A UNIFORMLY ACCURATE (UA) MULTISCALE TIME INTEGRATOR PSEUDOSPECTRAL METHOD FOR THE NONLINEAR DIRAC EQUATION IN THE NONRELATIVISTIC LIMIT REGIME[☆]

YONGYONG CAI AND YAN WANG*

Abstract. A multiscale time integrator Fourier pseudospectral (MTI-FP) method is proposed and rigorously analyzed for the nonlinear Dirac equation (NLDE), which involves a dimensionless parameter $\varepsilon \in (0, 1]$ inversely proportional to the speed of light. The solution to the NLDE propagates waves with wavelength $O(\varepsilon^2)$ and O(1) in time and space, respectively. In the nonrelativistic regime, *i.e.*, $0 < \varepsilon \ll 1$, the rapid temporal oscillation causes significantly numerical burdens, making it quite challenging for designing and analyzing numerical methods with uniform error bounds in $\varepsilon \in (0, 1]$. The key idea for designing the MTI-FP method is based on adopting a proper multiscale decomposition of the solution to the NLDE and applying the exponential wave integrator with appropriate numerical quadratures. Two independent error estimates are established for the proposed MTI-FP method as $h^{m_0} + \frac{\tau^2}{\varepsilon^2}$ and $h^{m_0} + \tau^2 + \varepsilon^2$, where *h* is the mesh size, τ is the time step and m_0 depends on the regularity of the solution. These two error bounds immediately suggest that the MTI-FP method converges uniformly and optimally in space with exponential convergence rate if the solution is smooth, and uniformly in time with linear convergence rate at $O(\tau)$ for all $\varepsilon \in (0, 1]$ and optimally with quadratic convergence rate at $O(\tau^2)$ in the regimes when either $\varepsilon = O(1)$ or $0 < \varepsilon \lesssim \tau$. Numerical results are reported to demonstrate that our error estimates are optimal and sharp.

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1. INTRODUCTION

We consider the d dimensional (d = 1, 2) two-component nonlinear Dirac equation (NLDE) [1, 2, 9, 23, 28, 38, 42] in the dimensionless form:

$$i\partial_t \Phi^{\varepsilon}(t, \mathbf{x}) = \left[-\frac{i}{\varepsilon} \sum_{j=1}^d \sigma_j \partial_j + \frac{1}{\varepsilon^2} \sigma_3 + V(t, \mathbf{x}) I_2 - \sum_{j=1}^d A_j(t, \mathbf{x}) \sigma_j + F(\Phi^{\varepsilon}) \right] \Phi^{\varepsilon}(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$
(1.1)

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Beijing Computational Science Research Center, Beijing 100193, P.R. China.

^{*} Corresponding author: matwyan@csrc.ac.cn

where $i = \sqrt{-1}$, t is time, $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d$ is the spatial coordinate vector, $\partial_k = \frac{\partial}{\partial x_k}$ $(k = 1, \ldots, d)$, $\Phi^{\varepsilon} := \Phi^{\varepsilon}(t, \mathbf{x}) = (\phi_1^{\varepsilon}(t, \mathbf{x}), \phi_2^{\varepsilon}(t, \mathbf{x}))^T \in \mathbb{C}^2$ is the complex-valued vector wave function of the "spinorfield", $\varepsilon \in (0, 1]$ is a dimensionless parameter inversely proportional to the speed of light. I_n is the $n \times n$ identity matrix for $n \in \mathbb{N}$, $V := V(t, \mathbf{x})$ is the real-valued electrical potential, and $\mathbf{A} := \mathbf{A}(t, \mathbf{x}) = (A_1(t, \mathbf{x}), \ldots, A_d(t, \mathbf{x}))^T$ is the real-valued magnetic potential vector. $F(\Phi^{\varepsilon}) \in \mathbb{C}^{2 \times 2}$ is the matrix nonlinearity, depending on real problems or applications [16, 23, 38]. $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
(1.2)

In this paper, we consider the following cubic nonlinearity [9]

$$F(\Phi^{\varepsilon})\Phi^{\varepsilon} = \left[\lambda_1\left((\Phi^{\varepsilon})^*\sigma_3\Phi^{\varepsilon}\right)\sigma_3 + \lambda_2|\Phi^{\varepsilon}|^2I_2\right]\Phi^{\varepsilon}, \quad \Phi^{\varepsilon} \in \mathbb{C}^2,$$
(1.3)

with $\lambda_1, \lambda_2 \in \mathbb{R}$ being two dimensionless constants, $(\Phi^{\varepsilon})^* = \overline{\Phi^{\varepsilon}}^T$ being the conjugate transpose of Φ^{ε} and $|\Phi^{\varepsilon}|$ being the Euclidean norm of Φ^{ε} . The initial data is given as

$$\Phi^{\varepsilon}(t=0,\mathbf{x}) = \Phi_0(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d, \qquad d=1,2.$$
(1.4)

For the simplicity of notations, we will omit the superscript ε and write Φ^{ε} as Φ in the rest of the paper, while we understand that the solution $\Phi(t, \mathbf{x})$ also depends on $\varepsilon \in (0, 1]$.

When $\lambda_1 \neq 0$ and $\lambda_2 = 0$, the corresponding NLDE is the famous Soler model [38] in quantum field theory and has been widely considered in the literature [1, 2, 13, 19, 28, 32, 42]. The nonlinearity with $\lambda_1 = 0$ and $\lambda_2 \neq 0$ is motivated from the Bose-Einstein condensates with a chiral confinement and/or spin-orbit coupling [16, 23, 31]. For simplicity, we take $\lambda_1 = 0$ and $\lambda_2 \neq 0$ in the following discussion, and our results are also suitable for other nonlinearities [34, 42].

We remark that in three dimensions, the nonlinear Dirac equation takes a four-component vector form as

$$i\partial_t \Phi(t, \mathbf{x}) = \left[-\frac{i}{\varepsilon} \sum_{j=1}^3 \alpha_j \partial_j + \frac{1}{\varepsilon^2} \beta + V(t, \mathbf{x}) I_4 - \sum_{j=1}^3 A_j(t, \mathbf{x}) \sigma_j + F(\Phi) \right] \Phi(t, \mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^3, \tag{1.5}$$

where Φ becomes a four-component vector function, $\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}$ (j = 1, 2, 3) are 4×4 matrices, $\beta = \text{diag}(1, 1, -1, -1)$ and $F(\Phi) = \lambda_1 (\Phi^* \beta \Phi) \beta + \lambda_2 |\Phi|^2 I_4$. (1.5) is quite similar to the two-component form (1.1) and it suffices to consider NLDE (1.1), while all the results and conclusions in the paper hold for the four-component form (1.5).

The NLDE (1.1) has been extensively studied in the literatures, both theoretically [4, 11, 13, 16, 30, 33, 39] and numerically [1, 9, 14, 21, 36, 40, 42]. In this paper, we are interested in developing a uniformly accurate (UA) scheme for the NLDE (1.1) in the "nonrelativistic regime", *i.e.*, $0 < \varepsilon \ll 1$, where the NLDE (1.1) converges to a coupled nonlinear Schrödinger system (NLS) as $\varepsilon \to 0^+$ [15, 33].

According to the theoretical analysis in [15, 33], the solution of the NLDE (1.1) propagates waves with wavelength $O(\varepsilon^2)$ in time and O(1) in space, which has also been validated numerically [9]. This implies that the solution would become highly oscillatory in time in the nonrelativistic regime where $\varepsilon \ll 1$. The highly oscillatory behavior of the solution brings severe burdens in practical computation. In fact, according to a rigorous analysis and detailed comparison in [9], the meshing strategies (ε -resolution) should be $h = O(\sqrt{\varepsilon}), \tau = O(\varepsilon^3)$ for the finite difference time domain (FDTD) methods and $h = O(1), \tau = O(\varepsilon^2)$ for the exponential wave integrator Fourier pseudospectral (EWI-FP) method and the time-splitting Fourier spectral (TSFP) methods, where h is the mesh size and τ is the time step. Therefore, it is a challenging issue to design efficient and accurate numerical methods for the NLDE (1.1) in the nonrelativistic limit regime, *i.e.*, $0 < \varepsilon \ll 1$.

Recently, a UA scheme has been proposed for the NLDE (1.1) based on a two-scale formulation approach in [28]. Although this UA scheme can achieve uniform second order accuracy in time, it causes significant increase in computational cost and memory since the reformulation increases the dimension of the problem. Apart from the two-scale formulation approach [17, 28], the multiscale expansion based approach is also widely considered for designing UA schemes [5, 7, 8, 20]. It is explicit, uniformly convergent in space and time, and efficient.

The main aim of this paper is to propose and analyze an efficient UA scheme for the NLDE (1.1) with $\varepsilon \in (0, 1]$. For the linear case ($\lambda_1 = \lambda_2 = 0$), such UA scheme [8] has been designed based on the superposition property and a natural decomposition of the solution. For the nonlinear case, the superposition property is no longer valid and it becomes a challenging problem to design and analyze UA schemes for the NLDE (1.1). More recently, we have studied an $O(\varepsilon^2)$ approximations of the NLDE (1.1) in the nonrelativistic limit regime [15], which enables possible design of a UA scheme. Based on this study of multiscale decomposition of the solution to NLDE [15], we propose a MTI-FP method for the problem (1.1) with the help of the exponential wave integrator (EWI) technique in solving highly oscillatory differential equations [5, 22, 24, 26]. Then, two independent error estimates, using the energy method, are established for the numerical method as $h^{m_0} + \frac{\tau^2}{\varepsilon^2}$ and $h^{m_0} + \tau^2 + \varepsilon^2$ with m_0 depending on the regularity of the solution. Thus the MTI-FP method can attain uniform convergence in both space and time with respect to $0 < \varepsilon \leq 1$.

This paper is organized as follows. First, we introduce a multiscale decomposition for the NLDE (1.1) and present the MTI-FP method in Section 2. Then in Section 3, we rigorously establish error estimates for the proposed MTI-FP method. At last, Section 4 is devoted to the numerical results of the MTI-FP method.

2. The MTI-FP Method

In this section, we mainly show the construction of the MTI-FP method based on a multiscale decomposition with $O(\varepsilon^2)$ accuracy [15]. For simplicity of notations, we only consider the NLDE (1.1) in 1D with nonlinearity $F(\Phi) = |\Phi|^2 I_2$, while all the notations and results can be easily generalized to (1.5) and (1.1) in higher dimensions without any extra work.

Similar to the linear case in [8], let us write the NLDE (1.1) in 1D as

$$i\partial_t \Phi(t,x) = \frac{1}{\varepsilon^2} \mathcal{T}\Phi(t,x) + W(t,x)\Phi(t,x) + F(\Phi)\Phi(t,x), \qquad x \in \mathbb{R},$$
(2.1)

where the operators \mathcal{T} and W(t, x) are defined as

$$\mathcal{T} = -i\varepsilon\sigma_1\partial_x + \sigma_3, \quad W(t,x) = V(t,x)I_2 - A_1(t,x)\sigma_1, \qquad x \in \mathbb{R}.$$
(2.2)

It is important to note that \mathcal{T} is diagonalizable in the phase space and can be decomposed as

$$\mathcal{T} = \sqrt{Id - \varepsilon^2 \Delta} \Pi_+ - \sqrt{Id - \varepsilon^2 \Delta} \Pi_-, \qquad (2.3)$$

where Δ is the Laplace operator, Id is the identity operator, Π_+ and Π_- are projectors [12, 33] defined as

$$\Pi_{+} = \frac{1}{2} \left[I_{2} + \left(Id - \varepsilon^{2} \Delta \right)^{-1/2} \mathcal{T} \right], \quad \Pi_{-} = \frac{1}{2} \left[I_{2} - \left(Id - \varepsilon^{2} \Delta \right)^{-1/2} \mathcal{T} \right].$$
(2.4)

It can be verified that $\Pi_{+} + \Pi_{-} = I_2$ and $\Pi_{+}\Pi_{-} = \Pi_{-}\Pi_{+} = \mathbf{0}$, $\Pi_{\pm}^2 = \Pi_{\pm}$. The projectors can be easily computed in Fourier domain. After truncating the NLDE (1.1) onto a bounded domain with periodic boundary conditions, the corresponding projectors are efficiently computable by fast Fourier transform, which is crucial to the success of our proposed numerical method. We remark that the method also works under homogeneous Dirichlet/Neumann boundary conditions using fast sine/cosine transform.



FIGURE 1. Residue function $E = \|\Phi(t, \cdot) - e^{-it/\varepsilon^2} \Psi_+(t, \cdot) - e^{it/\varepsilon^2} \Psi_-(t, \cdot)\|_{L^2}$ for different ε .

2.1. Multiscale decomposition

In [15], we have studied the (semi)-nonrelativistic limit of the NLDE (1.5) and obtained a second order approximation to the NLDE as $\varepsilon \to 0^+$ (we refer to Thm. 2.2 in [15] for more details):

$$\Phi(t,x) = e^{-it/\varepsilon^2} \Psi_+(t,x) + e^{it/\varepsilon^2} \Psi_-(t,x) + O(\varepsilon^2), \qquad (2.5)$$

where $\Psi_{\pm} := \Psi_{\pm}(t, x)$ satisfy the system

$$\begin{cases} i\partial_t \Psi_+ = \frac{\sqrt{Id - \varepsilon^2 \Delta} - Id}{\varepsilon^2} \Psi_+ + \Pi_+ \Big[\left(W + (|\Psi_+|^2 + |\Psi_-|^2)I_2 \right) \Psi_+ \Big], \quad \Psi_+(0, x) = \Pi_+ \Phi_0(x), \\ i\partial_t \Psi_- = -\frac{\sqrt{Id - \varepsilon^2 \Delta} - Id}{\varepsilon^2} \Psi_- + \Pi_- \Big[\left(W + (|\Psi_+|^2 + |\Psi_-|^2)I_2 \right) \Psi_- \Big], \quad \Psi_-(0, x) = \Pi_- \Phi_0(x). \end{cases}$$
(2.6)

To further validate this, we numerically solve the NLDE (1.1) and (2.6), and plot the residue function $E = \|\Phi(t, \cdot) - e^{-it/\varepsilon^2}\Psi_+(t, \cdot) - e^{it/\varepsilon^2}\Psi_-(t, \cdot)\|_{L^2}$ in Figure 1 for a 1D problem with $V(t, x) = \frac{1-x}{1+x^2}$, $A_1(t, x) = \frac{(1+x)^2}{1+x^2}$, $\lambda_1 = 0$, $\lambda_2 = 1$, $\Phi_0(x) = (e^{-x^2/2}, e^{-(x-1)^2/2})^T$. As can be seen that, the residue is of order $O(\varepsilon^2)$ and highly oscillatory with wavelength $O(\varepsilon^2)$ (shown in the zoom in plot).

