ESAIM: M2AN 52 (2018) 147–161
https://doi.org/10.1051/m2an/2018005

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

MIXED SCHEMES FOR QUAD-CURL EQUATIONS☆

Shuo Zhang*

Abstract. In this paper, mixed schemes are presented for two variants of quad-curl equations. Specifically, stable equivalent mixed formulations for the model problems are presented, which can be discretized by finite elements of low regularity and of low degree. The regularities of the mixed formulations and thus equivalently the primal formulations are established, and some finite elements examples are given which can exploit the regularity of the solutions to an optimal extent.

Mathematics Subject Classification. 65N30, 35Q60, 76E25, 76W05

Received March 16, 2017. Accepted January 11, 2018.

1. Introduction

In this paper, we study the boundary value problem of the quad-curl operator of type

$$(\mathbf{A})\quad \begin{cases}(\nabla\times)^4\mathbf{u}=\mathbf{f} \quad \text{in } \Omega;\\ \nabla\cdot\mathbf{u}=0 \quad \text{in } \Omega;\\ \mathbf{u}\times\mathbf{n}=\mathbf{0},\ (\nabla\times\mathbf{u})\times\mathbf{n}=\mathbf{0} \quad \text{on } \partial\Omega,\end{cases} \tag{1.1}$$

where $\mathrm{div}\mathbf{f}=0$, and of variant type

$$(\mathbf{B})\quad \begin{cases}(\nabla\times)^4\mathbf{u}+\mathbf{u}=\mathbf{f} \quad \text{in } \Omega;\\ \mathbf{u}\times\mathbf{n}=\mathbf{0},\ (\nabla\times\mathbf{u})\times\mathbf{n}=\mathbf{0} \quad \text{on } \partial\Omega.\end{cases} \tag{1.2}$$

For (1.2), it is not necessary that $\mathrm{div}\mathbf{f}=0$. But evidently, $\mathrm{div}\mathbf{u}=0$ when $\mathrm{div}\mathbf{f}=0$.

The quad-curl operator $(\nabla\times)^4$ arises in various sources of applied sciences, like in elasticity, in magnetohydrodynamics(MHD) and in the inverse electromagnetic scattering theory. In elasticity, the operator is used to model the effect of the couple stress (*cf.* [26, 36]); in MHD (*cf.* [41]), $(\nabla\times)^4\mathbf{B}$ is involved in the resistive system where $\mathbf{B}$ is the magnetic field as a primary variable, and in the inverse electromagnetic scattering theory (*cf.* [5, 6]), $(\nabla\times)^4$ appears in computing the transmission eigenvalue. The operator is also used

☆The author is supported partially by the National Natural Science Foundation of China with Grant No. 11471026 and National Centre for Mathematics and Interdisciplinary Sciences, Chinese Academy of Sciences.

Keywords and phrases: Quad-curl equation, mixed scheme, regularity analysis, finite element method.

LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100190, PR China.

* Corresponding author: szhang@lsec.cc.ac.cn

Article published by EDP Sciences

as the principal part of the Electron MHD model; see equation (1.1) of [7]. Some more applications of $(\nabla\times)^4$ can be found in their subsequent works. The divergence free condition and the boundary data as in (1.1) are generally imposed.

There have been a few works devoted to the discretization of the model problems. For the equation as in (1.2), two kinds of discretizations are discussed with respect to the primal variational formulation in literature, including a nonconforming element given by Zheng–Hu–Xu [41], and discretizations by Hong–Hu–Shu–Xu [16] with standard high order Nédélec's elements in the framework of discontinuous Galerkin method. Some alternative approach is to introduce and deal with mixed/order-reduced formulations. As Nédélec's edge elements have been well studied for mixed schemes (*cf.*, *e.g.*, [2, 3, 22] for a brief introduction), order reduced discretizations can be expected for the original problem. It is natural to consider possibly the operator splitting technique which introduces an intermediate variable and then reduce the original problem to a system of second order equations. This is the way adopted by Sun [28] for the equation as in (1.1). The associated eigenvalue problem is also discussed therein. Relevant discussions on mixed finite methods related to the quad-curl operator can be also found in Monk and Sun [21] and Sun and Xu [29]. Very recently, Brenner–Sun–Sung [4] studies the quad-curl problem individually in two dimension, and employ Hodge decomposition to decouple the original fourth order problem to second order problems; also, the multiply-connected characteristic of the domain is treated therein. Meanwhile, a coupled mixed formulation for quad-curl problem is also introduced in [39] based on non-orthogonal decompositions. Beyond these discussions, few results are known to us.

The operator $(\nabla\times)^4$ is of fourth order and not completely symmetric; this makes the model problems bear complicated intrinsic structure. Primarily, the high stiffness effects the property of the problems. Recently, Nicaise [25] studies the boundary value problem (1.1) and proves that the solution does not always belong to $\mathbf{H}^3(\Omega)$ on polyhedrons with $\mathbf{H}^{-1}(\Omega)$ data $\mathbf{f}$, and its $\mathbf{H}^2(\Omega)$ regularity is still open. The snarly stiffness makes the concentrative construction of finite element functions difficult. Zheng–Hu–Xu [41] studies finite element for the Sobolev space such that $\mathbf{u}\in\mathbf{H}(\mathrm{curl},\Omega)$ and $\nabla\times\mathbf{u}\in\mathbf{H}^1(\Omega)$ which is used to discretize the model problem. Tai and Winther [30] and Neilan [23] discussed finite elements of the Sobolev space such that $\mathbf{u}\in\mathbf{H}^1(\Omega)$ and $\nabla\times\mathbf{u}\in\mathbf{H}^1(\Omega)$; various finite elements of H^2 spaces can be found in, *e.g.*, Wang–Shi–Xu [31, 32, 33], Zenišek [37] and Zhang [38]; these methods can be expected to work for the model problems at different occasions. Finite element spaces with exactly $(\nabla\times)^2$ consistency, however, have been seldom discussed in literature. Moreover, the structure of these finite element spaces are complicated, which makes designing optimal solvers/multilevel methods difficult; there has been no discussion along this line. The order-reduced discretisation scheme in [28] suggests to solve the original problem with existing edge elements; this scheme can be viewed as an analogue of the Ciarlet–Raviart's scheme [12] for biharmonic equation in the context of quad-curl problem, and equivalence can be expected when $(\nabla\times)^2\mathbf{u}\in\mathbf{H}(\mathrm{curl},\Omega)$. However, the structure of the scheme has not shown friendlier. The stability analysis was not mentioned in [28], and thus the intrinsic topology is not clear and the convergence analysis is constructed in quite a technical way. Further, designing optimal solvers/multilevel methods for the scheme is still an issue.

In this paper, we develop new mixed formulations with a clear and flat structure. By bringing in auxiliary variables, we present mixed formulations for **(A)** and **(B)** which are stable in Babuška-Brezzi's sense on the spaces of L^2, $\mathbf{H}(\mathrm{curl})$ and H^1 types. For **(A)**, the divergence free condition makes the problem possess sixth-order essence. In our approach, the condition is imposed in a dual form; the same technique was used in, *e.g.*, Kikuchi [17]. The mixed problems admit routine discretisation with finite element spaces corresponding to L^2, $\mathbf{H}(\mathrm{curl})$ and H^1 under some mild conditions, and the theoretical convergence analysis can be done in a routine way. As the structures of the dicsretized L^2, $\mathbf{H}(\mathrm{curl})$ and H^1 spaces have been well-studied, the newly-developed discretisation scheme can be solved by the aid of existing optimal preconditioners [15, 27, 34, 35]. Moreover, it is easy to find finite element spaces that are nested on nested grids, algebraically and topologically, with respect to the mixed formulation; this can bring convenience in designing high-efficiency algorithms. We also establish some regularity results on convex polyhedrons for the mixed formulations. As the mixed formulations are equivalent to the primal ones, the regularity of $\mathbf{u}$ and $\nabla\times\mathbf{u}$ are established for **(A)** and **(B)**, and the assumptions adopted in [41] and [28] are confirmed. Several finite elements are presented to be fit for the regularity.

The mixed schemes presented can be viewed friendly to users. As only first-order spaces and first-order operators are going to be discretized, the mixed schemes can be implemented by many popular finite element packages. It can also be seen that the schemes are flexible on choosing different finite elements corresponding to the regularity of the respective variables. Besides, it is direct to design optimal solvers for the discretization schemes. As the stability of the scheme can be proved with respect to the topology of the L^2, $\mathbf{H}(\mathrm{curl})$ and H^1 spaces, optimal Poisson solver based preconditioner can be constructed in a routine way [15, 27, 35]. We note that, for the schemes, several additional unknowns are introduced, therefore, compared to existing primal discretizations, especially ones of DG type like [16], more unknowns and larger-scale discretized systems are expected to be dealt with. However, the numerical solution of the generated system is an easier task than of the primal discretizations. Model **(A)** can be decoupled to several second order problems which are easy to solve. Even though Model **(B)** is essentially different from Model **(A)** since the zero-order term appears and it can not be decoupled, the five-field problem (4.5) of **(B)** can be optimally solved in quite a routine way as well. The application of such mixed formulations is thus not that expensive, though it looks so.

We would emphasize the new variational problems (2.5) (for **(A)**) and (2.6) (for **(B)**) are the starting point of what we are going to do and what we are able to do. These new primal formulations arise from configurating the essential boundary conditions that should be satisfied by the solutions, and they are different from traditional ones, like ones discussed in [16, 25] or [28]. The variational formulation (2.6) is similar to the one used in [41], but the original boundary condition discussed in [41] is different from that of (1.2). The new variational formulations possess enough capacity for the essential boundary conditions, and make the sequel analysis smoother. The scheme discussed in the present paper is relevant to the one discussed in the recent paper [4]. In contrast to the approach in [4] which uses implicitly the Helmholtz decomposition of the source term (right hand side $\mathbf{f}$), however, our present paper utilize the approach of introducing Lagrangian multipliers.

The remaining of the paper is organised as follows. In Section 2, we present some preliminaries and new primal formulation of the model problems. We will particularly figure out the appropriate spaces of the model problem by clarifying the boundary conditions and specify the space whose capacity is fit for the boundary condition and the variational form. In Section 3, mixed formulations of the model problems are given with stability analysis. Section 4 is then devoted to the discretizations, including general discussion on the conditions to be satisfied, and also some specific examples. Finally in Section 5, concluding remarks are given.

