_1 - \psi_1\|_{H^s(A_e)} \le C\, h^{\gamma+1/2-s}\, p^{-2(\gamma+1/2-s)}\, (1 + \log(p/h))^{\beta}, \quad s \in [0, \min\{1, \gamma + 1/2\}). \tag{5.23}$$

Let us extend ψ_1 by zero onto $\Gamma \backslash A_e$. Then $\psi_1 \in V_0^{h,p}(\Gamma)$ and the norm $\|\varphi_1 - \psi_1\|_{H^s(\Gamma)}$ is obviously bounded as in (5.23) for $s = 0$. Due to Lemma 3.5, this conclusion is also true for any $s \in (1/2, \min\{1, \gamma+1/2\})$. Therefore, by using interpolation, we obtain for any $s \in [0, \min\{1, \gamma + 1/2\})$

$$\|\varphi_1 - \psi_1\|_{H^s(\Gamma)} \le C\, h^{\gamma+1/2-s}\, p^{-2(\gamma+1/2-s)}\, (1 + \log(p/h))^{\beta}. \tag{5.24}$$

To approximate the smooth part $\varphi_2 \in H^m(\Gamma) \cap H_0^1(\Gamma)$ of decomposition (5.21) we apply Proposition 4.1. There exists $\psi_2 \in V_0^{h,p}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \le Ch^{\mu-s} p^{-(k-\tilde{s})} \|\varphi_2\|_{H^k(\Gamma)}, \tag{5.25}$$

where $k \in (1, m]$ is integer, $\mu = \min\{k, p+1\}$, and $\tilde{s}$ is defined by (4.6).

Recalling the definition of the function χ_2^e in (5.20) (see Thm. 2.1), we conclude that $\operatorname{supp} \varphi_2 \subset \bar{\Gamma} \cap \bar{R}_3^h$, where $R_3^h = \{(x_1, x_2);\ h_0\delta_e < x_2 < 2\delta_e\}$. Hence we find by simple calculations

$$\|\varphi_2\|^2_{H^k(\Gamma)} \le C(\log(1/h))^{2\beta} \int_{h_0\delta_e}^{2\delta_e} x_2^{2(\gamma-k)} \mathrm{d}x_2.$$

Then for any k satisfying $k > 1$ and $\gamma + \frac{1}{2} \le k \le m$ we obtain by (5.25)

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \le Ch^{\gamma-k+1/2+\mu-s} p^{-(k-\tilde{s})} \log^{\beta+\nu}(1/h), \quad s \in [0,1], \tag{5.26}$$

where $\mu = \min\{k, p+1\}$, $\tilde{s}$ is defined by (4.6), $\nu = \frac{1}{2}$ if $k = \gamma + \frac{1}{2}$, and $\nu = 0$ if $k > \gamma + \frac{1}{2}$.

Now we set $u_{hp} := \psi_1 + \psi_2 \in V_0^{h,p}(\Gamma)$. Then combining estimates (5.24), (5.26), making use of decomposition (5.21) and the triangle inequality we obtain for any $s \in [0, \min\{1, \gamma + 1/2\})$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le Ch^{\gamma+1/2-s} \max\left\{p^{-2(\gamma+1/2-s)}, h^{\mu-k} p^{-(k-\tilde{s})}\right\} (1 + \log(p/h))^{\beta+\nu}. \tag{5.27}$$

Let $p > 2\gamma + \frac{1}{2}$. Since m is large enough, we can select an integer k satisfying

$$2\gamma + \tfrac{3}{2} < k \le \min\{m, p+1\}.$$

Then $\mu = \min\{k, p+1\} = k$, $\max\left\{p^{-2(\gamma+1/2-s)}, p^{-(k-\tilde{s})}\right\} = p^{-2(\gamma+1/2-s)}$ for any $s \in [0,1]$, and (5.27) leads to (5.18).