According to this study, we can construct the following multiscale decomposition. Choose the time step $\tau := \Delta t > 0$ and denote time steps as $t_n := n \tau$ for $n \ge 0$. We can write the solution $\Phi(t, x) = \Phi(t_n + s, x), s \in [0, \tau]$ to the NLDE (1.1) on the time interval $[t_n, t_{n+1}]$ as

$$\Phi(t_n + s, x) = e^{-is/\varepsilon^2} \Psi^n_+(s, x) + e^{is/\varepsilon^2} \Psi^n_-(s, x) + \mathbf{r}^n(s, x), \qquad x \in \mathbb{R}, \quad 0 \le s \le \tau,$$
(2.7)

where $(\Psi_+^n, \Psi_-^n) := (\Psi_+^n(s, x), \Psi_-^n(s, x))$ solves the coupled system for $x \in \mathbb{R}$ and $0 \le s \le \tau$ as

$$\begin{cases} i\partial_{s}\Psi_{+}^{n}(s,x) = \frac{\sqrt{Id-\varepsilon^{2}\Delta}-Id}{\varepsilon^{2}}\Psi_{+}^{n}(s,x) + \Pi_{+} \Big[\left(W^{n}+(|\Psi_{+}^{n}|^{2}+|\Psi_{-}^{n}|^{2})I_{2}\right)\Psi_{+}^{n}(s,x) \Big],\\ i\partial_{s}\Psi_{-}^{n}(s,x) = -\frac{\sqrt{Id-\varepsilon^{2}\Delta}-Id}{\varepsilon^{2}}\Psi_{-}^{n}(s,x) + \Pi_{-} \Big[\left(W^{n}+(|\Psi_{+}^{n}|^{2}+|\Psi_{-}^{n}|^{2})I_{2}\right)\Psi_{-}^{n}(s,x) \Big],\\ \Psi_{+}^{n}(0,x) = \Pi_{+}\Phi^{n}(x), \quad \Psi_{-}^{n}(0,x) = \Pi_{-}\Phi^{n}(x), \end{cases}$$
(2.8)

with $W^n := W^n(s) = W(t_n + s, x), \ \Phi^n(x) := \Phi(t_n, x).$

Differentiating (2.7) with respect to s, using (2.8) and (1.1), we can derive the equation for the residue $\mathbf{r}^n(s,x)$ as

$$\begin{cases} i\partial_s \mathbf{r}^n(s,x) = \frac{1}{\varepsilon^2} \mathcal{T} \mathbf{r}^n(s,x) + F_r(\Psi^n_+, \Psi^n_-, \mathbf{r}^n; s, x), \\ \mathbf{r}^n(0,x) = \mathbf{0}, \end{cases}$$
(2.9)

where

$$F_r(\Psi_+^n, \Psi_-^n, \mathbf{r}^n; s, x) = \sum_{k=-3}^3 e^{iks/\varepsilon^2} F_{r,k}(\Psi_+^n, \Psi_-^n, \mathbf{r}^n; s, x)$$
(2.10)

with $F_{r,k} := F_{r,k}(\Psi_{+}^{n}, \Psi_{-}^{n}, \mathbf{r}^{n}; s) = F_{r,k}(\Psi_{+}^{n}, \Psi_{-}^{n}, \mathbf{r}^{n}; s, x), \ k = -3, \dots, 3$ defined as follows

$$\begin{cases} F_{r,3} = ((\Psi_{+}^{n})^{*}\Psi_{-}^{n})\Psi_{-}^{n}, & F_{r,2} = ((\Psi_{+}^{n})^{*}\Psi_{-}^{n})\mathbf{r}^{n} + ((\Psi_{+}^{n})^{*}\mathbf{r}^{n})\Psi_{-}^{n} + ((\mathbf{r}^{n})^{*}\Psi_{-}^{n})\Psi_{-}^{n}, \\ F_{r,1} = \Pi_{+} \Big[\left(W^{n} + (|\Psi_{+}^{n}|^{2} + |\Psi_{-}^{n}|^{2})I_{2} \right)\Psi_{-}^{n} \Big] + ((\Psi_{+}^{n})^{*}\Psi_{-}^{n})\Psi_{+}^{n} + ((\Psi_{+}^{n})^{*}\mathbf{r}^{n} + (\mathbf{r}^{n})^{*}\Psi_{-}^{n})\mathbf{r}^{n} + |\mathbf{r}^{n}|^{2}\Psi_{-}^{n}, \\ F_{r,0} = \left(W^{n} + (|\Psi_{+}^{n}|^{2} + |\Psi_{-}^{n}|^{2} + |\mathbf{r}^{n}|^{2})I_{2} \right)\mathbf{r}^{n} + ((\Psi_{+}^{n})^{*}\mathbf{r}^{n} + (\mathbf{r}^{n})^{*}\Psi_{-}^{n})\Psi_{+}^{n} + ((\Psi_{-}^{n})^{*}\mathbf{r}^{n} + (\mathbf{r}^{n})^{*}\Psi_{+}^{n})\Psi_{-}^{n}, \\ F_{r,-1} = \Pi_{-} \Big[\left(W^{n} + (|\Psi_{+}^{n}|^{2} + |\Psi_{-}^{n}|^{2})I_{2} \right)\Psi_{+}^{n} \Big] + ((\Psi_{-}^{n})^{*}\Psi_{+}^{n})\Psi_{-}^{n} + ((\Psi_{-}^{n})^{*}\mathbf{r}^{n} + (\mathbf{r}^{n})^{*}\Psi_{+}^{n})\mathbf{r}^{n} + |\mathbf{r}^{n}|^{2}\Psi_{+}^{n}, \\ F_{r,-2} = ((\Psi_{-}^{n})^{*}\Psi_{+}^{n})\mathbf{r}^{n} + ((\Psi_{-}^{n})^{*}\mathbf{r}^{n})\Psi_{+}^{n} + ((\mathbf{r}^{n})^{*}\Psi_{+}^{n})\Psi_{+}^{n}, \qquad F_{r,-3} = ((\Psi_{-}^{n})^{*}\Psi_{+}^{n})\Psi_{+}^{n}. \end{aligned}$$

$$(2.11)$$

Notice that the above $F_{r,k}$ are computed for $F(\Phi) = |\Phi|^2 I_2$. For a general polynomial type nonlinearity $F(\Phi)$, one can compute them accordingly. Moreover, it should be noted that in each step we reset $\mathbf{r}^{n}(0,x) = \mathbf{0}, \Psi^{n}_{+}(0,x) = \Pi_{+}\Phi^{n}(x)$ to avoid error accumulation.

2.2. The MTI Fourier spectral discretization in 1D

In practical computation, similar to those in literatures [14, 25, 27, 40, 41], we truncate the NLDE (1.1) with d = 1 to a bounded interval $\Omega = (a, b)$ for $\Phi := \Phi(t, x) \in \mathbb{C}^2$,

$$i\partial_t \Phi(t,x) = \frac{1}{\varepsilon^2} \mathcal{T}\Phi(t,x) + W(t,x)\Phi(t,x) + F(\Phi)\Phi(t,x), \qquad x \in \Omega, \quad t > 0,$$
(2.12)

with periodic boundary conditions and initial condition as

$$\Phi(t,x) \text{ is } (b-a) \text{ periodic in } x, \quad t \ge 0; \qquad \Phi(0,x) = \Phi_0(x), \quad x \in \overline{\Omega},$$
(2.13)

where we use the same notation \mathcal{T} and W(t, x) here as those in the whole space case (2.2) by abuse of notation and we remark that the domain of \mathcal{T} here is $(H_p^1(\Omega))^2$ with $H_p^1(\Omega) = \{u \in H^1(\Omega) | u(a) = u(b)\}$. Similarly, systems (2.8) and (2.9) for the decomposition (2.7) are also truncated to the bounded domain

 $x \in \Omega$ with periodic boundary conditions.

Choose the mesh size $h := \Delta x = (b-a)/M$ with M being a positive even integer and denote the grid points as $x_j := a + j h \text{ for } j \in \mathcal{T}_M := \{0, 1, \dots, M\}. \text{ Denote } X_M = \{U = (U_0, U_1, \dots, U_M)^T \mid U_j \in \mathbb{C}^2, j \in \mathcal{T}_M, U_0 = U_M\}$ and the l^2 , l^∞ norms in X_M are given by $||U||_{l^2}^2 = h \sum_{j=0}^{M-1} |U_j|^2$ and $||U||_{l^\infty} = \max_{j \in \mathcal{T}_M} \{|U_j|\}.$ Introduce

$$Y_M = Z_M \times Z_M$$
, with $Z_M = \text{span}\left\{\phi_l(x) = e^{i\mu_l(x-a)}, \ \mu_l = \frac{2l\pi}{b-a}, \ l = -\frac{M}{2}, \dots, \frac{M}{2} - 1\right\}$.

Let $[C_p(a,b)]^2$ be the function space consisting of all periodic vector function U(x): $[a,b] \to \mathbb{C}^2$. For any $U(x) \in [L^2(a,b)]^2$ and $U \in X_M$, define $P_M : [L^2(a,b)]^2 \to Y_M$ as the standard projection operator, $I_M :$

 $[C_p(a,b)]^2 \to Y_M$ and $I_M: X_M \to Y_M$ as the standard interpolation operator, *i.e.*

$$(P_M U)(x) = \sum_{l=-M/2}^{M/2-1} \widehat{U}_l e^{i\mu_l(x-a)}, \quad (I_M U)(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{U}_l e^{i\mu_l(x-a)}, \qquad a \le x \le b,$$
(2.14)

with

$$\widehat{U}_{l} = \frac{1}{b-a} \int_{a}^{b} U(x) e^{-i\mu_{l}(x-a)} dx, \quad \widetilde{U}_{l} = \frac{1}{M} \sum_{j=0}^{M-1} U_{j} e^{-2ijl\pi/M}, \quad l = -\frac{M}{2}, \dots, \frac{M}{2} - 1, \quad (2.15)$$

where $U_j = U(x_j)$ when U is a function. Parseval's identity implies that

$$\|I_M(U)(\cdot)\|_{L^2} = \|U\|_{l^2}, \quad \forall U \in X_M.$$
(2.16)

The Fourier spectral discretization for (2.8) and (2.9) reads: Find $\Psi_{\pm,M}^n := \Psi_{\pm,M}^n(s) = \Psi_{\pm,M}^n(s,x)$, $\mathbf{r}_M^n := \mathbf{r}_M^n(s) = \mathbf{r}_M^n(s,x) \in Y_M$ $(0 \le s \le \tau, a \le x \le b)$, *i.e.*

$$\Psi_{\pm,M}^{n}(s,x) = \sum_{l=-M/2}^{M/2-1} \widehat{\left(\Psi_{\pm}^{n}\right)_{l}}(s) e^{i\mu_{l}(x-a)}, \quad \mathbf{r}_{M}^{n}(s,x) = \sum_{l=-M/2}^{M/2-1} \widehat{\left(\mathbf{r}^{n}\right)_{l}}(s) e^{i\mu_{l}(x-a)}, \tag{2.17}$$

such that for a < x < b and $0 \le s \le \tau$

$$\begin{cases} i\partial_{s}\Psi_{+,M}^{n} = \frac{\sqrt{Id - \varepsilon^{2}\Delta} - Id}{\varepsilon^{2}}\Psi_{+,M}^{n} + P_{M}\left(\Pi_{+}\left[\left(W^{n} + (|\Psi_{+,M}^{n}|^{2} + |\Psi_{-,M}^{n}|^{2})I_{2}\right)\Psi_{+,M}^{n}\right]\right), \\ i\partial_{s}\Psi_{-,M}^{n} = -\frac{\sqrt{Id - \varepsilon^{2}\Delta} - Id}{\varepsilon^{2}}\Psi_{-,M}^{n} + P_{M}\left(\Pi_{-}\left[\left(W^{n} + (|\Psi_{+,M}^{n}|^{2} + |\Psi_{-,M}^{n}|^{2})I_{2}\right)\Psi_{-,M}^{n}\right]\right), \\ i\partial_{s}\mathbf{r}_{M}^{n} = \frac{1}{\varepsilon^{2}}\mathcal{T}\mathbf{r}_{M}^{n} + P_{M}\left(F_{r}(\Psi_{+,M}^{n},\Psi_{-,M}^{n},\mathbf{r}_{M}^{n};s,x)\right), \\ \Psi_{+,M}^{n}(0) = P_{M}\left(\Pi_{+}\Phi(t_{n},x)\right), \quad \Psi_{-,M}^{n}(0) = P_{M}\left(\Pi_{-}\Phi(t_{n},x)\right), \quad \mathbf{r}_{M}^{n}(0) = \mathbf{0}. \end{cases}$$

$$(2.18)$$

For simplicity of notations, by omitting the spatial x variable, we introduce

$$f_{\pm}^{n}(s) = \left(W^{n}(s) + (|\Psi_{+,M}^{n}(s)|^{2} + |\Psi_{-,M}^{n}(s)|^{2})I_{2}\right)\Psi_{\pm,M}^{n}(s),$$

$$g_{k}^{n}(s) = F_{r,k}\left(\Psi_{+,M}^{n}(s), \Psi_{-,M}^{n}(s), \mathbf{r}_{M}^{n}(s); s\right), \qquad k = -3, \dots, 3.$$
(2.19)

Then we can obtain the equations for the Fourier coefficients with $0 \leq s \leq \tau$ as

$$\begin{cases} i\partial_{s}\widehat{\left(\Psi_{+}^{n}\right)}_{l}(s) = \frac{\delta_{l}-1}{\varepsilon^{2}}I_{2}\widehat{\left(\Psi_{+}^{n}\right)}_{l}(s) + \Pi_{l}^{+}\widehat{\left(f_{+}^{n}\right)}_{l}(s),\\ i\partial_{s}\widehat{\left(\Psi_{-}^{n}\right)}_{l}(s) = -\frac{\delta_{l}-1}{\varepsilon^{2}}I_{2}\widehat{\left(\Psi_{-}^{n}\right)}_{l}(s) + \Pi_{l}^{-}\widehat{\left(f_{-}^{n}\right)}_{l}(s), \qquad l = -\frac{M}{2}, \dots, \frac{M}{2} - 1, \qquad (2.20)\\ i\partial_{s}\widehat{\left(\mathbf{r}^{n}\right)}_{l}(s) = \frac{1}{\varepsilon^{2}}\mathcal{T}_{l}\widehat{\left(\mathbf{r}^{n}\right)}_{l}(s) + \sum_{k=-3}^{3}\widehat{\left(g_{k}^{n}\right)}_{l}(s), \end{cases}$$

where $\delta_l = \sqrt{1 + \varepsilon^2 \mu_l^2}$, and $\mathcal{T}_l = Q_l D_l(Q_l)^*$, Π_l^+ and Π_l^- are the corresponding Fourier representations of the operator \mathcal{T} and projectors Π_{\pm} as

$$\mathcal{T}_{l} = \begin{pmatrix} 1 & \varepsilon \mu_{l} \\ \varepsilon \mu_{l} & -1 \end{pmatrix}, \quad \Pi_{l}^{+} = \begin{pmatrix} \frac{1+\delta_{l}}{2\delta_{l}} & \frac{\varepsilon \mu_{l}}{2\delta_{l}} \\ \frac{\varepsilon \mu_{l}}{2\delta_{l}} & \frac{\varepsilon^{2} \mu_{l}^{2}}{2\delta_{l}(\delta_{l}+1)} \end{pmatrix}, \quad \Pi_{l}^{-} = \begin{pmatrix} \frac{\varepsilon^{2} \mu_{l}^{2}}{2\delta_{l}(\delta_{l}+1)} & -\frac{\varepsilon \mu_{l}}{2\delta_{l}} \\ -\frac{\varepsilon \mu_{l}}{2\delta_{l}} & \frac{1+\delta_{l}}{2\delta_{l}} \end{pmatrix}, \quad (2.21)$$

with

$$Q_{l} = \begin{pmatrix} \frac{1+\delta_{l}}{\sqrt{2\delta_{l}(1+\delta_{l})}} & -\frac{\varepsilon\mu_{l}}{\sqrt{2\delta_{l}(1+\delta_{l})}}\\ \frac{\varepsilon\mu_{l}}{\sqrt{2\delta_{l}(1+\delta_{l})}} & \frac{1+\delta_{l}}{\sqrt{2\delta_{l}(1+\delta_{l})}} \end{pmatrix}, \qquad D_{l} = \begin{pmatrix} \delta_{l} & 0\\ 0 & -\delta_{l} \end{pmatrix}.$$
(2.22)

Using the variation-of-constant formula, the solution to system (2.20) can be written as

$$\begin{split} \widehat{\left(\Psi_{+}^{n}\right)}_{l}(s) &= e^{-i(\delta_{l}-1)s/\varepsilon^{2}}\widehat{\left(\Psi_{+}^{n}\right)}_{l}(0) - i\int_{0}^{s} e^{-i(\delta_{l}-1)(s-w)/\varepsilon^{2}}\Pi_{l}^{+}\widehat{\left(f_{+}^{n}\right)}_{l}(w) \,\mathrm{d}w, \\ \widehat{\left(\Psi_{-}^{n}\right)}_{l}(s) &= e^{i(\delta_{l}-1)s/\varepsilon^{2}}\widehat{\left(\Psi_{-}^{n}\right)}_{l}(0) - i\int_{0}^{s} e^{i(\delta_{l}-1)(s-w)/\varepsilon^{2}}\Pi_{l}^{-}\widehat{\left(f_{-}^{n}\right)}_{l}(w), \\ \widehat{\left(\mathbf{r}^{n}\right)}_{l}(s) &= e^{-i\mathcal{T}_{l}s/\varepsilon^{2}}\widehat{\left(\mathbf{r}^{n}\right)}_{l}(0) - \sum_{k=-3}^{3} i\int_{0}^{s} e^{ikw/\varepsilon^{2}}e^{-i\mathcal{T}_{l}(s-w)/\varepsilon^{2}}\widehat{\left(g_{k}^{n}\right)}_{l}(w) \,\mathrm{d}w. \end{split}$$