2. Model problems: new primal formulations

2.1. Preliminaries: Sobolev spaces

Let $\Omega \subset \mathbb{R}^3$ be a simply connected polyhedral domain with simply connected boundary $\Gamma = \partial\Omega$, and unit outward norm vector $\mathbf{n}$. In this paper, we use the bold symbol for a vector in $\mathbb{R}^3$. We use $L^2_{(0)}(\Omega)$ and $H^t_{(0)}(\Omega)$ for $t = 1, 2, \ldots$ for the standard Lebesque space and Sobolev spaces, and ∇, div and curl for the standard gradient, div and curl operators. For a vector $\mathbf{w} = (w^1, w^2, w^3)^\top$, $\nabla\mathbf{w} = (\nabla w^1, \nabla w^2, \nabla w^3)$. Denote

$$\mathbf{H}^t(\mathrm{curl}, \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) := (L^2(\Omega))^3 : \mathrm{curl}^j\mathbf{v} \in \mathbf{L}^2(\Omega), \quad 1 \leqslant j \leqslant t\}, \quad t = 0, 1, 2, \ldots,$$

equipped with the inner product $(\mathbf{u}, \mathbf{v})_{H^t(\mathrm{curl},\Omega)} = (\mathbf{u}, \mathbf{v}) + \sum_{j=1}^t (\mathrm{curl}^j\mathbf{u}, \mathrm{curl}^j\mathbf{v})$, and the corresponding norm $\|\cdot\|_{H^t(\mathrm{curl},\Omega)}$. Particularly, $\mathbf{H}^1(\mathrm{curl}, \Omega) = \mathbf{H}(\mathrm{curl}, \Omega)$ the usually defined Sobolev space, and $\|\cdot\|_{\mathbf{H}(\mathrm{curl},\Omega)} = \|\cdot\|_{\mathrm{curl},\Omega}$. Similarly, define

$$\mathbf{H}(\mathrm{curl}^2, \Omega) := \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{curlcurl}\mathbf{v} \in \mathbf{L}^2(\Omega)\}, \tag{2.1}$$

equipped with the inner product $(\mathbf{u}, \mathbf{v})_{\mathbf{H}(\mathrm{curl}^2,\Omega)} = (\mathbf{u}, \mathbf{v}) + (\mathrm{curlcurl}\mathbf{u}, \mathrm{curlcurl}\mathbf{v})$ and with the corresponding norm. Corresponding to the boundary condition, define $\mathbf{H}^2_0(\mathrm{curl}, \Omega) := \{\mathbf{v} \in \mathbf{H}^2(\mathrm{curl}, \Omega) : \mathbf{v} \times \mathbf{n} = \mathbf{0}$ and $(\mathrm{curl}\mathbf{v}) \times \mathbf{n} = \mathbf{0}$ on $\Gamma\}$. In the sequel, we use $\nabla\times$ for curl in equations. As usual, denote $\mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) : \mathrm{div}\mathbf{v} \in L^2(\Omega)\}$, and $\mathbf{H}_0(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega) : \mathbf{v} \cdot \mathbf{n} = 0 \,\mathrm{on}\, \Gamma\}$.

These variational formulations are usually used in literature:

- for **(A)** (as used in, *e.g.*, [28]): given $\mathbf{f}$ with $\mathrm{div}\mathbf{f}=0$, find $\mathbf{u}\in\mathbf{H}_0^2(\mathrm{curl},\Omega)$ and $\mathrm{div}\mathbf{u}=0$, such that

$$(\nabla\times\nabla\times\mathbf{u},\nabla\times\nabla\times\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad \forall\,\mathbf{v}\in\mathbf{H}_0^2(\mathrm{curl},\Omega); \tag{2.2}$$

- for **(B)** (as used in, *e.g.*, [16]): given $\mathbf{f}$, to find $\mathbf{u}\in\mathbf{H}_0^2(\mathrm{curl},\Omega)$, such that

$$(\nabla\times\nabla\times\mathbf{u},\nabla\times\nabla\times\mathbf{v})+(\mathbf{u},\mathbf{v})=(\mathbf{f},\mathbf{v}),\quad \forall\,\mathbf{v}\in\mathbf{H}_0^2(\mathrm{curl},\Omega). \tag{2.3}$$

The well-posedness of the variational problems follows from Lemmas 2.1 and 2.3 below.

Lemma 2.1 (Friedrichs inequality, *cf.* Sect. 11.1.2 of [3])**.** *There exists a constant C, such that it holds for $\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)$ and $\mathrm{div}\mathbf{v}=0$ that*

$$\|\mathbf{v}\|_{0,\Omega}\leqslant C\|\nabla\times\mathbf{v}\|_{0,\Omega}. \tag{2.4}$$

Remark 2.2. $\nabla H_0^1(\Omega)$ is a closed subspace of $\mathbf{H}_0(\mathrm{curl},\Omega)$, and $\mathbf{H}_0(\mathrm{curl},\Omega)=\nabla H_0^1(\Omega)\oplus^{\perp}(\nabla H_0^1(\Omega))^{\perp}$, where $(\nabla H_0^1(\Omega))^{\perp}$ is the orthogonal complement of $\nabla H_0^1(\Omega)$ in $\mathbf{H}_0(\mathrm{curl},\Omega)$. Moreover, $(\nabla H_0^1(\Omega))^{\perp}=\{\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega):\mathrm{div}\,\mathbf{v}=0\}$.

Lemma 2.3 ([16], Lem. 2.1)**.** *$\mathbf{H}_0^2(\mathrm{curl},\Omega)$ is the closure of $(\mathcal{C}_0^\infty(\Omega))^3$ in $\mathbf{H}(\mathrm{curl}^2,\Omega)$, and*

$$\|\nabla\times\mathbf{v}\|_{0,\Omega}\leqslant\frac{1}{2}(\|\nabla\times\nabla\times\mathbf{v}\|_{0,\Omega}+\|\mathbf{v}\|_{0,\Omega})\quad \textit{on}\ \mathbf{H}_0^2(\mathrm{curl},\Omega).$$

2.2. New primal formulations

We begin with the fact below.

Lemma 2.4 (Lemma 2.5 of [13])**.** *If $\Omega\subset\mathbb{R}^3$ is bounded, simply connected with Liptschiz-continuous boundary, then*

$$\mathbf{H}_0^1(\Omega):=(H_0^1(\Omega))^3=\mathbf{H}_0(\mathrm{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div},\Omega),$$

and

$$(\nabla\mathbf{u},\nabla\mathbf{v})=(\nabla\times\mathbf{u},\nabla\times\mathbf{v})+(\mathrm{div}\mathbf{u},\mathrm{div}\mathbf{v})\quad \textit{for}\ \mathbf{u},\mathbf{v}\in\mathbf{H}_0^1(\Omega).$$

Define $\mathbf{H}_0^1(\mathrm{curl},\Omega):=\{\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega):\nabla\times\mathbf{v}\in\mathbf{H}_0^1(\Omega)\}$. Below is a crucial fact.

Lemma 2.5. $\mathbf{H}_0^1(\mathrm{curl},\Omega)=\mathbf{H}_0^2(\mathrm{curl},\Omega)$.

Proof. Evidently, $\mathbf{H}_0^1(\mathrm{curl},\Omega)\subset\mathbf{H}_0^2(\mathrm{curl},\Omega)$. On the other hand, given $\mathbf{v}\in\mathbf{H}_0^2(\mathrm{curl},\Omega)$, then $\nabla\times\mathbf{v}\in\mathbf{H}_0(\mathrm{div},\Omega)$ as $\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)$; namely, $\nabla\times\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div},\Omega)=\mathbf{H}_0^1(\Omega)$. This completes the proof. ☐

Evidently, $(\nabla(\nabla\times\mathbf{u}),\nabla(\nabla\times\mathbf{v}))=(\nabla\times\nabla\times\mathbf{u},\nabla\times\nabla\times\mathbf{v})$ on $\mathbf{H}_0^1(\mathrm{curl},\Omega)$.

Then we establish the variational form of the primal model problems as:

(A′) Given $\mathbf{f}$ with $\mathrm{div}\mathbf{f}=0$, find $\mathbf{u}\in\mathbf{H}_0^1(\mathrm{curl},\Omega)$, $\mathrm{div}\,\mathbf{u}=0$, such that

$$(\nabla(\nabla\times\mathbf{u}),\nabla(\nabla\times\mathbf{v}))=(\mathbf{f},\mathbf{v}),\quad \forall\,\mathbf{v}\in\mathbf{H}_0^1(\mathrm{curl},\Omega). \tag{2.5}$$

(B′) Given $\mathbf{f}$, find $\mathbf{u} \in \mathbf{H}_0^1(\mathrm{curl}, \Omega)$, such that

$$(\nabla(\nabla \times \mathbf{u}), \nabla(\nabla \times \mathbf{v})) + (\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\mathrm{curl}, \Omega). \tag{2.6}$$

The theorem below follows from Lemma 2.5.

Theorem 2.6. *The variational problems* **(A′)** *and* **(B′)** *are well-posed. They are equivalent to* (2.2) *and* (2.3)*, respectively.*

Remark 2.7. The variational problem on $\mathbf{H}_0^1(\mathrm{curl}, \Omega)$ has been discussed in [41], where the boundary condition is $\mathbf{u} \times \mathbf{n} = \nabla \times \mathbf{u} = \mathbf{0}$. Here we show that $\mathbf{H}_0^1(\mathrm{curl}, \Omega)$ is still the space fit for this simplified boundary condition.

3. Mixed formulation of model problems

In this section, we present mixed problems that are equivalent to **(A′)** and **(B′)**, thus to **(A)** and **(B)**, respectively. The stability and regularity of the mixed problems are given.

3.1. Mixedization of problem (A′)

We start with the observations below.

Lemma 3.1. *Given* $\mathbf{u} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$ *and* $\mathbf{y} \in \mathbf{H}_0^1(\Omega)$, $\nabla \times \mathbf{u} = \mathbf{y}$ *and* $\mathrm{div}\mathbf{u} = 0$ *if and only if there exists an* $\sigma \in H_0^1(\Omega)$*, such that*

$$\begin{cases} (\mathrm{div}\mathbf{y}, q) = 0, & \forall q \in L_0^2(\Omega), \\ (\nabla\sigma, \mathbf{s}) - (\nabla \times \mathbf{u}, \nabla \times \mathbf{s}) + (\mathbf{y}, \nabla \times \mathbf{s}) = 0, & \forall \mathbf{s} \in \mathbf{H}_0(\mathrm{curl}, \Omega), \\ (\mathbf{u}, \nabla\eta) = 0, & \forall \eta \in H_0^1(\Omega). \end{cases} \tag{3.1}$$

Proof. If $\mathrm{div}\mathbf{u} = 0$ and $\nabla \times \mathbf{u} = \mathbf{y}$, then (3.1) follow immediately. Indeed, $\sigma = 0$.

