

ON THE RATE OF CONVERGENCE OF A COLLOCATION PROJECTION OF THE KdV EQUATION^{*,**}

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Abstract. Based on estimates for the KdV equation in analytic Gevrey classes, a spectral collocation approximation of the KdV equation is proved to converge exponentially fast.

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1. INTRODUCTION

In this article, consideration is given to the error analysis of a spectral collocation projection of the periodic Korteweg-de Vries (KdV) equation

$$\partial_t u + u \partial_x u + \partial_x^3 u = 0 \quad (1.1)$$

on the interval $[0, 2\pi]$. It is proved that under appropriate assumptions on the initial data, the convergence of the numerical approximation is exponentially fast. This stands in contrast with previous results that achieved spectral convergence, or in other words super-polynomial convergence.

Since it was first derived by Boussinesq [5] and Korteweg and de Vries [20] as a model for water waves in a channel, the KdV equation has been useful as a model equation in a variety of contexts. The discovery by Zabusky and Kruskal of the elastic interaction of solitary waves [29], and the subsequent formulation of a solution algorithm by way of solving an inverse-scattering problem [1, 10], excited interest in the equation from both the mathematical and physical point of view. Along with the nonlinear Schrödinger equation, the KdV equation has subsequently become a paradigm for nonlinear wave equations featuring competing nonlinear and dispersive effects.

There have been a number of successful numerical schemes for the KdV equation. An interesting review of some of these methods is given by Taha and Ablowitz in [26]. Here we want to investigate the equation in the context of periodic boundary conditions, with a corresponding Fourier-collocation method. Since the discovery by Cooley and Tukey of a fast algorithm to compute the discrete Fourier transform [8], spectral methods based on the Fast Fourier Transform have become a popular choice for the spatial discretization of nonlinear partial differential equations. In particular, in wave propagation problems, spectral projection has been widely used in connection with the Fourier basis. In order to exploit the operational advantage of the fast algorithm, any

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nonlinear terms in the equations have to be implemented pseudospectrally. That means that even though derivatives are taken in transform space, nonlinearities are computed in physical space, making it necessary to perform a transform and an inverse transform at each time step. In this way, the convolution product which inherently takes $O(N^2)$ operations can be computed in $O(N \log N)$ operations. For large N , this represents a significant reduction in total operations.

In the Fourier basis, the pseudospectral method is in fact equivalent to a collocation projection. In connection with this, it becomes clear that instead of the usual Fourier coefficients $\hat{u}(k, t)$ of the solution $u(x, t)$, the discrete Fourier coefficients have to be used for the differentiation. These are given by the sum

$$\tilde{w}_N(k, t) = \frac{1}{2N+1} \sum_{j=0}^{2N} w_N(x_j, t) e^{-ikx_j}, \quad (1.2)$$

where w_N denotes the Fourier-collocation approximation, and $x_j = \frac{2\pi j}{2N+1}$ are the collocation points.

The convergence of both the Galerkin and collocation projections of the KdV equation has been proved by Maday and Quarteroni [23]. In particular, it was shown that for these approximations, spectral convergence is achieved. This means that for smooth solutions of (1.1), the approximants converge to the solution faster than any polynomial. It is our purpose in this article to improve the convergence result of Maday and Quarteroni by showing that if the initial data are analytic in a strip about the real axis, then the convergence rate is actually exponential. That is, if w_N denotes the Fourier-collocation approximation, there exist constants Λ_T and σ_T , depending on the initial data and the final time T , such that

$$\sup_{t \in [0, T]} \|u(\cdot, t) - w_N(\cdot, t)\|_{L^2}^2 \leq \Lambda_T e^{-\sigma_T N}. \quad (1.3)$$

A similar result for the Fourier-Galerkin method was obtained in a recent article of one of the authors [16]. However, as expounded above, collocation methods are more practical in the implementation, so that it is imperative to have a proof for the collocation method, as well. The exponential convergence of Galerkin schemes for parabolic equations has been previously advocated by Ferrari and Titi [11] and proved for the Ginzburg-Landau equation by Doelman and Titi [9]. In these papers, as is the case in our work, the proofs rely on existence results in analytic Gevrey classes. The study of the KdV equation in spaces of analytic functions was initiated by Kato and Masuda in [18]. The problem was subsequently studied by Hayashi [14, 15], and more recently by Bona and Grujić [2] and Bona *et al.* [3]. In particular, it was proved that the radius σ of spatial analyticity decreases at most exponentially over time [2].

All these studies have been in the context of the initial-value problem on the real line. For the periodic problem, existence, uniqueness and continuous dependence on the initial data of solutions to (1.1) have been studied by Temam [27], Kenig *et al.* [19], and more recently by Colliander *et al.* [7], and Kappeler and Topalov [17]. There does not appear to exist any work on the periodic problem in Gevrey-type function spaces.

Besides the Gevrey space analysis and the convergence results of Maday and Quarteroni already mentioned, our proof rests on previous work of Tadmor [25], who showed that for functions analytic in a strip, the Galerkin and collocation projections (to be defined in the next section) converge exponentially fast. This fact, combined with the estimates on solutions of the KdV equation provided in [2] and some techniques used by Maday and Quarteroni [23] will yield exponential convergence of the collocation projection of the KdV equation. In the present work, only a spatial discretization is considered, so that the resulting semi-discrete equation is a system of ordinary differential equations. Though time discretization is not addressed here, it goes without saying that this is also very active field, and time integration schemes for the KdV equation abound.

In the next section, the Galerkin and collocation projections of functions in Gevrey spaces will be discussed, and a version of Tadmor's theorem [25] proved. In Section 3, an estimate on the Gevrey norm of the solution u of (3.1) with periodic boundary conditions will be established. Finally, in Section 4, we put together all the pieces to prove the estimate (1.3). To close the introduction, we will introduce notation to be used throughout.