Using the initial conditions and choosing $s = \tau$, we can approximate the integrals *via* the Gautschi type quadrature rules [7, 22, 24, 26] or EWI [5, 7, 8]. Using Taylor expansion, we can approximate the first integral in the above equation as

$$-i\int_{0}^{\tau} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \Pi_{l}^{+}(\widehat{f_{+}^{n}})_{l}(s) \,\mathrm{d}s \approx -i\int_{0}^{\tau} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \Pi_{l}^{+}\left[\widehat{(f_{+}^{n})}_{l}(0) + s(\widehat{\partial_{s}f_{+}^{n}})_{l}(0)\right] \,\mathrm{d}s$$

$$= p_{l}(\tau) \,\Pi_{l}^{+}(\widehat{f_{+}^{n}})_{l}(0) + q_{l}(\tau) \,\Pi_{l}^{+}(\widehat{\partial_{s}f_{+}^{n}})_{l}(0),$$
(2.23)

where $(\partial_s f^n_+)(0)$ can be computed by using (2.18), and

$$p_{l}(\tau) := -i \int_{0}^{\tau} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} ds = -i\tau e^{\frac{-i\tau(\delta_{l}-1)}{2\varepsilon^{2}}} \operatorname{sinc}\left(\frac{(\delta_{l}-1)\tau}{2\varepsilon^{2}}\right),$$

$$q_{l}(\tau) := -i \int_{0}^{\tau} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} s \, ds = \frac{-\tau\varepsilon^{2}}{(\delta_{l}-1)} \left(1 - e^{\frac{-i\tau(\delta_{l}-1)}{2\varepsilon^{2}}} \operatorname{sinc}\left(\frac{(\delta_{l}-1)\tau}{2\varepsilon^{2}}\right)\right),$$
(2.24)

where $\operatorname{sin}(s) = \frac{\sin s}{s}$ with $\operatorname{sinc}(0) = 1$. According to the definition, it can be found that $p_l(\tau) = O(\tau)$ and $q_l(\tau) = O(\tau^2)$ for $l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1$, and for the special case l = 0, $p_0(\tau) = -i\tau$ and $q_0(\tau) = -i\frac{\tau^2}{2}$. Similarly, the other integrals can be approximated as

$$-i\int_{0}^{\tau} e^{i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \Pi_{l}^{-}(\widehat{f_{-}^{n}})_{l}(s) \,\mathrm{d}s \approx -\overline{p_{l}(\tau)} \,\Pi_{l}^{-}(\widehat{f_{-}^{n}})_{l}(0) - \overline{q_{l}(\tau)} \,\Pi_{l}^{-}(\widehat{\partial_{s}f_{-}^{n}})_{l}(0),$$

$$-i\int_{0}^{\tau} e^{iks/\varepsilon^{2}} e^{-i\mathcal{T}_{l}(\tau-s)/\varepsilon^{2}} \widehat{(g_{k}^{n})}_{l}(s) \,\mathrm{d}s \approx P_{l}^{(k)}(\tau) \widehat{(g_{k}^{n})}_{l}(0) + Q_{l}^{(k)}(\tau) \widehat{(\partial_{s}g_{k}^{n})}_{l}(0),$$

$$(2.25)$$

where $P_l^{(k)} = Q_l D_{l,P}^{(k)}(Q_l)^*$ and $Q_l^{(k)} = Q_l D_{l,Q}^{(k)}(Q_l)^*$ for k = -3, ..., 3 with Q_l defined in (2.22), $D_{l,P}^{(k)} = \text{diag}\left(p_{l,+}^{(k)}(\tau), \ p_{l,-}^{(k)}(\tau)\right), D_{l,Q}^{(k)} = \text{diag}\left(q_{l,+}^{(k)}(\tau), \ q_{l,-}^{(k)}(\tau)\right)$ and

$$p_{l,\pm}^{(k)} = -i\tau e^{\frac{i\tau(k\pm\delta_l)}{2\varepsilon^2}} \operatorname{sinc}\left(\frac{(k\pm\delta_l)\tau}{2\varepsilon^2}\right), \ q_{l,\pm}^{(k)} = \frac{-\tau\varepsilon^2}{\pm\delta_l+k} \left(e^{\frac{ik\tau}{\varepsilon^2}} - e^{\frac{-i\tau(\pm\delta_l-k)}{2\varepsilon^2}} \operatorname{sinc}\left(\frac{(k\pm\delta_l)\tau}{2\varepsilon^2}\right)\right).$$
(2.26)

Here, for the special cases where $\pm \delta_l + k = 0$, we have $q_{l,\pm}^{(k)} = -\frac{i\tau^2}{2} e^{\mp i\delta_l \tau/\varepsilon^2}$.

Therefore, we can obtain the solution to the system (2.20) as (for $l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1$)

$$\begin{cases} \widehat{(\Psi_{+}^{n})}_{l}(\tau) \approx e^{-i(\delta_{l}-1)\tau/\varepsilon^{2}} \widehat{(\Psi_{+}^{n})}_{l}(0) + p_{l}(\tau) \Pi_{l}^{+} \widehat{(f_{+}^{n})}_{l}(0) + q_{l}(\tau) \Pi_{l}^{+} \widehat{(\partial_{s}f_{+}^{n})}_{l}(0), \\ \widehat{(\Psi_{-}^{n})}_{l}(\tau) \approx e^{i(\delta_{l}-1)\tau/\varepsilon^{2}} \widehat{(\Psi_{-}^{n})}_{l}(0) - \overline{p_{l}(\tau)} \Pi_{l}^{-} \widehat{(f_{-}^{n})}_{l}(0) - \overline{q_{l}(\tau)} \Pi_{l}^{-} \widehat{(\partial_{s}f_{-}^{n})}_{l}(0), \\ \widehat{(\mathbf{r}^{n})}_{l}(\tau) \approx \sum_{k=-3}^{3} \left(P_{l}^{(k)}(\tau) \widehat{(g_{k}^{n})}_{l}(0) + Q_{l}^{(k)}(\tau) \widehat{(\partial_{s}g_{k}^{n})}_{l}(0) \right), \end{cases}$$
(2.27)

with initial values and derivatives determined from (2.18) as

$$\widehat{(\Psi_{+}^{n})}_{l}(0) = \Pi_{l}^{+}(\widehat{\Phi(t_{n})})_{l}, \quad \left(\widehat{\partial_{s}\Psi_{+,M}^{n}}\right)_{l}(0) \approx -i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau}\widehat{(\Psi_{+}^{n})}_{l}(0) - i\Pi_{l}^{+}\widehat{(f_{+}^{n})}_{l}(0), \\
\widehat{(\Psi_{-}^{n})}_{l}(0) = \Pi_{l}^{-}(\widehat{\Phi(t_{n})})_{l}, \quad \left(\widehat{\partial_{s}\Psi_{-,M}^{n}}\right)_{l}(0) \approx i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau}\widehat{(\Psi_{-}^{n})}_{l}(0) - i\Pi_{l}^{-}\widehat{(f_{-}^{n})}_{l}(0), \quad (2.28)$$

$$\widehat{(\mathbf{r}^{n})}_{l}(0) = 0, \quad \widehat{(\partial_{s}\mathbf{r}_{M}^{n})}_{l}(0) \approx \frac{\widehat{(\mathbf{r}^{n})}_{l}(\tau) - \widehat{(\mathbf{r}^{n})}_{l}(0)}{\tau} \approx \frac{\widehat{(\mathbf{r}^{n,*})}_{l} - \widehat{(\mathbf{r}^{n})}_{l}(0)}{\tau} = \frac{\widehat{(\mathbf{r}^{n,*})}_{l}}{\tau} = \frac{1}{\tau} \sum_{k=-3}^{3} P_{l}^{(k)}(\tau)\widehat{(g_{k}^{n})}_{l}(0).$$

Note that the derivatives $\partial_s \Psi_{\pm,M}^n(0)$ are approximated using (2.20), and filters $2\sin(\mu_l^2\tau/2)/\tau$ are used to replace μ_l^2 $(l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1)$, avoiding loss of regularity and accuracy, which is crucial in the error estimate (explained in Rem. 3.7); when approximating $(\mathbf{r}^n)_l(\tau)$ with the desired second order accuracy in time in (2.27), we need the information of $\partial_s \mathbf{r}_M^n(0)$, for which we first approximate $(\mathbf{r}^n)_l(\tau)$ by $(\mathbf{r}^{n,*})_l := \sum_{k=-3}^3 P_l^{(k)}(\tau)(\widehat{g_k^n})_l(0)$ with first order accuracy (throwing away high order terms involving $Q_l^{(k)}(\tau)$ in (2.27)), then use finite difference to approximate $\partial_s \mathbf{r}_M^n(0)$. From t_n to t_{n+1} , after obtaining numerical approximations of $\Psi_{\pm}^n(\tau)$ and $\mathbf{r}^n(\tau)$ from $\Phi(t_n, x)$, we use decomposition (2.7) to update the numerical solution $\Phi(t_{n+1}, x)$ at $t = t_{n+1}$.

2.3. The MTI-FP method in 1D

In practice, due to the difficulties of computing the Fourier transform coefficients in (2.15), (2.23)–(2.25) are usually approximated by the numerical quadratures. Let Φ_j^n be the numerical approximation of the exact solution $\Phi(t_n, x_j)$ to the NLDE (2.12) for $n \ge 0$ and $j \in \mathcal{T}_M$, and denote $\Phi^n \in X_M$ as the numerical solution vector at time $t = t_n$; in addition, let $\Psi_{\pm,j}^{n+1}$ (k = 1, 2) be the numerical approximation of $\Psi_{\pm}^n(\tau, x_j)$ for $j \in \mathcal{T}_M$ and $n \ge 0$, and denote $V_j^n = V(t_n, x_j)$, $A_{1,j}^n = A_1(t_n, x_j)$, $W_j^n = W(t_n, x_j) = V_j^n I_2 - A_{1,j}^n \sigma_1$, $\dot{W}_j^n = \partial_t W(t_n, x_j)$ for $j \in \mathcal{T}_M$ and $n \ge 0$. Choosing $\Phi_j^0 = \Phi_0(x_j)$ for $j \in \mathcal{T}_M$, then the multiscale time integrator Fourier pseudospectral (MTI-FP) method for discretizing the NLDE (1.1) in 1D reads for $n \ge 0$ and $j \in \mathcal{T}_M$ as:

$$\Phi_{j}^{n+1} = e^{-i\tau/\varepsilon^{2}}\Psi_{+,j}^{n+1} + e^{i\tau/\varepsilon^{2}}\Psi_{-,j}^{n+1} + \mathbf{r}_{j}^{n+1} = \sum_{l=-M/2}^{M/2-1} (\widetilde{\Phi^{n+1}})_{l} e^{i\mu_{l}(x_{j}-a)},$$
(2.29)

where
$$\Psi_{\pm,j}^{n+1} = \sum_{l=-M/2}^{M/2-1} (\widetilde{\Psi_{\pm}^{n+1}})_l e^{i\mu_l(x_j-a)}, \quad \mathbf{r}_j^{n+1} = \sum_{l=-M/2}^{M/2-1} (\widetilde{\mathbf{r}_{n+1}})_l e^{i\mu_l(x_j-a)} \text{ with}$$

$$\begin{cases} (\widetilde{\Psi_{\pm}^{n+1}})_l = e^{-i\frac{(\delta_l-1)\tau}{\epsilon^2}} (\widetilde{\Psi_{\pm}})_l + p_l(\tau) \Pi_l^+ (\widetilde{f_{\pm}})_l + q_l(\tau) \Pi_l^+ (\widetilde{f_{\pm}})_l, \\ (\widetilde{\Psi_{\pm}^{n+1}})_l = e^{i\frac{(\delta_l-1)\tau}{\epsilon^2}} (\widetilde{\Psi_{\pm}})_l - \overline{p_l(\tau)} \Pi_l^- (\widetilde{f_{\pm}})_l - \overline{q_l(\tau)} \Pi_l^- (\widetilde{f_{\pm}})_l, \\ (\widetilde{\mathbf{r}_{n+1}})_l = \sum_{k=-3}^3 \left(P_l^{(k)}(\tau) (\widetilde{g_k})_l + Q_l^{(k)}(\tau) (\widetilde{g_k})_l \right) \end{cases}$$
(2.30)

and

$$\begin{cases} f_{\pm,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(f_{\pm})}_l e^{i\mu_l(x_j-a)}, \quad \Psi_{\pm,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(\Psi_{\pm})}_l e^{i\mu_l(x_j-a)}, \quad \mathbf{r}_j = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathbf{r})}_l e^{i\mu_l(x_j-a)}, \\ \dot{f}_{\pm,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(f_{\pm})}_l e^{i\mu_l(x_j-a)}, \quad \dot{\Psi}_{\pm,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(\Psi_{\pm})}_l e^{i\mu_l(x_j-a)}, \quad \dot{\mathbf{r}}_j = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathbf{r})}_l e^{i\mu_l(x_j-a)}, \\ g_{k,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(g_k)}_l e^{i\mu_l(x_j-a)}, \quad \dot{g}_{k,j} = \sum_{l=-M/2}^{M/2-1} \widetilde{(g_k)}_l e^{i\mu_l(x_j-a)}, \quad j \in \mathcal{T}_M, \quad k = -3, \dots, 3, \end{cases}$$

with

$$\begin{cases} \widetilde{(\Psi_{+})}_{l} = \Pi_{l}^{+} \widetilde{(\Phi^{n})}_{l}, \quad \widetilde{(\Psi_{-})}_{l} = \Pi_{l}^{-} \widetilde{(\Phi^{n})}_{l}, \quad \widetilde{(\Psi_{+})}_{l} = -i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau} \widetilde{(\Psi_{+})}_{l} - i \Pi_{l}^{+} \widetilde{(f_{+})}_{l}, \\ \widetilde{(\Psi_{-})}_{l} = i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau} \widetilde{(\Psi_{-})}_{l} - i \Pi_{l}^{-} \widetilde{(f_{-})}_{l}, \quad \widetilde{(\mathbf{r})}_{l} = \mathbf{0}, \quad \widetilde{(\mathbf{r})}_{l} = \frac{1}{\tau} \sum_{k=-3}^{3} P_{l}^{(k)}(\tau) \widetilde{(g_{k})}_{l}, \\ f_{\pm,j} = \left(W_{j}^{n} + (|\Psi_{+,j}|^{2} + |\Psi_{-,j}|^{2})I_{2}\right)\Psi_{\pm,j}, \quad g_{k,j} = F_{r,k}(\Psi_{+,j}, \Psi_{-,j}, \mathbf{r}_{j}; 0, x_{j}), \quad k = -3, \dots, 3, \\ \dot{f}_{\pm,j} = \left(\dot{W}_{j}^{n} + 2Re\left[(\Psi_{+,j})^{*}\dot{\Psi}_{+,j} + (\Psi_{-,j})^{*}\dot{\Psi}_{-,j}\right]I_{2}\right)\Psi_{\pm,j} + \left(W_{j}^{n} + (|\Psi_{+,j}|^{2} + |\Psi_{-,j}|^{2})I_{2}\right)\dot{\Psi}_{\pm,j}, \\ \dot{g}_{3,j} = \left(\dot{\Psi}_{+,j}\right)^{*}\Psi_{-,j}\dot{\Psi}_{-,j} + \left(\dot{\Psi}_{+,j}\right)^{*}\dot{\Psi}_{-,j}\Psi_{-,j}, \\ \dot{g}_{1,j} = \Pi_{l}^{-}\left[\left(\dot{W}_{j}^{n} + 2Re\left[(\Psi_{+,j})^{*}\dot{\Psi}_{-,j} + (\Psi_{-,j})^{*}\dot{\Psi}_{-,j}\right]I_{2}\right)\Psi_{-,j} + \left(|\Psi_{+,j}|^{2} + |\Psi_{-,j}|^{2})\dot{\Psi}_{-,j} \\ + W_{j}^{n}\dot{\Psi}_{-,j}\right] + \left(\dot{\Psi}_{+,j}\right)^{*}\Psi_{-,j}\Psi_{+,j} + \left(\Psi_{-,j}\right)^{*}\dot{\Psi}_{-,j}U_{+,j} + \left(\Psi_{+,j}\right)^{*}\dot{\Psi}_{-,j}\psi_{+,j}, \\ \dot{g}_{0,j} = \left(W_{j}^{n} + (|\Psi_{+,j}|^{2} + |\Psi_{-,j}|^{2})I_{2}\right)\dot{\mathbf{r}}_{+} \left(\left(\Psi_{+,j}\right)^{*}\dot{\mathbf{r}}_{j} + \left(\dot{\Psi}_{-,j}\right)^{*}\dot{\Psi}_{-,j}\right)I_{2}\right)\Psi_{+,j} + \left(\left(\Psi_{-,j}\right)^{*}\dot{\mathbf{r}}_{j}\right)I_{2}\right)\Psi_{+,j} + \left(\left(\Psi_{-,j}\right)^{*}\dot{\mathbf{r}}_{j} + \left(\dot{\Psi}_{-,j}\right)^{2}\dot{\Psi}_{+,j}, \\ \dot{g}_{-1,j} = \Pi_{l}^{-}\left[\left(\dot{W}_{j}^{n} + 2Re\left[\left(\Psi_{+,j}\right)^{*}\dot{\Psi}_{+,j} + \left(\Psi_{-,j}\right)^{*}\dot{\Psi}_{-,j}\right]I_{2}\right)\Psi_{+,j} + \left(\left|\Psi_{+,j}\right|^{2} + \left|\Psi_{-,j}\right|^{2})\dot{\Psi}_{+,j} \\ + W_{j}^{n}\dot{\Psi}_{+,j}\right] + \left(\dot{\Psi}_{-,j}\right)^{*}\dot{\Psi}_{+,j} + \left(\Psi_{-,j}\right)^{*}\dot{\Psi}_{+,j}\Psi_{+,j}, \quad j \in \mathcal{T}_{M}. \end{cases}$$

$$(2.31)$$

where $F_{r,k}(\Psi_+, \Psi_-, \mathbf{r}; s, x)$ is given in (2.11) and $\operatorname{Re}(c)$ denotes the real part of the complex number c.