On the other hand, under the assumption (3.1), $\mathrm{div}\mathbf{y} = 0$ by the first equation, and thus there exists a unique $\mathbf{w} \in (\nabla H_0^1(\Omega))^\perp$, such that $\mathbf{y} = \nabla \times \mathbf{w}$. By the third equation of (3.1), $\mathrm{div}\mathbf{u} = 0$ and $\mathbf{u} \in (\nabla H_0^1(\Omega))^\perp$. Further, by the second equation, $(\nabla \times \mathbf{w}, \nabla \times \mathbf{s}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{s})$ for any $\mathbf{s} \in (\nabla H_0^1(\Omega))^\perp$, and thus $\mathbf{u} = \mathbf{w}$, namely $\mathbf{y} = \nabla \times \mathbf{u}$. The proof is completed. □

Now, let $\mathbf{u}$ be the solution of **(A′)** (thus **(A)**); if we define $\mathbf{y} := \nabla \times \mathbf{u}$, then $\mathbf{y} \in \mathbf{H}_0^1(\Omega)$. Thus the variational problem **(A′)** can be rewritten as: find $(\mathbf{u}, \mathbf{y}) \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times \mathbf{H}_0^1(\Omega)$, such that, $\mathrm{div}\mathbf{u} = 0$, $\nabla \times \mathbf{u} = \mathbf{y}$, and

$$(\nabla\mathbf{y}, \nabla\mathbf{z}) = (\mathbf{f}, \mathbf{v}), \tag{3.2}$$

for any $(\mathbf{v}, \mathbf{z}) \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times \mathbf{H}_0^1(\Omega)$, such that $\mathrm{div}\mathbf{v} = 0$ and $\nabla \times \mathbf{v} = \mathbf{z}$.

By Lemma 3.1, we can further introduce Lagrangian multipliers p, $\mathbf{r}$, g and a dual Lagrangian multiplier σ, to rewrite the equation (3.2) to a relaxed formulation. Namely, define

$$V := H_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega) \times \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega) \times H_0^1(\Omega), \tag{3.3}$$

and we consider to find $(\sigma,\mathbf{u},\mathbf{y},p,\mathbf{r},\xi)\in V$, such that for $(\tau,\mathbf{v},\mathbf{z},q,\mathbf{s},\eta)\in V$,

$$(\mathbf{A}'')\begin{cases}(\mathbf{r},\nabla\tau)=0,\\ -(\nabla\times\mathbf{r},\nabla\times\mathbf{v})+(\nabla\xi,\mathbf{v})=(\mathbf{f},\mathbf{v}),\\ (\nabla\mathbf{y},\nabla\mathbf{z})+(p,\mathrm{div}\mathbf{z})+(\nabla\times\mathbf{r},\mathbf{z})=0,\\ (\mathrm{div}\mathbf{y},q)=0,\\ (\nabla\sigma,\mathbf{s})-(\nabla\times\mathbf{u},\nabla\times\mathbf{s})+(\mathbf{y},\nabla\times\mathbf{s})=0,\\ (\mathbf{u},\nabla\eta)=0.\end{cases}\tag{3.4}$$

Lemma 3.2. *Given* $\mathbf{f}\in\mathbf{L}^2(\Omega)$*, Problem* (3.4) *admits a unique solution. Moreover,*

$$\|\sigma\|_{1,\Omega}+\|\mathbf{u}\|_{\mathrm{curl},\Omega}+\|\mathbf{y}\|_{1,\Omega}+\|p\|_{0,\Omega}+\|\mathbf{r}\|_{\mathrm{curl},\Omega}+\|\xi\|_{1,\Omega}\leqslant C\|\mathbf{f}\|_{(\mathbf{H}_0(\mathrm{curl},\Omega))'}.\tag{3.5}$$

Proof. We are going to verify conditions by Brezzi's theory. Define

$$a((\sigma,\mathbf{u},\mathbf{y}),(\tau,\mathbf{v},\mathbf{z})):=(\nabla\mathbf{y},\nabla\mathbf{z}),\tag{3.6}$$

and

$$b((\sigma,\mathbf{u},\mathbf{y}),(q,\mathbf{s},\eta)):=(\mathrm{div}\mathbf{y},q)+(\nabla\sigma,\mathbf{s})-(\nabla\times\mathbf{u},\nabla\times\mathbf{s})+(\mathbf{y},\nabla\times\mathbf{s})+(\mathbf{u},\nabla\eta).\tag{3.7}$$

Then $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous on $[H^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times\mathbf{H}^1_0(\Omega)]^2$ and $[H^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times\mathbf{H}^1_0(\Omega)]\times[L^2_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times H^1_0(\Omega)]$, respectively. Define $Z:=\Big\{(\sigma,\mathbf{u},\mathbf{y})\in[H^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times\mathbf{H}^1_0(\Omega)]:b((\sigma,\mathbf{u},\mathbf{y}),(q,\mathbf{s},\eta))=0,\ \forall(q,\mathbf{s},\eta)\in[L^2_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times H^1_0(\Omega)]\Big\}$. Then it remains for us to verify the coercivity of $a(\cdot,\cdot)$ on Z and inf-sup condition: given nonzero $(q,\mathbf{s},\eta)\in L^2_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times H^1_0(\Omega)$,

$$\sup_{(\sigma,\mathbf{u},\mathbf{y})\in H^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times\mathbf{H}^1_0(\Omega)\setminus\{\mathbf{0}\}}\frac{b((\sigma,\mathbf{u},\mathbf{y}),(q,\mathbf{s},\eta))}{\|\sigma\|_{1,\Omega}+\|\mathbf{u}\|_{\mathrm{curl},\Omega}+\|\mathbf{y}\|_{1,\Omega}}\geqslant C(\|q\|_{0,\Omega}+\|\mathbf{s}\|_{\mathrm{curl},\Omega}+\|\eta\|_{1,\Omega}).\tag{3.8}$$

Given $(\sigma,\mathbf{u},\mathbf{y})\in Z$, then $\sigma=0$. Since $(\mathbf{u},\nabla\eta)=0$ for any $\eta\in H^1_0(\Omega)$, by Lemma 2.1, we have $\|\mathbf{u}\|_{\mathrm{curl},\Omega}\leqslant C\|\nabla\times\mathbf{u}\|_{0,\Omega}$. As $(\nabla\times\mathbf{u},\nabla\times\mathbf{u})=(\mathbf{y},\nabla\times\mathbf{u})$, we have $\|\nabla\times\mathbf{u}\|_{0,\Omega}\leqslant\|\mathbf{y}\|_{0,\Omega}\leqslant C\|\nabla\mathbf{y}\|_{0,\Omega}$. This confirms the coercivity of $a(\cdot,\cdot)$ on Z.

Given $(q,\mathbf{s},\eta)\in L^2_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times H^1_0(\Omega)$, firstly, we decomose $\mathbf{s}=\mathbf{s}_1+\mathbf{s}_2$, such that $\mathbf{s}_1\in\nabla H^1_0(\Omega)$, and $\mathbf{s}_2\in(\nabla H^1_0(\Omega))^\perp$. Set $\mathbf{y}$ to be such that $(\mathrm{div}\mathbf{y},q)=(q,q)$ and $\|\mathbf{y}\|_{1,\Omega}\leqslant C\|\mathrm{div}\mathbf{y}\|_{0,\Omega}$, and σ to be such that $\nabla\sigma=\mathbf{s}_1$. Further, $\mathbf{u}$ is chosen to be $\mathbf{u}_1+\nabla\eta$, such that $(\mathbf{u}_1,\nabla\eta)=0$ for any $\eta\in H^1_0(\Omega)$ and $(\mathbf{y}-\nabla\times\mathbf{u}_1,\nabla\times\mathbf{v})=(\nabla\times\mathbf{s},\nabla\times\mathbf{v})$ for any $\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)$. Then

$$b((\sigma,\mathbf{u},\mathbf{y}),(q,\mathbf{s},\eta))=(q,q)+(\mathbf{s}_1,\mathbf{s}_1)+(\nabla\times\mathbf{s}_2,\nabla\times\mathbf{s}_2)+(\nabla\eta,\nabla\eta)\geqslant C(\|q\|_0^2+\|\mathbf{s}\|^2_{\mathrm{curl},\Omega}+\|\nabla\eta\|^2_{0,\Omega}),$$

where we have made use of Lemma 2.1 for $\mathbf{s}_2$. Meanwhile, $\|\nabla\sigma\|_{0,\Omega}=\|\mathbf{s}_1\|_{0,\Omega}\leqslant\|\mathbf{s}\|_{\mathrm{curl},\Omega}$, $\|\mathbf{y}\|_{1,\Omega}\leqslant C\|q\|_{0,\Omega}$, and $\|\mathbf{u}\|_{\mathrm{curl},\Omega}\leqslant C(\|\nabla\eta\|_{0,\Omega}+\|\mathbf{s}\|_{\mathrm{curl},\Omega}+\|\mathbf{y}\|_{0,\Omega})\leqslant C(\|\nabla\eta\|_{0,\Omega}+\|\mathbf{s}\|_{\mathrm{curl},\Omega}+\|q\|_{0,\Omega})$. This confirms (3.8) and completes the proof. □

Lemma 3.3. *The problem* (3.4) *is equivalent to the variational problem* $(\mathbf{A}')$.

Proof. Let $(\sigma,\mathbf{u},\mathbf{y},p,\mathbf{r},\xi)$ be the solution of (3.4). Then $\mathrm{div}\mathbf{u}=0$, $\nabla\times\mathbf{u}=\mathbf{y}$, and $(\nabla\mathbf{y},\nabla\mathbf{z})=(\mathbf{f},\mathbf{v})$ for any $\mathbf{z}\in\mathbf{H}^1_0(\Omega)$ and $\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)$ such that $\mathbf{z}=\mathrm{curl}\mathbf{v}$. This way, $\mathbf{u}\in\mathbf{H}^1_0(\Omega)$, and solves $(\mathbf{A}')$. As $(\mathbf{A}')$ admits only a unique solution, this confirms simultaneously that the solution of **(A′)** is part of the solution of (3.4). The proof is completed. □

Lemma 3.4. *The problem (3.4) can be decomposed to the three subsystems and solved sequentially:*

(1) given $\mathbf{f}$*, solve for* $\mathbf{r}\in\mathbf{H}_0(\mathrm{curl},\Omega)$ *and* $\xi\in H^1_0(\Omega)$ *that*

$$\begin{cases}(\nabla\times\mathbf{r},\nabla\times\mathbf{v})-(\nabla\xi,\mathbf{v})=-(\mathbf{f},\mathbf{v}), & \forall\,\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)\\ (\nabla\tau,\mathbf{r})=0, & \forall\,\tau\in H^1_0(\Omega);\end{cases}\tag{3.9}$$

(2) with $\mathbf{r}$ *obtained, solve for* $\mathbf{y}\in\mathbf{H}^1_0(\Omega)$ *and* $p\in L^2_0(\Omega)$ *that*

$$\begin{cases}(\nabla\mathbf{y},\nabla\mathbf{z})+(p,\mathrm{div}\mathbf{z})=-(\nabla\times\mathbf{r},\mathbf{z}), & \forall\,\mathbf{z}\in\mathbf{H}^1_0(\Omega)\\ (\mathrm{div}\mathbf{y},q)=0, & \forall\,q\in L^2_0(\Omega);\end{cases}\tag{3.10}$$

(3) with $\mathbf{y}$ *obtained, solve for* $\mathbf{u}\in\mathbf{H}_0(\mathrm{curl},\Omega)$ *and* $\sigma\in H^1_0(\Omega)$ *that*

$$\begin{cases}(\nabla\times\mathbf{u},\nabla\times\mathbf{s})-(\nabla\sigma,\mathbf{s})=(\mathbf{y},\nabla\times\mathbf{s}), & \forall\,\mathbf{s}\in\mathbf{H}_0(\mathrm{curl},\Omega)\\ (\nabla\eta,\mathbf{u})=0, & \forall\,\eta\in H^1_0(\Omega).\end{cases}\tag{3.11}$$

Proof. By the stable Helmholtz decomposition of $\mathbf{H}_0(\mathrm{curl},\Omega)$, we can verify the three subproblems are all well-posed. The proof follows then. □

The theorem below establishes the regularity of the mixed system, and thus the primal form of **(A)** in forms (1.1) and (2.2). It confirms the assumption in [28] (p. 190).

