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CONVERGENCE OF A VECTOR PENALTY PROJECTION SCHEME FOR THE NAVIER STOKES EQUATIONS WITH MOVING BODY

Vincent Bruneau[1], Adrien Doradoux[2,*] and Pierre Fabrie[2]

Abstract. In this paper, we analyse a *Vector Penalty Projection Scheme* (see [1]) to treat the displacement of a moving body in incompressible viscous flows in the case where the interaction of the fluid on the body can be neglected. The presence of the obstacle inside the computational domain is treated with a penalization method introducing a parameter η to enforce the velocity on the solid boundary. The incompressibility constraint is approached using a Vector Projection method which introduces a relaxation parameter ε. We show the stability of the scheme and that the pressure and velocity converge towards a limit when the relaxation parameter ε and the time step δt tend to zero with a proportionality constraint $\varepsilon = \lambda\delta t$. Finally, when η goes to 0, we show that the problem admits a weak limit which is a weak solution of the Navier-Stokes equations with no-slip condition on the solid boundary.

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1. Introduction

Simulation of complex flows such that Fluid Structure Interaction is a major challenge. Indeed, as the body moves, the fluid domain is not fixed but time dependent.

The first possibility to deal with this specifity is to move mesh nodes in function of the time, this is used in the Arbitrary Lagrangian Eulerian (ALE) [12, 21] method. It involves remeshing at each time step which can be very time consuming.

Immersed boundary methods are another way to apprehend the problem. The idea of this kind of methods, introduced by Peskin [22] to treat flow in a beating heart, is to consider a fixed grid which contains fluid and solid domains. Equations or discretization operators are then modified to take into account the presence of the solid in the computational domain. Many of these methods are presented in [19].

Among them, the penalization method adds a term in the Navier-Stokes equations to enforce the velocity in the solid region. For instance, in [7], the authors studied the coupling between penalized Navier-Stokes equations and solid dynamics to determine the solid velocity. In [6], fish like swimming is simulated using penalization and a level set method to localize the structure.

To deal with the incompressibility constraint, two families of methods have been developed. The first one aims to solve momentum and mass equations simultaneously. The resolution of this optimization problem is

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[1] Université de Bordeaux, IMB, CNRS UMR5251, 351 cours de la libération, 33405 Talence, France.

[2] Bordeaux INP, Institut de Mathématiques de Bordeaux, CNRS UMR5251, ENSEIRB-MATMECA, Talence, France.

*Corresponding author: adoradoux@math.u-bordeaux.fr

the Augmented Lagrangian strategy [11]. The second one is composed of scalar projection methods which were introduced by Chorin [9]. These methods reduce the saddle point problem to two distinct elliptic problems on the velocity and the pressure. We focus here on the Vector Penalty Projection scheme $(VPP)_\varepsilon$ introduced by Angot *et al.* [1] which belongs to the second family of methods described above and avoids many drawbacks of other projection methods. In particular, the pressure scalar does not need to be computed, which does not impose the resolution of a Poisson type equation that introduce boundary conditions on the pressure. The divergence of the velocity is controlled by an intrinsic parameter ε. In practice, this parameter is chosen as small as possible in order to approximate a divergence free condition. In [3] the authors obtained a second order convergence rate for pressure and velocity in space and time for a second order backward temporal scheme. The convergence towards the Navier-Stokes equations when the penalty parameter (on the divergence) tends to 0 has been studied in [4]. In this last article, the authors considered a continuous (in time) problem in div/curl formulation. In these two previous works the domain is fixed and no *space penalization* is included.

Here, the coupling between Vector Penalty Projection scheme and penalization method is analysed in the case where the solid is animated by its own velocity. We give new convergence results for $(VPP)_\varepsilon$ with a penalization term when the time step δt, the VPP parameter ε and the penalization parameter η tend to 0. From a stability result we first prove the weak convergence of the scheme towards the continuous incompressible Navier-Stokes problem with a penalization term when δt and ε go to 0. In the last section, we treat the convergence of this last continuous problem when the penalty parameter η goes to 0. At the limit process, we recover the Navier-Stokes equations on the time-dependent fluid domain with a no-slip condition on the solid boundary. It gives a new proof of an existence result in time dependent domains (see for instance [16,20] for analog results using Galerkin methods).

1.1. Notations

Let $T > 0$ and Ω be a simply connected bounded domain of $I\!R^d$ ($d = 2$ or 3) with a smooth boundary $\partial\Omega$. We use in the paper the usual functional setting for the unsteady Navier-Stokes Equations.

- For $p > 0$, $\mathbf{L}^p = L^p(\Omega)$, the classical Lebesgue space.
- For $p > 0$, $\mathbf{L}^p_0 = \{v \in \mathbf{L}^p;\ \int_\Omega v \,\mathrm{dx} = 0\}$.
- For $p \in I\!R$, $\mathbf{H}^p = H^p(\Omega)$, the classical Sobolev space.
- $\mathbf{H}^1_0 = \{v \in \mathbf{H}^1;\ v_{|\partial\Omega} = 0\}$.
- $\mathbf{H}^1_\nu = \{v \in \mathbf{H}^1;\ v_{|\partial\Omega}.\nu = 0\}$.
- $\mathbf{H} = \{v \in \mathbf{L}^2;\ \mathrm{div}(v) = 0 \text{ on } \Omega;\ (v.\nu)_{|\partial\Omega} = 0\}$.
- $\mathbf{H}_{\mathrm{div}} = \{v \in \mathbf{L}^2;\ \mathrm{div}(v) \in \mathbf{L}^2\}$.
- $\mathbf{H}_{\mathrm{div},\nu} = \{v \in \mathbf{L}^2;\ \mathrm{div}(v) \in \mathbf{L}^2;\ (v.\nu)_{|\partial\Omega} = 0\}$.
- $\mathbf{G} = \{v \in \mathbf{L}^2;\ \exists q \in \mathbf{H}^1;\ v = \nabla q\}$.
- $\mathbf{V} = \{v \in \mathbf{H}^1_0;\ \mathrm{div}(v) = 0 \text{ on } \Omega\}$.

where ν is the outward unit normal vector on $\partial\Omega$. For details on the definition of these spaces, we refer to [8].

1.2. Incompressible Navier-Stokes system

In Ω, we consider the smooth time dependent solid domain $\omega(t) \subset \Omega$, $t \in [0, T]$, and v_s its velocity. We focus on the incompressible Navier Stokes equation in $\Omega(t) := \Omega \setminus \omega(t)$:

$$\begin{aligned}
&\frac{\partial v}{\partial t} + (v.\nabla)v - \mathrm{div}(2\mu D(v)) + \nabla p = f \qquad \text{on} \qquad \Omega(t)\\
&\mathrm{div}(v) = 0 \qquad \text{on} \qquad \Omega(t)\\
&v = 0 \qquad \text{on} \qquad \partial\Omega\\
&v = v_s \qquad \text{on} \qquad \partial\omega(t)\\
&v(0, x) = v_0(x) \qquad \text{on} \qquad \Omega(0)
\end{aligned} \tag{1.1}$$

where D is the strain rate tensor and f is a given source term defined on $[0,T]\times\Omega$.

The associated penalized Navier Stokes problem reads:

$$\begin{aligned}
&\frac{\partial v}{\partial t}+(v.\nabla)v-\mathrm{div}(2\mu D(v))+\frac{1}{\eta}\chi_{\omega(t)}(v-v_s)+\nabla p=f \qquad \text{on} \qquad \Omega\\
&\mathrm{div}(v)=0 \qquad \text{on} \qquad \Omega\\
&v=0 \qquad \text{on} \qquad \partial\Omega\\
&v(0,x)=v_0(x) \qquad \text{on} \qquad \Omega
\end{aligned} \tag{1.2}$$

where $\chi_{\omega(t)}$ is the characteristic function of the solid domain $\omega(t)$.

Hypothesis.

$(\mathcal{H})$: We suppose that v_s is the restriction to $\bigcup_{t<T}\{t\}\times\omega(t)$ of a function ψ defined on $[0,T]\times\Omega$ such that:

$$\begin{cases}
\mathrm{div}(\psi)=0 \qquad \text{on} \qquad \Omega\\
\psi=0 \qquad \text{on} \qquad \partial\Omega\\
\psi\in L^\infty(]0,T[;\mathbf{H}^3)\\
\dfrac{\partial\psi}{\partial t}\in L^2(]0,T[;\mathbf{L}^2).
\end{cases} \tag{1.3}$$

The existence of such function is ensured as the regularity of v_s is sufficient and the moving body does not meet $\partial\Omega$ (see [20]).

1.3. The Vector Penalty Projection Scheme

Let $\delta t>0$ be the time step and $t^n=n\delta t$. For $p^0\in\mathbf{L}^2_0$ and $v^0=v_0\in\mathbf{H}^1_0\cap\mathbf{H}$, the Vector Penalty Projection Scheme is a fractional step method:

- A predicted velocity $\tilde{v}^{n+1}$ is first computed considering the pressure gradient at the previous time step t^n. At the end of this step, the velocity does not respect the free divergence condition.
- The velocity is then corrected such that $\mathrm{div}(v^{n+1})$ is approximately 0 at the end of the time step.
- The pressure gradient ∇p^{n+1} is finally actualized.

For all $n\in\mathbb{N}$ such that $n\delta t\le T$, the numerical scheme reads:

$$\frac{\tilde{v}^{n+1}-v^n}{\delta t}+B(v^n,\tilde{v}^{n+1})-\mathrm{div}(2\mu D(\tilde{v}^{n+1}))+\frac{1}{\eta}\chi_{\omega(t^{n+1})}(\tilde{v}^{n+1}-v_s^{n+1})+\nabla p^n=f^{n+1} \tag{1.4}$$

$$\frac{\varepsilon}{\delta t}\hat{v}^{n+1}-\nabla\left(\mathrm{div}(\hat{v}^{n+1})+\mathrm{div}(\tilde{v}^{n+1})\right)=0 \tag{1.5}$$

$$\nabla(p^{n+1}-p^n)+\frac{1}{\varepsilon}\nabla(\mathrm{div}(v^{n+1}))=0 \tag{1.6}$$

where $v^{n+1}=\tilde{v}^{n+1}+\hat{v}^{n+1}$ and $B(u,v)=(u.\nabla)v+\dfrac{1}{2}\mathrm{div}(u)\,v$.

It is completed by the following initial and boundary conditions on $\partial\Omega$:

$$\tilde{v}^{n+1}=0 \quad \text{on} \quad \partial\Omega,\quad \hat{v}^{n+1}.\nu=0 \quad \text{on} \quad \partial\Omega \tag{1.7}$$

$$\tilde{v}^0=v_0 \quad \text{in} \quad \Omega,\quad \hat{v}^0=0 \quad \text{in} \quad \Omega. \tag{1.8}$$

The original scheme has been completed by the penalization term which only appears in the prediction step.