This MTI-FP method (2.29)–(2.31) for the NLDE (2.12) is explicit, accurate, easy to implement and very efficient due to the discrete fast Fourier transform. The memory cost is O(M) and the computational cost per time step is $O(M \log M)$. As will be shown in the next section, it is uniformly convergent in space and time with respect to $\varepsilon \in (0, 1]$.

3. A UNIFORM ERROR BOUND

In this section, we rigorously establish two independent error estimates for the proposed MTI-FP method (2.29)-(2.31) via two different mathematical approaches. Let $0 < T < T_{\text{max}} < \infty$ with T_{max} being the common maximal existence time of the solution for the NLDE (2.12). Motivated by the results for the nonlinear Dirac equation (1.1) (or (1.5)) in [15, 33], we make the following assumptions on the electromagnetic potentials in (2.12)

(A)
$$||V||_{W^{2,\infty}([0,T];W_p^{m_0,\infty})} + ||A||_{W^{2,\infty}([0,T];W_p^{m_0,\infty})} \lesssim 1, \quad m_0 \ge 5,$$

and the exact solution $\Phi = \Phi(t) := \Phi(t, x)$ of the NLDE (2.12) with $\varepsilon \in (0, 1]$

$$(B) \quad \|\Phi\|_{L^{\infty}([0,T];(H_{p}^{m_{0}})^{2})} \lesssim 1, \qquad \|\partial_{t}\Phi\|_{L^{\infty}([0,T];(H_{p}^{m_{0}-1})^{2})} \lesssim \frac{1}{\varepsilon^{2}}, \qquad \|\partial_{tt}\Phi\|_{L^{\infty}([0,T];(L^{2})^{2})} \lesssim \frac{1}{\varepsilon^{4}},$$

where $H_p^m(\Omega) = \{u \mid u \in H^m(\Omega), \ \partial_x^l u(a) = \partial_x^l u(b), \ l = 0, \dots, m-1\}$ and $W_p^{m,\infty}(\Omega) = \{u \mid u \in W^{m,\infty}(\Omega), \ \partial_x^l u(a) = \partial_x^l u(b), \ l = 0, \dots, m-1\}$ for $m \in \mathbb{N}$. We remark here that the assumption (B) is equivalent to that the initial value $\Phi_0(x) \in (H_p^{m_0})^2$ [12, 33] under the assumption (A).

Denote

$$C_0 = \max_{\varepsilon \in (0,1]} \|\Phi\|_{L^{\infty}([0,T]; (L^{\infty}(\Omega))^2)}.$$
(3.1)

We have the following error estimates for the MTI-FP method (2.29)-(2.31).

Theorem 3.1. Let $\Phi^n \in X_M$ be the numerical approximation obtained from the MTI-FP method (2.29)– (2.31) for discretizing the NLDE (2.12) and $\Phi(t,x)$ be the exact solution of NLDE (2.12), and denote $\Phi^n_I(x) = I_M(\Phi^n)(x) \in Y_M$. Under the assumptions (A) and (B), there exist constants $0 < \tau_0, h_0 \leq 1$ sufficiently small and independent of ε , such that for any $0 < \varepsilon \leq 1$, when $0 < \tau \leq \tau_0$ and $0 < h \leq h_0$, we have

$$\|\Phi(t_{n},\cdot) - \Phi_{I}^{n}(\cdot)\|_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + \frac{\tau^{2}}{\varepsilon^{2}}, \quad \|\Phi(t_{n},\cdot) - \Phi_{I}^{n}(\cdot)\|_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + \tau^{2} + \varepsilon^{2}, \quad \|\Phi^{n}\|_{l^{\infty}} \le C_{0} + 1, \quad (3.2)$$

for $\sigma = 0, 1$ and $0 \le n \le \frac{T}{\tau}$, which yields the uniform error bound by taking minimum among the two error bounds for $\varepsilon \in (0, 1]$

$$\|\Phi(t_n,\cdot) - \Phi_I^n(\cdot)\|_{H^{\sigma}} \lesssim h^{m_0 - \sigma} + \min_{0 < \varepsilon \le 1} \left\{ \frac{\tau^2}{\varepsilon^2}, \tau^2 + \varepsilon^2 \right\} \lesssim h^{m_0 - \sigma} + \tau, \quad \sigma = 0, 1, \ 0 \le n \le \frac{T}{\tau}.$$
(3.3)

Remark 3.2. From the analysis point of view, we remark that the $W_p^{m_0,\infty}$ assumption in (A) is necessary such that the exact solution $\Phi(t,x)$ of the NLDE (2.12) remains in $(H_p^{m_0})^2$, which would give the spectral accuracy in space. In practice, as long as the solution of the NLDE (1.1) or (1.5) is well localized such that the error from the periodic truncation of potential term $W(t,x)\Phi(t,x)$ is negligible, the error estimates in the above theorem still hold.

Remark 3.3. The results in Theorem 3.1 can be generalized to higher dimensions d = 2, 3, and we refer to Remark 3.8 for more discussion.

Define the error function
$$\mathbf{e}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathbf{e}^{n})}_{l} e^{i\mu_{l}(x-a)} \in Y_{M}$$
 for $n \ge 0$ as
 $\mathbf{e}^{n}(x) = P_{M}(\Phi(t_{n}))(x) - \Phi_{I}^{n}(x) = P_{M}(\Phi(t_{n}))(x) - I_{M}(\Phi^{n})(x), \quad x \in \Omega.$
(3.4)

Applying assumption (B), triangle inequality and standard Fourier projection properties [37], we find for $0 \leq$ $n \leq \frac{\check{T}}{\tau},$

$$\|\Phi(t_n, \cdot) - \Phi_I^n(\cdot)\|_{H^{\sigma}} \le \|\Phi(t_n, \cdot) - P_M(\Phi(t_n))(\cdot)\|_{H^{\sigma}} + \|\mathbf{e}^n(\cdot)\|_{H^{\sigma}} \lesssim h^{m_0 - \sigma} + \|\mathbf{e}^n(\cdot)\|_{H^{\sigma}}, \ \sigma = 0, 1.$$
(3.5)

Hence, only $\|\mathbf{e}^{n}(\cdot)\|_{L^{2}}$ and $\|\mathbf{e}^{n}(\cdot)\|_{H^{1}}$ need to be estimated. To this purpose, we will respectively study the local truncation error and the error of nonlinear terms.

Before defining the local truncation error, let us pay attention to the following properties of the decomposition (2.8) and (2.9). Let $\Psi^n_{\pm}(s,x)$ and $\mathbf{r}^n(s,x)$ $(0 \le s \le \tau)$ be the solutions of the systems (2.8) and (2.9) for $t \in [t_n, t_{n+1}]$, and the decomposition (2.7) holds as

$$\Phi(t_n + s, x) = e^{-is/\varepsilon^2} \Psi^n_+(s, x) + e^{is/\varepsilon^2} \Psi^n_-(s, x) + \mathbf{r}^n(s, x), \quad x \in \Omega.$$
(3.6)

Then the error $\mathbf{e}^{n+1}(x)$ $(n \ge 0)$ (3.4) can be decomposed as

$$\mathbf{e}^{n+1}(x) = e^{-i\tau/\varepsilon^2} \mathbf{z}_{+}^{n+1}(x) + e^{i\tau/\varepsilon^2} \mathbf{z}_{-}^{n+1}(x) + \mathbf{z}_{r}^{n+1}(x), \quad x \in \Omega,$$
(3.7)

$$\mathbf{z}_{\pm}^{n+1}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathbf{z}_{\pm}^{n+1})}_{l} e^{i\mu_{l}(x-a)} = P_{M}(\Psi_{\pm}^{n}(\tau))(x) - I_{M}(\Psi_{\pm}^{n+1})(x),$$

$$\mathbf{z}_{r}^{n+1}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathbf{z}_{r}^{n+1})}_{l} e^{i\mu_{l}(x-a)} = P_{M}(\mathbf{r}^{n}(\tau))(x) - I_{M}(\mathbf{r}^{n+1})(x).$$
(3.8)

We have established the regularity results concerning the multiscale decomposition in [15] (see also the linear case in [12]:

Lemma 3.4. Under the assumptions (A) and (B), the exact solutions $\Psi^n_+(s,x)$ and $\mathbf{r}^n(s,x)$ ($0 \le s \le \tau$, $0 \le n \le \frac{T}{\tau} - 1$) of systems (2.8) and (2.9) satisfy

$$\|\Psi_{\pm}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{0}})^{2})} + \|\partial_{s}\Psi_{\pm}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{2}})^{2})} + \|\partial_{ss}\Psi_{\pm}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{4}})^{2})} \lesssim 1,$$
(3.9)

$$\|\mathbf{r}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{0}})^{2})} \lesssim 1, \qquad \|\mathbf{r}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{2}})^{2})} \lesssim \varepsilon^{2}, \qquad m_{l} = m_{0} - l, \quad l = 1, 2, 3, 4,$$
(3.10)

$$\|\partial_{s}\mathbf{r}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{3}})^{2})} \lesssim 1, \quad \|\partial_{ss}\mathbf{r}^{n}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{4}})^{2})} \lesssim \frac{1}{\varepsilon^{2}}.$$
(3.11)

We notice that (3.11) can be derived from (3.9) to (3.10) and the equation for \mathbf{r}^n . Now, we can define the local truncation error $\xi_{\pm}^n(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\xi_{\pm}^n)}_l e^{i\mu_l(x-a)}$ and $\xi_r^n = \sum_{l=-M/2}^{M/2-1} \widehat{(\xi_r^n)}_l e^{i\mu_l(x-a)}$ $(x \in \Omega, n \ge 0)$ for the MTI-FP method (2.29)–(2.31) as

$$\begin{cases} \widehat{(\xi_{+}^{n})}_{l} = (\widehat{\Psi_{+}^{n}(\tau)})_{l} - e^{-i\frac{(\delta_{l}-1)\tau}{\varepsilon^{2}}} (\widehat{\Psi_{+}^{n}(0)})_{l} - p_{l}(\tau)\Pi_{l}^{+} (\widehat{f_{+}(0)})_{l} - q_{l}(\tau)\Pi_{l}^{+} (\widehat{f_{+}(0)})_{l}, \\ \widehat{(\xi_{-}^{n})}_{l} = (\widehat{\Psi_{-}^{n}(\tau)})_{l} - e^{i\frac{(\delta_{l}-1)\tau}{\varepsilon^{2}}} (\widehat{\Psi_{-}^{n}(0)})_{l} + \overline{p_{l}(\tau)}\Pi_{l}^{-} (\widehat{f_{-}(0)})_{l} + \overline{q_{l}(\tau)}\Pi_{l}^{-} (\widehat{f_{-}(0)})_{l}, \quad 0 < s \le \tau, \\ \widehat{(\xi_{r}^{n})}_{l} = (\widehat{\mathbf{r}^{n}(\tau)})_{l} - \sum_{k=-3}^{3} \left(P_{l}^{(k)}(\tau) (\widehat{g_{k}(0)})_{l} + Q_{l}^{(k)}(\tau) (\widehat{g_{k}(0)})_{l} \right), \quad l = -\frac{M}{2}, \dots, \frac{M}{2} - 1, \end{cases}$$

$$(3.12)$$

with

where $f_{\pm}(s), g_k(s)$ is defined the same as $f_{\pm}^n(s), g_k^n(s)$ as in (2.19) with $\Psi_{\pm,M}^n, \mathbf{r}_M^n$ being replaced by $\Psi_{\pm}^n, \mathbf{r}^n$ and

$$\begin{split} \widehat{\left(\Psi_{+}^{n}\right)_{l}}(0) &= \Pi_{l}^{+}(\widehat{\Phi(t_{n})})_{l}, \ \widehat{\left(\Psi_{-}^{n}\right)_{l}}(0) = \Pi_{l}^{-}(\widehat{\Phi(t_{n})})_{l}, \ \widehat{\left(r^{n}\right)_{l}}(0) = \mathbf{0}, \ \widehat{\left(r^{n,*}\right)_{l}} = \sum_{k=-3}^{3} P_{l}^{(k)}(\tau)(\widehat{g_{k}(0)})_{l}, \\ \dot{f}_{\pm}(s) &= \left(W^{n}(s) + (|\Psi_{+}^{n}(s)|^{2} + |\Psi_{-}^{n}(s)|^{2})_{L}\right) \dot{\Psi}_{\pm}^{n}(s) + \partial_{s}W^{n}(0)\Psi_{\pm}^{n}(s) \\ &+ 2Re\left(\left(\Psi_{+}^{n}(s)\right)^{*}\dot{\Psi}_{+}^{n}(s) + \left(\Psi_{-}^{n}(s)\right)^{*}\dot{\Psi}_{-}^{n}(s)\right) \Psi_{\pm}^{n}(s), \\ \dot{g}_{3}(s) &= (\dot{\Psi}_{+}^{n}(s))^{*}\Psi_{-}^{n}(s)\Psi_{-}^{n}(s) + (\Psi_{+}^{n}(s))^{*}\Psi_{-}^{n}(s) + (\Psi_{+}^{n}(s))^{*}\Psi_{-}^{n}(s) \dot{\Psi}_{-}^{n}(s), \\ \dot{g}_{2}(s) &= \frac{1}{\tau}\left[\left(\Psi_{+}^{n}(s)\right)^{*}\Psi_{-}^{n}(s)r^{n,*} + \left(\Psi_{+}^{n}(s)\right)^{*}r^{n,*}\Psi_{-}^{n}(s) + \left(r^{n,*}\right)^{*}\Psi_{-}^{n}(s)\Psi_{-}^{n}(s)\right], \\ \dot{g}_{1}(s) &= \Pi_{l}^{+}\left[\left(\partial_{s}W^{n}(0) + 2Re\left[\left(\Psi_{+}^{n}(s)\right)^{*}\dot{\Psi}_{+}^{n}(s) + \left(\Psi_{-}^{n}(s)\right)^{*}\dot{\Psi}_{-}^{n}(s)\right]I_{2}\right)\Psi_{-}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2} + |\Psi_{-}^{n}(s)|^{2})I_{2}\right)\dot{\Psi}_{-}^{n}(s)\right] + \left(\dot{\Psi}_{+}^{n}(s)\right)^{*}\Psi_{-}^{n}(s)\Psi_{+}^{n}(s) \\ &+ \left(\Psi_{-}^{n}(s)\right)^{*}\dot{\Psi}_{-}^{n}(s)\Psi_{+}^{n}(s) + \left(\Psi_{-}^{n}(s)\right)^{*}\Psi_{-}^{n}(s)\right]I_{2}\right)\Psi_{+}^{n}(s) \\ \dot{g}_{0}(s) &= \frac{1}{\tau}\left[\left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2} + |\Psi_{-}^{n}(s)|^{2})I_{2}\right)\dot{\mathbf{r}}^{n,*} + \left(\Psi_{+}^{n}(s)\right)^{*}r^{n,*}\Psi_{+}^{n}(s)\right]I_{2}\right)\Psi_{+}^{n}(s) \\ &+ \left(r^{n,*}\right)^{*}\Psi_{-}^{n}(s)\Psi_{+}^{n}(s) + \left(\left(\Psi_{-}^{n}(s)\right)^{*}r^{n,*} + \left(r^{n,*}\right)^{*}\Psi_{+}^{n}(s)\right)I_{2}\right)\Psi_{+}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2} + |\Psi_{-}^{n}(s)|^{2})I_{2}\right)\dot{\Psi}_{+}^{n}(s)\right]I_{2}\right)\Psi_{+}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2} + |\Psi_{-}^{n}(s)\right)^{*}\Psi_{+}^{n}(s)H_{+}^{n}(s)\right)^{*}\Psi_{+}^{n}(s)\Psi_{+}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2}r^{n,*} + \left(\Psi_{-}^{n}(s)\right)^{*}\Psi_{+}^{n}(s), \\ \dot{g}_{-1}(s) = \prod_{i=1}^{n}\left[\left(\dot{\Phi}_{s}W^{n}(s) + \left(\dot{\Psi}_{-}^{n}(s)\right)^{*}\Psi_{+}^{n}(s)\right]I_{2}\right)\dot{\Psi}_{+}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2}H_{+}^{n}(s)\right)^{*}\Psi_{+}^{n}(s)\right]I_{2}\right)\dot{\Psi}_{+}^{n}(s) \\ &+ \left(W^{n}(0) + \left(|\Psi_{+}^{n}(s)|^{2}H_{+}^{n}(s)\right)^{*}\Psi_{+}^{n}(s)I_{+}^{n}(s), \\ \dot{g}_{-2}(s) = \frac{1}{\tau}\left[\left(\Psi_{-}^{n}(s)\right$$

and
$$\dot{\Psi}^{n}_{\pm}(s,x) = \sum_{l=-M/2}^{M/2-1} \left(\widetilde{\dot{\Psi}^{n}_{\pm}(s)} \right)_{l} e^{i\mu_{l}(x-a)} \ (0 \le s \le \tau)$$
 with

$$\left(\widetilde{\Psi_{+}^{n}(s)}\right)_{l} = -i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau} \left(\widetilde{\Psi_{+}^{n}(s)}\right)_{l} - i\,\Pi_{l}^{+}(\widetilde{f_{+}(s)})_{l}, \ \left(\widetilde{\Psi_{-}^{n}(s)}\right)_{l} = i\frac{2\sin(\mu_{l}^{2}\tau/2)}{(\delta_{l}+1)\tau} \left(\widetilde{\Psi_{-}^{n}(s)}\right)_{l} - i\,\Pi_{l}^{-}(\widetilde{f_{-}(s)})_{l}.$$
(3.14)