Theorem 3.5. *Let* Ω *be a convex polyhedron, and* $\mathbf{f}\in(L^2(\Omega))^3$ *such that* $\mathrm{div}\mathbf{f}=0$*. Let* $(\sigma,\mathbf{u},\mathbf{y},p,\mathbf{r},\xi)$ *be the solution of (3.4). Then*

$$\sigma=0;\tag{3.12}$$
$$\mathbf{u}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}_0(\mathrm{curl},\Omega),\ \nabla\times\mathbf{u}\in\mathbf{H}^2(\Omega);\tag{3.13}$$
$$\mathbf{y}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}^1_0(\Omega);\tag{3.14}$$
$$p\in H^1(\Omega)\cap L^2_0(\Omega);\tag{3.15}$$
$$\mathbf{r}\in\mathbf{H}^2(\Omega)\cap\mathbf{H}_0(\mathrm{curl},\Omega);\tag{3.16}$$
$$\xi=0.\tag{3.17}$$

Moreover, if further $\mathbf{f}\in\mathbf{H}^1(\Omega)$*, then* $\nabla\times\mathbf{r}\in\mathbf{H}^2(\Omega)$*.*

We postpone the proof of Theorem 3.5 after the technical lemma below. From this point onwards, $\lesssim$, $\gtrsim$, and $\eqsim$ respectively denote $\leqslant$, $\geqslant$, and $=$ up to a constant. The hidden constants depend on the domain, and, when triangulation is involved, they also depend on the shape-regularity of the triangulation, but they do not depend on h or any other mesh parameter.

Lemma 3.6. *Let* Ω *be a convex simply connected polyhedron.*

(1) (Sects. 3.4 and 3.5 of [13])

$$\mathbf{H}_0(\mathrm{div},\Omega)\cap\mathbf{H}(\mathrm{curl},\Omega)\subset\mathbf{H}^1(\Omega),\ \ \text{and}\ \ \mathbf{H}_0(\mathrm{curl},\Omega)\cap\mathbf{H}(\mathrm{div},\Omega)\subset\mathbf{H}^1(\Omega).$$

(2) *(Lem. 3.2 of [10]) For functions* $\mathbf{z}\in\mathbf{H}(\mathrm{div},\Omega)\cap\mathbf{H}_0(\mathrm{curl},\Omega)$ *or* $\mathbf{H}_0(\mathrm{div},\Omega)\cap\mathbf{H}(\mathrm{curl},\Omega)$ *satisfying* $\mathrm{curl}\mathbf{z}\in\mathbf{H}^1(\Omega)$ *and* $\mathrm{div}\mathbf{z}\in\mathbf{H}^1(\Omega)$. *Then*

$$\mathbf{z}\in\mathbf{H}^2(\Omega),\ \ \textit{and}\ \|\mathbf{z}\|_2\lesssim\|\mathrm{curl}\mathbf{z}\|_{1,\Omega}+\|\mathrm{div}\mathbf{z}\|_{1,\Omega}.$$

Proof of Theorem 3.5. By Lemma 3.4, we will show the regularity result by dealing with the subsystems (3.9), (3.10) and (3.11) sequentially.

Since $\mathrm{div}\mathbf{f}=0$, by (3.9), it holds that $\xi=0$, and $(\nabla\times\mathbf{r},\nabla\times\mathbf{v})=(\mathbf{f},\mathbf{v})$ for any $\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)$. Thus $\nabla\times\nabla\times\mathbf{r}=\mathbf{f}\in\mathbf{L}^2(\Omega)$, namely $\nabla\times\mathbf{r}\in\mathbf{H}(\mathrm{curl},\Omega)$. Also, $\nabla\times\mathbf{r}\in\mathbf{H}_0(\mathrm{div},\Omega)$, and thus $\nabla\times\mathbf{r}\in\mathbf{H}(\mathrm{curl},\Omega)\cap\mathbf{H}_0(\mathrm{div},\Omega)\subset\mathbf{H}^1(\Omega)$. Note that by (3.9), $\mathrm{div}\mathbf{r}=0$, thus by Lemma 3.6, $\mathbf{r}\in\mathbf{H}^2(\Omega)$. This proves (3.16) and (3.17).

Substitute $\nabla\times\mathbf{r}$ into the system (3.10), standardly we obtain the estimate (3.14) and (3.15). Here we refer to [24] for the regularity analysis of the 3D Stokes problem.

Further, substitute $\mathbf{y}$ into the system (3.11), we obtain $\sigma=0$, $\mathrm{div}\mathbf{u}=0$, $\mathbf{u}\in\mathbf{H}^1(\Omega)\cap\mathbf{H}_0(\mathrm{curl},\Omega)$ and $\nabla\times\mathbf{u}\in\mathbf{H}^1(\Omega)$. By Lemma 3.6 again, it holds that $\mathbf{u}\in\mathbf{H}^2(\Omega)$. Moreover, $\nabla\times\mathbf{u}=\mathbf{y}$; this leads to $\nabla\times\mathbf{u}\in\mathbf{H}^2(\Omega)$.

When $\mathbf{f}\in\mathbf{H}^1$, $\nabla\times\mathbf{r}\in\mathbf{H}^2(\Omega)$ by Lemma 3.6. The proof is completed. □

Remark 3.7. As $\sigma=\xi=0$ for any $\mathbf{f}$ with $\mathrm{div}\mathbf{f}=0$, the system (3.4) can be simplified to: find $(\mathbf{u},\mathbf{y},p,r)\in V'':=(\nabla H^1_0(\Omega))^\perp\times\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)\times(\nabla H^1_0(\Omega))^\perp$, such that, for any $(\mathbf{v},\mathbf{z},q,\mathbf{s})\in V''$,

$$\begin{cases}-(\nabla\times\mathbf{r},\nabla\times\mathbf{v})=(\mathbf{f},\mathbf{v}),\\ (\nabla\mathbf{y},\nabla\mathbf{z})+(p,\mathrm{div}\mathbf{z})+(\nabla\times\mathbf{r},\mathbf{z})=0,\\ (\mathrm{div}\mathbf{y},q)=0,\\ -(\nabla\times\mathbf{u},\nabla\times\mathbf{s})+(\mathbf{y},\nabla\times\mathbf{s})=0.\end{cases}$$

This new system is equivalent to (3.4), and simpler formally. However, its dicretization needs finite element space for $(\nabla H^1_0(\Omega))^\perp$, and we will not discuss it much here.

3.2. Mixedization of problem (B′)

We simply repeat the procedure for Problem **(A′)**. Define

$$U:=H^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times\mathbf{H}^1_0(\Omega)\times\mathbf{H}_0(\mathrm{curl},\Omega)\times L^2_0(\Omega).$$

The mixed formulation is to find $(\sigma,\mathbf{u},\mathbf{y},\mathbf{r},p)\in U$, such that for $(\tau,\mathbf{v},\mathbf{z},\mathbf{s},q)\in U$,

$$(\mathbf{B}'')\begin{cases}(\mathbf{r},\nabla\tau)=0,\\ (\mathbf{u},\mathbf{v})-(\nabla\times\mathbf{r},\nabla\times\mathbf{v})=(\mathbf{f},\mathbf{v}),\\ (\nabla\mathbf{y},\nabla\mathbf{z})+(\nabla\times\mathbf{r},\mathbf{z})+(p,\mathrm{div}\mathbf{z})=0,\\ (\nabla\sigma,\mathbf{s})-(\nabla\times\mathbf{u},\nabla\times\mathbf{s})+(\mathbf{y},\nabla\times\mathbf{s})=0,\\ (\mathrm{div}\mathbf{y},q)=0.\end{cases}\tag{3.18}$$

Lemma 3.8. *Given* $\mathbf{f}\in\mathbf{L}^2(\Omega)$, *Problem* (3.18) *admits a unique solution. Moreover,*

$$\|\sigma\|_{1,\Omega}+\|\mathbf{u}\|_{\mathrm{curl},\Omega}+\|\mathbf{y}\|_{1,\Omega}+\|\mathbf{r}\|_{\mathrm{curl},\Omega}+\|p\|_{0,\Omega}\leqslant C\|\mathbf{f}\|_{(\mathbf{H}_0(\mathrm{curl},\Omega))'}.\tag{3.19}$$

Proof. Again, we are going to verify Brezzi's conditions. Define $a((\sigma,\mathbf{u},\mathbf{y}),(\tau,\mathbf{v},\mathbf{z})):=(\mathbf{u},\mathbf{v})+(\nabla\mathbf{y},\nabla\mathbf{z})$, and $b((\sigma,\mathbf{u},\mathbf{y}),(\mathbf{s},q)):=(\nabla\sigma,\mathbf{s})-(\nabla\times\mathbf{u},\nabla\times\mathbf{s})+(\mathbf{y},\nabla\times\mathbf{s})+(\mathrm{div}\mathbf{y},q)$. The continuity of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ associated with U follow immediately.

Define $Z := \Big\{(\sigma, \mathbf{u}, \mathbf{y}) \in H_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega) \times \mathbf{H}_0^1(\Omega) : b((\sigma, \mathbf{u}, \mathbf{y}), (\mathbf{s}, q)) = 0, \forall (\mathbf{s}, q) \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times L_0^2(\Omega)\Big\}$. Given $(\sigma, \mathbf{u}, \mathbf{y}) \in Z$, we have $\sigma = 0$ and $\mathbf{y} = \nabla \times \mathbf{u}$. Thus the coercivity of $a(\cdot, \cdot)$ on Z follows.