To quantify the domain of analyticity, we use the class of periodic Gevrey spaces as introduced by Foias and Temam in [12]. Here we follow the notation of Ferrari and Titi [11]. For $\sigma \geq 0$ and $s \geq 0$, we define the Gevrey norm $\|\cdot\|_{G_{\sigma,s}}$ by

$$\|f\|_{G_{\sigma,s}}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s e^{2\sigma\sqrt{1+|k|^2}} |\hat{f}(k)|^2,$$

where the Fourier coefficients $\hat{f}(k)$ of the function f , periodic on the interval $[0, 2\pi]$ are defined by

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x) dx.$$

A Paley-Wiener type argument shows that functions in the space $G_{\sigma,s}$ are analytic in a strip of width 2σ about the real axis. Note that by setting σ equal to zero, we recover the usual periodic Sobolev spaces H^s . These norms are written as

$$\|f\|_{H^s}^2 = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s |\hat{f}(k)|^2.$$

In particular, for $\sigma = 0$ and $s = 0$, the space $L^2(0, 2\pi)$ appears. For simplicity, the L^2 -norm is written without any subscript, so that $\|f\| = \|f\|_{H^0}$. In the sequel, we will often use the inner product on this space, given by

$$(f, g) = \int_0^{2\pi} f(x) \overline{g(x)} dx.$$

We will also have occasion to use the inner product on $G_{\sigma,s}$, given by

$$(f, g)_{G_{\sigma,s}} = \sum_{k \in \mathbb{Z}} (1 + |k|^2)^s e^{2\sigma\sqrt{1+|k|^2}} \hat{f}(k) \overline{\hat{g}(k)}.$$

For functions $f \in H^s$ with $s > \frac{1}{2}$, we have the Sobolev inequality, namely

$$\|f\|_{L^\infty} = \sup_x |f(x)| \leq C \|f\|_{H^s}$$

for some constant C . The space of continuous functions from the interval $[0, T]$ into H^s or $G_{\sigma,s}$ is denoted by $C([0, T], H^s)$ or $C([0, T], G_{\sigma,s})$, respectively.

2. PROJECTION AND INTERPOLATION OPERATORS

The subspace of $L^2(0, 2\pi)$ spanned by the set

$$\left\{ e^{ikx} \mid k \in \mathbb{Z}, -N \leq k \leq N \right\}$$

is denoted by S_N . The self-adjoint operator P_N denotes the orthogonal projection from L^2 onto S_N , defined by

$$P_N f(x) = \sum_{-N \leq k \leq N} e^{ikx} \hat{f}(k).$$

Observe that P_N may also be characterized by the property that, for any $f \in L^2$, $P_N f$ is the unique element in S_N such that

$$\int_0^{2\pi} (P_N f - f) \phi dx = 0,$$

for all $\phi \in S_N$. Using a straightforward calculation, the following inequality can be proved.

$$\|f - P_N f\|_{H^r} \leq N^{r-s} \|f\|_{H^s}, \quad (2.1)$$

for $0 \leq r \leq s$. Moreover, it appears immediately that when $f \in G_{\sigma,s}$, the inequality

$$\|f - P_N f\|_{H^r} \leq N^{r-s} e^{-\sigma N} \|f\|_{G_{\sigma,s}} \quad (2.2)$$

holds for $0 \leq r \leq s$ and $\sigma > 0$. The proof is given by the following computation.

$$\begin{aligned} \|f - P_N f\|_{H^r}^2 &\leq \sum_{|k| \geq N} (1 + |k|^2)^r |\hat{f}(k)|^2 \\ &\leq \sup_{|k| \geq N} \left\{ \frac{1}{(1 + |k|^2)^{s-r}} e^{-2\sigma k} \right\} \sum_{|k| \geq N} e^{2\sigma|k|} (1 + |k|^2)^s |\hat{f}(k)|^2 \\ &\leq \left(\frac{1}{N^{s-r}} e^{-\sigma N} \right)^2 \|f\|_{G_{\sigma,s}}^2. \end{aligned}$$

Finally note the inverse inequality

$$\|\phi\|_{H^r} \leq (2N)^{r-s} \|\phi\|_{H^s}, \quad (2.3)$$

which holds for $r > s \geq 0$ and $\phi \in S_N$. The proof of this estimate proceeds along the lines of the proof of (2.2).

We now turn to the interpolation operator I_N . Let the collocation points be $x_j = \frac{2\pi j}{2N+1}$ for $j = 0, 1, \dots, 2N$. Then, given a continuous periodic function u , $I_N u$ is the unique element in S_N , such that $I_N u(x_j) = u(x_j)$ for $j = 0, 1, \dots, 2N$. Note that S_N is an invariant subspace with respect to I_N . In other words, we have

$$I_N P_N = P_N. \quad (2.4)$$

In connection with the interpolation operator I_N , we also consider the discrete semi-inner product on the space of continuous periodic function on $[0, 2\pi]$, defined as

$$(\phi, \psi)_N = \frac{2\pi}{2N+1} \sum_{j=0}^{2N} \phi(x_j) \overline{\psi(x_j)}. \quad (2.5)$$

Recall that for functions $\phi, \psi \in S_N$, this inner product is equal to the L^2 -inner product, as shown by the identity

$$(\phi, \psi)_N = (\phi, \psi). \quad (2.6)$$

It follows immediately from (2.5), (2.6), and the definition of I_N that

$$(f, g)_N = (I_N f, I_N g) \quad (2.7)$$

for any two functions $f, g \in L^2$. The corresponding semi-norm is defined by

$$\|f\|_N^2 = (f, f)_N.$$

An estimate corresponding to (2.1) also holds for the interpolation operator. It has been proved in [21, 24] that when $f \in H^s$ with $s \geq 1$, and $0 \leq r \leq s$, then there exists a constant C_I , such that

$$\|f - I_N f\|_r \leq C_I N^{r-s} \|f\|_s. \quad (2.8)$$

Moreover, when $f \in G_{\sigma,s}$ for some $\sigma > 0$ and $s > r$, then the difference between f and its interpolant $I_N f$ is exponentially decreasing. This is established in the following lemma.

Lemma 2.1. *Let $\sigma > 0$ and $s > r \geq 0$. For any $f \in G_{\sigma,s}$, the estimate*

$$\|f - I_N f\|_{H^r} \leq C_I N^{r-s} e^{-\sigma N} \|f\|_{G_{\sigma,s}}, \quad (2.9)$$

holds for some constant C_I which only depends on σ , r and s .

Proof. The function $I_N f$ can be expressed in terms of the discrete Fourier coefficients of f as

$$I_N f(x) = \sum_{|p| \leq N} \tilde{f}(p) e^{ipx},$$

where

$$\tilde{f}(p) = \frac{1}{2N+1} \sum_{j=0}^{2N} f(x_j) e^{-ipx_j}. \quad (2.10)$$

The discrete Fourier coefficients $\tilde{f}(p)$ are related to the usual Fourier coefficients $\hat{f}(k)$ by the aliasing relation

$$\tilde{f}(p) = \sum_{k \in \mathbb{Z}} \hat{f}(p + (2N+1)k).$$

In order to prove (2.9), it is convenient to split the norm into two parts.

$$\|f - I_N f\|_{H^r}^2 = \sum_{|p| \leq N} (1 + |p|^2)^r \left| \sum_{k \neq 0} \hat{f}(p + (2N+1)k) \right|^2 + \sum_{|p| > N} (1 + |p|^2)^r |\hat{f}(p)|^2. \quad (2.11)$$