According to the definition of $f_{\pm}(s)$, $g_k(s)$, it can be checked that $\partial_s f_{\pm}(s)$, $\partial_s g_k(s)$, $k = \pm 1, \pm 3$ do not contain $\mathbf{r}^n(s)$ terms, while $\partial_s g_k(s)$, $k = 0, \pm 2$ do. Recalling Lemma 3.4 that $\mathbf{r}^n(s)$ is highly oscillatory, then we have

$$\begin{aligned} \|\partial_{s}f_{\pm}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{2}})^{2})} + \|\partial_{ss}f_{\pm}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{4}})^{2})} &\lesssim 1, \\ \|\partial_{s}g_{k}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{2}})^{2})} + \|\partial_{ss}g_{k}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{4}})^{2})} &\lesssim 1, \quad \text{for} \quad k = \pm 1, \pm 3, \\ \|\partial_{s}g_{k}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{3}})^{2})} &\lesssim 1, \quad \|\partial_{ss}g_{k}\|_{L^{\infty}([0,\tau];(H_{p}^{m_{4}})^{2})} &\lesssim \frac{1}{\varepsilon^{2}}, \quad \text{for} \quad k = 0, \pm 2, \end{aligned}$$

$$(3.15)$$

with m_l , l = 2, 3, 4 defined in Lemma 3.4.

To deal with nonlinear terms, the following inequalities [5, 7] will be frequently used

$$\|I_M(U)(\cdot)\|_{H^1} \lesssim \|U\|_{l^2} + \|\delta_x^+ U\|_{l^2} \lesssim \|I_M(U)(\cdot)\|_{H^1}, \qquad \forall U \in X_M,$$
(3.16)

where the difference operator $\delta_x^+ U \in X_M$ is defined as $\delta_x^+ U_j = \frac{U_{j+1} - U_j}{h}$ for $j = 0, \dots, M-1$ and $\delta_x^+ U_M = \delta_x^+ U_0$. In addition, Parseval's identity imply that $||U||_{l^2} = ||I_M(U)||_{L^2}$.

We have the following estimates for the local truncation error (3.12).

Lemma 3.5. Under the assumptions (A) and (B), there exist constants $0 < \tau_0, h_0 \leq 1$ sufficiently small and independent of ε , such that for any $0 < \varepsilon \leq 1$, when $0 < \tau \leq \tau_0$ and $0 < h \leq h_0$, we have the error estimates

for the local truncation error $\xi_{\pm}^n, \, \xi_r^n \in Y_M$ in (3.12) for $0 \le n < \frac{T}{\tau}$ and $\sigma = 0, 1$ as

$$\|\xi^n_{\pm}(\cdot)\|_{H^{\sigma}} \lesssim \tau \left(h^{m_0-\sigma} + \tau^2\right), \ \|\xi^n_r(\cdot)\|_{H^{\sigma}} \lesssim \tau \left(h^{m_0-\sigma} + \frac{\tau^2}{\varepsilon^2}\right), \ \|\xi^n_r(\cdot)\|_{H^{\sigma}} \lesssim \tau \left(h^{m_0-\sigma} + \tau^2 + \varepsilon^2\right).$$
(3.17)

Proof. According to the definitions of $p_l(\tau), q_l(\tau), p_{\pm,l}^{(k)}(\tau)$ and $q_{\pm,l}^{(k)}(\tau)$ $(l = -M/2, \ldots, M/2 - 1)$, we have

$$|p_l(\tau)| \lesssim \tau, \quad |p_{\pm,l}^{(k)}(\tau)| \lesssim \tau, \quad |q_l(\tau)| \lesssim \tau^2, \quad |q_{\pm,l}^{(k)}(\tau)| \lesssim \tau^2.$$
 (3.18)

Multiplying $e^{-i\mu_l(x-a)}$ on both sides of the equations in the system (2.8) and integrating over Ω , we can recover the equations for $(\widehat{\Psi^n_{\pm}})_l(s)$ and $(\widehat{\mathbf{r}^n})_l(s)$, which are exactly the same as (2.20) with $\Psi^n_{\pm,M}$, \mathbf{r}^n_M being replaced by Ψ^n_{\pm} and \mathbf{r}^n , respectively.

Following the derivation of the MTI-FP method, one can find that the local truncation error comes from the approximations in the integrals (2.23) and (2.25). In particular, for $l = -M/2, \ldots, M/2 - 1$, we have

$$\widehat{(\xi_{+}^{n})}_{l} = -i \int_{0}^{\tau} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \prod_{l}^{+} \widehat{(f_{+}(s))}_{l} \,\mathrm{d}s - p_{l}(\tau) \prod_{l}^{+} \widetilde{(f_{+}(0))}_{l} - q_{l}(\tau) \prod_{l}^{+} \widetilde{(f_{+}(0))}_{l}, \tag{3.19}$$

$$\widehat{(\xi_{-}^{n})}_{l} = -i \int_{0}^{\tau} e^{i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \prod_{l} \widehat{(f_{-}(s))}_{l} \,\mathrm{d}s + \overline{p_{l}(\tau)} \prod_{l} \widehat{(f_{-}(0))}_{l} + \overline{q_{l}(\tau)} \prod_{l} \widehat{(f_{-}(0))}_{l}, \tag{3.20}$$

$$\widehat{(\xi_r^n)}_l = \sum_{k=-3}^3 \left(-i \int_0^\tau e^{iks/\varepsilon^2} e^{-i\mathcal{T}_l(\tau-s)/\varepsilon^2} \, \widehat{(g_k(s))}_l \, \mathrm{d}s - P_l^{(k)}(\tau) \, \widetilde{(g_k(0))}_l - Q_l^{(k)}(\tau) \, \widetilde{(\dot{g}_k(0))}_l \right). \tag{3.21}$$

Type I estimate. We first prove the estimates for ξ^n_{\pm} and the first kind estimate $O(\tau^2/\varepsilon^2)$ for ξ^n_r in (3.17). Using Taylor expansion, we have

$$\widehat{\left(\xi_{+}^{n}\right)}_{l} = -i \int_{0}^{\tau} \int_{0}^{s} \int_{0}^{s_{1}} e^{-i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \Pi_{l}^{+} (\partial_{s_{2}s_{2}}\widehat{f_{+}}(s_{2}))_{l} \,\mathrm{d}s_{2} \mathrm{d}s_{1} \mathrm{d}s + \widehat{\left(\eta_{+}^{n}\right)}_{l}, \tag{3.22}$$

$$\widehat{(\xi_{-}^{n})}_{l} = -i \int_{0}^{\tau} \int_{0}^{s} \int_{0}^{s_{1}} e^{i(\delta_{l}-1)(\tau-s)/\varepsilon^{2}} \Pi_{l}^{-} (\partial_{s_{2}s_{2}}f_{-}(s_{2}))_{l} \,\mathrm{d}s_{2} \mathrm{d}s_{1} \mathrm{d}s + \widehat{(\eta_{-}^{n})}_{l},$$
(3.23)

$$\widehat{(\xi_r^n)}_l = \sum_{k=-3}^3 \left[-i \int_0^\tau \int_0^s \int_0^{s_1} e^{iks/\varepsilon^2} e^{-i\mathcal{T}_l(\tau-s)/\varepsilon^2} (\widehat{\partial_{s_2s_2}g_k(s_2)})_l \,\mathrm{d}s_2 \mathrm{d}s_1 \mathrm{d}s + \widehat{(\eta_{r,k}^n)_l} \right],\tag{3.24}$$

where $\eta_{\pm}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\eta_{\pm}^{n})}_{l} e^{i\mu_{l}(x-a)}, \ \eta_{r,k}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\eta_{r,k}^{n})}_{l} e^{i\mu_{l}(x-a)}, \ k = -3, \dots, 3$ with

$$\begin{split} \widehat{(\eta_{+}^{n})}_{l} &= p_{l}(\tau) \ \Pi_{l}^{+} \left(\widehat{(f_{+}(0))}_{l} - \widetilde{(f_{+}(0))}_{l} \right) + q_{l}(\tau) \ \Pi_{l}^{+} \left(\widehat{(\dot{f}_{+}(0))}_{l} - \widetilde{(\dot{f}_{+}(0))}_{l} \right), \\ \widehat{(\eta_{-}^{n})}_{l} &= \overline{p_{l}(\tau)} \ \Pi_{l}^{-} \left(\widetilde{(f_{-}(0))}_{l} - \widetilde{(f_{-}(0))}_{l} \right) + \overline{q_{l}(\tau)} \ \Pi_{l}^{-} \left(\widetilde{(\dot{f}_{-}(0))}_{l} - \widetilde{(\dot{f}_{-}(0))}_{l} \right), \\ \widehat{(\eta_{r,k}^{n})}_{l} &= P_{l}^{(k)}(\tau) \left(\widehat{(g_{k}(0))}_{l} - \widetilde{(g_{k}(0))}_{l} \right) + Q_{l}^{(k)}(\tau) \left(\widetilde{(\dot{g}_{k}^{n}(0))}_{l} - \widetilde{(\dot{g}_{k}(0))}_{l} \right), \end{split}$$

and $\dot{f}^n_{\pm}(s) = \partial_s f_{\pm}(s)$ and $\dot{g}^n_k(s) = \partial_s g_k(s), k = -3, \dots, 3.$

Noticing (3.18) and $\|\Pi_l^{\pm}\|_{l^2} \leq 1$ $(l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1)$ with $\|Q\|_{l^2}$ being the standard l^2 norm of the matrix Q, the triangle inequality gives

$$\left| \widehat{(\eta_{+}^{n})}_{l} \right| \lesssim \tau \left| \widehat{(f_{+}(0))}_{l} - (\widehat{f_{+}(0)})_{l} \right| + \tau^{2} \left| \widehat{\left(\widehat{f_{+}^{n}(0)} \right)}_{l} - \left(\widehat{f_{+}^{n}(0)} \right)_{l} \right| + \tau^{2} \left| \widehat{\left(\widehat{f_{+}^{n}(0)} \right)}_{l} - \left(\widehat{f_{+}(0)} \right)_{l} \right|.$$
(3.25)

Recalling assumptions (A) and (B) and noticing Lemma 3.4, we have

$$\begin{aligned} f_{\pm}(0) &= \left(W^{n}(0) + (|\Psi_{+}^{n}(0)|^{2} + |\Psi_{-}^{n}(0)|^{2})I_{2} \right) \Psi_{\pm}^{n}(0) \in H_{p}^{m_{0}}, \\ \dot{f}_{\pm}^{n}(0) &= \left(W^{n}(0) + (|\Psi_{+}^{n}(0)|^{2} + |\Psi_{-}^{n}(0)|^{2})I_{2} \right) \partial_{s} \Psi_{\pm}^{n}(0) + \partial_{s} W^{n}(0) \Psi_{\pm}^{n}(0) \\ &+ 2Re\left(\left(\Psi_{+}^{n}(0) \right)^{*} \partial_{s} \Psi_{+}^{n}(0) + \left(\Psi_{-}^{n}(0) \right)^{*} \partial_{s} \Psi_{-}^{n}(0) \right) \Psi_{\pm}^{n}(0) \in H_{p}^{m_{0}-2}. \end{aligned}$$

Applying (2.16) and Cauchy inequality, for $m_0 \ge 4$, we can obtain

$$\begin{aligned} \|\eta_{+}^{n}(\cdot)\|_{L^{2}} &\lesssim \tau \|P_{M}(f_{+}(0)) - I_{M}(f_{+}(0))\|_{L^{2}} + \tau^{2} \|P_{M}(\dot{f}_{+}^{n}(0)) - I_{M}(\dot{f}_{+}^{n}(0))\|_{L^{2}} + \tau^{2} \|I_{M}(\dot{f}_{+}^{n}(0)) - I_{M}(\dot{f}_{+}(0))\|_{L^{2}} \\ &\lesssim \tau h^{m_{0}} + \tau^{2} h^{m_{0}-2} + \tau^{2} \|\dot{f}_{+}^{n}(0,x_{j}) - \dot{f}_{+}(0,x_{j})\|_{l^{2}} \\ &\lesssim \tau h^{m_{0}} + \tau^{2} h^{m_{0}-2} + \tau^{2} \left(\|\partial_{s}\Psi_{+}^{n}(0,x_{j})) - \dot{\Psi}_{+}^{n}(0,x_{j})\|_{l^{2}} + \|\partial_{s}\Psi_{-}^{n}(0,x_{j}) - \dot{\Psi}_{-}^{n}(0,x_{j})\|_{l^{2}} \right) \\ &\lesssim \tau (h^{m_{0}} + \tau^{2}) + \tau^{2} \left(\|P_{M}(\partial_{s}\Psi_{+}^{n}(0)) - \dot{\Psi}_{+}^{n}(0)\|_{L^{2}} + \|P_{M}(\partial_{s}\Psi_{-}^{n}(0)) - \dot{\Psi}_{-}^{n}(0)\|_{L^{2}} \right), \end{aligned}$$

$$(3.26)$$

where we have used Parseval identity together with the fact that $|\dot{f}_{+}^{n}(0,x_{j}) - \dot{f}_{+}(0,x_{j})| \leq |\partial_{s}\Psi_{+}^{n}(0,x_{j}) - \dot{\Psi}_{+}^{n}(0,x_{j})| + |\partial_{s}\Psi_{-}^{n}(0,x_{j}) - \dot{\Psi}_{-}^{n}(0,x_{j})|$ under the assumptions (A) and (B). Similarly, we can estimate the H^{1} norm of η_{+}^{n} by multiplying μ_{l} to (3.25) and then taking the corresponding l^{2} norm of the Fourier modes. Using inequality (3.16) and the fact that (essentially established in [5])

$$\begin{split} \|\delta_x^+(\dot{f}_+^n(0,x_j) - \dot{f}_+(0,x_j))\|_{l^2} &\lesssim \|\delta_x^+(\partial_s \Psi_+^n(0,x_j) - \dot{\Psi}_+^n(0,x_j))\|_{l^2} + \|\partial_s \Psi_+^n(0,x_j) - \dot{\Psi}_+^n(0,x_j)\|_{l^2} \\ &+ \|\delta_x^+(\partial_s \Psi_-^n(0,x_j) - \dot{\Psi}_-^n(0,x_j))\|_{l^2} + \|\partial_s \Psi_-^n(0,x_j) - \dot{\Psi}_-^n(0,x_j)\|_{l^2}, \end{split}$$

we could derive (similar to the L^2 norm case)

$$\begin{aligned} \|\eta_{+}^{n}(\cdot)\|_{H^{1}} &\lesssim \tau h^{m_{0}-1} + \tau^{2} h^{m_{0}-3} + \tau^{2} \left(\|I_{M}(\partial_{s}\Psi_{+}^{n}(0)) - \dot{\Psi}_{+}^{n}(0)\|_{H^{1}} + \|I_{M}(\partial_{s}\Psi_{-}^{n}(0)) - \dot{\Psi}_{-}^{n}(0)\|_{H^{1}} \right) \\ &\lesssim \tau (h^{m_{0}-1} + \tau^{2}) + \tau^{2} \left(\|P_{M}(\partial_{s}\Psi_{+}^{n}(0)) - \dot{\Psi}_{+}^{n}(0)\|_{H^{1}} + \|P_{M}(\partial_{s}\Psi_{-}^{n}(0)) - \dot{\Psi}_{-}^{n}(0)\|_{H^{1}} \right). \end{aligned}$$
(3.27)