If $\gamma - \frac{1}{2} < p \le 2\gamma + \frac{1}{2}$ (*i.e.*, p is bounded), we select an integer $k \in \Big(\max\{1, \gamma + \frac{1}{2}\}, p+1\Big]$, and if $p = \gamma - \frac{1}{2}$, then we choose $k = \gamma + \frac{1}{2} = p + 1$. In both these cases $\mu = k$, $p^{-(k-\tilde{s})} \le C(\gamma)\, p^{-2(\gamma+1/2-s)}$ for any $s \in [0,1]$, and (5.18) is again deduced from (5.27).

If $1 \le p < \gamma - \frac{1}{2}$, then $u \in H^m(\Gamma) \cap H_0^1(\Gamma)$ with $1 < m < \gamma + \frac{1}{2}$. In this case we apply Proposition 4.1 directly to the function u: there exists $u_{hp} \in V_0^{h,p}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le Ch^{\mu-s} \|u\|_{H^m(\Gamma)}, \quad \mu = \min\{m, p+1\}.$$

Hence, selecting $m \in [p+1, \gamma + \frac{1}{2})$ we prove (5.19). □

Remark 5.1. Observe that the proof of Theorem 5.2 also applies to the edge singularity terms given by (2.4). In fact, adjusting the support of the cut-off function χ_1^e in (2.4) it is easy to obtain $\chi_1^e = 0$ on the edges l_v, $l_w \subset \partial A_e$. Therefore each component of u^e can be written in the more general form (5.20) and the statement of Theorem 5.2 remains valid for $u = u^e$.

6. Approximation of vertex singularities

Let v be a vertex of Γ and let A_v be the union of elements Γ_j such that $v \in \bar{\Gamma}_j$.

Theorem 6.1. *Let $u = u^v$ be given by (2.5). Then there exists $u_{hp} \in V_0^{h,p}(\Gamma)$ with $p \ge \lambda$ such that for $0 \le s \le 1$,*

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^{\beta+\nu}, \tag{6.1}$$

where $\lambda = \lambda_1^v > 0$, $\beta = q_1^v \ge 0$ is integer, $\nu = \frac{1}{2}$ if $p = \lambda$, and $\nu = 0$ otherwise.

If $1 \le p < \lambda$, then there exists $u_{hp} \in V_0^{h,p}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C\, h^{p+1-s}. \tag{6.2}$$

Proof. The idea and arguments in the proof below are the same as in the proofs of Lemma 4.2, Theorems 5.1 and 5.2. That is why we outline the proof omitting inessential details.

Let

$$u = r^{\lambda} |\log r|^{\beta} \chi^v(r) w(\theta), \tag{6.3}$$

where $\lambda = \lambda_1^v > 0$, $\beta \ge 0$ is integer, χ^v is the same as in (2.5), $w \in H^m(0, \omega_v) \cap H_0^1(0, \omega_v)$, ω_v denotes the interior angle on Γ at v, and m is as large as required. Note that $u \in H_0^1(\Gamma)$, because $\lambda > 0$.

We decompose u as $u = \varphi_1 + \varphi_2$, where

$$\varphi_1 := u\chi^v(r/h_0), \quad \varphi_2 := u(1-\chi^v(r/h_0)), \quad h_0 = (\sigma_1\sigma_2)^{-1}h. \tag{6.4}$$

The singular function φ_1 has small support, $\operatorname{supp}\varphi_1 \subset \bar{A}_v$. Let $K^h = \Gamma_j \subset A_v$ and let $K \subset \mathbb{R}^2$ be a triangle or parallelogram such that $K^h = M(K)$ under the affine mapping $M : x_i = h\hat{x}_i,\ i = 1, 2,\ x \in K^h,\ \hat{x} \in K$. Then $O = (0, 0)$ is a vertex of K and for $h < \frac{1}{2}$ we have

$$\hat{\varphi}_1(\hat{x}) = \varphi_1(h\hat{x}_1, h\hat{x}_2) = h^\lambda \hat{r}^\lambda \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k |\log \hat{r}|^{\beta-k} \chi^v(\sigma_1\sigma_2\hat{r}) w(\hat{\theta}).$$