Note that since $v^{n+1}.\nu=0$, then $\mathrm{div}(v^{n+1})\in\mathbf{L}^2_0$ and since $p^0\in\mathbf{L}^2_0$ we can show recursively that p^{n+1} has a null average on Ω for all $n\in\mathbb{N}$, solving (1.6) in the space of null average functions. We finally obtain:

$$\varepsilon(p^{n+1}-p^n)+\mathrm{div}(v^{n+1})=0. \tag{1.9}$$

Proposition 1.1 (Existence of the iterates)**.**
We suppose that $v^0 \in \mathbf{H}_0^1 \cap \mathbf{H}$, $p^0 \in \mathbf{L}_0^2$ *and* $f \in L^2(]0,T[;\mathbf{L}^2)$. *Then, for all* $n \in \mathbb{N}$ *such that* $n\delta t \leq T$, (1.4)–(1.9) *defined recursively:*

$$(\tilde{v}^n, \hat{v}^n, p^n) \in \mathbf{H}_0^1 \times \mathbf{H}_\nu^1 \times \mathbf{L}_0^2$$

with $v^n = \tilde{v}^n + \hat{v}^n$.

Proof. Suppose that $(\tilde{v}^n, \hat{v}^n) \in \mathbf{H}_0^1 \times \mathbf{H}_\nu^1$ and show that $(\tilde{v}^{n+1}, \hat{v}^{n+1}) \in \mathbf{H}_0^1 \times \mathbf{H}_\nu^1$. Equation (1.4) is a linear advection diffusion problem, therefore we obtain that $\tilde{v}^{n+1}$ exists and lies in $\mathbf{H}_0^1$. We now show the existence of $\hat{v}^{n+1}$. In the space $\mathbf{H}_{\mathrm{div},\nu}$, (1.5) is a linear and coercive problem, then $\hat{v}^{n+1}$ exists and is unique in $\mathbf{H}_{\mathrm{div},\nu}$. Moreover, $\hat{v}^{n+1}$ is a gradient, therefore $\mathrm{curl}(\hat{v}^{n+1}) = 0$. We deduce that $\hat{v}^{n+1}$ lies in $\mathbf{H}_\nu^1$ (see Prop. 2.2).

Using (1.6), for $p^n \in \mathbf{L}_0^2$ we deduce that ∇p^{n+1} lies in $\mathbf{H}^{-1}$. As p^{n+1} has a null average on Ω, we conclude using the Poincaré Wirtinger inequality that $p^{n+1} \in \mathbf{L}_0^2$. $\square$

1.4. Organization of the article

In Section 3, the stability of the scheme introduced above is demonstrated using a set of energy estimates as in [2] where no space penalization is included. It gives a bound for the velocity in the space $L^\infty(]0,T[;\mathbf{L}^2) \cap L^2(]0,T[;\mathbf{H}_\nu^1)$. The final inequality is quite similar to the one obtained in [2] but contains an additional term which is only active in the solid region and ensures that the difference $\tilde{v}^n - v_s$ is of the order $\eta^{\frac{1}{2}}$ for the $L^2(]0,T[,L^2(\omega(t)))$-norm. We then obtain an upper bound on the velocity divergence which depends on ε. Finally, we give an estimate on the velocity translation in the space $\mathbf{H}^{-1}$ which is useful to demonstrate the strong convergence of the velocity in Section 4.

Section 4 aims to establish the convergence when the parameters ε and δt tend to 0 with the proportionality constraint $\varepsilon = \lambda\delta t$. To do so, the weak convergence of the velocity is first obtained. The strong convergence is then demonstrated using a compactness result from Aubin Lions Simon. To the best of our knowledge, there is no previous convergence result on the discretized $(VPP)_\varepsilon$ scheme. Finally, in Section 5 we state a Theorem that deals with the weak convergence when the penalization parameter η goes to 0. At the limit process, the solution verifies (in the weak sense) the Navier-Stokes equations in the moving fluid domain with no-slip boundary condition on the solid.

2. Mathematical recalls

In this article, we will need the following standard results and notations. Here and in the following sections, the different constants are always denoted C.

To deal with the nonlinear convective term, we use the bilinear form B introduced by Temam (see [25, 26]).

Definition 2.1. For $u \in \mathbf{H}^1$ and $v \in \mathbf{H}_0^1$, we define the bilinear form B by:

$$B(u,v) = (u.\nabla)v + \frac{1}{2}\mathrm{div}(u)v. \tag{2.1}$$

Taking the scalar product of $B(u,v)$ by $w \in \mathbf{H}_0^1$ and integrating by part the second term, we obtain the associated trilinear form b:

$$b(u,v,w) = \frac{1}{2}\int_\Omega (u.\nabla)v.w \,\mathrm{dx} - \frac{1}{2}\int_\Omega (u.\nabla)w.v \,\mathrm{dx}. \tag{2.2}$$

The trilinear form b satisfies the antisymmetry property $b(u,v,w) = -b(u,w,v)$ and $b(u,v,v) = 0$.

Proposition 2.2 [10, 14]**.** *Under the previous hypothesis and notations, one has the following decomposition*:

$$\mathbf{L}^2 = \mathbf{H} \oplus \mathbf{G},$$
$$\mathrm{Ker}(\mathrm{curl}) = \mathbf{G}.$$

Moreover, there exists a constant $C > 0$ depending only on Ω such that:

$$||u||^2_{\mathbf{H}^1} = ||u||^2_{\mathbf{L}^2} + ||\nabla u||^2_{\mathbf{L}^2} \leq C \left(||u||^2_{\mathbf{L}^2} + ||\mathrm{div}(u)||^2_{\mathbf{L}^2} + ||\mathrm{curl}(u)||^2_{\mathbf{L}^2}\right), \forall u \in \mathbf{H}^1_\nu.$$

Besides, if we suppose that the open set Ω is simply-connected, there exist two constants λ_0 and λ_1 depending only on Ω such that:

$$||u||^2_{\mathbf{L}^2} \leq \lambda_0 \left(||\mathrm{div}(u)||^2_{\mathbf{L}^2} + ||\mathrm{curl}(u)||^2_{\mathbf{L}^2}\right), \forall u \in \mathbf{H}^1_\nu,$$
$$||u||^2_{\mathbf{L}^2} + ||\nabla u||^2_{\mathbf{L}^2} \leq \lambda_1 \left(||\mathrm{div}(u)||^2_{\mathbf{L}^2} + ||\mathrm{curl}(u)||^2_{\mathbf{L}^2}\right), \forall u \in \mathbf{H}^1_\nu.$$

Remark 2.3. In the domain Ω, a Poincaré-type inequality holds since Ω is simply connected. At the limit process $\eta \to 0$, we are in the domain $\Omega(t) = \Omega \setminus \omega(t)$ which is not simply connected anymore and the last inequality is not verified.

We now recall the discrete Gronwall Lemma (see [13, 15, 23]).

Lemma 2.4 Discrete Gronwall Lemma [15]**.**
Let (y_n), (f_n) and (g_n) three non-negative sequences such that:

$$y_n \leq f_n + \sum_{k=0}^{n-1} g_k\, y_k \quad \textit{for} \quad n \geq 0.$$

Then,

$$y_n \leq f_n + \sum_{k=0}^{n-1} f_k g_k \exp\left(\sum_{j=k+1}^{n-1} g_j\right) \quad \textit{for} \quad n \geq 0.$$

Estimates on inertia terms, developed in Section 4, will use the following interpolation properties between L^p-spaces.

Proposition 2.5 ([8], Thm. II.5.5)**.** *Let I be an interval of $I\!R$, let Ω be an open subset of $I\!R^d$, and let p_1, q_1, p_2, q_2 be four real numbers in $[1, +\infty]$. If $f \in L^{p_1}(I, L^{q_1}(\Omega)) \cap L^{p_2}(I, L^{q_2}(\Omega))$ then for all $\theta \in]0, 1[$, the function f belongs to $L^p(I, L^q(\Omega))$ for p and q defined by*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \textit{and} \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}$$

and we have

$$||f||_{L^p(I, L^q(\Omega))} \leq ||f||^{\theta}_{L^{p_1}(I, L^{q_1}(\Omega))} ||f||^{1-\theta}_{L^{p_2}(I, L^{q_2}(\Omega))}.$$

In order to prove the convergence of the velocity we will need the following analysis result ([8], Prop. II.5.11). Let X and Y two Banach spaces, let $T > 0$ and p, q satisfying $1 \leq p, q \leq +\infty$. We denote:

$$E_{p,q}(X, Y) = \left\{ u \in L^p(]0, T[, X),\ \frac{\mathrm{d}u}{\mathrm{d}t} \in L^q(]0, T[, Y) \right\}.$$

Proposition 2.6. *Suppose that X is embedded in a continuous and dense way into Y. Any element u of $E_{p,q}(X,Y)$ (defined almost everywhere) possesses a continuous representation on $[0,T]$ with values in Y, and the embedding of $E_{p,q}(X,Y)$ into $\mathcal{C}^0([0,T],Y)$ is continuous.*

Moreover, for all $t_1, t_2 \in [0,T]$ we have:

$$u(t_2) - u(t_1) = \int_{t_1}^{t_2} \frac{\mathrm{d}u}{\mathrm{d}t}\,\mathrm{dt}$$

where it is understood that we have identified u and its continuous representation.

Finally, let us formulate an important compactness theorem, which will be useful to prove the strong convergence of the velocity in Section 4.

Lemma 2.7 Aubin−Lions−Simon [5, 24]**.**
Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let $T > 0$ and p, r such that $1 \leq p, r \leq +\infty$. Then,

(i) *If $p < +\infty$, the embedding of $E_{p,r}(B_0, B_2)$ in $L^p(]0,T[;B_1)$ is compact.*
(ii) *If $p = +\infty$ and if $r > 1$, the embedding of $E_{p,r}(B_0, B_2)$ in $\mathcal{C}(]0,T[;B_1)$ is compact.*

3. Stability analysis

In this section the stability of the numerical scheme is obtained considering energy estimates. Then we obtain an upper bound on the velocity divergence in the space $L^2(]0,T[;\mathbf{L}^2)$.

To each sequence $(v^k)_k$ defined on Ω we will associate a sequence of functions $(v_{\delta t})_{\delta t}$ which are the step functions in time $v_{\delta t}$ defined by:

$$v_{\delta t}(t) = v^k \text{ if } t \in [t^k, t^{k+1}[. \tag{3.1}$$

We denote $(\cdot\,,\cdot)_{\mathbf{L}^2}$ the usual scalar product on $\mathbf{L}^2$ and $\langle\cdot\,,\cdot\rangle_{E',E}$ the duality bracket.