Comparing (2.20) and (3.14), and with the help of the term $i \frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l+1)\tau} \left((\widehat{\Psi_{\pm}^0(0)})_l - (\widehat{\Psi_{\pm}^0(0)})_l \right)$, we can get

$$\begin{split} (\widehat{\partial_s \Psi_+^n})_l(0) &- (\widehat{\Psi_+^n})_l(0) = -i\frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l + 1)\tau} \left((\widehat{\Psi_+^n(0)})_l - (\widehat{\Psi_+^n(0)})_l \right) \\ &- i \Pi_l^+ \left((\widehat{f_+(0)})_l - (\widehat{f_+(0)})_l \right) - i \left(\frac{\delta_l - 1}{\varepsilon^2} - \frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l + 1)\tau} \right) \left(\widehat{\Psi_+^n(0)} \right)_l, \\ (\widehat{\partial_s \Psi_-^n})_l(0) &- (\widetilde{\Psi_-^n})_l(0) = i\frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l + 1)\tau} \left((\widehat{\Psi_-^n(0)})_l - (\widehat{\Psi_-^n(0)})_l \right) \\ &- i \Pi_l^- \left((\widehat{f_-(0)})_l - (\widehat{f_-(0)})_l \right) + i \left(\frac{\delta_l - 1}{\varepsilon^2} - \frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l + 1)\tau} \right) \left(\widehat{\Psi_-^n(0)} \right)_l. \end{split}$$

Since $|\sin(s) - s| \le \frac{s^2}{2}$ $(s \in \mathbb{R})$, we have

$$\left|\frac{\delta_l - 1}{\varepsilon^2} - \frac{2\sin(\mu_l^2 \tau/2)}{(\delta_l + 1)\tau}\right| = \frac{2}{(\delta_l + 1)} \left|\frac{1}{2}\mu_l^2 - \frac{\sin(\mu_l^2 \tau/2)}{\tau}\right| \lesssim \mu_l^4 \tau, \quad l = -\frac{M}{2}, \dots, \frac{M}{2} - 1.$$

So, we can obtain

$$\left| \left(\widehat{\partial_s \Psi^n_{\pm}} \right)_l(0) - \left(\widecheck{\Psi^n_{\pm}} \right)_l(0) \right| \lesssim \frac{1}{\tau} \left| \left(\widehat{\Psi^n_{\pm}(0)} \right)_l - \left(\widetilde{\Psi^n_{\pm}(0)} \right)_l \right| + \left| \left(\widehat{f_{\pm}(0)} \right)_l - \left(\widetilde{f_{\pm}(0)} \right)_l \right| + \tau \mu_l^4 \left| \left(\widehat{\Psi^n_{\pm}(0)} \right)_l \right|,$$

and for $m_0 \geq 5$,

$$\begin{split} \|P_M(\partial_s \Psi^n_{\pm}(0)) - I_M(\dot{\Psi}^n_{\pm}(0))\|_{L^2} \\ \lesssim \frac{1}{\tau} \|P_M(\Psi^n_{\pm}(0)) - I_M(\Psi^n_{\pm}(0))\|_{L^2} + \|P_M(f_{\pm}(0)) - I_M(f_{\pm}(0))\|_{L^2} + \tau \|P_M(\Psi^n_{\pm}(0))\|_{H^4} \\ \lesssim h^{m_0}/\tau + h^{m_0} + \tau, \\ \|P_M(\partial_s \Psi^n_{\pm}(0)) - I_M(\dot{\Psi}^n_{\pm}(0))\|_{H^1} \lesssim h^{m_0 - 1}/\tau + h^{m_0 - 1} + \tau. \end{split}$$

Combining the above estimates with (3.26) and (3.27), we get

$$\|\eta_{+}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau(h^{m_{0}-\sigma}+\tau^{2}) + \tau^{2}(h^{m_{0}-\sigma}/\tau+h^{m_{0}-\sigma}+\tau) \lesssim \tau(h^{m_{0}-\sigma}+\tau^{2}), \ \sigma = 0, 1.$$
(3.28)

By the same procedure, we can obtain the same estimates for $\eta_{-}^{n}(\cdot)$, *i.e.*,

$$\|\eta_{-}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau (h^{m_{0}-\sigma} + \tau^{2}), \quad \sigma = 0, 1.$$
 (3.29)

Thus, Lemma 3.4 together with (3.22), (3.23), (3.26), (3.27) and (3.29) implies

$$\|\xi_{\pm}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau^{3} \|\partial_{ss} f_{\pm}(\cdot)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} + \|\eta_{\pm}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau \left(h^{m_{0}-\sigma} + \tau^{2}\right), \quad \sigma = 0, 1.$$
(3.30)

Similarly, according to assumptions (A), (B), Lemma 3.4, (3.18) and the definitions of $P_l^{(k)}(\tau)$ and $Q_l^{(k)}(\tau)$, we have

$$g_k(0) \in H_p^{m_0}, \quad \dot{g}_k^n(0) \in H_p^{m_0-2}, \quad \|P_l^{(k)}(\tau)\|_{l^2} \le \tau, \quad \|Q_l^{(k)}(\tau)\|_{l^2} \le \tau^2, \qquad k = -3, \dots, 3,$$

and for $\sigma = 0, 1,$

$$\begin{aligned} \|\eta_{r,k}^{n}(\cdot)\|_{H^{\sigma}} &\lesssim \tau \|P_{M}(g_{k}(0)) - I_{M}(g_{k}(0))\|_{H^{\sigma}} + \tau^{2} \|P_{M}(\dot{g}_{k}^{n}(0)) - I_{M}(\dot{g}_{k}^{n}(0))\|_{H^{\sigma}} + \tau^{2} \|I_{M}(\dot{g}_{k}^{n}(0)) - I_{M}(\dot{g}_{k}(0))\|_{H^{\sigma}} \\ &\lesssim \begin{cases} \tau h^{m_{0}-\sigma} + \tau^{2} h^{m_{0}-2-\sigma} + \tau^{2} \|P_{M}(\partial_{s}\Psi_{+}^{n}(0)) - \dot{\Psi}_{+}^{n}(0)\|_{H^{\sigma}} \\ + \tau^{2} \|P_{M}(\partial_{s}\Psi_{-}^{n}(0)) - \dot{\Psi}_{-}^{n}(0)\|_{H^{\sigma}}, \qquad k = \pm 1, \pm 3, \\ \tau h^{m_{0}-\sigma} + \tau^{2} h^{m_{0}-2-\sigma} + \tau^{2} \|P_{M}(\partial_{s}\mathbf{r}^{n}(0)) - P_{M}(\mathbf{r}^{n,*}/\tau)\|_{H^{\sigma}}, \qquad k = 0, \pm 2, \end{cases}$$

$$(3.31)$$

Noting that

$$\widehat{(\mathbf{r}^{n}(\tau))}_{l} - \widehat{(\mathbf{r}^{n,*})}_{l} = \sum_{k=-3}^{3} \left[-i \int_{0}^{\tau} \int_{0}^{s} e^{iks/\varepsilon^{2}} e^{-i\mathcal{T}_{l}(\tau-s)/\varepsilon^{2}} \partial_{s_{1}}(\widehat{g_{k}(s_{1})})_{l} \,\mathrm{d}s_{1} \mathrm{d}s \right], \qquad (3.32)$$
$$\partial_{s} \mathbf{r}^{n}(0) - \mathbf{r}^{n}(\tau)/\tau = -\tau \int_{0}^{1} \int_{0}^{s} \partial_{s_{1}s_{1}} \mathbf{r}^{n}(\tau s_{1}) \,\mathrm{d}s_{1} \mathrm{d}s,$$

we can obtain

$$\begin{aligned} \|P_{M}(\partial_{s}\mathbf{r}^{n}(0)) - P_{M}(\mathbf{r}^{n,*}/\tau)\|_{H^{\sigma}} &\leq \|P_{M}(\partial_{s}\mathbf{r}^{n}(0)) - P_{M}(\mathbf{r}^{n}(\tau)/\tau)\|_{H^{\sigma}} + \frac{1}{\tau}\|P_{M}(\mathbf{r}^{n}(\tau)) - P_{M}(\mathbf{r}^{n,*})\|_{H^{\sigma}} \\ &\lesssim \tau \|\partial_{ss}\mathbf{r}^{n}(\cdot)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} + \frac{1}{\tau}\tau^{2}\sum_{k=-3}^{3} \|\partial_{s}g_{k}(\cdot)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} \lesssim \frac{\tau}{\varepsilon^{2}} + \tau, \end{aligned}$$

where we have used Lemma 3.4 and (3.15) to deduce the estimates on $\partial_{ss} \mathbf{r}^{n}(\cdot)$ and $\partial_{s} g_{k}(\cdot)$.

Hence, $\eta_{r,k}^n(\cdot)$ can be estimated from (3.31) as

$$\|\eta_{r,k}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \begin{cases} \tau(h^{m_{0}-\sigma}+\tau^{2}), & k=\pm 1,\pm 3, \\ \tau(h^{m_{0}-\sigma}+\frac{\tau^{2}}{\varepsilon^{2}}), & k=0,\pm 2, \end{cases} \quad \sigma = 0,1,$$
(3.33)

and therefore we have

$$\begin{aligned} \|\xi_{r}^{n}(\cdot)\|_{H^{\sigma}} &\lesssim \sum_{k=-3}^{3} \left(\tau^{3} \|\partial_{ss} g_{k}(s)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} + \|\eta_{r,k}^{n}(\cdot)\|_{H^{\sigma}}\right) \\ &\lesssim \tau^{3} \|\partial_{ss} \mathbf{r}^{n}(s)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} + \tau(h^{m_{0}-\sigma} + \frac{\tau^{2}}{\varepsilon^{2}}) \lesssim \tau(h^{m_{0}-\sigma} + \frac{\tau^{2}}{\varepsilon^{2}}). \end{aligned}$$
(3.34)

Type II estimate. Next, we prove the second kind estimate $O(\varepsilon^2 + \tau^2)$ for $\xi_r^n(x)$ in (3.17). Let us move back to (3.21), treating $k = \pm 1, \pm 3$ terms in the same way as in proving (3.34), and leaving the rest terms as

$$\widehat{(\xi_r^n)}_l = \widehat{(\zeta_r^n)}_l - \sum_{k=0,\pm 2} \left[i \int_0^\tau e^{iks/\varepsilon^2} e^{-i\mathcal{T}_l(\tau-s)/\varepsilon^2} \widehat{(g_k(s))}_l \,\mathrm{d}s + Q_l^{(k)}(\tau) \widetilde{(g_k(0))}_l \right],$$

with $\zeta_r^n(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\zeta_r^n)}_l e^{i\mu_l(x-a)}$ satisfying

$$\|\zeta_r^n(\cdot)\|_{L^2} \lesssim \tau(h^{m_0} + \tau^2), \quad \|\zeta_r^n(\cdot)\|_{H^1} \lesssim \tau(h^{m_0-1} + \tau^2).$$

We here omit the proof of the above error bounds for $\zeta_r^n(x)$ since it is identical to the proof of (3.34). Applying triangle inequality and (3.18), we have

$$\left| \left(\widehat{\xi_r^n} \right)_l \right| \le \left| \left(\widehat{\zeta_r^n} \right)_l \right| + \sum_{k=0,\pm 2} \left(\int_0^\tau \left| \left(\widehat{g_k(s)} \right)_l \right| \, \mathrm{d}s + \tau^2 \left| \left(\widehat{g_k(0)} \right)_l \right| \right),$$

and there holds

$$\|\xi_r^n(\cdot)\|_{H^{\sigma}} \lesssim \|\zeta_r^n(\cdot)\|_{H^{\sigma}} + \sum_{k=0,\pm 2} \left(\tau \|g_k(\cdot)\|_{L^{\infty}([0,\tau];(H^{\sigma})^2)} + \tau^2 \|I_M(\dot{g}_k(0))\|_{H^{\sigma}}\right), \quad \sigma = 0, 1.$$
(3.35)

Recalling Lemma 3.4 which shows that $\|\mathbf{r}^{n}(\cdot)\|_{L^{\infty}([0,\tau];(H_{p}^{m_{0}-2})^{2})} \lesssim \varepsilon^{2}$ $(s \in [0,\tau])$, it is easy to check that $\|g_{k}(\cdot)\|_{L^{\infty}([0,\tau];(H_{p}^{m_{0}-2})^{2})} \lesssim \varepsilon^{2}$ $(k = 0, \pm 2)$. According to the definition of $\dot{g}_{k}(s)$ in (3.13), noticing (3.32), and using Lemma 3.4, (3.15) and triangle inequality, we can bound $\|I_{M}(\dot{g}_{k}(0))\|_{H^{\sigma}}$ $(k = 0, \pm 2)$ as

$$\begin{aligned} \|I_{M}(\dot{g}_{k}(0))\|_{H^{\sigma}} \lesssim &\frac{1}{\tau} \|I_{M}(\mathbf{r}^{n,*})\|_{H^{\sigma}} \lesssim \frac{1}{\tau} \left(\|I_{M}(\mathbf{r}^{n,*}) - P_{M}(\mathbf{r}^{n}(\tau))\|_{H^{\sigma}} + \|P_{M}(\mathbf{r}^{n}(\tau))\|_{H^{\sigma}} \right) \\ \lesssim &\frac{1}{\tau} \left(\tau^{2} \sum_{k=-3}^{3} \|\partial_{s}g_{k}(\cdot)\|_{L^{\infty}([0,\tau];(H^{\sigma})^{2})} + \varepsilon^{2} \right) \lesssim \frac{1}{\tau} \left(\tau^{2} + \varepsilon^{2} \right), \ \sigma = 0, 1. \end{aligned}$$

Now, (3.35) implies

$$\|\xi_r^n(\cdot)\|_{H^{\sigma}} \lesssim \tau(h^{m_0-\sigma}+\tau^2) + \tau\varepsilon^2 + \frac{\tau^2}{\tau}(\tau^2+\varepsilon^2) \lesssim \tau(h^{m_0-\sigma}+\tau^2+\varepsilon^2), \quad \sigma = 0, 1.$$
(3.36)

Hence, we complete the proof for (3.17).