An application of the Cauchy-Schwarz inequality yields

$$\left| \sum_{k \neq 0} \hat{f}(p + (2N+1)k) \right|^2 \leq \sum_{\substack{j=p+(2N+1)k \\ k \neq 0}} (1 + |j|^2)^{-s} e^{-2\sigma\sqrt{1+|j|^2}} \sum_{\substack{j=p+(2N+1)k \\ k \neq 0}} (1 + |j|^2)^s e^{2\sigma\sqrt{1+|j|^2}} |\hat{f}(j)|^2,$$

where the sums on the right are over $k \in \mathbb{Z} \setminus \{0\}$ and j is a function of k . Let us estimate the first factor appearing on the right-hand side, keeping p fixed for the moment. For any j that can be written as $j = p + (2N+1)k$ with $|p| \leq N$ and $k \neq 0$, we have $|j| \geq (2|k| - 1)N \geq 0$. Hence,

$$\begin{aligned} \sum_{\substack{j=p+(2N+1)k \\ k \neq 0}} (1 + |j|^2)^{-s} e^{-2\sigma\sqrt{1+|j|^2}} &\leq \sum_{k \neq 0} (1 + (2|k| - 1)^2 N^2)^{-s} e^{-2\sigma\sqrt{(1+(2|k|-1)^2 N^2)}} \\ &\leq \sum_{k \neq 0} N^{-2s} e^{-2\sigma(2|k|-1)N} \\ &= 2N^{-2s} e^{2\sigma N} \sum_{k > 0} e^{-4\sigma k N} \\ &\leq C_1 N^{-2s} e^{-2\sigma N}, \end{aligned}$$

where the constant $C_1 = \frac{2}{1-e^{-4\sigma}}$ only depends on σ . Continuing the estimate in (2.11), we sum over $|p| \leq N$ to obtain

$$\sum_{|p| \leq N} (1 + |p|^2)^r \left| \sum_{k \neq 0} \hat{f}(p + (2N + 1)k) \right|^2 \leq C_1 (1 + |N|^2)^r \sum_{\substack{j=p+(2N+1)k \\ |p| \leq N \\ k \neq 0}} N^{-2s} e^{-2\sigma N} (1 + |j|^2)^s e^{2\sigma \sqrt{1+|j|^2}} |\hat{f}(j)|^2.$$

In the sum above, for a given $j \in \mathbb{Z}$, there exists at most one couple (p, k) with $|p| \leq N$ such that $j = p + (2N + 1)k$. Hence, there appears

$$\sum_{|p| \leq N} (1 + |p|^2)^r \left| \sum_{k \neq 0} \hat{f}(p + (2N + 1)k) \right|^2 \leq C_1 (1 + N^2)^r N^{-2s} e^{-2\sigma N} \|f\|_{G_{\sigma,s}}^2.$$

Finally, the last term in (2.11) can be estimated as in the proof of (2.2). \square

The error estimate in Lemma 2.1 will be key in establishing the exponential convergence of the collocation approximation to a solution of the KdV equation. However it is only applicable if the solution u of (1.1) can be shown to be bounded in a corresponding Gevrey norm. Obtaining such a bound will be on the agenda in the next section.

3. ESTIMATES IN GEVREY SPACES

It will be now shown that if initial data u_0 are taken to be analytic in a strip around the real axis, then for any t , the solution $u(\cdot, t)$ of (1.1) can also be continued analytically to a (possible smaller) strip around the real axis. The key estimate was proved by Bona and Grujić in the case of the real line [2]. Here we outline a corresponding proof for the initial-value problem on the interval $[0, 2\pi]$ with periodic boundary conditions. The periodic initial value problem associated to equation (1.1) is

$$\begin{cases} \partial_t u + u \partial_x u + \partial_x^3 u = 0, & x \in [0, 2\pi], t \geq 0, \\ u(0, t) = u(2\pi, t), & t \geq 0, \\ u(x, 0) = u_0(x). \end{cases} \quad (3.1)$$

As mentioned in the introduction, existence, uniqueness and continuous dependence on the initial data of this problem in the usual periodic Sobolev classes have been well documented. For our purposes, the following theorem suffices.

Theorem 3.1. *Suppose $s \geq 1$, and $u_0 \in H^s$. Then there exists a solution $u \in C([0, \infty), H^s)$ of (3.1). Moreover, there is a constant κ_s depending on $\|u_0\|_s$, such that u satisfies the estimate*

$$\sup_{t \in [0, \infty)} \|u(\cdot, t)\|_{H^s} \leq \kappa_s. \quad (3.2)$$

In order to gain estimates in Gevrey norms, we use a standard approximation procedure based on a Galerkin projection of the KdV equation. It should be noted here, that the Galerkin procedure is only used as a tool to obtain existence and estimates for the solution of the KdV equation. The numerical scheme in focus in this article is the collocation projection of the KdV equation, which is treated in Section 4. The Galerkin projection of (1.1) is defined as the solution of the equation

$$\begin{cases} (\partial_t u_N + \frac{1}{2} \partial_x (u_N^2) + \partial_x^3 u_N, \phi) = 0, & t \in [0, T], \\ u_N(0) = P_N u_0, \end{cases} \quad (3.3)$$

for all $\phi \in S_N$. The following theorem was proved by Maday and Quarteroni [23].

Theorem 3.2. *Assume that u_0 belongs to H^s , for some $s \geq 2$, and u is a solution of (3.1). Then for N large enough, there exists a solution of (3.3). Moreover, there exists a constant $c > 0$ independent of N , but depending on T and $\|u_0\|_{H^s}$, such that*

$$\sup_{t \in [0, T]} \|u_N(\cdot, t) - u(\cdot, t)\|_{H^1} \leq cN^{2-s}. \quad (3.4)$$

In connection with Theorem 3.1, we can state the following corollary.

Corollary 3.3. *Assume that u_0 belongs to H^s , for some $s > 1$, and u is a solution of (3.1). Let $0 \leq r < s - 1$. Then for N large enough,*

$$\sup_{t \in [0, T]} \|u_N(\cdot, t)\|_{H^r} \leq 2\kappa_s. \quad (3.5)$$

Proof. Using the triangle inequality, the inverse inequality (2.3), and the estimates (2.1) and (3.4), it follows that

$$\begin{aligned} \|u_N - u\|_{H^r} &\leq \|u_N - P_N u\|_{H^r} + \|P_N u - u\|_{H^r} \\ &\leq (2N)^{r-1} \|u_N - P_N u\|_{H^1} + N^{r-s} \|u\|_{H^s} \\ &\leq (2N)^{r-1} \|u_N - u\|_{H^1} + (2N)^{r-1} \|u - P_N u\|_{H^1} + N^{r-s} \kappa_s \\ &\leq (2N)^{r-1} cN^{2-s} + (2N)^{r-1} N^{1-s} \kappa_s + N^{r-s} \kappa_s. \end{aligned}$$

Thus it can be seen that

$$\begin{aligned} \sup_{t \in [0, T]} \|u_N(\cdot, t)\|_{H^r} &\leq \sup_{t \in [0, T]} \|u_N(\cdot, t) - u(\cdot, t)\|_{H^r} + \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^r} \\ &\leq \kappa_s + \kappa_s \end{aligned}$$

for N large enough. □

The next step is the derivation of *a priori* estimates in Gevrey norms for each of the approximants u_N . As the estimates turn out to be independent of N , a standard argument will yield estimates on the limit function u . The main result of this section is the following theorem.