Given $(\mathbf{s}, q) \in \mathbf{H}_0(\mathrm{curl}, \Omega) \times L_0^2(\Omega)$, decompose $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$ with $\mathbf{s}_1 \in \nabla H_0^1(\Omega)$, and $\mathbf{s}_2 \in (\nabla H_0^1(\Omega))^\perp$. Set $\mathbf{y} \in \mathbf{H}_0^1(\Omega)$, such that $(\mathrm{div}\mathbf{y}, q) = (q, q)$ and $\|\mathbf{y}\|_1 \leqslant C\|q\|_0$, $\nabla\sigma = \mathbf{s}_1$, and $\mathbf{u} \in (\nabla H_0^1(\Omega))^\perp$ such that $(\mathbf{y} - \nabla \times \mathbf{u}, \nabla \times \mathbf{v}) = (\nabla \times \mathbf{s}, \nabla \times \mathbf{v})$ for any $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$. Then $b((\sigma, \mathbf{u}, \mathbf{y}), (\mathbf{s}, q)) = (\mathbf{s}_1, \mathbf{s}_1) + (\nabla \times \mathbf{s}, \nabla \times \mathbf{s}) + (q, q)$, and $\|\sigma\|_1 + \|\mathbf{u}\|_{\mathrm{curl}} + \|\mathbf{y}\|_{1,\Omega} \leqslant C(\|\mathbf{s}\|_{\mathrm{curl},\Omega} + \|q\|_{0,\Omega})$. This leads to the inf-sup condition and completes the proof. □

Similar to Lemma 3.3, we can obtain the equivalence result below.

Lemma 3.9. *The problem* (3.18) *is equivalent to the primal problem* **(B′)**.

Theorem 3.10. *Let Ω be a convex polyhedron and $(\sigma, \mathbf{u}, \mathbf{y}, \mathbf{r}, p)$ be the solution of* (3.18)*. Then $\sigma = 0$, $\nabla \times \mathbf{u} \in \mathbf{H}^2(\Omega)$, $\mathbf{y} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $\mathbf{r} \in \mathbf{H}^1(\Omega) \cap \mathbf{H}_0(\mathrm{curl}, \Omega)$, $\nabla \times \mathbf{r} \in \mathbf{H}^1(\Omega)$ and $p \in L_0^2(\Omega) \cap H^1(\Omega)$. Further, if $\mathrm{div}\mathbf{f} = 0$, then $\mathbf{u} \in \mathbf{H}^2(\Omega)$.*

Proof. By the stability of the system and since $\mathrm{div}\mathbf{r} = 0$, we obtain $\mathbf{r} \in \mathbf{H}^1(\Omega)$. As $(\nabla \times \mathbf{r}, \nabla \times \mathbf{v}) = (\mathbf{u} - \mathbf{f}, \mathbf{v})$ for any $\mathbf{v} \in \mathbf{H}_0(\mathrm{curl}, \Omega)$, we have $\nabla \times \nabla \times \mathbf{r} = \mathbf{u} - \mathbf{f}$, thus $\nabla \times \mathbf{r} \in \mathbf{H}(\mathrm{curl}, \Omega) \cap \mathbf{H}_0(\mathrm{div}, \Omega) \subset \mathbf{H}^1(\Omega)$. Then, standardly, we obtain $\mathbf{y} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, and $p \in H^1(\Omega) \cap L_0^2(\Omega)$. By the second last line of the system, $\sigma = 0$ and $\nabla \times \mathbf{u} = \mathbf{y}$; thus $\nabla \times \mathbf{u} \in \mathbf{H}^2(\Omega)$. Further, if $\mathrm{div}\mathbf{f} = 0$, then $\mathrm{div}\mathbf{u} = 0$; this combined with that $\mathrm{curl}\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ leads to that $\mathbf{u} \in \mathbf{H}^2(\Omega)$. The proof is completed. □

Remark 3.11. Theorem 3.10 constructs regularity of **(B)** in forms (1.2) and (3.18). It also confirms the validity of assumptions in Lemma 3.2 and Theorem 3.12 of [41] by showing that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and $\nabla \times \mathbf{u} \in \mathbf{H}^2(\Omega)$ when $\mathbf{f} \in \mathbf{L}^2(\Omega)$ with $\mathrm{div}\mathbf{f} = 0$. This proof can be repeated onto Theorem 3.5. The difference between the two proofs is that we do not try to decompose (3.18) to subsystems sequentially for Theorem 3.10.

4. Finite element discretizations of the mixed formulations

Given finite element spaces $H_{h0}^1 \subset H_0^1(\Omega)$, $\mathbf{H}_{h0}(\mathrm{curl}) \subset \mathbf{H}_0(\mathrm{curl}, \Omega)$, $\mathbf{H}_{h0}^1 \subset \mathbf{H}_0^1(\Omega)$ and $L_{h0}^2 \subset L_0^2(\Omega)$, we can use them to replace the respective Sobolev spaces in the mixed systems to generate a discretisation scheme. Particularly, the spaces $H_0^1(\Omega)$ and $\mathbf{H}_0(\mathrm{curl}, \Omega)$ may appear more than once in the mixed formulation; this hints us to use different $H_{h0}^{1,a}$ and $H_{h0}^{1,b}$ and different $\mathbf{H}_{h0}^a(\mathrm{curl})$ and $\mathbf{H}_{h0}^b(\mathrm{curl})$ when convenient. In this section, we present some conditions of the well-posed-ness of the discretised system, construct generally its convergence analysis, and give some examples.

4.1. Discretize problem (A″)

Let $H_{h0}^{1,a} \times \mathbf{H}_{h0}^a(\mathrm{curl})$ and $H_{h0}^{1,b} \times \mathbf{H}_{h0}^b(\mathrm{curl})$, identical or not, be two finite element subspaces of $H_0^1(\Omega) \times \mathbf{H}_0(\mathrm{curl}, \Omega)$, and finite element space $\mathbf{H}_{h0}^1 \times L_{h0}^2 \subset \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$. Define $V_h := H_{h0}^{1,a} \times \mathbf{H}_{h0}^a(\mathrm{curl}) \times \mathbf{H}_{h0}^1 \times L_{h0}^2 \times \mathbf{H}_{h0}^b(\mathrm{curl}) \times H_{h0}^{1,b}$, and $V_h' := H_{h0}^{1,b} \times \mathbf{H}_{h0}^b(\mathrm{curl}) \times \mathbf{H}_{h0}^1 \times L_{h0}^2 \times \mathbf{H}_{h0}^a(\mathrm{curl}) \times H_{h0}^{1,a}$. The discretized formulation of **(A′)** is to find $(\sigma_h, \mathbf{u}_h, \mathbf{y}_h, p_h, \mathbf{r}_h, \xi_h) \in V_h$, such that, for any $(\tau_h, \mathbf{v}_h, \mathbf{z}_h, q_h, \mathbf{s}_h, \eta_h) \in V_h'$,

$$\begin{cases} a((\sigma_h, \mathbf{u}_h, \mathbf{y}_h), (\tau_h, \mathbf{v}_h, \mathbf{z}_h)) + b((p_h, \mathbf{r}_h, \xi_h), (\tau_h, \mathbf{v}_h, \mathbf{z}_h)) = (\mathbf{f}, \mathbf{v}_h), \\ b((\sigma_h, \mathbf{u}_h, \mathbf{y}_h), (q_h, \mathbf{s}_h, \eta_h)) = 0, \end{cases} \tag{4.1}$$

where $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ follows the definitions (3.6) and (3.7). The system is symmetric indefinite when $V_h = V_h'$, and unsymmetric otherwise.

For the well-posedness of (4.1), we set up some assumptions below.

A1: The exact relation holds: $\nabla H^{1,\alpha}_{h0} = \{\mathbf{s}_h \in \mathbf{H}^{\alpha}_{h0}(\mathrm{curl}) : \mathrm{curl}\mathbf{s}_h = 0\}$, $\alpha = a, b$.

A2: The discrete Friedrichs inequality holds: $\|\mathbf{s}_h\|_{0,\Omega} \leqslant C\|\nabla \times \mathbf{s}_h\|_{0,\Omega}$ for $\mathbf{s}_h \in (\nabla H^{1,\alpha}_{h0})^{\perp}$. Here $(\nabla H^{1,\alpha}_{h0})^{\perp}$ is the orthogonal completion of $(\nabla H^{1,\alpha}_{h0})$ in $\mathbf{H}^{\alpha}_{h0}(\mathrm{curl})$ in L^2 inner product, $\alpha = a, b$.

A3: The inf-sup condition holds: $\inf_{q_h \in L^2_{h0}\setminus\{0\}} \sup_{\mathbf{z}_h \in \mathbf{H}^1_{h0}\setminus\{\mathbf{0}\}} \frac{(\mathrm{div}\mathbf{z}_h, q_h)}{\|q_h\|_{0,\Omega}\|\mathbf{z}_h\|_{1,\Omega}} \geqslant C.$

Remark 4.1. The assumptions **A1** and **A2** are imposed on the space pair $H^{1,\alpha}_{h0}$ and $\mathbf{H}^{\alpha}_{h0}(\mathrm{curl})$, $\alpha = a, b$, and the assumption **A3** is imposed on the space pair $\mathbf{H}^1_{h0}$ and L^2_{h0}. This allows us to choose the three space pairs independently. Given the assumption **A3** itself, the system (4.1) can be decomposed to three subproblems and solved sequentially.

Lemma 4.2. *Under the assumptions **A1**–**A3**, the problem (4.1) is well-posed.*

Proof. When $H^{1,a}_{h0} = H^{1,b}_{h0}$ and $\mathbf{H}^a_{h0}(\mathrm{curl}) = \mathbf{H}^b_{h0}(\mathrm{curl})$, namely $V_h = V_h'$, the proof is essentially the same as that of Lemma 3.2; otherwise, we can decompose the system to three subproblems and analyse them one by one, and the result can be proved. We only emphasize that $\|\mathbf{u}_h\|_{\mathrm{curl},\Omega} \leqslant C\|\mathbf{y}_h\|_{0,\Omega}$. Actually, the problem (4.1) can be decomposed to the three subsystems and solved sequentially:

(1) given $\mathbf{f}$, solve for $\mathbf{r}_h \in \mathbf{H}^b_{h0}(\mathrm{curl})$ and $\xi_h \in H^{1,b}_{h0}$ that

$$\begin{cases}(\nabla \times \mathbf{r}_h, \nabla \times \mathbf{v}_h) - (\nabla \xi_h, \mathbf{v}_h) = -(\mathbf{f}, \mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{H}^b_{h0}(\mathrm{curl}) \\ (\nabla \tau_h, \mathbf{r}_h) = 0 & \forall\, \tau_h \in H^{1,b}_{h0};\end{cases} \tag{4.2}$$

(2) with $\mathbf{r}$ obtained, solve for $\varphi \in \mathbf{H}^1_{h0}$ and $p \in L^2_{h0}$ that

$$\begin{cases}(\nabla \mathbf{y}_h, \nabla \mathbf{z}_h) + (p_h, \mathrm{div}\mathbf{z}_h) = -(\nabla \times \mathbf{r}_h, \mathbf{z}_h) & \forall\, \mathbf{z}_h \in \mathbf{H}^1_{h0} \\ (\mathrm{div}\mathbf{y}_h, q_h) = 0 & \forall\, q_h \in L^2_{h0};\end{cases} \tag{4.3}$$

(3) with $\mathbf{y}_h$ obtained, solve for $\mathbf{u}_h \in \mathbf{H}^a_{h0}(\mathrm{curl})$ and $\sigma_h \in H^{1,a}_{h0}$ that

$$\begin{cases}(\nabla \times \mathbf{u}_h, \nabla \times \mathbf{s}_h) - (\nabla \sigma_h, \mathbf{s}_h) = (\mathbf{y}_h, \nabla \times \mathbf{s}_h) & \forall\, \mathbf{s}_h \in \mathbf{H}^a_{h0}(\mathrm{curl}) \\ (\nabla y_h, \mathbf{u}_h) = 0 & \forall\, y_h \in H^{1,a}_{h0}.\end{cases} \tag{4.4}$$

All the three subsystems are well-posed. Indeed, (4.2) and (4.4) are well-posed by Assumptions **A1** and **A2**, and (4.3) is well-posed by Assumption **A3**. The stability estimate follows immediately. ☐

The convergence of the scheme is surveyed in the lemma below.