Next, we derive the equations governing the growth of the error functions. Subtracting (2.30) from (3.12), noticing (2.31) and (3.13), we get error equations for $\mathbf{z}_{\pm}^{n+1}(x)$ and $\mathbf{z}_{r}^{n+1}(x)$ in (3.8) as

$$\widetilde{\left(\mathbf{z}_{\pm}^{n+1}\right)}_{l} = e^{\mp i \frac{(\delta_{l}-1)\tau}{\varepsilon^{2}}} \Pi_{l}^{\pm} \widetilde{\left(\mathbf{e}^{n}\right)}_{l} + \widetilde{\left(\mathcal{F}_{\pm}^{n}\right)}_{l} + \widetilde{\left(\xi_{\pm}^{n}\right)}_{l}, \qquad \widetilde{\left(\mathbf{z}_{r}^{n+1}\right)}_{l} = \widetilde{\left(\mathcal{F}_{r}^{n}\right)}_{l} + \widetilde{\left(\xi_{r}^{n}\right)}_{l}, \qquad (3.37)$$

where $\mathcal{F}^{n}_{\pm}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathcal{F}^{n}_{\pm})}_{l} e^{i\mu_{l}(x-a)}, \ \mathcal{F}^{n}_{r}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\mathcal{F}^{n}_{r})}_{l} e^{i\mu_{l}(x-a)}$ are given by

$$\begin{cases} \widetilde{\left(\mathcal{F}_{+}^{n}\right)}_{l} = p_{l}(\tau)\Pi_{l}^{+}\widetilde{\left(F_{+}^{n}\right)}_{l} + q_{l}(\tau)\Pi_{l}^{+}\widetilde{\left(\dot{F}_{+}^{n}\right)}_{l}, & \widetilde{\left(\mathcal{F}_{-}^{n}\right)}_{l} = -\overline{p_{l}(\tau)}\Pi_{l}^{-}\widetilde{\left(F_{-}^{n}\right)}_{l} - \overline{q_{l}(\tau)}\Pi_{l}^{-}\widetilde{\left(\dot{F}_{-}^{n}\right)}_{l}, \\ \widetilde{\left(\mathcal{F}_{r}^{n}\right)}_{l} = \sum_{k=-3}^{3} \left(P_{l}^{(k)}(\tau)\widetilde{\left(G_{k}^{n}\right)}_{l} + Q_{l}^{(k)}(\tau)\widetilde{\left(\dot{G}_{k}^{n}\right)}_{l} \right), \end{cases}$$
(3.38)

with $F_{\pm}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(F_{\pm}^{n})}_{l} e^{i\mu_{l}(x-a)} \in Y_{M}, \ \dot{F}_{\pm}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\dot{F}_{\pm}^{n})}_{l} e^{i\mu_{l}(x-a)} \in Y_{M},$ $G_{k}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(G_{k}^{n})}_{l} e^{i\mu_{l}(x-a)} \in Y_{M}, \ \text{and} \ \dot{G}_{k}^{n}(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\dot{G}_{k}^{n})}_{l} e^{i\mu_{l}(x-a)} \in Y_{M} \ \text{defined as}$

$$\widetilde{(F_{\pm}^n)}_l = \widetilde{(f_{\pm}(0))}_l - \widetilde{(f_{\pm})}_l, \qquad \widetilde{(F_{\pm}^n)}_l = \widetilde{(f_{\pm}(0))}_l - \widetilde{(f_{\pm})}_l, \qquad l = -\frac{M}{2}, \dots, \frac{M}{2} - 1, \\
\widetilde{(G_k^n)}_l = \widetilde{(g_k(0))}_l - \widetilde{(g_k)}_l, \qquad \widetilde{(G_k^n)}_l = \widetilde{(g_k(0))}_l - \widetilde{(g_k)}_l, \qquad k = -3, \dots, 3.$$
(3.39)

For the nonlinear error part $\mathcal{F}^n_{\pm}(x)$, $\mathcal{F}^n_r(x)$ $(0 \le n \le \frac{T}{\tau} - 1)$, we have the lemma below.

Lemma 3.6. Assuming that $\|\Phi^n\|_{l^{\infty}} \leq C_0 + 1$, then under the assumptions (A) and (B), the nonlinear error part $\mathcal{F}^n_{\pm}(x), \mathcal{F}^n_r(x) \in Y_M$ ($0 \leq n \leq \frac{T}{\tau} - 1$) defined in (3.38) with (3.39) satisfies the bounds for $\sigma = 0, 1$ as

$$\|F_{\pm}^{n}(\cdot)\|_{H^{\sigma}} + \|G_{k}^{n}(\cdot)\|_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}, \quad \|\dot{F}_{\pm}^{n}(\cdot)\|_{H^{\sigma}} + \|\dot{G}_{k}^{n}(\cdot)\|_{H^{\sigma}} \lesssim \frac{1}{\tau} \left(h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}\right),$$

for $k = -3, \ldots, 3$, which implies that

$$\|\mathcal{F}^{n}_{\pm}(\cdot)\|_{H^{\sigma}} \lesssim \tau(h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}), \quad \|\mathcal{F}^{n}_{r}(\cdot)\|_{H^{\sigma}} \lesssim \tau(h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}), \tag{3.40}$$

and the constants in front of $\|\mathbf{e}^n\|_{H^{\sigma}}$ in the inequality depend on C_0 (3.1).

Proof. Define $f_{+}(0), f_{+}, \Psi_{+}^{n}(0), \Psi_{+} \in X_{M}$ with $f_{+,j}(0) = f_{+}(0, x_{j}), \Psi_{+,j}^{n}(0) = \Psi_{+}^{n}(0, x_{j})$ and $f_{+,j}, \Psi_{+,j}, j \in \mathcal{T}_{M}$ as in (2.31). Recalling the assumptions (A), (B) and $\|\Phi^{n}\|_{l^{\infty}} \leq C_{0} + 1$, Lemma 3.4, (3.13) and (2.31), we have

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$$\begin{aligned} |f_{+,j}(0) - f_{+,j}|^2 &\lesssim |\Psi_{+,j}^n(0) - \Psi_{+,j}|^2 + |\Psi_{-}^n(0,x_j) - \Psi_{-,j}|^2, \\ |\delta_x^+ f_{+,j}(0) - \delta_x^+ f_{+,j}|^2 &\lesssim \sum_{k=j,j+1} \left(|\Psi_{+,k}^n(0) - \Psi_{+,k}|^2 + |\Psi_{-,k}^n(0) - \Psi_{-,k}|^2 \right) \\ &+ |\delta_x^+ \Psi_{+,j}^n(0) - \delta_x^+ \Psi_{+,j}|^2 + |\delta_x^+ \Psi_{-,j}^n(0) - \delta_x^+ \Psi_{-,j}|^2, \quad j \in \mathcal{T}_M \end{aligned}$$

Applying the Parseval's identity, triangle inequality, (2.16) and (3.16), we have

$$\begin{split} \|F_{+}^{n}(\cdot)\|_{L^{2}}^{2} &= \|f_{+}(0) - f_{+}\|_{l^{2}}^{2} \lesssim \|I_{M}(\Psi_{+}^{n}(0)) - I_{M}(\Psi_{+})\|_{L^{2}}^{2} + \|I_{M}(\Psi_{-}^{n}(0)) - I_{M}(\Psi_{-})\|_{L^{2}}^{2} \\ &\lesssim h^{2m_{0}} + \|P_{M}(\Phi(t_{n})) - I_{M}(\Phi^{n})\|_{L^{2}}^{2} \lesssim h^{2m_{0}} + \|\mathbf{e}^{n}(\cdot)\|_{L^{2}}^{2}, \\ \|F_{+}^{n}(\cdot)\|_{H^{1}}^{2} &= \|f_{+}(0) - f_{+}\|_{l^{2}}^{2} + \|\delta_{x}^{+}f_{+}(0) - \delta_{x}^{+}f_{+}\|_{l^{2}}^{2} \\ &\lesssim \|I_{M}(\Psi_{+}^{n}(0)) - I_{M}(\Psi_{+})\|_{H^{1}}^{2} + \|I_{M}(\Psi_{-}^{n}(0)) - I_{M}(\Psi_{-})\|_{H^{1}}^{2} \\ &\lesssim h^{2m_{0}-2} + \|P_{M}(\Phi(t_{n})) - I_{M}(\Phi^{n})\|_{H^{1}}^{2} \lesssim h^{2m_{0}-2} + \|\mathbf{e}^{n}(\cdot)\|_{H^{1}}^{2}, \end{split}$$

which gives $||F_{+}^{n}(\cdot)||_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + ||\mathbf{e}^{n}(\cdot)||_{H^{\sigma}}$ ($\sigma = 0, 1$). Similarly, we can obtain

$$\|F_{-}^{n}(\cdot)\|_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}, \quad \|G_{k}^{n}(\cdot)\|_{H^{\sigma}} \lesssim h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}, \quad k = -3, \dots, 3$$

and

$$\begin{aligned} \|\dot{F}^{n}_{\pm}(\cdot)\|_{L^{2}} &\lesssim \|I_{M}(\Psi^{n}_{+}(0)) - I_{M}(\Psi_{+})\|_{L^{2}} + \|I_{M}(\Psi^{n}_{-}(0)) - I_{M}(\Psi_{-})\|_{L^{2}} \\ &+ \|I_{M}(\dot{\Psi}^{n}_{+}(0)) - I_{M}(\dot{\Psi}_{+})\|_{L^{2}} + \|I_{M}(\dot{\Psi}^{n}_{-}(0)) - I_{M}(\dot{\Psi}_{-})\|_{L^{2}}, \end{aligned}$$

$$(3.41)$$

$$\begin{aligned} \|\dot{G}_{k}^{n}(\cdot)\|_{L^{2}} &\lesssim \|I_{M}(\Psi_{+}^{n}(0)) - I_{M}(\Psi_{+})\|_{L^{2}} + \|I_{M}(\Psi_{-}^{n}(0)) - I_{M}(\Psi_{-})\|_{L^{2}} \\ &+ \|I_{M}(\mathbf{r}^{n,*}/\tau) - I_{M}(\dot{\mathbf{r}})\|_{L^{2}}, \end{aligned}$$

$$(3.42)$$

$$\begin{aligned} \|\dot{G}_{k}^{n}(\cdot)\|_{L^{2}} &\lesssim \|I_{M}(\Psi_{+}^{n}(0)) - I_{M}(\Psi_{+})\|_{L^{2}} + \|I_{M}(\Psi_{-}^{n}(0)) - I_{M}(\Psi_{-})\|_{L^{2}} \\ &+ \|I_{M}(\dot{\Psi}_{+}^{n}(0)) - I_{M}(\dot{\Psi}_{+})\|_{L^{2}} + \|I_{M}(\dot{\Psi}_{-}^{n}(0)) - I_{M}(\dot{\Psi}_{-})\|_{L^{2}}, \quad k = \pm 1, \pm 3. \end{aligned}$$

$$(3.43)$$

By comparing (3.13)–(3.14) with (2.31) and applying Parseval's identity, we find that

$$\|I_{M}(\dot{\Psi}_{\pm}^{n}(0)) - I_{M}(\dot{\Psi}_{\pm})\|_{L^{2}} \lesssim \frac{1}{\tau} \|I_{M}(\Psi_{\pm}^{n}(0)) - I_{M}(\Psi_{\pm})\|_{L^{2}} + \|I_{M}(f_{\pm}(0)) - I_{M}(f_{\pm})\|_{L^{2}} \lesssim \frac{1}{\tau} (h^{m_{0}} + \|\mathbf{e}^{n}(\cdot)\|_{L^{2}}) + h^{m_{0}} + \|\mathbf{e}^{n}(\cdot)\|_{L^{2}},$$

$$\|I_{M}(\mathbf{r}^{n,*}/\tau) - I_{M}(\dot{\mathbf{r}})\|_{L^{2}} \lesssim \frac{1}{\tau} (h^{m_{0}} + \sum_{k=-3}^{3} \|I_{M}(g_{k}(0)) - I_{M}(g_{k})\|_{L^{2}}) \lesssim \frac{1}{\tau} (h^{m_{0}} + \|\mathbf{e}^{n}(\cdot)\|_{L^{2}}).$$
(3.44)

Therefore, we can obtain the bounds for $\|\dot{F}^n_{\pm}(\cdot)\|_{L^2}$ and $\|\dot{G}^n_k(\cdot)\|_{L^2}$ in view of (3.41)–(3.43). We omit the proof for the H^1 -norm bounds since it is quite similar to the L^2 -norm bounds. Combining all the above results, recalling (3.39) and properties of the coefficients $p_l(\tau)$, $q_l(\tau)$, $p_{\pm,l}^{(k)}(\tau)$ and $q_{\pm,l}^{(k)}(\tau)$ in (3.18), we conclude that (3.40) holds.

Remark 3.7. From (3.44), we can find that it is crucial to introduce the filter $2\sin(\mu_l^2 \tau)/\tau$ to replace μ_l^2 $(l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1)$ when approximating $\partial_s \Psi_{\pm,M}^n(0)$. The direct use of μ_l^2 will cause a second-order decreasing in the spatial accuracy, then a severe CFL condition is needed for the proof of error bounds [7]. By using the

filter, the accuracy is now controlled by the time step τ (cf. (3.44)). We remark here that one can choose other type filters.

Now, we are ready to prove the main theorem.

Proof of Theorem 3.1. The theorem will be proved by the method of mathematical induction. We remark that the constants in the inequalities are independent of time step n. For n = 0, we have

$$\|\Phi^0\|_{l^{\infty}} \le C_0, \ \|\mathbf{e}^0\|_{L^2} = \|P_M(\Phi_0) - I_M(\Phi_0)\|_{L^2} \le h^{m_0}, \ \|\mathbf{e}^0\|_{H^1} = \|P_M(\Phi_0) - I_M(\Phi_0)\|_{H^1} \le h^{m_0-1}.$$

By (3.5), we know that (3.2) is true for n = 0.

Assuming that (3.2) is valid for $0 \le n \le m \le T/\tau - 1$, we next prove that (3.2) will hold for n = m + 1. Recalling the decomposition (3.7) and the error equation (3.37), we get for $l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1$,

$$\widetilde{(\mathbf{e}^{n+1})}_{l} = e^{-i\frac{\tau}{\varepsilon^{2}}} \widetilde{(\mathbf{z}^{n+1}_{+})}_{l} + e^{i\frac{\tau}{\varepsilon^{2}}} \widetilde{(\mathbf{z}^{n+1}_{-})}_{l} + \widetilde{(\mathbf{z}^{n+1}_{r})}_{l} = (e^{-i\frac{\delta_{l}\tau}{\varepsilon^{2}}} \Pi_{l}^{+} \widetilde{(\mathbf{e}^{n})}_{l} + e^{i\frac{\delta_{l}\tau}{\varepsilon^{2}}} \Pi_{l}^{-} \widetilde{(\mathbf{e}^{n})}_{l}) + \widetilde{(\chi^{n})}_{l}, \qquad (3.45)$$

with $\chi^n(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{(\chi^n)}_l e^{i\mu_l(x-a)} \in Y_M$ given as

$$\widetilde{(\chi^n)}_l = e^{-i\tau/\varepsilon^2} \left(\widetilde{(\mathcal{F}_+^n)}_l + \widetilde{(\xi_+^n)}_l \right) + e^{i\tau/\varepsilon^2} \left(\widetilde{(\mathcal{F}_-^n)}_l + \widetilde{(\xi_-^n)}_l \right) + \left(\widetilde{(\mathcal{F}_r^n)}_l + \widetilde{(\xi_r^n)}_l \right).$$
(3.46)

Therefore, utilizing Lemma 3.6 and Parserval's identity, there holds for $0 < \tau \le 1, 0 \le n \le m$,

$$\|\chi^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau(h^{m_{0}-\sigma} + \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}) + \|\xi^{n}_{+}(\cdot)\|_{H^{\sigma}} + \|\xi^{n}_{-}(\cdot)\|_{H^{\sigma}} + \|\xi^{n}_{r}(\cdot)\|_{H^{\sigma}}, \quad \sigma = 0, 1.$$

From Lemma 3.5 on the local truncation error $\xi_{\pm}^n(x)$ and $\xi_r^n(x)$, we get for $0 \le n \le m$ and $\sigma = 0, 1$,

$$\|\chi^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}} + \tau (h^{m_{0}-\sigma} + \frac{\tau^{2}}{\varepsilon^{2}}), \quad \|\chi^{n}(\cdot)\|_{H^{\sigma}} \lesssim \tau \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}} + \tau (h^{m_{0}-\sigma} + \tau^{2} + \varepsilon^{2}).$$
(3.47)

Now, taking the l^2 norm of the vectors on both sides of (3.45), making use of the orthogonal properties of Π_l^{\pm} where $|e^{i\theta_1}\Pi_l^{\pm}\mathbf{v} + e^{i\theta_2}\Pi_l^{-}\mathbf{v}|^2 = |\Pi_l^{\pm}\mathbf{v}|^2 + |\Pi_l^{-}\mathbf{v}|^2 = |\mathbf{v}|^2$ for all $\mathbf{v} \in \mathbb{C}^2, \theta_1, \theta_2 \in \mathbb{R}$, we can have

$$\begin{split} \left| \widetilde{(\mathbf{e}^{n+1})}_l \right|^2 &= \left| e^{-i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^+ \widetilde{(\mathbf{e}^n)}_l + e^{i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^- \widetilde{(\mathbf{e}^n)}_l \right|^2 + |\widetilde{(\chi^n)}_l|^2 + 2\mathrm{Re} \left((e^{-i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^+ \widetilde{(\mathbf{e}^n)}_l + e^{i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^- \widetilde{(\mathbf{e}^n)}_l)^* \widetilde{(\chi^n)}_l \right) \\ &= \left| \widetilde{(\mathbf{e}^n)}_l \right|^2 + |\widetilde{(\chi^n)}_l|^2 + 2\mathrm{Re} \left((e^{-i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^+ \widetilde{(\mathbf{e}^n)}_l + e^{i\frac{\tau\delta_l}{\varepsilon^2}} \Pi_l^- \widetilde{(\mathbf{e}^n)}_l)^* \widetilde{(\chi^n)}_l \right). \end{split}$$