Theorem 3.4. *Suppose that $u \in C([0, T], H^s)$ is a solution of (3.3) with initial data $u_0 \in G_{\sigma_0, s}$ for some $\sigma_0 > 0$ and $s > \frac{5}{2}$. Then $u(\cdot, t)$ extends uniquely to a function in $G_{\sigma(t), s}$ with $\sigma(t)$ given by*

$$\sigma(t) = \sigma_0 e^{-ct \|u_0\|_{G_{\sigma_0, s}} e^{-ct^{3/2}}}, \quad (3.6)$$

for some constant c independent of t . Moreover, for any $\tau \in (0, T]$, we have $u \in C([0, \tau], G_{\sigma(\tau), s})$, and the estimate

$$\|u(\cdot, t)\|_{G_{\sigma(\tau), s}} \leq \|u_0\|_{G_{\sigma_0, s}} + c\sqrt{t}, \quad (3.7)$$

holds for another constant c independent of t .

Remark 3.5. Note that exponential decay of the radius of analyticity is not an optimal result, and could lead to the perception of non-analyticity in a short time. Recently, the algebraic decrease of the radius of analyticity has been proved for the KdV equation on the real line [3]. While it is very likely that a similar result holds for periodic boundary conditions, it has not yet been established.

The proof of the theorem builds on the following auxiliary results which can be found in [2, 12, 22].

Lemma 3.6. *Let $\sigma \geq 0$ and $s > \frac{1}{2}$. Then there exists a constant $c(s)$, not depending on σ , such that*

$$\|fg\|_{G_{\sigma,s}} \leq c(s) \|f\|_{G_{\sigma,s}} \|g\|_{G_{\sigma,s}}, \quad (3.8)$$

for any f and g in $G_{\sigma,s}$.

Lemma 3.7. *Let $\sigma > 0$, $s > 0$ and $r > 0$. Then there exists a constant c , not depending on σ , s or r , such that*

$$\|f\|_{G_{\sigma,s}} \leq c \|f\|_{H^s} + c\sigma^r \|f\|_{G_{\sigma,s+r}},$$

for any $f \in G_{\sigma,s+r}$.

Lemma 3.8. *Let $\sigma \geq 0$ and $s > \frac{3}{2}$. There exists a constant $c(s)$, not depending on σ , such that*

$$(u\partial_x v, v)_{G_{\sigma,s}} \leq c(s) \|u\|_{H^{s+1}} \|v\|_{H^s}^2 + \sigma c(s) \|u\|_{G_{\sigma,s+1}} \|v\|_{G_{\sigma,s+1/2}}^2,$$

for $u \in G_{\sigma,s+1}$ and $v \in G_{\sigma,s+1/2}$.

Proof of Theorem 3.4. To prove the theorem, it is more convenient to work with $v = u_x$. The discrete counterpart v_N satisfies the problem

$$\begin{cases} (\partial_t v_N + v_N^2 + u_N \partial_x v_N + \partial_x^3 v_N, \phi) &= 0, & t \in [0, T], \\ v_N(0) &= P_N \partial_x u_0, \end{cases} \quad (3.9)$$

for all $\phi \in S_N$. Note that u_N exists on the time interval $[0, T]$ by Theorem 3.2, and that it is bounded by Corollary 3.3. Since v_N is sought in $C([0, T], S_N)$, this is equivalent to a finite-dimensional system of ordinary differential equations for the Fourier coefficients $\hat{v}_N(k, t)$ of $v_N(x, t)$. Short-time existence can be proved using a standard contraction argument since the nonlinearity clearly satisfies the Lipschitz condition. It is remarked here that each v_N is a member of $G_{\sigma,s}$ for all $s > 0$ and $\sigma > 0$, and for all times t where the solution exists. Taking the function $\phi(\cdot, t) \in S_N$ defined by its Fourier coefficients $\hat{\phi}(k, t) = (1 + |k|^2)^{s-1} e^{2\sigma\sqrt{(1+|k|^2)}} \hat{v}_N(k, t)$ as a test function in (3.9), there appears the equation

$$(\partial_t v_N + v_N^2 + u_N \partial_x v_N + \partial_x^3 v_N, v_N)_{G_{\sigma,s-1}} = 0.$$

Now if σ is allowed to depend on t , it is plain from the definition of v_N and the inner product on $G_{\sigma,s-1}$ that

$$\frac{d}{dt} (v_N, v_N)_{G_{\sigma,s-1}} = 2(\partial_t v_N, v_N)_{G_{\sigma,s-1}} + 2\dot{\sigma} (v_N, v_N)_{G_{\sigma,s-1/2}}.$$

Since the third derivative operator is skew-symmetric, the equation

$$\frac{1}{2} \frac{d}{dt} (v_N, v_N)_{G_{\sigma,s-1}} - \dot{\sigma} (v_N, v_N)_{G_{\sigma,s-1/2}} + (v_N^2, v_N)_{G_{\sigma,s-1}} + (u_N \partial_x v_N, v_N)_{G_{\sigma,s-1}} = 0$$

appears. Using Cauchy-Schwarz and the previous inequalities, one may now estimate the third and fourth terms of this equation in order to arrive at the differential inequality

$$\frac{d}{dt} \|v_N(\cdot, t)\|_{G_{\sigma(t),s-1}}^2 - (\dot{\sigma} + c\sigma \|v_N\|_{G_{\sigma,s-1}}) \|v_N\|_{G_{\sigma,s-1/2}}^2 \leq C \|v_N\|_{H^{s-1}}^3,$$

for two constants c and C . Recall that it was established in Corollary 3.3 that $u_N(\cdot, t)$ is bounded in the H^s -norm for $t \in [0, T]$. Thus the right hand side of the differential inequality is strictly less than a constant, say K^2 . Accordingly, we may write

$$\frac{d}{dt} \|v_N(\cdot, t)\|_{G_{\sigma(t),s-1}}^2 - (\dot{\sigma} + c\sigma \|v_N\|_{G_{\sigma,s-1}}) \|v_N\|_{G_{\sigma,s-1/2}}^2 \leq C \|u_N\|_{H^s}^3 < K^2. \quad (3.10)$$