Let $\mathcal{A} = \{l_i\}$ contain those sides $l_i \subset \partial K$ for which $O \in \bar{l}_i$, and let $\mathcal{B}$ be the union of the other sides of K. Then applying Proposition 4.2 to each function $\hat{r}^\lambda |\log \hat{r}|^k \chi^v(\sigma_1\sigma_2\hat{r}) w(\hat{\theta}),\ k = 0, \ldots, \beta$, we find a polynomial $\hat{\phi} \in \mathcal{P}_p(K)$ such that $\hat{\phi}(0, 0) = 0$, $\hat{\phi} = 0$ on $\mathcal{B}$,

$$\|\hat{\varphi}_1 - \hat{\phi}\|_{H^s(K)} \le C(\beta)\, h^\lambda\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad s = 0, 1, \tag{6.5}$$

$$\|\hat{\varphi}_1 - \hat{\phi}\|_{L_2(l)} \le C(\beta)\, h^\lambda\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta \quad \text{for every } l \in \mathcal{A}. \tag{6.6}$$

Let us define $\phi_j := \hat{\phi} \circ M^{-1}$. Then $\phi_j \in \mathcal{P}_p(\Gamma_j)$, $\phi_j = 0$ at the vertex v and on the sides $l_i^h \in \mathcal{B}_j = M(\mathcal{B})$. Furthermore, making use of Lemma 3.1, we obtain by (6.5), (6.6)

$$\|\varphi_1 - \phi_j\|_{H^s(\Gamma_j)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad s = 0, 1, \tag{6.7}$$

$$\|\varphi_1 - \phi_j\|_{L_2(l^h)} \le C\, h^{\lambda+1/2}\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta \quad \text{for every } l^h \in \mathcal{A}_j = M(\mathcal{A}). \tag{6.8}$$

Suppose that $\Gamma_i, \Gamma_j \subset A_v$ are two elements having the common edge $l^h = \bar{\Gamma}_i \cap \bar{\Gamma}_j$. Let $\phi_i \in \mathcal{P}_p(\Gamma_i)$ and $\phi_j \in \mathcal{P}_p(\Gamma_j)$ be the approximations of φ_1 constructed above and satisfying estimates (6.7), (6.8). Then the jump $g = (\phi_j - \phi_i)|_{l^h}$ vanishes at the end points of l^h and

$$\|g\|_{L_2(l^h)} \le C\, h^{\lambda+1/2}\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta.$$

Hence, due to Lemma 3.4, there exists $z \in \mathcal{P}_p(\Gamma_i)$ such that $z = g$ on l^h, $z = 0$ on $\partial\Gamma_i \backslash l^h$, and

$$\|z\|_{H^s(\Gamma_i)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad s = 0, 1.$$

Setting $\tilde{\phi} = \phi_i + z$ on Γ_i and $\tilde{\phi} = \phi_j$ on Γ_j we find a continuous piecewise polynomial $\tilde{\phi}$ such that the norm $\|\varphi_1 - \tilde{\phi}\|_{H^s(\Gamma_i \cup \Gamma_j)}$ is bounded as in (6.7) for $s = 0, 1$.

Let e_1, e_2 be the edges of Γ meeting at the vertex v. Since $w(0) = w(\omega_v) = 0$, the function φ_1 vanishes on e_1, e_2. Therefore, using the same arguments as above we can adjust ϕ_i on each element $\Gamma_i \subset A_v \cap (A_{e_1} \cup A_{e_2})$. Then we construct a polynomial $\tilde{\phi} \in \mathcal{P}_p(\Gamma_i)$ vanishing on $\partial\Gamma_i \cap \bar{e}_k$ with $k = 1$ or 2 as appropriate.