Applying Cauchy inequality, we find

$$\left| \widetilde{(\mathbf{e}^{n+1})}_l \right|^2 - \left| \widetilde{(\mathbf{e}^n)}_l \right|^2 \lesssim \tau |\widetilde{(\mathbf{e}^n)}_l|^2 + \frac{1}{\tau} |\widetilde{(\chi^n)}_l|^2, \quad l = -M/2, \dots, M/2 - 1.$$
(3.48)

Multiplying both sides of (3.48) by $(1 + \mu_l^{2\sigma})$ ($\sigma = 0, 1$) and then summing up for $l = -\frac{M}{2}, \ldots, \frac{M}{2} - 1$ and using Parseval's identity, we obtain

$$\|\mathbf{e}^{n+1}(\cdot)\|_{H^{\sigma}}^{2} - \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}^{2} \lesssim \tau \|\mathbf{e}^{n}(\cdot)\|_{H^{\sigma}}^{2} + \frac{1}{\tau} \|\chi^{n}(\cdot)\|_{H^{\sigma}}^{2}, \quad 0 \le n \le m.$$
(3.49)

Summing (3.49) for indices $0, 1, \ldots, n$, respectively, and using (3.47), we derive that for $n \leq m$,

$$\begin{aligned} \|\mathbf{e}^{n+1}(\cdot)\|_{H^{\sigma}}^{2} - \|\mathbf{e}^{0}(\cdot)\|_{H^{\sigma}}^{2} &\lesssim \tau \sum_{k=0}^{n} \|\mathbf{e}^{k}(\cdot)\|_{H^{\sigma}}^{2} + n\tau (h^{m_{0}-\sigma} + \tau^{2}/\varepsilon^{2})^{2}, \\ \|\mathbf{e}^{n+1}(\cdot)\|_{H^{\sigma}}^{2} - \|\mathbf{e}^{0}(\cdot)\|_{H^{\sigma}}^{2} &\lesssim \tau \sum_{k=0}^{n} \|\mathbf{e}^{k}(\cdot)\|_{H^{\sigma}}^{2} + n\tau (h^{m_{0}-\sigma} + \tau^{2} + \varepsilon^{2})^{2}, \quad \sigma = 0, 1. \end{aligned}$$

Since $\|\mathbf{e}^0(\cdot)\|_{H^{\sigma}} \lesssim h^{m_0 - \sigma}$, Gronwall's inequality will lead to the conclusion when $0 < \tau \le \tau_1 \le 1$ and $0 < h \le h_1 \le 1$ for some sufficiently small $\tau_1, h_1 < 1$

$$\|\mathbf{e}^{m+1}(\cdot)\|_{H^{\sigma}}^{2} \lesssim \left(h^{m_{0}-\sigma} + \tau^{2}/\varepsilon^{2}\right)^{2}, \quad \|\mathbf{e}^{m+1}(\cdot)\|_{H^{\sigma}}^{2} \lesssim (h^{m_{0}-\sigma} + \tau^{2} + \varepsilon^{2})^{2}, \quad \sigma = 0, 1.$$
(3.50)

It remains to show the l^{∞} bound of Φ^{m+1} such that Lemma 3.6 is valid during the mathematical induction process. Above estimates imply

$$\|\mathbf{e}^{m+1}(\cdot)\|_{H^1} \lesssim h^{m_0-1} + \min\{\tau^2/\varepsilon^2, \tau^2 + \varepsilon^2\} \lesssim h^{m_0-1} + \tau.$$
(3.51)

Applying the inverse inequality, and discrete Sobolev inequality and assumption (B), we have

$$\begin{split} \|\Phi^{m+1}\|_{l^{\infty}} &\leq \sup_{j \in \mathcal{T}_{M}} |\Phi_{j}^{m+1} - \Phi(t_{m+1}, x_{j})| + \|\Phi(t_{m+1})\|_{L^{\infty}} \\ &\leq \|\mathbf{e}^{m+1}\|_{l^{\infty}} + C_{1}h^{-1/2}\|P_{M}(\Phi(t_{m+1}))(\cdot) - I_{M}(\Phi(t_{m+1}))(\cdot)\|_{L^{2}} + C_{0} \\ &\leq C_{0} + C_{2}h^{m_{0}-1} + C_{3}(h^{m_{0}-1} + \tau), \end{split}$$

where C_1, C_2 and C_3 are independent of m. Therefore, for sufficiently small $0 < \tau_2, h_2 \leq 1$, when $\tau < \tau_2$ and $h < h_2$, we have $\|\Phi^{m+1}\|_{l^{\infty}} \leq C_0 + 1$, *i.e.* (3.2) holds for n = m + 1 with $\tau \leq \min\{\tau_1, \tau_2\}, h \leq \min\{h_1, h_2\}$. Thus the proof is complete.

Remark 3.8. We remark that the above proof as well as the conclusions in Theorem 3.1 can be easily extended to higher dimensions d = 2, 3. For 3D, the NLDE takes the form of (1.5). The key idea is to use Sobolev inequality in 2D and 3D for mesh functions ψ_h [5, 9] as

$$\|\psi_h\|_{l^{\infty}} \lesssim C_d(h)(\|\psi_h\| + \|\delta_x^+\psi_h\|), \quad C_2(h) = |\log h|, \quad C_3(h) = h^{-1/2}.$$

Thus, under the additional requirement that $\tau = o(1/|\log h|)$ (d = 2) or $\tau = o(h^{1/2})$ (d = 3), the current proof and conclusions are valid in 2D and 3D cases. It is also possible to establish higher order Sobolev norm estimates for the error, *e.g.* H^2 estimates, such that no requirements on the relation between τ and h would be needed in 2D and 3D. The proof would be quite similar to the one presented here and is left to interested readers.

4. Numerical results

In this section, we present numerical tests on our MTI-FP method (2.29) to show the uniform convergence in both space and time. To this purpose, we take d = 1 and solve the NLDE (2.12) with the electromagnetic potentials $A_1(t,x) = (x+1)^2/(1+x^2)$, $V(t,x) = (1-x)/(1+x^2)$, $x \in \mathbb{R}$, $t \ge 0$, and the initial data $\Phi_0(x) = (\phi_1(x), \phi_2(x))^T = \left(e^{-x^2/2}, e^{-(x-1)^2/2}\right)^T$, $x \in \mathbb{R}$.

The problem is solved on a bounded interval $\Omega = (-16, 16)$, *i.e.*, a = -16, b = 16, with periodic boundary conditions. Denote $\Phi_{h,\tau}^n$ as the numerical solution obtained by the MTI-FP method with mesh size h and time step τ . To quantify the convergence, we introduce

$$e_{h,\tau}(t_n) = \sqrt{\|\Phi^n - \Phi(t_n, \cdot)\|_{l^2}^2 + \|\delta_x^+ (\Phi^n - \Phi(t_n, \cdot))\|_{l^2}^2}.$$
(4.1)

$e_{h,\tau}(1.0)$	$h_0 = 2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\varepsilon_0 = 1$	1.44	5.88E - 1	8.26E - 2	$3.51E{-4}$	2.28E - 9
$\varepsilon_0/2$	$9.92E{-1}$	$4.32E{-1}$	$3.99E{-2}$	2.07 E - 4	5.32E - 9
$\varepsilon_0/2^2$	7.65 E - 1	$3.18E{-1}$	$1.37E{-2}$	$2.20E{-5}$	$2.54E{-}10$
$\varepsilon_0/2^3$	$7.85 \text{E}{-1}$	2.62 E - 1	6.64 E - 3	7.33E - 6	$2.23E{-}10$
$\varepsilon_0/2^4$	$8.56E{-1}$	$2.63E{-1}$	$6.29E{-3}$	5.15E-6	$2.18E{-}10$

TABLE 1. Spatial error analysis of the MTI-FP method for the NLDE in 1D.

Table 1 shows the spatial errors at T = 1, *i.e.*, $e_{h,\tau}(t = 1.0)$, with $\tau = 10^{-6}$ for varying ε and h; Table 2 lists the temporal errors at T = 1, *i.e.*, $e_{h,\tau}(t = 1.0)$ with h = 1/8 for varying ε and τ . The "reference" solution $\Phi(t, x) = (\phi_1(t, x), \phi_2(t, x))$ is obtained numerically by the MTI-FP method with very fine mesh h = 1/32 and time step $\tau = 1 \times 10^{-6}$. For a better observation of the convergence, we also show the spatial and temporal errors in Figure 2. From Tables 1 and 2, Figure 2 and extensive additional results not shown here for brevity, we can make the following observations:

- (i) In space, the MTI-FP method is uniformly spectrally accurate for all $\varepsilon \in (0, 1]$ (*cf.* Tab. 1 and left figure in Fig. 2).
- (ii) In time, the MTI-FP method is uniformly convergent with a linear rate at $O(\tau)$ for $\varepsilon \in (0,1]$ (cf. Tab. 2 and right figure in Fig. 2). For any fixed $0 < \varepsilon \leq 1$, when τ is small enough, *i.e.*, $\tau \leq \varepsilon^2$, (cf. upper triangle part, *i.e.*, above bold values, of Tab. 2), and when ε is small, *i.e.*, $\varepsilon \leq \tau$ (cf. lower triangle part, *i.e.*, below italic values, of Tab. 2), the MTI-FP method converges quadratically at $O(\tau^2)$. However, near the diagonal part where $\tau \sim \varepsilon^2$ (cf. the underline part of Tab. 2), degeneracy of the convergence rate is observed. In particular, the underline errors degenerate in the parameter regime $\tau \sim \varepsilon^2$, which has been predicted by our error estimates when ε^2 is comparable to τ^2/ε^2 .
- (iii) The MTI-FP method is uniformly accurate for all $\varepsilon \in (0, 1]$ under the meshing strategy (or ε -scalability) $\tau = O(1)$ and h = O(1).

Comparing to the methods proposed in [9] and [28], it can be found that (i) there is almost no difference in space since all the methods, except for the finite different method in [9], are uniformly accurate with spectral accuracy without any constraints on the mesh size; (ii) In time, the methods in [9] are not uniformly accurate, therefore, the method in [28] and ours are much better for small ε since there is no time step restriction. The method in [28] could achieve uniform first/second order convergence rate. However, they have introduced a new independent variable, therefore, the dimension of problem increases and this may enlarge computational cost and memory cost significantly. Our method converges uniformly with a linear rate and is of optimal second order accuracy when $\varepsilon = O(1)$ or $\varepsilon \leq \tau$, while the computational cost is comparable to the classical methods presented in [9] (roughly two times more).

5. CONCLUSION

A MTI-FP method was proposed and rigorously analyzed for the NLDE with a dimensionless parameter $\varepsilon \in (0, 1]$ which is inversely proportional to the speed of light. As studied in the literatures, the main difficulty for solving the NLDE is that the solution highly oscillates with $O(\varepsilon^2)$ wavelength in time when $0 < \varepsilon \ll 1$. The MTI-FP method is designed to capture the highly oscillation and compute the 'correct' solution efficiently, and the key ideas included adopting a proper multiscale decomposition of the NLDE and the Gautschi type exponential wave integrator in time discretization. Rigorous error analysis showed that the MTI-FP method converges (i) uniformly in space with spectral accuracy, (ii) uniformly in time with linear order for $\varepsilon \in (0, 1]$, (iii) quadratically in time when either $\varepsilon = O(1)$ or $\varepsilon \leq \tau$. This result significantly improves the error bounds of the existing numerical methods for the NLDE in the nonrelativistic limit regime. Numerical results confirmed the error estimates and suggested our error bounds are sharp and optimal.

$e_{h,\tau}(1.0)$	$\tau_0 = 0.2$	$ au_0/2^2$	$ au_{0}/2^{4}$	$ au_{0}/2^{6}$	$ au_{0}/2^{8}$	$ au_0/2^{10}$	$ au_{0}/2^{12}$
$\varepsilon_0 = 1$	$3.57E{-1}$	$1.69E{-2}$	$9.81 \text{E}{-4}$	$6.03E{-}5$	3.76E - 6	$2.35E{-7}$	1.49E - 8
Order	_	2.20	2.05	2.01	2.00	2.00	1.99
$\varepsilon_0/2$	$5.39\mathrm{E}{-1}$	2.35E-2	$1.35E{-3}$	8.24E-5	$5.12 \text{E}{-6}$	$3.20E{-7}$	2.07E - 8
Order	_	2.26	2.06	2.01	2.00	2.00	1.97
$\varepsilon_0/2^2$	$4.74E{-1}$	$3.04\mathrm{E}{-2}$	1.51E - 3	$8.71 E{-5}$	$5.34\mathrm{E}{-6}$	$3.32E{-7}$	2.08E - 8
Order	_	1.98	2.17	2.06	2.01	2.00	2.00
$\varepsilon_0/2^3$	$3.49E{-1}$	8.08E - 2	$4.97E{-3}$	$1.42E{-4}$	6.32E - 6	$3.61E{-7}$	2.20E - 8
Order	_	1.05	2.01	2.56	2.25	2.06	2.02
$\varepsilon_0/2^4$	$3.11E{-1}$	2.21E - 2	1.84E - 2	$1.14\mathrm{E}{-3}$	$2.20E{-5}$	$5.75E{-7}$	2.56E - 8
Order	_	1.91	0.13	2.01	2.84	2.63	2.25
$\varepsilon_0/2^5$	$3.06E{-1}$	1.42E - 2	4.36E - 3	4.40E - 3	$2.79\mathrm{E}{-4}$	4.82E - 6	8.68E - 8
Order	_	2.22	0.85	-0.01	1.99	2.93	2.90
$\varepsilon_0/2^6$	$3.05E{-1}$	$1.41E{-2}$	1.23E - 3	1.11E - 3	1.11E - 3	$6.96\mathrm{E}{-5}$	1.17E - 6
Order	_	2.22	1.76	0.08	-0.00	2.00	2.95
$\varepsilon_0/2^7$	$3.06E{-1}$	$1.42E{-2}$	8.30E-4	$2.68E{-4}$	$2.79E{-4}$	$2.78E{-4}$	$1.74\mathrm{E}{-5}$
Order	_	2.21	2.05	0.81	-0.03	$\overline{0.00}$	2.00
$\varepsilon_0/2^8$	$3.07E{-1}$	1.43E - 2	$8.36E{-4}$	$7.59E{-}5$	6.84 E - 5	6.99 E - 5	6.96E - 5
Order	_	2.21	2.05	1.73	0.07	-0.02	0.00
$\varepsilon_0/2^9$	$3.07E{-1}$	$1.43E{-2}$	$8.47E{-4}$	5.10E - 5	1.67 E - 5	$1.72E{-}5$	$1.75E{-5}$
Order	_	2.21	2.04	2.03	0.80	-0.02	-0.01
$\varepsilon_0/2^{11}$	$3.07E{-1}$	$1.44E{-2}$	$8.51 \text{E}{-4}$	5.26E - 5	3.44E - 6	$6.26E{-7}$	$5.49 E{-7}$
Order	_	2.21	2.04	2.01	1.97	1.23	0.09
$\varepsilon_0/2^{13}$	$3.07E{-1}$	$1.44E{-2}$	$8.52E{-4}$	5.25E - 5	3.28E - 6	2.15E-7	2.86E - 8
Order	_	2.21	2.04	2.01	2.00	1.96	1.46
$e_{\tau,h}^{\infty}$	$5.39E{-1}$	$8.08E{-2}$	$1.84E{-2}$	$4.40E{-3}$	$1.11E{-3}$	$2.78E{-4}$	$6.96E{-}5$
Order	_	1.37	1.07	1.03	1.00	1.00	1.00

TABLE 2. Temporal error analysis of the MTI-FP method for the NLDE in 1D. The convergence order is calculated as $\log_4(e_{h,4\tau}/e_{h,\tau})$. $e_{\tau,h}^{\infty}$ is the maximum error in each column.



FIGURE 2. Left: Spatial error of the MTI-FP method with N = (b - a)/h = 32/h. Right: Temporal error of the MTI-FP method.

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