Now define

$$\sigma(t) = \sigma_0 e^{-ct \|\partial_x u_0\|_{G_{\sigma_0, s}}} e^{-\frac{2}{3}cKt^{3/2}}, \quad (3.11)$$

and note that for $t = 0$, we have

$$\frac{d}{dt} \|v_N(\cdot, t)\|_{G_{\sigma(t), s-1}}^2 < K^2,$$

so that at least for a short time $t \in [0, \tilde{t})$

$$\|v_N(\cdot, t)\|_{G_{\sigma(t), s-1}} < \|\partial_x u_0\|_{G_{\sigma_0, s}} + K\sqrt{\tilde{t}}. \quad (3.12)$$

Let \tilde{t} be the largest time such that (3.12) is true for all $t \in [0, \tilde{t})$. Then, if we assume that $\tilde{t} < T$, we have

$$\|v_N(\cdot, \tilde{t})\|_{G_{\sigma(\tilde{t}), s-1}} = \|\partial_x u_0\|_{G_{\sigma_0, s}} + K\sqrt{\tilde{t}}. \quad (3.13)$$

For all $t \in [0, \tilde{t}]$, it follows from (3.11) and (3.12), that

$$\dot{\sigma}(t) \leq -c\sigma(t) \|v_N(\cdot, t)\|_{G_{\sigma(t), s-1}},$$

so that (3.10) implies that $\frac{d}{dt} \|v_N(\cdot, t)\|_{G_{\sigma(t), s-1}}^2 < K^2$ and, after integrating,

$$\|v_N(\cdot, t)\|_{G_{\sigma(t), s-1}} < \|\partial_x u_0\|_{G_{\sigma_0, s}} + K\sqrt{t},$$

for all $t \in [0, \tilde{t}]$, which contradicts (3.13). Thus we can conclude that (3.12) holds for all $t \in [0, T]$, and hence $\|u_N(\cdot, t)\|_{G_{\sigma(t), s}} < \|\partial_x u_0\|_{G_{\sigma_0, s}} + K\sqrt{t}$. Since this estimate is uniform in N , it appears that a compactness argument can be used to conclude that the sequence u_N has a subsequence that converges strongly in $C([0, T], G_{\sigma(T), s-\epsilon})$ for any $\epsilon > 0$. By uniqueness, the limit is a classical solution of (3.3). Moreover, it can be shown by an elementary argument that the limit function is bounded by the same constant in the space $C([0, T], G_{\sigma(T), s})$. \square

All the pieces are now in place to proceed to the proof of the main convergence theorem in the next section.

4. THE FOURIER-COLLOCATION METHOD

The collocation approximation to (3.1) is given by a function w_N from $[0, T]$ to S_N , such that

$$\begin{cases} \partial_t w_N + \frac{1}{2} \partial_x I_N(w_N^2) + \partial_x^3 w_N = 0, & t \in [0, T], \\ w_N(0) = I_N u_0. \end{cases} \quad (4.1)$$

Thus we assume that the solution is written as the sum

$$w_N(x, t) = \sum_{-N \leq k \leq N} \tilde{w}_N(k, t) e^{ikx},$$

where the $\tilde{w}_N(k, t)$ are the discrete Fourier coefficients of $w_N(x, t)$ as defined in (1.2).

Theorem 4.1. *Let u be the solution of the periodic initial-value problem (3.1) with initial data $u_0 \in G_{\sigma_0, s}$, where $\sigma_0 > 0$ and $s > \frac{5}{2}$, and let $T > 0$ be given. For N large enough, there exists a unique solution w_N of the finite-dimensional problem (4.1) on the time interval $[0, T]$. Moreover, there exists constants Λ_T and σ_T , depending only on T and $\|u_0\|_{G_{\sigma_0, s}}$, such that*

$$\sup_{t \in [0, T]} \|u(\cdot, t) - w_N(\cdot, t)\| \leq \Lambda_T N^{3-s} e^{-\sigma_T N}. \quad (4.2)$$

The remainder of this section is devoted to the proof of this theorem. The short-time existence of a maximal solution of (4.1) is proved using the contraction mapping principle, and the solution is unique on its maximal interval of definition, $[0, t_N^m)$, where t_N^m is possibly equal to T . Since the argument is standard, the proof is omitted here. Note that as stated in Theorem 3.1, the standard theory of the KdV equation yields the existence of a constant κ_s , such that

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^s} \leq \kappa_s.$$

The main ingredient in the proof of Theorem 4.1 is a local error estimate which will be established by the following lemma.

Lemma 4.2. *Let u be the solution of the periodic initial-value problem (3.1) with initial data $u_0 \in G_{\sigma_0, s}$, where $\sigma_0 > 0$ and $s > \frac{5}{2}$. Suppose there is a solution w_N of (4.1) which exists on the time interval $[0, t_N^*]$ and satisfies $\sup_{t \in [0, t_N^*]} \|w_N(\cdot, t)\|_4 \leq \lambda$, for some $\lambda > 0$. Then there exist two constants σ_T and Λ_T , which only depend on T , $\|u_0\|_{G_{\sigma_0, s}}$ and λ , such that*

$$\sup_{t \in [0, t_N^*]} \|u(\cdot, t) - w_N(\cdot, t)\| \leq \Lambda_T N^{3-s} e^{-\sigma_T N}. \quad (4.3)$$

Proof. Let w_N be a solution of (4.1) which exists on the time interval $[0, t_N^*]$ and satisfies¹ $\sup_{t \in [0, t_N^*]} \|w_N(\cdot, t)\|_{H^4} \leq \lambda$. In the remainder of this proof, we will always consider $t \in [0, t_N^*]$. For the sake of readability, the t -dependence will be suppressed whenever possible. The constant σ_T is given by (3.6) with $t = \tau = T$. We denote by $C_{\lambda, T}$ a generic constant that depends only on λ , T and $\|u_0\|_{G_{\sigma_0, s}}$ but not on N . Using this notation, (3.7) can be written as

$$\sup_{t \in [0, T]} \|u(\cdot, t)\|_{G_{\sigma_T, s}} \leq C_{\lambda, T}.$$

Let $h = w_N - P_N u$. We apply the projection operator P_N to (3.1) and take the scalar product of the resulting equation with an arbitrary function $\psi \in S_N$. We obtain

$$(P_N u_t, \psi) + \frac{1}{2} (P_N \partial_x(u^2), \psi) + (P_N \partial_x^3 u, \psi) = 0. \quad (4.4)$$

After taking the scalar product with ψ , we subtract (4.1) from (4.4) and, since P_N commutes with ∂_x , we get

$$(\partial_t h + \partial_x^3 h + \frac{1}{2} \partial_x I_N(w_N^2) - \frac{1}{2} P_N \partial_x(u^2), \psi) = 0 \quad (4.5)$$

for all $\psi \in S_N$. For $\psi = h$, (4.5) yields

$$(h_t, h) - \frac{1}{2} (P_N \partial_x(u^2) - \partial_x I_N(w_N^2), h) + (\partial_x^3 h, h) = 0.$$

By integrating by parts, one easily checks that $(\partial_x^3 h, h) = 0$. Hence,

$$\begin{aligned} 2(\partial_t h, h) &= (\partial_x(P_N(u^2) - I_N(w_N^2)), h) \\ &= (\partial_x(P_N(u^2) - I_N((P_N u)^2)), h) + (\partial_x I_N((P_N u)^2 - w_N^2), h) \\ &= (\partial_x(P_N(u^2) - I_N((P_N u)^2)), h) - (I_N((P_N u)^2) - I_N(w_N^2), h_x), \end{aligned}$$

after one integration by parts, and

$$2(\partial_t h, h) \leq \|h\| \|P_N(u^2) - I_N((P_N u)^2)\|_{H^1} + \|h_x\| \|I_N((P_N u)^2 - w_N^2)\|. \quad (4.6)$$

¹As can be anticipated, λ will be chosen to be $\lambda = 2\kappa_4$.