Note that the number ν_v of elements in A_v is independent of h ($\nu_v \le \frac{\omega_v}{\theta_0}$, where θ_0 is the minimal angle of elements in the mesh). Therefore, repeating the above procedure we construct a continuous function ψ_1 such that $\psi_1 \in \mathcal{P}_p(\Gamma_j)$ for each $\Gamma_j \subset A_v$, $\psi_1 = 0$ on ∂A_v, and the norm $\|\varphi_1 - \psi_1\|_{H^s(A_v)}$ for $s = 0, 1$ is bounded as in (6.7). Extending ψ_1 by zero onto $\Gamma \backslash A_v$ we obtain $\psi_1 \in V_0^{h,p}(\Gamma)$ satisfying for $s = 0, 1$

$$\|\varphi_1 - \psi_1\|_{H^s(\Gamma)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta. \tag{6.9}$$

By interpolation we prove that (6.9) holds for $0 \le s \le 1$.

For the function φ_2 (see (6.4)) one has

$$\varphi_2 = r^\lambda |\log r|^\beta \chi^v(r)(1-\chi^v(r/h_0))w(\theta) \in H^m(\Gamma) \cap H_0^1(\Gamma),$$

$$\operatorname{supp}\varphi_2 \subset \bar\Gamma \cap \bar R^h, \quad \text{where } R^h = \{(x_1,x_2);\ \tau_v h_0 < r < 2\tau_v\}.$$

Hence, using (5.9) and (5.10) we find by simple calculations

$$\|\varphi_2\|^2_{H^k(\Gamma)} \le C(\log(1/h))^{2\beta} \int_{\tau_v h_0}^{2\tau_v} r^{2(\lambda-k)}\, r\mathrm{d}r, \quad 0 \le k \le m. \tag{6.10}$$

Further, due to Proposition 4.1, there exists $\psi_2 \in V_0^{h,p}(\Gamma)$ such that for $s \in [0,1]$

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \le C h^{\mu-s} p^{-(k-\tilde s)} \|\varphi_2\|_{H^k(\Gamma)}, \tag{6.11}$$

where $k \in (1,m]$ is integer, $\mu = \min\{k, p+1\}$, and $\tilde s$ is defined by (4.6).

If $\lambda + 1 \le k \le m$, then (6.10) and (6.11) yield

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \le C h^{\mu-s+\lambda-k+1} p^{-(k-\tilde s)} \log^{\beta+\nu}(1/h), \quad s \in [0,1], \tag{6.12}$$

where $\nu = \frac12$ if $k = \lambda+1$, and $\nu = 0$ if $k > \lambda+1$.

If $p \ge \lambda$, then similarly as in the proofs of Theorems 5.1 and 5.2 we select an integer k such that $\mu = k$ in (6.12) and $p^{-(k-\tilde s)} \le C(\lambda)\, p^{-2(\lambda+1-s)}$ for any $s \in [0,1]$. Then combination of (6.9) and (6.12) gives (6.1) with $u_{hp} := \psi_1 + \psi_2 \in V_0^{h,p}(\Gamma)$.

The proof of estimate (6.2) is analogous to the proof of the corresponding results in Theorems 5.1 and 5.2. $\square$

7. General approximation result and proof of Theorem 2.2

By combination of the approximation results for singularities from Sections 5 and 6 we obtain a general approximation result for the function u given by (2.3)–(2.7).

Theorem 7.1. *Let the function u be given by (2.3)–(2.7) on Γ with $\gamma_1^e > 0$ and $\lambda_1^v > 0$. Also, let $v_0 \in V$, $e_0 \in E(v_0)$ be such that $\min\{\lambda_1^{v_0} + 1/2, \gamma_1^{e_0}\} = \min_{v\in V, e\in E(v)} \min\{\lambda_1^v + 1/2, \gamma_1^e\}$, with λ_1^v and γ_1^e being as in (2.4)–(2.7). Then, for any $h > 0$ and every $p \ge \min\{\lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$, there exists a function $u_{hp} \in V_0^{h,p}$ such that for $0 \le s < \min\{1, \lambda_1^{v_0}+1, \gamma_1^{e_0}+1/2\}$*

$$\begin{aligned}\|u - u_{hp}\|_{H^s(\Gamma)} \ \le\ C \max\Big\{ & h^{\min\{k,p+1\}-s}\, p^{-(k-\tilde s)},\\ & h^{\min\{\lambda_1^{v_0}+1,\gamma_1^{e_0}+1/2\}-s}\, p^{-2(\min\{\lambda_1^{v_0}+1,\gamma_1^{e_0}+1/2\}-s)} (1+\log(p/h))^{\beta+\nu} \Big\},\end{aligned} \tag{7.1}$$

where β and ν are defined by (2.11) and (2.12), respectively, $\tilde s = s$ if the mesh Δ_h on Γ does not contain triangles, and $\tilde s$ is defined by (4.6) for meshes containing triangles.