Lemma 4.3. *Let $(\sigma, \mathbf{u}, \varphi, p, \mathbf{r}, \xi)$ and $(\sigma_h, \mathbf{u}_h, \mathbf{y}_h, p_h, \mathbf{r}_h, \xi_h)$ be the solutions of (3.4) and (4.1), respectively. Under the assumptions* **A1**–**A3**, *it holds that*

(1) $\|\mathbf{r} - \mathbf{r}_h\|_{\mathrm{curl},\Omega} \leqslant C \inf_{\mathbf{s}_h \in \mathbf{H}^b_{h0}(\mathrm{curl})} \|\mathbf{r} - \mathbf{s}_h\|_{\mathrm{curl},\Omega}$;

(2) $\|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega}$
$\leqslant C\big[\inf_{\mathbf{z}_h \in \mathbf{H}^1_{h0}, q_h \in L^2_{h0}} (\|\mathbf{y} - \mathbf{z}_h\|_{1,\Omega} + \|p - q_h\|_{0,\Omega}) + \|\mathbf{r} - \mathbf{r}_h\|_{0,\Omega}\big]$;

(3) $\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{curl},\Omega} \leqslant C\big[\inf_{\mathbf{v}_h \in \mathbf{H}^a_{h0}(\mathrm{curl})} \|\mathbf{u} - \mathbf{v}_h\|_{\mathrm{curl},\Omega} + \|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}\big]$;

(4) $\sigma_h = 0 = \sigma$, $\xi_h = 0 = \xi$.

Proof. Similar to Lemma 3.4, we can decompose (4.1) to three subproblems and solve them sequentially. The lemma follows directly from the Ceá lemma and the second Strang lemma. We only have to note that, as div$\mathbf{f}=0$, it follows that $\xi=0$ and $\sigma=0$, and $\inf_{\eta_h\in H^1_{h0}}\|\xi-\eta_h\|_{1,\Omega}=\inf_{\tau_h\in H^1_{h0}}\|\sigma-\tau_h\|_{1,\Omega}=0$. ☐

Remark 4.4. By Lemma 4.3, the error $\|\sigma-\sigma_h\|_{1,\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl},\Omega}$ could be comparable to $\|\mathbf{y}-\mathbf{y}_h\|_{0,\Omega}$. This implies that when $H^{1,a}_{h0}$ and $\mathbf{H}^a_{h0}(\mathrm{curl})$ are chosen appropriately, a higher order convergence rate of $\mathbf{u}$ may be expected. Indeed, for general $\mathbf{f}$ (not assuming $\mathbf{f}\in\mathbf{H}^1(\Omega)$), that we choose $\mathbf{H}^a_{h0}(\mathrm{curl})$ bigger than $\mathbf{H}^b_{h0}(\mathrm{curl})$ (meanwhile $H^{1,a}_{h0}$ bigger than $H^{1,b}_{h0}$) coincides with the higher regularity of $\mathbf{u}$ than that of $\mathbf{r}$ on convex polyhedrons. However, if $H^{1,a}_{h0}\neq H^{1,b}_{h0}$ and $V_h\neq V_h'$, the problem is no longer symmetric, which may bring extra difficulties. This is why we still discuss the choice $V_h=V_h'$ in many applications like, *e.g.*, eigenvalue computation.

4.2. Discretize problem (B″)

Let $H^1_{h0}\subset H^1_0(\Omega)$, $\mathbf{H}_{h0}(\mathrm{curl})\subset\mathbf{H}_0(\mathrm{curl},\Omega)$, $\mathbf{H}^1_{h0}\subset\mathbf{H}^1_0(\Omega)$ and $L^2_{h0}\subset L^2_0(\Omega)$ be respective finite element spaces. Define

$$U_h:=H^1_{h0}\times\mathbf{H}_{h0}(\mathrm{curl})\times\mathbf{H}^1_{h0}\times\mathbf{H}_{h0}(\mathrm{curl})\times L^2_{h0}.$$

The discretized mixed formulation is to find $(\sigma_h,\mathbf{u}_h,\mathbf{y}_h,\mathbf{r}_h,p_h)\in U_h$, such that, for any $(\tau_h,\mathbf{v}_h,\mathbf{z}_h,\mathbf{s}_h,q_h)\in U_h$,

$$\begin{cases}(\mathbf{r}_h,\nabla\tau_h)=0\\(\mathbf{u}_h,\mathbf{v}_h)-(\nabla\times\mathbf{r}_h,\nabla\times\mathbf{v}_h)=(\mathbf{f},\mathbf{v}_h)\\(\nabla\mathbf{y}_h,\nabla\mathbf{z}_h)+(\nabla\times\mathbf{r}_h,\mathbf{z}_h)+(p_h,\mathrm{div}\mathbf{z}_h)=0\\(\nabla\sigma_h,\mathbf{s}_h)-(\nabla\times\mathbf{u}_h,\nabla\times\mathbf{s}_h)(\mathbf{y}_h,\nabla\times\mathbf{s}_h)=0\\(\mathrm{div}\mathbf{y}_h,q_h)=0.\end{cases}\tag{4.5}$$

The two lemmas below are for the stability and the convergence of the scheme.

Lemma 4.5. *Under the assumptions **A1**∼**A3**, the problem* (4.5) *is well-posed on* U_h.

Lemma 4.6. *Let* $(m,\mathbf{u},\mathbf{y},\mathbf{r},p)$ *and* $(\sigma_h,\mathbf{u}_h,\mathbf{y}_h,\mathbf{r}_h,p_h)$ *be the solutions of* (3.18) *and* (4.5), *respectively.*

(1) $\sigma_h=0$;
(2) $\|\mathbf{y}-\mathbf{y}_h\|_{1,\Omega}+\|p-p_h\|_{0,\Omega}\leqslant C\inf_{(\tau_h,\mathbf{v}_h,\mathbf{z}_h,\mathbf{s}_h,q_h)\in U_h}[\|\mathbf{u}-\mathbf{v}_h\|_{\mathrm{curl},\Omega}+\|\mathbf{y}-\mathbf{z}_h\|_{1,\Omega}+\|\mathbf{r}-\mathbf{s}_h\|_{\mathrm{curl},\Omega}+\|p-q_h\|_{0,\Omega}]$;
(3) $\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl},\Omega}+\|\mathbf{r}-\mathbf{r}_h\|_{\mathrm{curl},\Omega}$
$\leqslant C\Big[\inf_{\mathbf{v}_h,\mathbf{s}_h\in\mathbf{H}_{h0}(\mathrm{curl})}(\|\mathbf{u}-\mathbf{v}_h\|_{\mathrm{curl},\Omega}+\|\mathbf{r}-\mathbf{s}_h\|_{\mathrm{curl},\Omega})+\|\mathbf{y}-\mathbf{y}_h\|_{0,\Omega}\Big]$.

Proof. The first item follows from **A1**, and the second follows from the Cea lemma. Note that $(\sigma,\mathbf{u},\mathbf{r})$ solves the problem

$$\begin{cases}(\mathbf{r},\nabla\tau)=0 & \forall\,\tau\in H^1_0(\Omega)\\(\mathbf{u},\mathbf{v})-(\nabla\times\mathbf{r},\nabla\times\mathbf{v})=(\mathbf{f},\mathbf{v}) & \forall\,\mathbf{v}\in\mathbf{H}_0(\mathrm{curl},\Omega)\\(\nabla\sigma,\mathbf{s})-(\nabla\times\mathbf{u},\nabla\times\mathbf{s})=-(\mathbf{y},\nabla\times\mathbf{s}) & \forall\,\mathbf{s}\in\mathbf{H}_0(\mathrm{curl},\Omega),\end{cases}\tag{4.6}$$

and $(\sigma_h, \mathbf{u}_h, \mathbf{r}_h)$ solves the finite element problem:

$$\begin{cases}(\mathbf{r}_h, \nabla\tau_h) = 0 & \forall\, \tau_h \in H^1_{h0}\\ (\mathbf{u}_h, \mathbf{v}_h) - (\nabla\times\mathbf{r}_h, \nabla\times\mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h) & \forall\, \mathbf{v}_h \in \mathbf{H}_{h0}(\mathrm{curl})\\ (\nabla\sigma_h, \mathbf{s}_h) - (\nabla\times\mathbf{u}_h, \nabla\times\mathbf{s}_h) = -(\mathbf{y}_h, \nabla\times\mathbf{s}_h) & \forall\, \mathbf{s}_h \in \mathbf{H}_{h0}(\mathrm{curl}).\end{cases} \tag{4.7}$$

The third item follows from the Ceá lemma and the second Strang lemma. □

Lemma 4.6 reveals that the error $\|\mathbf{u}-\tilde{\mathbf{u}}_h\|_{\mathrm{curl},\Omega} + \|\mathbf{r}-\tilde{\mathbf{r}}_h\|_{\mathrm{curl},\Omega}$ can be comparable with $\|\mathbf{y}-\mathbf{y}_h\|_0$. This hints us to use some bigger $\mathbf{H}_{h0}(\mathrm{curl})$ to expect higher accuracy of $\mathbf{u}$ and $\mathbf{r}$ than that of $\mathbf{y}$ and p.