To perform calculus, we need to build a lifting of the velocity. In the following, to the function v (resp. v^n, $\tilde{v}^n$, $\hat{v}^n$, $v_{\delta t}$) we will associate the function w (resp. w^n, $\tilde{w}^n$, $\hat{w}^n$, $w_{\delta t}$) defined by substraction of the function ψ (resp. $\psi^n = \psi(t^n)$), introduced in (1.3):

$$w = v - \psi, \quad w^n = v^n - \psi^n, \quad \tilde{w}^n = \tilde{v}^n - \psi^n, \quad \hat{w}^n = \hat{v}^n, \quad w_{\delta t} = v_{\delta t} - \psi. \tag{3.2}$$

The system (1.4)−(1.8) becomes:

$$\begin{aligned} \frac{\tilde{w}^{n+1} - w^n}{\delta t} &+ B(w^n, \tilde{w}^{n+1}) + B(\psi^n, \tilde{w}^{n+1}) \\ &- \mathrm{div}(2\mu D(\tilde{w}^{n+1})) + \frac{1}{\eta}\chi_{\omega(t^{n+1})}\tilde{w}^{n+1} + \nabla p^n = F^{n+1} - B(w^n, \psi^{n+1}) \end{aligned} \tag{3.3}$$

$$\frac{\hat{w}^{n+1}}{\delta t} - \frac{1}{\varepsilon}\nabla(\mathrm{div}(\tilde{w}^{n+1} + \hat{w}^{n+1})) = 0 \tag{3.4}$$

$$\nabla(p^{n+1} - p^n) + \frac{1}{\varepsilon}\nabla(\mathrm{div}(w^{n+1})) = 0 \tag{3.5}$$

$$\tilde{w}^{n+1} = 0 \qquad \text{on} \quad \partial\Omega, \quad \hat{w}^{n+1}.\nu = 0 \qquad \text{on} \quad \partial\Omega \tag{3.6}$$

$$\tilde{w}^0 = v_0 - \psi^0 \qquad \text{in} \quad \Omega, \hat{w}^0 = 0 \qquad \text{in} \quad \Omega \tag{3.7}$$

where $F^{n+1} = f^{n+1} - \dfrac{\psi^{n+1} - \psi^n}{\delta t} + \mathrm{div}(2\mu D(\psi^{n+1})) - B(\psi^n, \psi^{n+1})$.

Performing estimates on the prediction, the correction and pressure equations respectively given by (3.3)−(3.5), we establish the following result:

Proposition 3.1 Stability**.**
Let $v^0 \in \mathbf{H}^1_0 \cap \mathbf{H}$, $p^0 \in \mathbf{L}^2_0$ *such that* $\nabla p^0 \in \mathbf{L}^2$ *and* $f \in L^2(]0,T[;\mathbf{L}^2)$. *We assume that hypothesis* $(\mathcal{H})$ *is verified. Then, there exists a constant* $C > 0$ *independent of* ε *and* δt *such that:*

(i) $||w_{\delta t}||_{L^\infty(]0,T[;\mathbf{L}^2)} \le C$.
(ii) $||\tilde{w}_{\delta t}||_{L^2(]0,T[;\mathbf{H}^1_0)} \le C$.
(iii) $||w_{\delta t}||_{L^2(]0,T[;\mathbf{H}^1_\nu)} \le C\left(1+\dfrac{\delta t}{\varepsilon}\right)$.

Proof of Proposition 3.1. We prove the result using several energy estimates as in [2] for homogeneous Navier-Stokes flows. In our estimates, an additional term appears due to the penalization term on the moving body.

Taking $\tilde{w}^{n+1}$ as a test function in (3.3), we obtain:

$$\begin{aligned}\frac{1}{\delta t}\left(\tilde{w}^{n+1}-w^n,\tilde{w}^{n+1}\right)_{\mathbf{L}^2} + 2\mu\left(D(\tilde{w}^{n+1}):D(\tilde{w}^{n+1})\right)_{\mathbf{L}^2} + \frac{1}{\eta}\int_\Omega \chi_{\omega(t^{n+1})}|\tilde{w}^{n+1}|^2\,\mathrm{dx}\\ + b(w^n,\tilde{w}^{n+1},\tilde{w}^{n+1}) + b(\psi^n,\tilde{w}^{n+1},\tilde{w}^{n+1}) + \left(\nabla p^n,\tilde{w}^{n+1}\right)_{\mathbf{L}^2}\\ = \left(F^{n+1},\tilde{w}^{n+1}\right)_{\mathbf{L}^2} - b(w^n,\psi^{n+1},\tilde{w}^{n+1}).\end{aligned}$$

The diffusion term is integrated by parts, the Korn inequality ([8, 17, 18], Rem. IV.7.3) is then used in $\mathbf{H}^1_0$, to obtain the lower bound:

$$||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2} \le 2||D(\tilde{w}^{n+1})||^2_{\mathbf{L}^2}.$$

The convective terms $b(w^n,\tilde{w}^{n+1},\tilde{w}^{n+1})$ and $b(\psi^n,\tilde{w}^{n+1},\tilde{w}^{n+1})$ vanish by antisymmetry of the trilinear form b. By definition of b given in (2.2) and using Hölder's inequality it comes:

$$\begin{aligned}2|b(w^n,\psi^{n+1},\tilde{w}^{n+1})| &\le ||(w^n.\nabla)\psi^{n+1}.\tilde{w}^{n+1}||_{\mathbf{L}^1} + ||(w^n.\nabla)\tilde{w}^{n+1}.\psi^{n+1}||_{\mathbf{L}^1}\\ &\le ||w^n||_{\mathbf{L}^2}||\nabla\psi^{n+1}||_{\mathbf{L}^\infty}||\tilde{w}^{n+1}||_{\mathbf{L}^2} + ||w^n||_{\mathbf{L}^2}||\nabla\tilde{w}^{n+1}||_{\mathbf{L}^2}||\psi^{n+1}||_{\mathbf{L}^\infty}.\end{aligned} \tag{3.8}$$

Since $\tilde{w}^{n+1} \in \mathbf{H}^1_0$, from the Poincaré inequality there exists a constant $C(\Omega)$ such that:

$$||\tilde{w}^{n+1}||_{\mathbf{L}^2} \le C(\Omega)||\nabla\tilde{w}^{n+1}||_{\mathbf{L}^2}.$$

Going back to (3.8) and using finally Young's inequality, it yields:

$$\begin{aligned}|b(w^n,\psi^{n+1},\tilde{w}^{n+1})| &\le C(\Omega,\mu)||w^n||^2_{\mathbf{L}^2}||\nabla\psi^{n+1}||^2_{\mathbf{L}^\infty} + \frac{\mu}{8}||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2} + C(\mu)||w^n||^2_{\mathbf{L}^2}||\psi^{n+1}||^2_{\mathbf{L}^\infty} + \frac{\mu}{8}||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2}\\ &\le \frac{\mu}{4}||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2} + C(\Omega,\mu)||w^n||^2_{\mathbf{L}^2}\left(||\psi^{n+1}||^2_{\mathbf{L}^\infty} + ||\nabla\psi^{n+1}||^2_{\mathbf{L}^\infty}\right).\end{aligned}$$

The part of $\tilde{w}^{n+1}$ in $\left(F^{n+1},\tilde{w}^{n+1}\right)$ is absorbed thanks to the diffusion term using again the Poincaré inequality.

We finally use the following equality:

$$(a-b,a) = \frac{1}{2}\left(||a||^2 - ||b||^2 + ||a-b||^2\right). \tag{3.9}$$

The following estimate is obtained:

$$\begin{aligned}&\frac{1}{2\delta t}\left(||\tilde{w}^{n+1}||^2_{\mathbf{L}^2} - ||w^n||^2_{\mathbf{L}^2} + ||\tilde{w}^{n+1}-w^n||^2_{\mathbf{L}^2}\right) + \frac{\mu}{2}||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2}\\ &+ \frac{1}{\eta}\int_\Omega \chi_{\omega(t^{n+1})}|\tilde{w}^{n+1}|^2\,\mathrm{dx} + (\nabla p^n,\tilde{w}^{n+1})_{\mathbf{L}^2} \le C||F^{n+1}||^2_{\mathbf{L}^2} + C||w^n||^2_{\mathbf{L}^2},\end{aligned} \tag{3.10}$$

where C depends on μ, Ω and ψ.

From (3.4) and (3.5), we have:

$$\frac{w^{n+1}-\tilde{w}^{n+1}}{\delta t}+\nabla(p^{n+1}-p^n)=0. \tag{3.11}$$

By taking w^{n+1} as a test function in (3.11) and using again (3.9) we obtain:

$$\frac{1}{2\delta t}\left(||w^{n+1}||^2_{\mathbf{L}^2}-||\tilde{w}^{n+1}||^2_{\mathbf{L}^2}+||w^{n+1}-\tilde{w}^{n+1}||^2_{\mathbf{L}^2}\right)+\left(\nabla p^{n+1}-\nabla p^n, w^{n+1}\right)_{\mathbf{L}^2}=0. \tag{3.12}$$

We choose p^{n+1} as a test function in (1.9) and exploit that $\operatorname{div}(\psi)=0$. The boundary term vanishes in the integration by part since $w^{n+1}.\nu=0$ on $\partial\Omega$ and we have:

$$\frac{\varepsilon}{2}\left(||p^{n+1}||^2_{\mathbf{L}^2}-||p^n||^2_{\mathbf{L}^2}+||p^{n+1}-p^n||^2_{\mathbf{L}^2}\right)-\left(\nabla p^{n+1}, w^{n+1}\right)_{\mathbf{L}^2}=0. \tag{3.13}$$

At last, taking ∇p^{n+1} as a test function in (3.11) we obtain:

$$\frac{\delta t}{2}\left(||\nabla p^{n+1}||^2_{\mathbf{L}^2}-||\nabla p^n||^2_{\mathbf{L}^2}+||\nabla p^{n+1}-\nabla p^n||^2_{\mathbf{L}^2}\right)+\left(\nabla p^{n+1}, w^{n+1}-\tilde{w}^{n+1}\right)_{\mathbf{L}^2}=0. \tag{3.14}$$

Finally, these four estimates (3.10), (3.12)−(3.14) are summed up. The sum of the scalar products reduces to $\left(\nabla p^{n+1}-\nabla p^n, w^{n+1}-\tilde{w}^{n+1}\right)$, which is bounded using Young inequality:

$$|\left(\nabla(p^{n+1}-p^n), w^{n+1}-\tilde{w}^{n+1}\right)_{\mathbf{L}^2}|\le\frac{\delta t}{2}||\nabla p^{n+1}-\nabla p^n||^2_{\mathbf{L}^2}+\frac{1}{2\delta t}||w^{n+1}-\tilde{w}^{n+1}||^2_{\mathbf{L}^2}.$$