Let us estimate the terms on the right-hand side of (4.6). First note that

$$\begin{aligned} \|I_N((P_N u)^2 - w_N^2)\| &= \|I_N(h(P_N u + w_N))\| \\ &= \|(h(P_N u + w_N))\|_N \quad (\text{see (2.7)}) \\ &\leq \|h\|_N \|P_N u + w_N\|_{L^\infty} \\ &= \|h\| \|P_N u + w_N\|_{L^\infty}, \end{aligned}$$

because $\|h\|_N = \|h\|$ as $h \in S_N$, see (2.6). Using the convention we introduced earlier for $C_{\lambda,T}$, the last inequality reads

$$\|I_N((P_N u)^2 - w_N^2)\| \leq C_{\lambda,T} \|h\|, \quad (4.7)$$

as $\|P_N u\|_{L^\infty} \leq C \|u\|_{H^1} \leq C_{\lambda,T}$ (the first inequality corresponding to the Sobolev embedding of H^1 into L^∞) and $\|w_N\|_{L^\infty} \leq C \|w_N\|_{H^1} \leq C\lambda = C_{\lambda,T}$. Using the triangle inequality, the first term on the right in (4.6) may be estimated as

$$\|P_N(u^2) - I_N((P_N u)^2)\|_{H^1} \leq \|P_N(u^2) - (P_N u)^2\|_{H^1} + \|(P_N u)^2 - I_N((P_N u)^2)\|_{H^1}.$$

Lemma 2.1 yields

$$\|(P_N u)^2 - I_N((P_N u)^2)\|_1 \leq C_I N^{1-s} e^{-\sigma N} \|(P_N u)^2\|_{G_{\sigma,s}}.$$

Recall Lemma 3.6 which states that $G_{\sigma,s}$ is a continuous algebra for $s > \frac{1}{2}$. Accordingly, it follows that

$$\begin{aligned} \|(P_N u)^2 - I_N((P_N u)^2)\|_{H^1} &\leq C_{Ic}(s) N^{1-s} e^{-\sigma T N} \|P_N u\|_{G_{\sigma,s}}^2 \\ &\leq C_{\lambda,T} N^{1-s} e^{-\sigma T N}. \end{aligned}$$

Using the triangle inequality again, we have

$$\|(P_N u)^2 - P_N(u^2)\|_{H^1} \leq \|(P_N u)^2 - u^2\|_{H^1} + \|u^2 - P_N(u^2)\|_{H^1}.$$

The second term on the right may be estimated using (2.2), so that

$$\begin{aligned} \|u^2 - P_N(u^2)\|_{H^1} &\leq N^{1-s} e^{-\sigma N} \|u^2\|_{G_{\sigma,s}} \\ &\leq C_{\lambda,T} N^{1-s} e^{-\sigma T N}. \end{aligned}$$

Similarly, it appears that

$$\begin{aligned} \|(P_N u)^2 - u^2\|_{H^1} &\leq C \|P_N u - u\|_{H^1} \|P_N u + u\|_{H^1} \quad (H^1 \text{ is a continuous algebra}) \\ &\leq C_{\lambda,T} N^{1-s} e^{-\sigma T N} 2\kappa_1. \end{aligned}$$

Finally, using the last six inequalities, it is immediate that

$$\|P_N(u^2) - I_N((P_N u)^2)\|_{H^1} \leq C_{\lambda,T} N^{1-s} e^{-\sigma T N}. \quad (4.8)$$

Then the inequalities (4.6), (4.7) and (4.8), yield

$$2(\partial_t h, h) \leq C_{\lambda,T} \|h\| (\|\partial_x h\| + N^{1-s} e^{-\sigma T N}).$$

In conclusion, we obtain the differential inequality

$$\frac{d}{dt} \|h\| \leq C_{\lambda,T} (\|\partial_x h\| + N^{1-s} e^{-\sigma T N}). \quad (4.9)$$

Our next task is to estimate $\|\partial_x h\|$, which appears on the right-hand side of (4.9). For this purpose, we take $\psi = \partial_x^2 h$ in (4.5). Since $(\partial_x^3 h, \partial_x^2 h) = 0$, we obtain

$$(\partial_{xt} h, \partial_x h) = -\frac{1}{2} (I_N(w_N^2) - P_N(u^2), \partial_x^3 h), \quad (4.10)$$

after integrating by parts. Next, we take $\psi = \frac{1}{2} I_N(w_N^2) - \frac{1}{2} P_N(u^2)$ in (4.5) and get

$$\frac{1}{2} (\partial_t h, I_N(w_N^2) - P_N(u^2)) + \frac{1}{2} (\partial_x^3 h, I_N(w_N^2) - P_N(u^2)) + (\partial_x \psi, \psi) = 0. \quad (4.11)$$

The last term in (4.11) vanishes. Comparing (4.10) and (4.11), we obtain

$$2 (\partial_{xt} h, \partial_x h) = (\partial_t h, I_N(w_N^2) - P_N(u^2)),$$

or equivalently

$$\frac{d}{dt} \|\partial_x h\|^2 = (\partial_t h, I_N(w_N^2) - P_N(u^2)).$$

An integration with respect to time now yields

$$\|\partial_x h(\cdot, t)\|^2 - \|\partial_x h(\cdot, 0)\|^2 = \int_0^t (\partial_t h, I_N(w_N^2) - P_N(u^2))(\tau) d\tau.$$

Note also that

$$\begin{aligned} (\partial_t h, I_N(w_N^2) - P_N(u^2)) &= (\partial_t h, w_N^2 - P_N(u^2))_N \\ &= (\partial_t h, h(w_N + P_N u))_N + (\partial_t h, (P_N u)^2 - P_N(u^2))_N \\ &= \frac{1}{2} (\partial_t (h^2), (w_N + P_N u))_N + (\partial_t h, (P_N u)^2 - P_N(u^2))_N. \end{aligned}$$