4.3. Examples of finite element quartos

Let Ω be subdivided to tetrahedrons which form a grid $\mathcal{T}_h$. We impose the shape regularity assumption on $\mathcal{T}_h$. On the grid, finite element spaces can be constructed. We refer to [11, 13, 20] for the context of finite element methods. We only recall these familiar finite element spaces, $k \geqslant 1$:

- continuous Lagrangian element space of kth degree: subspace of $H^1(\Omega)$, consist of piecewise kth-degree polynomials; denoted by W_h^k; $W_{h0}^k = W_h^k \cap H_0^1(\Omega)$;
- Nédélec edge element of first family of kth degree [22]: subspace of $\mathbf{H}(\mathrm{curl}, \Omega)$, consist of piecewise polynomials of the form $\mathbf{u}+\mathbf{v}$, with $\mathbf{u} \in (P_{k-1})^3$ and $\mathbf{v} \in \mathbf{x}\times(\hat{P}_{k-1})^3$, where P_{k-1} is the space of $(k-1)$th degree polynomials, and $\hat{P}^3_{k-1}$ is the space of homogeneous $(k-1)$th degree polynomials; denoted by $\mathbf{N}_h^k$; $\mathbf{N}_{h0}^k = \mathbf{N}_h^k \cap \mathbf{H}_0(\mathrm{curl}, \Omega)$.

4.3.1. Examples for problem $(\mathbf{A}'')$

For problem $(\mathbf{A}'')$, we choose

$$\begin{aligned} H^{1,a}_{h0} := W^2_{h0}, \quad \mathbf{H}^a_{h0}(\mathrm{curl}) := \mathbf{N}^2_{h0}, \quad \mathbf{H}^1_{h0} := (W^2_{h0})^2,\\ H^{1,b}_{h0} := W^1_{h0}, \quad \mathbf{H}^b_{h0}(\mathrm{curl}) := \mathbf{N}^1_{h0}, \quad L^2_{h0} := W^1_h \cap L^2_0(\Omega). \end{aligned} \tag{4.8}$$

The assumptions **A1**–**A3** can be verified. Particularly, **A1** and **A3** can be found in [3], and **A2** can be found in [1, 14, 18]. Its convergence then follows from Lemma 4.3. We here present a specific estimate on convex polyhedrons.

Lemma 4.7. *Let Ω be a convex polyhedron and* $\mathrm{div}\mathbf{f} = 0$. *Let $(\sigma, \mathbf{u}, \mathbf{y}, p, \mathbf{r}, \xi)$ and $(\sigma_h, \mathbf{u}_h, \mathbf{y}_h, p_h, \mathbf{r}_h, \xi_h)$ be the solutions of* (3.4) *and* (4.1), *respectively. Then*

(1) $\xi_h = 0 = \xi$ *and* $\sigma_h = 0 = \sigma$;
(2) $\|\mathbf{r}-\mathbf{r}_h\|_{\mathrm{curl},\Omega} \leqslant Ch(\|\mathbf{r}\|_{1,\Omega} + \|\nabla\times\mathbf{r}\|_{1,\Omega}) \leqslant Ch\|\mathbf{f}\|_{0,\Omega}$;
(3) $\|\mathbf{y}-\mathbf{y}_h\|_{1,\Omega} + \|p-p_h\|_{0,\Omega}$
$\leqslant Ch(\|\mathbf{y}\|_{2,\Omega} + \|p\|_{1,\Omega} + \|\mathbf{r}\|_{1,\Omega} + \|\nabla\times\mathbf{r}\|_{1,\Omega}) \leqslant Ch\|\mathbf{f}\|_{0,\Omega}$;
(4) $\|\mathbf{y}-\mathbf{y}_h\|_0 \leqslant Ch^2\|\mathbf{f}\|_{0,\Omega}$;
(5) $\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl},\Omega} \leqslant C(h^2(\|\mathbf{u}\|_{2,\Omega} + \|\nabla\times\mathbf{u}\|_{2,\Omega}) + \|\mathbf{y}-\mathbf{y}_h\|_{0,\Omega}) \leqslant Ch^2\|\mathbf{f}\|_{0,\Omega}$.

Proof. We only prove the estimate of $\|\mathbf{y}-\mathbf{y}_h\|_{0,\Omega}$ by dual argument, and the remaining follows from Lemma 4.3 directly. Define $\hat{V}_h := H^{1,b}_{h0} \times \mathbf{H}^b_{h0}(\mathrm{curl}) \times \mathbf{H}^1_{h0} \times L^2_{h0} \times \mathbf{H}^b_{h0}(\mathrm{curl}) \times H^{1,b}_{h0}$, and let $(\hat{\sigma}_h, \hat{\mathbf{u}}_h, \hat{\mathbf{y}}_h, \hat{p}_h, \hat{\mathbf{r}}_h, \hat{\xi}_h) \in \hat{V}_h$ be such that

$$\begin{cases} a((\hat{\sigma}_h, \hat{\mathbf{u}}_h, \hat{\mathbf{y}}_h), (\hat{\tau}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h)) + b((\hat{p}_h, \hat{\mathbf{r}}_h, \hat{\xi}_h), (\hat{\tau}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h)) = (\mathbf{f}, \hat{\mathbf{v}}_h),\\ b((\hat{\sigma}_h, \hat{\mathbf{u}}_h, \hat{\mathbf{y}}_h), (\hat{q}_h, \hat{\mathbf{s}}_h, \hat{\eta}_h)) = 0, \end{cases} \tag{4.9}$$

for any $(\hat{\tau}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h, \hat{q}_h, \hat{\mathbf{s}}_h, \hat{\eta}_h) \in \hat{V}_h$, where $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ follows the definitions (3.6) and (3.7).Then $\hat{\sigma}_h = \hat{\xi}_h = 0$, and $(\hat{\mathbf{y}}_h, \hat{p}_h, \hat{\mathbf{r}}_h) = (\mathbf{y}_h, p_h, \mathbf{r}_h)$. Let $(\tilde{\sigma}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}, \tilde{p}, \tilde{\mathbf{r}}, \tilde{\xi}) \in V$ be such that, for any $(\tau, \mathbf{v}, \psi, q, \mathbf{s}, \eta) \in V$,

$$\begin{cases} a((\tilde{\sigma}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}), (\eta, \mathbf{v}, \psi)) + b((\tilde{p}, \tilde{\mathbf{r}}, \tilde{\xi}), (\tau, \mathbf{v}, \psi)) = (\mathbf{y} - \mathbf{y}_h, \psi), \\ b((\tilde{\sigma}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}), (q, \mathbf{s}, \eta)) = 0. \end{cases} \tag{4.10}$$

Then $\tilde{\mathbf{r}} = \mathbf{0}$, $\tilde{\xi} = 0$, $\tilde{\sigma} = 0$, and, by the same virtue as that of Theorem 3.5,

$$\|\tilde{\mathbf{y}}\|_{2,\Omega} + \|\tilde{p}\|_{1,\Omega} + \|\tilde{\mathbf{u}}\|_{2,\Omega} + \|\mathrm{curl}\tilde{\mathbf{u}}\|_{2,\Omega} \leqslant C\|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}.$$

Thus substituting $(\tau, \mathbf{v}, \psi, q, \mathbf{s}, \eta) = (\sigma - \hat{\sigma}_h, \mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{y} - \hat{\mathbf{y}}_h, p - \hat{p}_h, \mathbf{r} - \hat{\mathbf{r}}_h, \xi - \hat{\xi}_h)$ into (4.10), we have

$$\begin{aligned} \|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}^2 = {} & a((\tilde{\sigma}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}), (\sigma - \hat{\sigma}_h, \mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{y} - \hat{\mathbf{y}}_h)) + b((\tilde{p}, \tilde{\mathbf{r}}, \tilde{\xi}), (\sigma - \hat{\sigma}_h, \mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{y} - \hat{\mathbf{y}}_h)) \\ & + b((\tilde{\sigma}, \tilde{\mathbf{u}}, \tilde{\mathbf{y}}), (p - \hat{p}_h, \mathbf{r} - \hat{\mathbf{r}}_h, \xi - \hat{\xi}_h)). \end{aligned}$$

Further, for any $(\hat{\tau}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h, \hat{q}_h, \hat{\mathbf{s}}_h, \hat{\eta}_h) \in \hat{V}_h$, the orthogonality holds that

$$\begin{aligned} \|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}^2 = {} & a((\tilde{\sigma} - \hat{\tau}_h, \tilde{\mathbf{u}} - \hat{\mathbf{v}}_h, \tilde{\mathbf{y}} - \hat{\mathbf{z}}_h), (\sigma - \hat{\sigma}_h, \mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{y} - \hat{\mathbf{y}}_h)) + b((\tilde{p} - \hat{q}_h, \tilde{\mathbf{r}} - \hat{\mathbf{s}}_h, \tilde{\xi} - \hat{\eta}_h), \\ & \times (\sigma - \hat{\sigma}_h, \mathbf{u} - \hat{\mathbf{u}}_h, \mathbf{y} - \hat{\mathbf{y}}_h)) + b((\tilde{\sigma} - \hat{\tau}_h, \tilde{\mathbf{u}} - \hat{\mathbf{s}}_h, \tilde{\mathbf{y}}), (p - \hat{p}_h, \mathbf{r} - \hat{\mathbf{r}}_h, \xi - \hat{\xi}_h)). \end{aligned}$$

Thus

$$\begin{aligned} \|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}^2 \leqslant {} & C(\|\sigma - \hat{\sigma}_h\|_{1,\Omega} + \|\mathbf{u} - \hat{\mathbf{u}}_h\|_{\mathrm{curl},\Omega} + \|\mathbf{y} - \hat{\mathbf{y}}_h\|_{1,\Omega} + \|p - \hat{p}_h\|_{0,\Omega} + \|\mathbf{r} - \hat{\mathbf{r}}_h\|_{\mathrm{curl},\Omega}) \\ & \times \inf_{(\hat{\tau}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{z}}_h, \hat{q}_h, \hat{\mathbf{s}}_h, \hat{\eta}_h) \in \hat{V}_h} (\|\tilde{\sigma} - \hat{\tau}_h\|_{1,\Omega} + \|\tilde{\mathbf{u}} - \hat{\mathbf{v}}_h\|_{\mathrm{curl},\Omega} + \|\tilde{\mathbf{y}} - \hat{\mathbf{z}}_h\|_{1,\Omega} + \|\tilde{p} - \hat{q}_h\|_{0,\Omega} + \|\tilde{\mathbf{r}} - \hat{\mathbf{s}}_h\|_{\mathrm{curl},\Omega}). \end{aligned}$$

Then by finite element estimate,

$$\|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}^2 \leqslant Ch\|\mathbf{f}\|_{0,\Omega} \cdot h\|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega}, \tag{4.11}$$

which leads to that $\|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega} \leqslant Ch^2\|\mathbf{f}\|_{0,\Omega}$. This completes the proof. □

Remark 4.8. The convergence analysis in terms of the regularity of the solution follows directly from Lemma 4.3. We here construct a convergence analysis with respect to $\|\mathbf{f}\|_{0,\Omega}$, which can exploit the regularity of solution functions to a full extent with economical complexity. When further $\mathbf{f} \in \mathbf{H}^1(\Omega)$, we can set $H_{h0}^{1,b} := W_{h0}^2$ and $\mathbf{H}_{h0}^b(\mathrm{curl}) := \mathbf{N}_{h0}^2$, and obtain higher accuracy of $\|\mathbf{r} - \mathbf{r}_h\|_{\mathrm{curl},\Omega}$.