Therefore,

$$\begin{aligned}&\frac{1}{2\delta t}\left(||w^{n+1}||^2_{\mathbf{L}^2}-||w^n||^2_{\mathbf{L}^2}+||\tilde{w}^{n+1}-w^n||^2_{\mathbf{L}^2}\right)+\frac{\mu}{2}||\nabla\tilde{w}^{n+1}||^2_{\mathbf{L}^2}\\&+\frac{\varepsilon}{2}\left(||p^{n+1}||^2_{\mathbf{L}^2}-||p^n||^2_{\mathbf{L}^2}+||p^{n+1}-p^n||^2_{\mathbf{L}^2}\right)\\&+\frac{\delta t}{2}\left(||\nabla p^{n+1}||^2_{\mathbf{L}^2}-||\nabla p^n||^2_{\mathbf{L}^2}\right)+\frac{1}{\eta}\int_\Omega\chi_{\omega(t^{n+1})}|\tilde{w}^{n+1}|^2\,\mathrm{dx}\le C||F^{n+1}||^2_{\mathbf{L}^2}+C||w^n||^2_{\mathbf{L}^2}.\end{aligned}$$

This last equation is multiplied by $2\delta t$ and written in k instead of n. Finally, the equations are summed from $k=0$ to $n-1$ with $n\le N=E(\dfrac{T}{\delta t})$ where E denotes the floor function, and we deduce:

$$\begin{aligned}&||w^n||^2_{\mathbf{L}^2}+\delta t\varepsilon||p^n||^2_{\mathbf{L}^2}+\delta t^2||\nabla p^n||^2_{\mathbf{L}^2}+\sum_{k=0}^{n-1}||\tilde{w}^{k+1}-w^k||^2_{\mathbf{L}^2}\\&+\mu\sum_{k=0}^{n-1}\delta t||\nabla\tilde{w}^{k+1}||^2_{\mathbf{L}^2}+\varepsilon\sum_{k=0}^{n-1}\delta t||p^{k+1}-p^k||^2_{\mathbf{L}^2}+\frac{2}{\eta}\sum_{k=0}^{n-1}\delta t\int_\Omega\chi_{\omega(t^{k+1})}|\tilde{w}^{k+1}|^2\,\mathrm{dx}\\&\le||w^0||^2_{\mathbf{L}^2}+\varepsilon\delta t||p^0||^2_{\mathbf{L}^2}+\delta t^2||\nabla p^0||^2_{\mathbf{L}^2}+2C\sum_{k=0}^{n-1}\delta t||F^{k+1}||^2_{\mathbf{L}^2}+2C\sum_{k=0}^{n-1}\delta t||w^k||^2_{\mathbf{L}^2}.\end{aligned} \tag{3.15}$$

It implies:

$$||w^n||^2_{\mathbf{L}^2}\le f_n+\sum_{k=0}^{n-1}g_k||w^k||^2_{\mathbf{L}^2}$$

with,

$$\begin{cases} f_n = ||w^0||^2_{\mathbf{L}^2} + \varepsilon\delta t||p^0||^2_{\mathbf{L}^2} + \delta t^2||\nabla p^0||^2_{\mathbf{L}^2} + 2C\sum_{k=0}^{n-1}\delta t||F^{k+1}||^2_{\mathbf{L}^2}, \\ g_k = 2C\delta t. \end{cases}$$

The discrete Gronwall Lemma 2.4 thus gives the following upper bound on $||w^n||_{\mathbf{L}^2}$:

$$||w^n||^2_{\mathbf{L}^2} \leq f_n(1 + 2CT\exp(2CT)).$$

Going back to (3.15), from the assumptions on w^0, p^0, ψ and f, we deduce that there exists $C > 0$, independent of $n < N$ and δt, such that:

$$\begin{aligned} &\delta t\varepsilon||p^n||^2_{\mathbf{L}^2} + \delta t^2||\nabla p^n||^2_{\mathbf{L}^2} + \sum_{k=0}^{n-1}||\tilde{w}^{k+1} - w^k||^2_{\mathbf{L}^2} \\ &+ \mu\sum_{k=0}^{n-1}\delta t||\nabla\tilde{w}^{k+1}||^2_{\mathbf{L}^2} + \varepsilon\sum_{k=0}^{n-1}\delta t||p^{k+1} - p^k||^2_{\mathbf{L}^2} + \frac{2}{\eta}\sum_{k=0}^{n-1}\delta t\int_{\Omega}\chi_{\omega(t^{k+1})}|\tilde{w}^{k+1}|^2\,\mathrm{dx} \leq C, \end{aligned} \tag{3.16}$$

which demonstrates the two first points of Proposition 3.1.

To prove the last point, we take $\hat{w}^{k+1}$ as a test function in the correction step (3.4) and obtain:

$$\begin{aligned} ||\hat{w}^{k+1}||^2_{\mathbf{L}^2} + \frac{\delta t}{\varepsilon}||\mathrm{div}(\hat{w}^{k+1})||^2_{\mathbf{L}^2} &= -\frac{\delta t}{\varepsilon}\left(\mathrm{div}(\tilde{w}^{k+1}), \mathrm{div}(\hat{w}^{k+1})\right)_{\mathbf{L}^2} \\ &\leq \frac{\delta t}{2\varepsilon}||\mathrm{div}(\hat{w}^{k+1})||^2_{\mathbf{L}^2} + \frac{\delta t}{2\varepsilon}||\mathrm{div}(\tilde{w}^{k+1})||^2_{\mathbf{L}^2}. \end{aligned}$$

We thus deduce an estimate on the corrected velocity $\hat{w}^{k+1}$ and its divergence:

$$||\hat{w}^{k+1}||^2_{\mathbf{L}^2} + \frac{\delta t}{2\varepsilon}||\mathrm{div}(\hat{w}^{k+1})||^2_{\mathbf{L}^2} \leq \frac{\delta t}{2\varepsilon}||\mathrm{div}(\tilde{w}^{k+1})||^2_{\mathbf{L}^2}. \tag{3.17}$$

Using that for functions of $\mathbf{H}^1_\nu$, the norm $||\cdot||_{\mathbf{H}^1}$ is equivalent to the norm $\left(||\cdot||^2_{\mathbf{L}^2} + ||\mathrm{div}(\cdot)||^2_{\mathbf{L}^2} + ||\mathrm{curl}(\cdot)||^2_{\mathbf{L}^2}\right)^{\frac{1}{2}}$ (see Prop. 2.2) and $\mathrm{curl}(\hat{w}^{k+1}) = 0$, we obtain:

$$\begin{aligned} ||\hat{w}^{k+1}||^2_{\mathbf{H}^1} &\leq C\Big(||\hat{w}^{k+1}||^2_{\mathbf{L}^2} + ||\mathrm{div}(\hat{w}^{k+1})||^2_{\mathbf{L}^2}\Big) \\ &\leq C\left(\frac{\delta t}{2\varepsilon} + 1\right)||\nabla\tilde{w}^{k+1}||^2_{\mathbf{L}^2}. \end{aligned}$$

The previous inequalities are summed up from $k = 0$ to $N - 1$. The predicted velocity gradient is bounded using the stability result (3.16). Then, we find the claimed upper bound on the total velocity gradient:

$$\begin{aligned} \sum_{k=0}^{N-1}||\nabla w^{k+1}||^2_{\mathbf{L}^2}\delta t &\leq 2\sum_{k=0}^{N-1}\delta t||\nabla\tilde{w}^{k+1}||^2_{\mathbf{L}^2} + 2\sum_{k=0}^{N-1}\delta t||\nabla\hat{w}^{k+1}||^2_{\mathbf{L}^2} \\ &\leq 2C + 2\,C\left(1 + \frac{\delta t}{2\varepsilon}\right)\sum_{k=0}^{N-1}\delta t||\nabla\tilde{w}^{k+1}||^2_{\mathbf{L}^2} \\ &\leq 2\left(2 + \frac{\delta t}{2\varepsilon}\right)C. \end{aligned}$$

□

Remark 3.2. The bounds found for $w_{\delta t}$ and $\tilde{w}_{\delta t}$ are independant of η. Equation (3.16) contains the penalization term which is only active in the solid region. It ensures that the difference between the predicted velocity given by the scheme and the solid velocity v_s is of the order $\eta^{\frac{1}{2}}$:

$$\frac{2}{\eta}\int_0^T ||\tilde{v}_{\delta t} - v_{s,\delta t}||^2_{L^2(\omega(\tau))} d\tau \leq C. \tag{3.18}$$

Lemma 3.3. *Under the hypothesis of Proposition 3.1, we have*:

(i) *the divergence of $w_{\delta t}$ lies in $L^2(]0,T[;\mathbf{L}^2)$ and there exists $C > 0$ such that for any $\varepsilon > 0$,*

$$||\mathrm{div}\ w_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)} \leq C\sqrt{\varepsilon}. \tag{3.19}$$

As ψ is divergence free, the same inequality holds for $v_{\delta t}$ which implies the strong convergence of $\mathrm{div}(v_{\delta t})$ *towards* 0 *when ε tends to* 0.