If $1 \le p < \min\{\lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$, then for any $h > 0$ there exists $u_{hp} \in V_0^{h,p}$ such that for $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C\, h^{\min\{k,p+1\}-s}. \tag{7.2}$$

Proof. To approximate the smooth part $u_{\mathrm{reg}} \in H^k(\Gamma) \cap H_0^1(\Gamma)$ of decomposition (2.3) we use Proposition 4.1, and applying Theorems 5.1, 5.2, and 6.1 we find piecewise polynomial approximations for the singularities u^{ev}, u^v, and u^e on Γ (see also Rem. 5.1). Then combining the corresponding error estimates from these statements we obtain (7.1) and (7.2). $\square$

Proof of Theorem 2.2. Due to the regularity result of Theorem 2.1 and the quasi-optimal convergence of the BEM (see, *e.g.*, [29]), one needs to find piecewise polynomial functions approximating the solution u in (2.3) and satisfying the upper bounds stated in (2.10), (2.13).

Let $p \ge \min\{\lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$. Then applying Theorem 7.1 we find $v_{hp} \in V_0^{h,p}(\Gamma)$ satisfying the upper bound given by (7.1). Since $(u - v_{hp}) \in H_0^{s'}(\Gamma)$ for some $s' \in (\frac{1}{2}, 1)$, we obtain by interpolation between $H^0(\Gamma)$ and $H_0^{s'}(\Gamma)$

$$\|u - v_{hp}\|_{\tilde H^{1/2}(\Gamma)} \le C \max\Big\{ h^{\min\{k,p+1\}-1/2} p^{-(k-1/2-\varepsilon)}, \; h^{\min\{\lambda_1^{v_0}+1/2, \gamma_1^{e_0}\}} p^{-2\min\{\lambda_1^{v_0}+1/2,\gamma_1^{e_0}\}} (1 + \log(p/h))^{\beta+\nu} \Big\}, \qquad (7.3)$$

where $\varepsilon > 0$ and β, ν are the same as in (7.1).

Let us select $k > 2\min\{\lambda_1^{v_0} + \frac{1}{2}, \gamma_1^{e_0}\} + \frac{1}{2} \ge \frac{3}{2}$. Then for sufficiently small $\varepsilon > 0$

$$h^{\min\{k,p+1\}-1/2} p^{-(k-1/2-\varepsilon)} \le h^{\min\{\lambda_1^{v_0}+1/2,\gamma_1^{e_0}\}} p^{-2\min\{\lambda_1^{v_0}+1/2,\gamma_1^{e_0}\}},$$

and the desired error bound (see (2.10)) follows from (7.3).

If $1 \le p < \min\{\lambda_1^{v_0}, \gamma_1^{e_0} - 1/2\}$, then $u \in H^m(\Gamma) \cap H_0^1(\Gamma)$ with $1 < m < \min\{\lambda_1^{v_0} + 1, \gamma_1^{e_0} + \frac{1}{2}\}$. Selecting $m \in [p+1, \min\{\lambda_1^{v_0} + 1, \gamma_1^{e_0} + \frac{1}{2}\})$ and applying Proposition 4.1 we find $v_{hp} \in V_0^{h,p}(\Gamma)$ such that

$$\|u - v_{hp}\|_{\tilde H^{1/2}(\Gamma)} \le C\, h^{\min\{m,p+1\}-1/2} \|u\|_{H^m(\Gamma)} \le C\, h^{p+1/2},$$

which proves (2.13). $\square$

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