4.3.2. Examples for problem $(\mathbf{B}'')$

For problem $(\mathbf{B}'')$, we choose

$$H_{h0}^1 := W_{h0}^2, \quad \mathbf{H}_{h0}(\mathrm{curl}) := \mathbf{N}_{h0}^2, \quad \mathbf{H}_{h0}^1 := (W_{h0}^2)^2, \quad L_{h0}^2 := W_h^1 \cap L_0^2(\Omega). \tag{4.12}$$

The assumptions **A1**–**A3** can be verified. The lemma below is for the convergence.

Lemma 4.9. *Let Ω be a convex polyhedron and $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Let $(m, \mathbf{u}, \mathbf{y}, \mathbf{r}, p)$ and $(\sigma_h, \mathbf{u}_h, \mathbf{y}_h, \mathbf{r}_h, p_h)$ be the solutions of (3.18) and (4.5), respectively. Then*

(1) $\sigma_h = 0 = \sigma$;
(2) $\|\mathbf{y} - \mathbf{y}_h\|_{1,\Omega} + \|p - p_h\|_{0,\Omega} \leqslant Ch\|\mathbf{f}\|_{0,\Omega}$, $\|\mathbf{y} - \mathbf{y}_h\|_0 \leqslant Ch^2\|\mathbf{f}\|_{0,\Omega}$;

(3) $\|\mathbf{r}-\mathbf{r}_h\|_{\mathrm{curl},\Omega}+\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl},\Omega}\leqslant Ch\|\mathbf{f}\|_{0,\Omega}$;
(4) *if* $\mathbf{f}\in\mathbf{H}^1(\Omega)$, *then* $\|\mathbf{u}-\mathbf{u}_h\|_{\mathrm{curl},\Omega}+\|\mathbf{r}-\mathbf{r}_h\|_{\mathrm{curl},\Omega}\leqslant Ch^2\|\mathbf{f}\|_{1,\Omega}$.

Proof. Similar to Lemma 4.7, we only have to construct the estimate $\|\mathbf{y}-\mathbf{y}_h\|_{0,\Omega}$, which can be done by repeating the dual argument in the proof of Lemma 4.7, and we omit the details here. The proof is completed by following from Lemma 4.6 directly. □

5. Concluding remarks

In this paper, we study the quad curl equations and develop for them friendly mixed schemes. We construct equivalent mixed formulation of the two variant model problems, which are stable on standard L^2, $\mathbf{H}(\mathrm{curl})$ and H^1 spaces. Existing finite element quartos that satisfy some quite mild assumptions can then lead to stable discretisation schemes, and the convergence then follow in a routine way. Regularities are established for the mixed formulations, and then the primal ones. Some finite element examples which are optimal with respect to the regularity are given. The newly developed schemes are easy to build, to analyze and to design optimal solvers for. The schemes can be implemented with various finite element packages. The discussion also shows some possibility to analyse the model problems under the framework of finite element exterior calculus(*cf.*, *e.g.*, [1]). Further discussions on related topics such as adaptive algorithms can be expected.

In constructing of the equivalent formulations, we first configurate the boundary conditions that $\nabla\times\mathbf{u}$ has to satisfy. Particularly, $(\nabla\times\mathbf{u})\cdot\mathbf{n}=0$ is a condition satisfied by the variable $\nabla\times\mathbf{u}$ essentially. This forces us to bring $\mathbf{H}_0^1(\Omega)$ which has bigger capacity for boundary conditions into discussion. This partially illustrates the importance of the boundary condition for the variational problem. Discussions and comparisons can be extended to problems with other boundary conditions.

As only low-order Sobolev spaces are involved in the mixed forms, it is possible to construct finite element schemes that are nested algebraically and topologically on nested grids; this will provide convenience in designing multilevel methods, and can be utilised in practice (*cf.*, *e.g.*, [8, 9, 19, 40]). Beyond the two familiar variants boundary value problems considered in this paper, in many contexts, second order operators also appear in the boundary value problem, and parameters of various scales may appear in front of operators of different orders; see the model problem in [41]. Designing parameter-robust discretisation is interesting and practically important, and will be discussed in future. Also, we focus ourselves on source problems in the present paper. The utilisation of the mixed scheme presented in this paper onto eigenvalue computation and analysis, especially in designing multilevel algorithm (*cf.* [40]), will be discussed in future.

References

[1] D.N Arnold, R.S. Falk and R. Winther, Finite element exterior calculus, homological techniques, and applications. *Acta Numer.* **15** (2006) 1–155.
[2] D. Boffi, Fortin operator and discrete compactness for edge elements. *Numer. Math.* **87** (2000) 229–246.
[3] D. Boffi, F. Brezzi and M. Fortin. Mixed Finite Element Methods and Applications, Vol. 44. Springer (2013).
[4] S.C. Brenner, J. Sun and L.-Y. Sung, Hodge decomposition methods for a quad-curl problem on planar domains. *J. Sci. Comput.* **73** (2017) 495–513.
[5] F. Cakoni and H. Haddar, A variational approach for the solution of the electromagnetic interior transmission problem for anisotropic media. *Inverse Probl. Imaging* **1** (2007) 443.
[6] F. Cakoni, D. Colton, P. Monk and J. Sun, The inverse electromagnetic scattering problem for anisotropic media. *Inverse Probl.* **26** (2010) 074004.
[7] L. Chacón, A.N. Simakov and A. Zocco, Steady-state properties of driven magnetic reconnection in 2D electron magnetohydrodynamics. *Phys. Rev. Lett.* **99** (2007) 235001.
[8] L. Chen, Multigrid methods for saddle point systems using constrained smoothers. *Comput. Math. Appl.* **70** (2015) 2854–2866.
[9] L. Chen, X. Hu, M. Wang and J. Xu, A multigrid solver based on distributive smoother and residual overweighting for Oseen problems. *Numer. Math. Theory Methods Appl.* **8** (2015) 237–252.
[10] L. Chen, Y. Wu, L. Zhong and J. Zhou, Multigrid Preconditioners for Mixed Finite Element Methods of Vector Laplacian. Preprint arXiv:1601.04095 (2016).
[11] P.G. Ciarlet., The Finite Element Method for Elliptic Problems. (1978).
[12] P.G. Ciarlet and P.-A. Raviart, A mixed finite element method for the biharmonic equation, in *Proc. of Symposium on Mathematical Aspects of Finite Elements in PDE* (1974) 125–145.

[13] V. Girault and P.-A. Raviart, Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms, Vol. 5. Springer (1986).
[14] R. Hiptmair, Finite elements in computational electromagnetism. *Acta Numer.* **11** (2002) 237–339.
[15] R. Hiptmair and J. Xu, Nodal auxiliary space preconditioning in $\mathbf{H}$(curl) and $\mathbf{H}$(div) spaces. *SIAM J. Numer. Anal.* **45** (2007) 2483–2509.
[16] Q. Hong, J. Hu, S. Shu and J. Xu, A discontinuous Galerkin method for the fourth-order curl problem. *J. Comput. Math.* **30** (2012) 565–578.
[17] F. Kikuchi, Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism. *Comput. Methods Appl. Mech. Eng.* **64** (1987) 509–521.
[18] F. Kikuchi, Mixed formulations for finite element analysis of magnetostatic and electrostatic problems. *Jpn. J. Appl. Math.* **6** (1989) 209–221.
[19] Z. Li and S. Zhang, A stable mixed element method for biharmonic equation with first-order function spaces. *Comput. Methods Appl. Math.* **17** (2017) 601–616.
[20] P. Monk, Finite Element Methods for Maxwell's Equations. Oxford University Press (2003).
[21] P. Monk and J. Sun, Finite element methods for Maxwell's transmission eigenvalues. *SIAM J. Sci. Comput.* **34** (2012) B247–B264.
[22] J.-C. Nédélec, Mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **35** (1980) 315–341.
[23] M. Neilan, Discrete and conforming smooth de Rham complexes in three dimensions. *Math. Comput.* **84** (2015) 2059–2081.
[24] S. Nicaise, Regularity of the solutions of elliptic systems in polyhedral domains. *Bull. Belg. Math. Soc. Simon Stevin* **4** (1997) 411–430.
[25] S. Nicaise, Singularities of the Quad Curl Problem. HALpreprint (2016).
[26] S.K. Park and X.-L. Gao. Variational formulation of a modified couple stress theory and its application to a simple shear problem. *Z. Angew. Math. Phys.* **59** (2008) 904–917.
[27] T. Rusten and R. Winther, A preconditioned iterative method for saddlepoint problems. *SIAM J. Matrix Anal. Appl.* **13** (1992) 887–904.
[28] J. Sun, A mixed fem for the quad-curl eigenvalue problem. *Numer. Math.* **132** (2016) 185–200.
[29] J. Sun and L. Xu, Computation of Maxwell's transmission eigenvalues and its applications in inverse medium problems. *Inverse Probl.* **29** (2013) 104013.
[30] X.-C. Tai and R. Winther, A discrete de Rham complex with enhanced smoothness. *Calcolo* **43** (2006) 287–306.
[31] M. Wang and J. Xu, Nonconforming tetrahedral finite elements for fourth order elliptic equations. *Math. Comput.* **76** (2007) 1–18.
[32] M. Wang, Z.-C. Shi and J. Xu, A new class of Zienkiewicz-type non-conforming element in any dimensions. *Numer. Math.* **106** (2007) 335–347.
[33] M. Wang, Z.-C. Shi and J. Xu, Some n-rectangle nonconforming elements for fourth order elliptic equations. *J. Comput. Math.* **25** (2007) 408–420.
[34] J. Xu, Iterative methods by space decomposition and subspace correction. *SIAM Rev.* **34** (1992) 581–613.
[35] J. Xu, Fast poisson-based solvers for linear and nonlinear PDEs, in *Proc. of the International Congress of Mathematics* (2010) Vol. 4, 2886–2912.
[36] F. Yang, A.C.M. Chong, D.C.C. Lam and P. Tong, Couple stress based strain gradient theory for elasticity. *Int. J. Solids Struct.* **39** (2002) 2731–2743.
[37] A. Ženíšek, Polynomial approximation on tetrahedrons in the finite element method. *J. Approx. Theory* **7** (1973) 334–351.
[38] S. Zhang, A family of 3D continuously differentiable finite elements on tetrahedral grids. *Appl. Numer. Math.* **59** (2009) 219–233.
[39] S. Zhang, Regular decomposition and a framework of order reduced methods for fourth order problems. *Numer. Math.* **138** (2018) 241–271.
[40] S. Zhang, Y. Xi and X. Ji, A multi-level mixed element method for the eigenvalue problem of biharmonic equation. *J. Sci. Comput.* (2017).
[41] B. Zheng, Q. Hu and J. Xu, A nonconforming finite element method for fourth order curl equations in $\mathbb{R}^3$. *Math. Comput.* **80** (2011) 1871–1886.