(ii) *$\hat{w}_{\delta t}$ is bounded in $L^2(]0,T[;\mathbf{H}^{-1})$ with*:

$$||\hat{w}_{\delta t}||^2_{L^2(]0,T[;\mathbf{H}^{-1})} \leq C\delta t\frac{\delta t}{\varepsilon}\cdot$$

Proof of (i). From the pressure equation (1.9) we have $\varepsilon(p^{n+1} - p^n) = -\mathrm{div}(w^{n+1})$. Then we have:

$$\sum_{k=0}^{N-1} \delta t||\mathrm{div}(w^{k+1})||^2_{\mathbf{L}^2} = \varepsilon^2 \sum_{k=0}^{N-1} \delta t||p^{k+1} - p^k||^2_{\mathbf{L}^2},$$

and we deduce (3.19) exploiting the stability result (3.16). □

Proof of (ii). The second point is proved using the correction equation (3.4). Taking the $\mathbf{H}^{-1}$-norm we obtain:

$$||\hat{w}^{k+1}||_{\mathbf{H}^{-1}} \leq \frac{\delta t}{\varepsilon}||\mathrm{div}(w^{k+1})||_{\mathbf{L}^2}. \tag{3.20}$$

Therefore, summing the square of this inequality from $k = 0$ to $N-1$ and using the bound of the velocity's divergence (3.19), we finally obtain:

$$\sum_{k=0}^{N-1} \delta t||\hat{w}^{k+1}||^2_{\mathbf{H}^{-1}} \leq \frac{C\delta t^2}{\varepsilon}\cdot \tag{3.21}$$

□

Lemma 3.4. *Under the hypothesis of Proposition 3.1, the velocity translation satisfies:*

$$\sum_{k=0}^{N-1} ||w^{k+1} - w^k||^2_{\mathbf{H}^{-1}} \leq C\left(\frac{\delta t}{\varepsilon} + 1\right)\cdot \tag{3.22}$$

Proof. The stability result (3.16) gives a bound on the difference between the predicted velocity at the current time step and the velocity at the previous time step. Using the embedding of $\mathbf{L}^2$ in $\mathbf{H}^{-1}$ we deduce:

$$\sum_{k=0}^{N-1} ||\tilde{w}^{k+1} - w^k||^2_{\mathbf{H}^{-1}} \leq C.$$

Then, combining it with (3.21) the following inequality holds:

$$\sum_{k=0}^{N-1} ||w^{k+1} - w^k||^2_{\mathbf{H}^{-1}} \leq 2 \sum_{k=0}^{N-1} \left[||w^{k+1} - \tilde{w}^{k+1}||^2_{\mathbf{H}^{-1}} + ||\tilde{w}^{k+1} - w^k||^2_{\mathbf{H}^{-1}}\right]$$
$$\leq C \left(\frac{\delta t}{\varepsilon} + 1\right) \cdot$$

□

Remark 3.5. The ratio $\frac{\delta t}{\varepsilon}$ appears in some bounds found in this section (see Prop. 3.1, Lem. 3.3 or Lem. 3.4). Therefore, to pass to the limit on ε, we impose that ε and δt tend to 0 with the proportionality constraint $\varepsilon = \lambda \delta t$. In fact, it would be sufficient to assume that $\delta t = O(\varepsilon)$ with ε tending to 0.

Remark 3.6. All the results obtained in this section are independent of η and will allow us to pass to the limit when η goes to 0 in Section 5 using the lower semicontinuity of the norm for the weak topology.

4. Convergence analysis when ε and δt tend to 0

A stability result has been obtained in the previous section. The main purpose of this section is to establish the following convergence theorem, when the parameters ε and δt tend to 0 with the constraint $\varepsilon = \lambda \delta t$, $\lambda > 0$ fixed.

Theorem 4.1 Convergence when ε and δt tend to 0**.**

Let $\Omega \in \mathbb{R}^d$ ($d = 2$ ou 3) be a simply connected bounded domain, $v^0 \in \mathbf{H}^1_0 \cap \mathbf{H}$ and $f \in L^2(]0, T[; \mathbf{L}^2)$.

We suppose that hypothesis $(\mathcal{H})$ is verified.

Then, up to a subsequence, $(v_{\delta t}, p_{\delta t})_{\delta t}$ the sequence of step functions defined by (1.4)–(1.8) and (3.1) converges towards (v, p), weak solution of the penalized Navier Stokes problem (1.2), when ε and δt tend to 0 with $\varepsilon = \lambda \delta t$, $\lambda > 0$ fixed. Furthermore, v and p satisfy:

$$v \in L^\infty(]0, T[; \mathbf{H}) \cap L^2(]0, T[; \mathbf{H}^1_0), \qquad p \in W^{-1,\infty}(]0, T[; \mathbf{L}^2_0).$$

Moreover, this solution is unique in two dimensional space.

4.1. Weak convergence of the velocity

We first establish the following result:

Lemma 4.2. *Under the hypothesis of Proposition 3.1, there exists $v \in L^2(]0, T[; \mathbf{H}^1_0)$ such that, up to a subsequence, $(v_{\delta t})_{\delta t}$ and $(\tilde{v}_{\delta t})_{\delta t}$ weakly converge towards v when ε and δt tend to 0 with $\varepsilon = \lambda \delta t$:*

(i) $(\tilde{v}_{\delta t})_{\delta t} \rightharpoonup v$ *weakly in* $L^2(]0, T[; \mathbf{H}^1_0)$

(ii) $(v_{\delta t})_{\delta t} \rightharpoonup v$ *weakly in* $L^2(]0, T[; \mathbf{H}^1_\nu)$.

Proof. This result directly comes from the stability study. Indeed, the properties of ψ (1.3) and Proposition 3.1 ensure that $(\tilde{v}_{\delta t})_{\delta t}$ is bounded in $L^2(]0, T[; \mathbf{H}^1_0)$. Then we can extract a subsequence that weakly converges towards v in $L^2(]0, T[; \mathbf{H}^1_0)$.

By definition, $v_{\delta t} = \tilde{v}_{\delta t} + \hat{w}_{\delta t}$, where from Lemma 3.3, when $\varepsilon = \lambda \delta t$, $\hat{w}_{\delta t}$ satisfies:

$$\hat{w}_{\delta t} \underset{\delta t \to 0}{\longrightarrow} 0 \text{ strongly in } L^2(]0, T[; \mathbf{H}^{-1}). \tag{4.1}$$

Moreover, Proposition 3.1 ensures that $\hat{w}_{\delta t} = w_{\delta t} - \tilde{w}_{\delta t}$ is bounded in $L^2(]0, T[; \mathbf{H}^1_\nu)$. Therefore, we can extract a subsequence such that $\hat{w}_{\delta t}$ weakly converges in $L^2(]0, T[; \mathbf{H}^1_\nu)$. Finally (4.1) implies that this limit is zero and we deduce the weak convergence of $v_{\delta t}$ towards v, the limit of $\tilde{v}_{\delta t}$. □

4.2. Strong convergence of the velocity

We demonstrate here the strong convergence of $w_{\delta t}$ towards $w = v - \psi$. To do so, we shall apply Aubin Lions Simon's Lemma 2.7 with $B_0 = \mathbf{F}$, $B_1 = \mathbf{H}$ and $B_2 = \mathbf{V}'$ where $\mathbf{F} = \left\{ v \in \mathbf{H}^1_\nu;\ \mathrm{div}(v) = 0 \text{ on } \Omega \right\}$.

In order to perform estimates on the time derivative, we introduce the piecewise linear function $w_{1,\delta t}$ defined by:

$$w_{1,\delta t}(t) = \frac{(t - t^k)w^{k+1} + (t^{k+1} - t)w^k}{\delta t} \text{ for } t \in [t^k, t^{k+1}[,\ k \le N - 1. \tag{4.2}$$

Aubin Lions Simon's Lemma cannot be directly applied to $w_{1,\delta t}$ since it does not belong to $\mathbf{H}$. To bypass this difficulty, we introduce the orthogonal projection P from $\mathbf{L}^2$ onto $\mathbf{H}$ in order to apply Aubin Lions Simon's Lemma on $P(w_{1,\delta t})$. The operator P verifies some continuity properties, there exists $C > 0$ such that:

$$\begin{aligned} &||P(v)||_{\mathbf{L}^2} \le C||v||_{\mathbf{L}^2} \text{ for } v \in \mathbf{L}^2 \\ &||P(v)||_{\mathbf{H}^1} \le C||v||_{\mathbf{H}^1} \text{ for } v \in \mathbf{H}^1. \end{aligned} \tag{4.3}$$

Using the Helmhotz−Hodge decomposition [26], there exists $q_{\delta t} \in L^2(]0,T[;\mathbf{H}^1)$ such that:

$$w_{1,\delta t} = P(w_{1,\delta t}) + \nabla q_{\delta t}. \tag{4.4}$$

Proposition 4.3. Strong convergence of the velocity. *Let* $p \in [2,+\infty[$. *Under the assumptions and notations of Lemma 4.2, if* δt *and* ε *tend to* 0 *with* $\varepsilon = \lambda \delta t$, $\lambda > 0$ *then* $(v_{\delta t})_{\delta t}$ *strongly converges towards* v *in* $L^p(]0,T[;\mathbf{L}^2)$.

Proof. We first consider the case where $p = 2$.

The proof of the Proposition is composed of three steps. First, we demonstrate the strong convergence of $P(w_{1,\delta t})$ towards a function g in $L^2(]0,T[;\mathbf{L}^2)$ when ε and δt tend to 0 using Aubin Lions Simon's Lemma 2.7. Then we show that the gradient term of the Helmholtz decomposition tends to 0 in $L^2(]0,T[;\mathbf{L}^2)$. We finally prove that the convergence applies for the step functions $w_{\delta t}$ and that $g = w = v - \psi$ so that $v_{\delta t}$ strongly converges towards v in $L^2(]0,T[;\mathbf{L}^2)$ when ε and δt tend to 0, $\varepsilon = \lambda\delta t$, $\lambda > 0$ fixed.

Step 1. The embedding of $\mathbf{F}$ in $\mathbf{H}$ is compact. Moreover, we also have $\mathbf{H} \subset \mathbf{V}'$ with continuous embedding by density of $\mathbf{V}$ in $\mathbf{H}$. By continuity of the orthogonal projection (4.3) and according to the stability result Proposition 3.1, $P(w_{1,\delta t})$ is bounded in $L^2(]0,T[;\mathbf{F})$. Finally Lemma A.2 (given in appendix) ensures that $\dfrac{\partial P(w_{1,\delta t})}{\partial t}$ is bounded in $L^{\frac{4}{3}}(]0,T[;\mathbf{V}')$. Therefore, applying the compactness result from Aubin Lions Simon Lemma 2.7 with $B_0 = \mathbf{F}$, $B_1 = \mathbf{H}$ and $B_2 = \mathbf{V}'$, we obtain the strong convergence of $P(w_{1,\delta t})$ towards a function g in $L^2(]0,T[;\mathbf{H})$.

Step 2. Let us establish the strong convergence of $\nabla q_{\delta t} = w_{1,\delta t} - P(w_{1,\delta t})$ to 0 in $L^2(]0,T[;\mathbf{L}^2)$. From (4.4), we have $\mathrm{div}(\nabla q_{\delta t}) = \mathrm{div}(w_{1,\delta t}) \in L^2(]0,T[;\mathbf{L}^2)$ so that $\nabla q_{\delta t}.\nu$ makes sense in $\mathbf{H}^{-\frac{1}{2}}(\partial\Omega)$. Thus $q_{\delta t}$ verifies the following Neumann problem:

$$\begin{cases} -\Delta q_{\delta t} + \mathrm{div}(w_{1,\delta t}) = 0 \quad \text{on} \quad [0,T] \times \Omega \\ (\nabla q_{\delta t}.\nu)_{\partial\Omega} = 0 \quad \text{on} \quad [0,T] \times \partial\Omega. \end{cases} \tag{4.5}$$

Since $\mathrm{div}(w_{\delta t}) \underset{\varepsilon \to 0}{\longrightarrow} 0$ in $L^2(]0,T[;\mathbf{L}^2)$ (see Lem. 3.3), then $\mathrm{div}(w_{1,\delta t})$ also tends to 0 in the same space. Thus, by ellipticity of the Neumann problem one has $q_{\delta t} \longrightarrow 0$ in $L^2(]0,T[;\mathbf{H}^1)$. We finally deduce that $w_{1,\delta t}$ strongly converges towards g in the space $L^2(]0,T[;\mathbf{L}^2)$.