Combining the last two identities, and integrating by parts with respect to time leads to the following formula

$$\begin{aligned} \|\partial_x h(\cdot, t)\|^2 - \|\partial_x h(\cdot, 0)\|^2 &= \frac{1}{2} [(h^2, w_N + P_N u)_N(\tau)]_{\tau=0}^{\tau=t} \\ &\quad - \frac{1}{2} \int_0^t (h^2, \partial_t (w_N + P_N u))_N(\tau) d\tau + [(h, (P_N u)^2 - P_N(u^2))_N(\tau)]_{\tau=0}^{\tau=t} \\ &\quad - \int_0^t (h, \partial_t ((P_N u)^2 - P_N(u^2)))_N(\tau) d\tau. \end{aligned} \quad (4.12)$$

In order to conclude this part of the proof, all terms on the right-hand side of (4.12) have to be estimated. For the first one, we simply have

$$\begin{aligned} |(h^2, w_N + P_N u)_N| &\leq \|w_N + P_N u\|_{L^\infty} \|h\|_N^2 \\ &\leq C_{\lambda, T} \|h\|^2, \end{aligned} \quad (4.13)$$

because $\|w_N\|_{L^\infty} \leq C \|w_N\|_1 \leq \lambda C = C_{\lambda, T}$. For the second one, note that

$$|(h^2, \partial_t (w_N + P_N u))_N| \leq \|h\|^2 \|\partial_t (w_N + P_N u)\|_{L^\infty}. \quad (4.14)$$

Since

$$\begin{aligned} \|\partial_t w_N\|_{H^1} &= \|\partial_x I_N(w_N^2) + \partial_x^3 w_N\|_{H^1}, && \text{from (4.1),} \\ &\leq \|I_N(w_N^2)\|_{H^2} + \|w_N\|_{H^4} \\ &\leq C \|w_N^2\|_{H^2} + \|w_N\|_{H^4} = C_{\lambda, T}, \end{aligned}$$

and similarly $\|\partial_t P_N w_N\|_{H^1} \leq C_{\lambda, T}$, the inequality (4.14) implies

$$(h^2, \partial_t(w_N + P_N u))_N \leq C_{\lambda, T} \|h\|^2. \quad (4.15)$$

Using the estimate $ab \leq \frac{1}{2}(a^2 + b^2)$ for $a, b \in \mathbb{R}$, it can be inferred that

$$|(h, \partial_t((P_N u)^2 - P_N(u^2)))_N| \leq \frac{1}{2} \left(\|h\|_N^2 + \|\partial_t((P_N u)^2 - P_N(u^2))\|_N^2 \right), \quad (4.16)$$

and that

$$\begin{aligned} \|\partial_t((P_N u)^2 - P_N(u^2))\|_N &= 2 \|P_N(u)P_N(\partial_t u) - P_N(u\partial_t u)\|_N \\ &= 2 \|I_N(P_N(u)P_N(\partial_t u)) - P_N(u\partial_t u)\|, \text{ see (2.7) and (2.4)} \\ &\leq 2 \|I_N(P_N(u)P_N(\partial_t u)) - P_N(u)P_N(\partial_t u)\| \\ &\quad + 2 \|P_N(u)P_N(\partial_t u) - P_N(u\partial_t u)\| \\ &\leq 2N^{3-s} e^{-\sigma T N} \|P_N(u)P_N(\partial_t u)\|_{G_{\sigma, s-3}} \\ &\quad + 2 \|P_N(u)P_N(\partial_t u) - P_N(u\partial_t u)\|. \end{aligned} \quad (4.17)$$

Using (3.8) and (3.1), we get

$$\begin{aligned} \|P_N u P_N(\partial_t u)\|_{G_{\sigma, s-3}} &\leq \|P_N(u)\|_{G_{\sigma, s-3}} \|P_N(\partial_t u)\|_{G_{\sigma, s-3}} \\ &\leq \|P_N(u)\|_{G_{\sigma, s-3}} \|P_N(u\partial_x u) + P_N\partial_x^3 u\|_{G_{\sigma, s-3}} \\ &\leq \|u\|_{G_{\sigma, s-3}} (\|u\|_{G_{\sigma, s-3}} \|u\|_{G_{\sigma, s-2}} + \|u\|_{G_{\sigma, s}}) \\ &\leq C_{\lambda, T}. \end{aligned}$$

One may get a bound of the same type for $\|P_N(u)P_N(\partial_t u) - P_N(u\partial_t u)\|$ in (4.17) by writing

$$\|P_N(u)P_N(\partial_t u) - P_N(u\partial_t u)\| \leq \|P_N(u)P_N(\partial_t u) - P_N(u)\partial_t u\| + \|P_N(u)\partial_t u - u\partial_t u\| + \|u\partial_t u - P_N(u\partial_t u)\|.$$

In summary, there appears the estimate

$$\|\partial_t((P_N u)^2 - P_N(u^2))\|_N \leq C_{\lambda, T} N^{3-s} e^{-\sigma T N}.$$

Inserting this into (4.16), we obtain

$$|(h, \partial_t((P_N u)^2 - P_N(u^2)))_N| \leq C_{\lambda, T} (\|h\|^2 + (N^{3-s} e^{-\sigma T N})^2). \quad (4.18)$$

The only term which remains to be estimated in (4.12) is $(h, (P_N u)^2 - P_N(u^2))_N$. This is done in a similar way as (4.18). In fact, it is even more straight forward since it does not involve a time derivative. The result is that

$$|(h, (P_N u)^2 - P_N(u^2))_N| \leq C_{\lambda, T} (\|h\|^2 + (N^{1-s} e^{-\sigma T N})^2). \quad (4.19)$$

Let us now define $v(t) = \sup_{\tau \in [0, t]} \|h(\cdot, \tau)\|$. From (4.12), using (4.13), (4.15), (4.18) and (4.19), we obtain

$$\begin{aligned} \|\partial_x h(\cdot, t)\|^2 &\leq \|\partial_x h(\cdot, 0)\|^2 + C_{\lambda, T}((N^{1-s}e^{-\sigma T N})^2 + \|h(\cdot, t)\|^2) \\ &\quad + C_{\lambda, T} \int_0^t (\|h(\cdot, \tau)\|^2 + (N^{3-s}e^{-\sigma T N})^2) d\tau \\ &\leq \|\partial_x h(\cdot, 0)\|^2 + C_{\lambda, T}(v(t)^2 + (N^{3-s}e^{-\sigma T N})^2). \end{aligned}$$

Hence

$$\|\partial_x h(\cdot, t)\| \leq C_{\lambda, T}(\|\partial_x h(\cdot, 0)\| + v(t) + N^{3-s}e^{-\sigma T N}),$$

and since $\|\partial_x h(\cdot, 0)\| \leq \|h(\cdot, 0)\|_1 \leq CN^{1-s}e^{-\sigma T N}$, it follows that

$$\|h_x(\cdot, t)\| \leq C_{\lambda, T}(v(t) + N^{3-s}e^{-\sigma T N}). \quad (4.20)$$