Step 3. The two previous points show that $w_{1,\delta t}$ strongly converges towards a function g in the space $L^2(]0,T[;\mathbf{L}^2)$. Moreover, applying Lemma 4.2 we know that $w_{\delta t}$ weakly converges to $w = v - \psi$ in $L^2(]0,T[;\mathbf{H}^1)$

and from Lemma A.1 (given in appendix) $w_{1,\delta t} - w_{\delta t}$ strongly converges towards 0 in $L^2(]0,T[;\mathbf{L}^2)$ when δt tends to 0. It gives the strong convergence of $w_{\delta t}$ towards w in $L^2(]0,T[;\mathbf{L}^2)$ when δt tends to 0. Moreover, $\psi_{\delta t} \underset{\delta t \to 0}{\longrightarrow} \psi$. Consequently $v_{\delta t}$ strongly converges towards v in $L^2([0,T];\mathbf{L}^2)$.

Finally, $v_{\delta t}$ is bounded in $L^\infty(]0,T[;\mathbf{L}^2)$. Therefore, using Lemma 4.2, the following result holds:

$$\begin{cases} v_{\delta t} \underset{\delta t \to 0}{\longrightarrow} v \text{ strongly in } L^2(]0,T[;\mathbf{L}^2) \\ v_{\delta t} \underset{\delta t \to 0}{\rightharpoonup} v \text{ weakly-} \star \text{ in } L^\infty(]0,T[;\mathbf{L}^2). \end{cases}$$

Therefore, we deduce the strong convergence of $v_{\delta t}$ towards v when ε and δt tend to 0 in $L^p(]0,T[;\mathbf{L}^2)$ for any $p \in [2,+\infty[$ using interpolation properties. □

4.3. Weak convergence of the inertia terms

We study the convergence of the inertia term $B(v_{\delta t}(\cdot - \delta t), \tilde{v}_{\delta t}(\cdot))$.

Lemma 4.4. *If δt and ε tend to 0 with $\varepsilon = \lambda \delta t$ then*:

(i) $\mathrm{div}(v_{\delta t})\tilde{v}_{\delta t}$ *tends to* 0 *in* $L^1(]0,T[;\mathbf{L}^{\frac{3}{2}})$.
(ii) $(v_{\delta t}.\nabla)\tilde{v}_{\delta t}$ *weakly converges towards* $(v.\nabla)v$ *in* $L^p(]0,T[;\mathbf{L}^q)$, *with* $(p,q) = (\frac{4}{3},\frac{4}{3})$ *in two dimensions and* $(p,q) = (\frac{4}{3},\frac{6}{5})$ *in three dimensions.*

Proof of (i). From Lemma 3.3 $\mathrm{div}(v_{\delta t})$ tends to 0 in $L^2(]0,T[;\mathbf{L}^2)$. Then (i) is a consequence of the boundedness of $\tilde{v}_{\delta t}$ in $L^2(]0,T[;\mathbf{H}^1_0)$ exploiting the embedding of $\mathbf{H}^1$ into $\mathbf{L}^6$. □

Proof of (ii). We distinguish the cases of two and three dimensional spaces.

4.3.1. The three dimensional case

For $d=3$, from Hölder's inequalities, we have:

$$||(v_{\delta t}.\nabla)\tilde{v}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^1)} \leq ||v_{\delta t}||_{L^\infty(]0,T[;\mathbf{L}^2)} ||\nabla \tilde{v}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)}$$

and,

$$||(v_{\delta t}.\nabla)\tilde{v}_{\delta t}||_{L^1(]0,T[;\mathbf{L}^{\frac{3}{2}})} \leq ||v_{\delta t}||_{L^2(]0,T[;\mathbf{L}^6)} ||\nabla \tilde{v}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)}.$$

Thanks to Proposition 3.1 and Sobolev embeddings in three dimensions, the r.h.s. in the above estimates is uniformly bounded. Then by interpolation (see Prop. 2.5), we deduce that $(v_{\delta t}.\nabla)\tilde{v}_{\delta t}$ is uniformly bounded in $L^{\frac{4}{3}}(]0,T[;\mathbf{L}^{\frac{6}{5}})$. Therefore, in this space, the sequence $((v_{\delta t}.\nabla)\tilde{v}_{\delta t})_{\delta t}$ weakly converges towards a function g that remains to determine.

From the convergences of Lemma 4.2 and Proposition 4.3, we deduce the following weak convergence:

$$(v_{\delta t}.\nabla)\tilde{v}_{\delta t} \rightharpoonup (v.\nabla)v \text{ in } L^1(]0,T[;\mathbf{L}^1).$$

The weak convergence in the smaller space $L^{\frac{4}{3}}(]0,T[;\mathbf{L}^{\frac{6}{5}})$ yields $g = (v.\nabla)v$ and:

$$(v_{\delta t}.\nabla)\tilde{v}_{\delta t} \rightharpoonup (v.\nabla)v \text{ weakly in } L^{\frac{4}{3}}(]0,T[;\mathbf{L}^{\frac{6}{5}}). \tag{4.6}$$

Thus for $d=3$ Lemma 4.4 follows.

4.3.2. The two dimensional case

For a two dimensional space, we can demonstrate the convergence in a higher regularity space.

Indeed, as $v_{\delta t} \in L^\infty(]0,T[;\mathbf{L}^2) \cap L^2(]0,T[;\mathbf{H}^1)$ then by interpolation (see Prop. 2.5), $v_{\delta t}$ lies in $L^4(]0,T[;\mathbf{H}^{\frac{1}{2}})$. Yet, $\mathbf{H}^{\frac{1}{2}}$ is embedded into $\mathbf{L}^4$, therefore $v_{\delta t} \in L^4(]0,T[;\mathbf{L}^4)$.

Then $(v_{\delta t}.\nabla)\tilde{v}_{\delta t}$ is bounded in $L^{\frac{4}{3}}(]0,T[;\mathbf{L}^{\frac{4}{3}})$. Using the same arguments as above we deduce:

$$(v_{\delta t}.\nabla)\tilde{v}_{\delta t} \rightharpoonup (v.\nabla)v \text{ weakly in } L^{\frac{4}{3}}\left(]0,T[;\mathbf{L}^{\frac{4}{3}}\right)$$

and we conclude the proof of Lemma 4.4 in the two dimensional case. □

4.4. Proof of Theorem 4.1

We can now pass to the limit in the numerical scheme. Let $\phi \in \mathbf{V}$. The numerical scheme (1.4)-(1.8) reads in variational formulation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(v_{\delta t}(t),\phi) + 2\mu(D(\tilde{v}_{\delta t}) : D(\phi)) + (B(v_{\delta t}(t-\delta t),\tilde{v}_{\delta t}(t)),\phi) + \frac{1}{\eta}(\chi_{\omega(t)}(\tilde{v}_{\delta t}(t) - v_s(t)),\phi) = (f(t),\phi).$$

We multiply by a function $\theta \in \mathcal{C}^1(0,T)$ such that $\theta(T)=0$ and we integrate from 0 to T. We do not have any information on the time derivative of the velocity. Therefore, the temporal term is integrated by part so that the time derivative holds on θ:

$$\begin{aligned}
&-\int_0^T (v_{\delta t}(t),\phi)\,\theta'(t)\,\mathrm{d}t - (v_{\delta t}(0),\phi)\,\theta(0) + 2\mu\int_0^T (D(\tilde{v}_{\delta t}(t)) : D(\phi))\,\theta(t)\,\mathrm{d}t\\
&+\int_0^T ((v_{\delta t}(t-\delta t).\nabla)\tilde{v}_{\delta t}(t),\phi)\,\mathrm{d}t + \frac{1}{2}\int_0^T (\mathrm{div}(v_{\delta t}(t-\delta t))\tilde{v}_{\delta t}(t),\phi)\,\theta(t)\,\mathrm{d}t\\
&+\frac{1}{\eta}\int_0^T (\chi_{\omega(t)}(\tilde{v}_{\delta t}(t) - v_s(t)),\phi)\,\theta(t)\,\mathrm{d}t\\
&=\int_0^T (f_{\delta t},\phi)\,\theta(t)\,\mathrm{d}t.
\end{aligned}\tag{4.7}$$

We pass to the limit $\delta t \to 0$ in this last equation with $\varepsilon = \lambda\delta t$ using Lemmas 4.2 and 4.4. It gives:

$$\begin{aligned}
&-\int_0^T (v(t),\phi)\,\theta'(t)\,\mathrm{d}t - (v_0,\phi)\,\theta(0) + 2\mu\int_0^T (D(v(t)),D(\phi))\,\theta(t)\,\mathrm{d}t\\
&+\int_0^T ((v(t).\nabla)v(t),\phi)\,\theta(t)\,\mathrm{d}t + \frac{1}{\eta}\int_0^T (\chi_{\omega(t)}(v(t) - v_s(t)),\phi)\,\theta(t)\,\mathrm{d}t\\
&=\int_0^T (f(t),\phi)\,\theta(t)\,\mathrm{d}t.
\end{aligned}\tag{4.8}$$

Since ε also tends to 0 then from Lemma 3.3 and (1.3), we have at the limit:

$$\mathrm{div}(v) = 0 \qquad \text{on} \qquad \Omega.$$

Applying the above equality for $\theta \in \mathcal{D}(0,T)$, we deduce the following equality in $\mathbf{V}'$:

$$-\int_0^T v(t)\,\theta'(t)\,\mathrm{d}t = \int_0^T \mathrm{div}(2\mu D(v(t)))\,\theta(t)\,\mathrm{d}t - \frac{1}{\eta}\int_0^T \chi_{\omega(t)}(v(t)-v_s)\,\theta(t)\,\mathrm{d}t - \int_0^T (v(t).\nabla)v(t)\,\theta(t)\,\mathrm{d}t + \int_0^T f(t)\,\theta(t)\,\mathrm{d}t. \tag{4.9}$$

The operator $L : u \mapsto \mathrm{div}(2\mu D(u))$ (respectively $B : (u,v) \mapsto B(u,v)$) is continuous from $\mathbf{V}$ to $\mathbf{V}'$ (resp. from $\mathbf{V}\times\mathbf{V}$ to $\mathbf{V}'$). Therefore, there exists $C > 0$ such that:

$$\int_0^T ||\mathrm{div}(2\mu D(v))||_{\mathbf{V}'}\,\mathrm{d}t \le C\int_0^T ||v||_{\mathbf{V}}\,\mathrm{d}t \le C\sqrt{T}||v||_{L^2(]0,T[;\mathbf{V})}.$$
$$\int_0^T ||(v.\nabla)v||_{\mathbf{V}'}\,\mathrm{d}t \le C\int_0^T ||v||^2_{\mathbf{V}}\,\mathrm{d}t \le C||v||^2_{L^2(]0,T[;\mathbf{V})}. \tag{4.10}$$

As (4.9) is valid for any $\theta \in \mathcal{D}(]0,T[)$ we deduce that v has a weak derivative in time which lies in $L^1(]0,T[;\mathbf{V}')$ and for almost every $t \in]0,T[$:

$$\frac{\partial v}{\partial t} - \mathrm{div}(2\mu D(v)) + (v.\nabla)v + \frac{1}{\eta}\chi_{\omega(t)}(v-v_s) = f \quad \text{in} \quad \mathbf{V}'. \tag{4.11}$$

We now need to recover the initial data. Since $\dfrac{\partial v}{\partial t}$ belongs to $L^1(]0,T[;\mathbf{V}')$ and v belongs to $L^2(]0,T[;\mathbf{V})$, we show using Proposition 2.6 that v is continuous with values in $\mathbf{V}'$ for the strong topology. Furthermore, by hypothesis $v(0) = v_0$ in the weak continuity sense with values in $\mathbf{V}'$. Therefore, the initial condition $v(0) = v_0$ is verified in the strong sense because the weak limit is unique.

From De Rham theorem, we can now deduce the existence of the pressure. Let $G(t)$ be defined by:

$$G(t) = -\mathrm{div}(2\mu D(v)) + (v.\nabla)v + \frac{1}{\eta}\chi_{\omega(t)}(v-v_s) - f.$$

Thanks to (4.11), for almost every $t \in]0,T[$,

$$\left\langle \frac{\mathrm{d}v}{\mathrm{d}t}, \phi \right\rangle_{\mathbf{V}',\mathbf{V}} + \langle G(t), \phi\rangle_{\mathbf{H}^{-1},\mathbf{H}^1_0} = 0.$$

We integrate this last equation from 0 to t. It gives:

$$\langle v(t), \phi\rangle_{\mathbf{L}^2} - \langle v(0), \phi\rangle_{\mathbf{L}^2} + \left\langle \int_0^t G(\tau)\,\mathrm{d}\tau, \phi \right\rangle = 0.$$

It can be written under the form:

$$\langle K(t), \phi\rangle_{\mathbf{H}^{-1},\mathbf{H}^1_0} = 0,$$

where

$$K(t) = v(t) - v(0) - \int_0^t \mathrm{div}(2\mu D(v))\,\mathrm{d}\tau + \int_0^t (v.\nabla)v\,\mathrm{d}\tau + \frac{1}{\eta}\int_0^t \chi_{\omega(\tau)}(v-v_s)\,\mathrm{d}\tau - \int_0^t f\,\mathrm{d}\tau.$$

Note that K is weakly continuous in time with values in $\mathbf{H}^{-1}$. Therefore, for all $t \in]0,T[$ we deduce from De Rham theorem the existence of $\pi(t) \in \mathbf{L}^2_0$ such that:

$$K(t) = -\nabla \pi(t).$$

Following the work of [8] (chapter V), we show that $t \mapsto \pi(t)$ is weakly continuous in time with values in $\mathbf{L}^2$. In particular, π lies in the space $L^\infty(]0,T[;\mathbf{L}^2_0)$. Indeed, if $g \in \mathbf{L}^2$, there exists $h \in \mathbf{H}^1_0$ such that $\mathrm{div}(h) = g - m(g)$ where $m(g)$ denotes the mean value of g on Ω. Then,

$$\begin{aligned}
(\pi(t), g)_{\mathbf{L}^2} &= (\pi(t), g - m(g))_{\mathbf{L}^2} \text{ because } m(\pi) = 0 \\
&= (\pi(t), \mathrm{div}(h))_{\mathbf{L}^2} \\
&= -(\nabla \pi(t), h)_{\mathbf{H}^{-1},\mathbf{H}^1_0} \\
&= (K(t), h)_{\mathbf{H}^{-1},\mathbf{H}^1_0}.
\end{aligned}$$

This quantity is continuous because K is weakly continuous in time with values in $\mathbf{H}^{-1}$. We can then introduce the distribution $p = \dfrac{\partial \pi}{\partial t}$ which lies in the space $W^{-1,\infty}(]0,T[;\mathbf{L}^2_0)$. Taking test functions under the form $\dfrac{\partial \phi}{\partial t}$ with $\phi \in \mathcal{D}(]0,T[\times\Omega)$, we show that the equation

$$\frac{\partial v}{\partial t} - \mathrm{div}(2\mu D(v)) + (v.\nabla)v + \frac{1}{\eta}\chi_{\omega(t)}(v - v_s) + \nabla p = f \tag{4.12}$$

is satisfied in the sense of distributions.

In two dimensional space we can show the uniqueness of solutions of this equation using classical results (see [8] Chapt. V).

5. Convergence towards the Navier-Stokes equations

The aim of this section is to study the convergence when η tends to 0. To do so, we consider the weak limit of the scheme when ε and δt tend to 0 which verifies (4.12) and indice the solution by η.

First, inspired by ([20], Def. 2.1) (see also [16]), let us define a weak solution of (1.1), the Navier Stokes equation in the moving domain $\Omega(t) = \Omega\backslash\omega(t)$. We assume that $(\mathcal{H})$ holds and there exists a C^∞-diffeomorphism between $\bigcup_{t<T} \{t\} \times \Omega(t)$ and a cylindrical domain $[0,T] \times \tilde{\Omega}$ for some bounded domain $\tilde{\Omega} \subset I\!R^d$. Then as for Ω (see Notations in the Introduction), we can define $\tilde{\mathbf{H}}$ (resp. $\tilde{\mathbf{V}}$) on $\tilde{\Omega}$ and using the pull back by the C^∞-diffeomorphism we also define $\mathbf{H}_t$ (resp. $\mathbf{V}_t$) the associated space on $\Omega(t)$. Moreover, using this diffeomorphism, we are able to transport any function, any space and any equation on $[0,T] \times \tilde{\Omega}$ onto $\bigcup_{t<T} \{t\} \times \Omega(t)$. For ψ given by $(\mathcal{H})$, we will say that v is a weak solution of (1.1) if, for any $\phi \in \mathbf{V}_t$ and any $\theta \in \mathcal{C}^1(0,T)$, with $\theta(T) = 0$, the function $v - \psi \in L^\infty(]0,T[;\mathbf{H}_t) \cap L^2(]0,T[;\mathbf{V}_t)$ satisfies:

$$\begin{aligned}
-\int_0^T (v(t), \phi)_t\, \theta'(t)\,\mathrm{dt} - (v_0, \phi)_0\, \theta(0) + 2\mu \int_0^T (D(v(t)), D(\phi))_t\, \theta(t)\,\mathrm{dt} \\
+ \int_0^T ((v(t).\nabla)v(t), \phi)_t\, \theta(t)\,\mathrm{dt} = \int_0^T (f(t), \phi)_t\, \theta(t)\,\mathrm{dt}.
\end{aligned}$$

where $(\cdot,\cdot)_t$ denotes the scalar product in $\mathbf{L}^2(\Omega(t))$.

Theorem 5.1 Convergence when η tends to 0**.**

Under the above assumptions, when η tends to 0*, the sequence $(v_\eta)_\eta$, weak solution of the penalized Navier Stokes problem* (1.2) *weakly converges towards v a weak solution of* (1.1)*, the Navier Stokes problem on the time dependent domain $\Omega(t) = \Omega \setminus \omega(t)$.*

We first prove the following lemma:

Lemma 5.2. *For all $t \in]0, T[$, and $\eta \in]0, 1]$, we have*:

$$\int_0^t ||v_\eta - v_s||^2_{L^2(\partial\omega(\tau))} \,\mathrm{d}\tau \le C\eta^{\frac{1}{2}}.$$

Proof. For a.e. $t \in]0, T[$,

$$||v_\eta - v_s||^2_{L^2(\partial\omega(t))} \le C||v_\eta - v_s||_{L^2(\omega(t))}||v_\eta - v_s||_{H^1(\omega(t))}.$$

Going back to the fixed domain and using norm equivalence properties (see [16,20]), we show that the constant C is uniform in time. We integrate this last inequality from 0 to t and obtain:

$$\begin{aligned}\int_0^t ||v_\eta - v_s||^2_{L^2(\partial\omega(\tau))} \,\mathrm{d}\tau &\le C \int_0^t ||v_\eta - v_s||_{L^2(\omega(\tau))}||v_\eta - v_s||_{H^1(\omega(\tau))} \,\mathrm{d}\tau \\ &\le C \left(\int_0^t ||v_\eta - v_s||^2_{L^2(\omega(\tau))} \,\mathrm{d}\tau \right)^{\frac{1}{2}} \left(\int_0^t ||v_\eta - v_s||^2_{H^1(\omega(\tau))} \,\mathrm{d}\tau \right)^{\frac{1}{2}} \\ &\le C \left(\int_0^t ||v_\eta - v_s||^2_{L^2(\omega(\tau))} \,\mathrm{d}\tau \right)^{\frac{1}{2}} ||w_\eta||_{L^2(]0,T[;\mathbf{H}^1)}.\end{aligned}$$

From Proposition 3.1 (see also Rem. 3.2), $w_{\delta t}$ is uniformly bounded in $L^2(]0, T[; \mathbf{H}^1)$, with respect to $\eta \in]0, 1]$ and to $\dfrac{\delta t}{\varepsilon}$. Using the lower semicontinuity of the norm for the weak topology, we obtain that w_η is also bounded in $\mathbf{H}^1$. Finally, from the energy estimates of Section 3 (see Rem. 3.2), we obtain:

$$\int_0^t ||v_\eta - v_s||^2_{L^2(\partial\omega(\tau))} \,\mathrm{d}\tau \le C\eta^{\frac{1}{2}}. \qquad \square$$

Therefore, when $\eta \to 0$, the velocity on the immersed boundary $\partial\omega(t)$ tends towards the obstacle velocity $v = v_s$ in the space $L^2(]0, T[; L^2(\partial\omega(t)))$.