Evidently, the preceding argument was inspired by the corresponding computation for the third conservation law of the KdV equation. However, the fact that the interpolation operator I_N does not commute with the derivative ∂_x makes the proof that much more complicated. Having in hand an estimate for $\|\partial_x h\|$, we can return to the main thread of the proof. We integrate (4.9) with respect to time, and use the bound for $\|\partial_x h\|$ given by (4.20) to obtain

$$\begin{aligned} \|h(\cdot, t)\| &\leq \|h(\cdot, 0)\| + C_{\lambda, T} \int_0^t (v(\tau) + N^{3-s}e^{-\sigma T N}) d\tau \\ &\leq C_{\lambda, T}N^{3-s}e^{-\sigma T N} + C_{\lambda, T} \int_0^t v(\tau) d\tau. \end{aligned}$$

Since v is positive, the function on the right is increasing, so that

$$\|h(\cdot, \tau)\| \leq C_{\lambda, T}N^{3-s}e^{-\sigma T N} + C_{\lambda, T} \int_0^t v(\tau') d\tau', \text{ for any } \tau \in [0, t].$$

Taking the supremum over $[0, t]$, it transpires that

$$v(t) \leq C_{\lambda, T}N^{3-s}e^{-\sigma T N} + C_{\lambda, T} \int_0^t v(\tau) d\tau.$$

Gronwall's Lemma now yields

$$v(t) \leq C_{\lambda, T}N^{3-s}e^{-\sigma T N} e^{C_{\lambda, T}t},$$

which implies that

$$\|h(\cdot, t)\| \leq C_{\lambda, T}N^{3-s}e^{-\sigma T N} \quad (4.21)$$

for all $t \in [0, t_N^*]$. Finally, we have

$$\begin{aligned} \|u - w_N\| &\leq \|u - P_N u\| + \|P_N u - w_N\| \\ &\leq N^{-s}e^{-\sigma T N} \|u\|_{G_{\sigma, s}} + \|h\| \\ &\leq C_{\lambda, T}N^{3-s}e^{-\sigma T N}. \end{aligned} \quad (4.22)$$

Since (4.22) holds for any $t \in [0, t_N^*]$, the estimate (4.3) follows directly from (4.22), with Λ_T chosen to be the constant $C_{\lambda, T}$ appearing in (4.22). \square

Proof of Theorem 4.1. We want to extend the estimate (4.3) to the time interval $[0, T]$. Let

$$\lambda = 2 \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^4} \leq 2\kappa_4. \quad (4.23)$$

We now define t_N^* by

$$t_N^* = \sup \{t \in [0, T] \mid \text{for all } t' \leq t, \|w_N(\cdot, t')\|_{H^4} \leq \lambda\}. \quad (4.24)$$

Thus the time t_N^* corresponds to the largest time in $[0, T]$ for which the H^4 -norm of w_N is uniformly bounded by λ and Lemma 4.2 applies for this particular λ . From (2.9), we obtain that

$$\begin{aligned} \|I_N u_0\|_{H^4} &\leq \|u_0\|_{H^4} + C_I N^{4-s} e^{-\sigma_0 N} \|u_0\|_{G_{\sigma_0, s}} \\ &\leq \frac{\lambda}{2} + C_I N^{4-s} e^{-\sigma_0 N} \|u_0\|_{G_{\sigma_0, s}} \\ &< \lambda \end{aligned}$$

for N large enough. Hence, $\|w_N(\cdot, 0)\|_{H^4} = \|I_N u_0\|_{H^4} < \lambda$ and $t_N^* > 0$ for all large enough N . Note that t_N^* is necessarily smaller than the maximum time of existence. On the other hand, we are going to prove that there exists \bar{N} such that

$$t_N^* = T \quad \text{for all } N \geq \bar{N}, \quad (4.25)$$

and therefore the supremum in (4.3) holds on $[0, T]$. By definition (4.24), we either have $t_N^* = T$ or $t_N^* < T$ and in this case, since $\|w_N(t)\|_{H^4}$ is a continuous function of time, $\|w_N(t_N^*)\| = \lambda$. Assume that $t_N^* < T$. Using the triangle inequality, we have

$$\begin{aligned} \lambda &= \|w_N(\cdot, t_N^*)\|_{H^4} \\ &\leq \|w_N(\cdot, t_N^*) - u(\cdot, t_N^*)\|_4 + \sup_{t \in [0, T]} \|u(\cdot, t)\|_{H^4} \\ &= \|w_N(\cdot, t_N^*) - u(\cdot, t_N^*)\|_{H^4} + \frac{\lambda}{2}, \end{aligned}$$

by the definition of λ . Hence, $\frac{\lambda}{2} \leq \|w_N(\cdot, t_N^*) - u(\cdot, t_N^*)\|_{H^4}$. By (4.21), the triangle inequality and the inverse inequality (2.3), it follows that

$$\lambda \leq C N^{7-s} e^{-\sigma_T N} \quad (4.26)$$

for some constant C independent of N . However, since $\lim_{N \rightarrow \infty} N^{7-s} e^{-\sigma_T N} = 0$, there exists \bar{N} such that for all $N \geq \bar{N}$, $N^{7-s} e^{-\sigma_T N} < \lambda/C$. For such $N \geq \bar{N}$, (4.26) does not hold and therefore we cannot have $t_N^* < T$. Thus it is plain that $t_N^* = T$, and the claim (4.25) is proved. It follows that for $N \geq \bar{N}$ the solution w_N of (4.1) is defined on $[0, T]$ and, from (4.3), we get

$$\sup_{t \in [0, T]} \|u(\cdot, t) - w_N(\cdot, t)\| \leq \Lambda_T N^{3-s} e^{-\sigma_T N}. \quad (4.27)$$

□

It appears that when the initial data have sufficient Gevrey-class regularity, then the power of N on the right-hand side of (4.27) can be eschewed, and the result advertised in the introduction appears. The statement is summarized in the following corollary.

Corollary 4.3. *Let u be the solution of the periodic initial-value problem (3.1) with initial data $u_0 \in G_{\sigma_0, s}$, where $\sigma_0 > 0$ and $s \geq 3$; and let $T > 0$ be given. Then for N large enough, there exists a unique solution w_N of*

the finite-dimensional problem (4.1) on $[0, T]$. Moreover, there exist constants Λ_T and σ_T , depending only on T and $\|u_0\|_{G_{\sigma_0, s}}$, such that

$$\sup_{t \in [0, T]} \|u(\cdot, t) - w_N(\cdot, t)\| \leq \Lambda_T e^{-\sigma_T N}.$$

Similar error estimates in higher Sobolev norms can be proved by using the triangle inequality, the inverse inequality (2.3), and the estimates (2.2) and (4.21).

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