Proof of Theorem 5.1. Let $\phi \in \mathbf{V}_t$ and $\theta \in \mathcal{C}^1(0, T)$ such that $\theta(T) = 0$. Since $\phi(t, .) \in \mathbf{H}^1_0(\Omega(t))$, it can be extended to a function $\overline{\phi} \in \mathbf{V}$ such that $\overline{\phi}(t, .) = 0$ on $\omega(t)$. Then taking $\overline{\phi}$ instead of ϕ in (4.8) we deduce:

$$\begin{aligned}-\int_0^T (v_\eta(t), \overline{\phi})\, \theta'(t) \,\mathrm{d}t - (v_0, \overline{\phi})\, \theta(0) + 2\mu \int_0^T (D(v_\eta(t)), D(\overline{\phi}))\, \theta(t) \,\mathrm{d}t \\ + \int_0^T ((v_\eta(t).\nabla)v_\eta(t), \overline{\phi})\, \theta(t) \,\mathrm{d}t = \int_0^T (f(t), \overline{\phi})\, \theta(t) \,\mathrm{d}t,\end{aligned} \tag{5.1}$$

where we used that $\overline{\phi}$ and $\chi_{\omega(t)}$ have disjoint supports. Let us mention that this above substitutions can be justified coming back to a fixed cylindrical domain $[0,T]\times\tilde{\Omega}$ (see above or [20] for more details).

Now, following the proof of Theorem 4.1 (in particular estimates of Prop. 3.1) we obtain that $(v_\eta)_\eta$ is bounded in $L^\infty(]0,T[;\mathbf{H})\cap L^2(]0,T[;\mathbf{V})$ (see also Rem. 3.2). Thus, up to a subsequence, $(v_\eta)_\eta$ weakly converge in $L^\infty(]0,T[;\mathbf{H})\cap L^2(]0,T[;\mathbf{V})$ to a fonction v which, thanks to Lemma 5.2, satisfies $(v-\psi)_{|\partial\omega(t)}=0$.

We conclude the proof taking the limit as η goes to 0 in (5.1) and using that $\overline{\phi}(t,.)$ being supported in $\Omega(t)$, we have $(v(t),\overline{\phi}(t,.))=(v(t),\phi(t,.))_t$. $\square$

Appendix A.

In this appendix we use the assumptions and the notations of Section 4.2. We prove auxiliary results for the proof of Proposition 4.3.

Lemma A.1. *The difference between the piecewise linear function and the step function $w_{1,\delta t}-w_{\delta t}$ strongly converges to* 0 *in the space $L^2(]0,T[;\mathbf{L}^2)$, when δt goes to* 0.

Proof. We first show that $w_{1,\delta t}-w_{\delta t}$ strongly converges to 0 in $L^2(]0,T[;\mathbf{H}^{-1})$ when δt tends to 0. Using Lemma 3.4 we have:

$$\begin{aligned}||w_{1,\delta t}-w_{\delta t}||^2_{L^2(]0,T[;\mathbf{H}^{-1})} &\leq \sum_{k=0}^{N-1}\delta t||w^{k+1}-w^k||^2_{\mathbf{H}^{-1}}\\ &\leq C\delta t \end{aligned}\tag{A.1}$$

which gives the strong convergence of the difference $w_{1,\delta t}-w_{\delta t}$ to 0 in $L^2(]0,T[;\mathbf{H}^{-1})$ when δt goes to 0. Moreover, as both functions $w_{\delta t}$ and $w_{1,\delta t}$ are bounded in $L^2(]0,T[;\mathbf{H}^1)$ (see Prop. 3.1), the difference $w_{1,\delta t}-w_{\delta t}$ is bounded in $L^2(]0,T[;\mathbf{H}^1)$. We then deduce the strong convergence $w_{1,\delta t}-w_{\delta t}\longrightarrow 0$ when δt goes to 0 in $L^2(]0,T[;\mathbf{L}^2)$. $\square$

The following Lemma gives a bound on the time derivative in the space $L^{\frac{4}{3}}(]0,T[;\mathbf{V}')$.

Lemma A.2. *Using the definition of the piecewise linear functions $w_{1,\delta t}$* (4.2)*, there exists a constant $C>0$ such that the time derivative of $P(w_{1,\delta t})$ verifies*:

$$\left|\left|\frac{\partial P(w_{1,\delta t})}{\partial t}\right|\right|_{L^{\frac{4}{3}}(]0,T[;\mathbf{V}')}\leq C.$$

Proof. To obtain the bound in the dual space of $\mathbf{V}$, we have to show that there exists a constant $C>0$ such that for any $\phi\in\mathbf{V}$:

$$\left|\int_0^T\left\langle\frac{\partial P(w_{1,\delta t})}{\partial t},\phi\right\rangle_{\mathbf{V}',\mathbf{V}}\mathrm{d}t\right|\leq C||\phi||_{L^4(]0,T[;\mathbf{V})}.$$

Let $\phi\in\mathbf{V}$. We perform estimates on the variational formulation of the problem (3.3)−(3.5). Since ϕ vanishes on the boundary $\partial\Omega$ and $\mathrm{div}(\phi)=0$, the pressure cancelled.

The temporal term is decomposed using the orthogonal projection P. There exists a function $q_{\delta t}\in\mathbf{H}^1$ such that $w_{1,\delta t}=P(w_{1,\delta t})+\nabla q_{\delta t}$. Since we work at a discrete level, the time derivative of the piecewise linear function $w_{1,\delta t}$ is equal to $\dfrac{w^{k+1}-w^k}{\delta t}$ on each interval $[t^k,t^{k+1}[$ $(k\leq N-1)$ and commutes with P. Then we can integrate by part the gradient term $\nabla q_{\delta t}$ of the Helmholtz decomposition without consideration on the time derivative. Finally this last term vanishes using again that $\mathrm{div}(\phi) = 0$ and $\phi = 0$ on the boundary and the variational formulation reads:

$$\int_0^T < \frac{\partial P(w_{1,\delta t})}{\partial t}, \phi >_{\mathbf{V}',\mathbf{V}} \,\mathrm{d}t + 2\mu \int_0^T D(w_{\delta t}) : D(\phi)\,\mathrm{d}t + \int_0^T B(w_{\delta t}(t-\delta t), \tilde{w}_{\delta t}(t)).\phi\,\mathrm{d}t$$

$$+ \int_0^T B(\psi_{\delta t}(t-\delta t), \tilde{w}_{\delta t}).\phi\,\mathrm{d}t + \int_0^T B(w_{\delta t}(t-\delta t), \psi(t)).\phi\,\mathrm{d}t + \frac{1}{\eta}\int_0^T \chi_{\omega(t)}\tilde{w}_{\delta t}.\phi\,\mathrm{d}t = \int_0^T F.\phi\,\mathrm{d}t. \tag{A.2}$$

We now have to perform estimates on inertia terms. We detail here the estimates for the convective term $b(w_{\delta t}, \tilde{w}_{\delta t}, \phi)$. The same process applies on a simpler way for the two other terms.

Going back to the definition of the convective term (2.2), the trilinear form reads:

$$b(w_{\delta t}, \tilde{w}_{\delta t}, \phi) = \frac{1}{2}\int_\Omega (w_{\delta t}.\nabla)\tilde{w}_{\delta t}.\phi\,\mathrm{d}x + \frac{1}{2}\int_\Omega (w_{\delta t}.\nabla)\phi.\tilde{w}_{\delta t}\,\mathrm{d}x.$$

Using Hölder's inequality then Sobolev embeddings $\mathbf{H}^1_\nu \hookrightarrow \mathbf{H}^{\frac{1}{2}} \hookrightarrow \mathbf{L}^3$ and $\mathbf{H}^1 \hookrightarrow \mathbf{L}^6$ for $d \le 3$, there exists a constant $C > 0$ such that:

$$\begin{aligned}||(w_{\delta t}.\nabla)\tilde{w}_{\delta t}.\phi||_{\mathbf{L}^1} &\le C||w_{\delta t}||_{\mathbf{L}^3}||\nabla\tilde{w}_{\delta t}||_{\mathbf{L}^2}||\phi||_{\mathbf{L}^6}\\ &\le C||w_{\delta t}||^{\frac{1}{2}}_{\mathbf{L}^2}||\nabla w_{\delta t}||^{\frac{1}{2}}_{\mathbf{L}^2}||\nabla\tilde{w}_{\delta t}||_{\mathbf{L}^2}||\nabla\phi||_{\mathbf{L}^2}.\end{aligned}$$

This last inequality is integrated in time using again Hölder's inequality. The factor $||\nabla w_{\delta t}||^{\frac{1}{2}}_{\mathbf{L}^2}$ lies in $L^4(]0,T[)$ so that we have:

$$\begin{aligned}||(w_{\delta t}.\nabla)\tilde{w}_{\delta t}.\phi||_{L^1(]0,T[;\mathbf{L}^1)} &\le ||w_{\delta t}||^{\frac{1}{2}}_{L^\infty(]0,T[;\mathbf{L}^2)}\left|\left|\left(||\nabla w_{\delta t}||^{\frac{1}{2}}_{\mathbf{L}^2}\right)\right|\right|_{L^4(]0,T[)}||\nabla\tilde{w}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)}||\phi||_{L^4(]0,T[;\mathbf{V})}\\ &\le ||w_{\delta t}||^{\frac{1}{2}}_{L^\infty(]0,T[;\mathbf{L}^2)}||\nabla w_{\delta t}||^{\frac{1}{2}}_{L^2(]0,T[;\mathbf{L}^2)}||\nabla\tilde{w}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)}||\phi||_{L^4(]0,T[;\mathbf{V})}.\end{aligned}$$

Using the same arguments, we obtain a same expression for the second term:

$$||(w_{\delta t}.\nabla)\phi.\tilde{w}_{\delta t}||_{L^1(]0,T[;\mathbf{L}^1)} \le ||w_{\delta t}||^{\frac{1}{2}}_{L^\infty(]0,T[;\mathbf{L}^2)}||\nabla w_{\delta t}||^{\frac{1}{2}}_{L^2(]0,T[;\mathbf{L}^2)}||\nabla\tilde{w}_{\delta t}||_{L^2(]0,T[;\mathbf{L}^2)}||\phi||_{L^4(]0,T[;\mathbf{V})}.$$

Finally, using Proposition 3.1, the following inequality is obtained:

$$\left|\int_0^T \langle\frac{\partial P(w_{1,\delta t})}{\partial t}, \phi\rangle_{\mathbf{V}',\mathbf{V}}\,\mathrm{d}t\right| \le C||\phi||_{L^4(]0,T[;\mathbf{V})}.$$

which ensures that $\dfrac{\partial P(w_{1,\delta t})}{\partial t}$ is bounded in the space $L^{\frac{4}{3}}(]0,T[;\mathbf{V}')$. □

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