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ON THE STEKLOV PROBLEM IN A DOMAIN PERFORATED ALONG A PART OF THE BOUNDARY *

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Abstract. We study the asymptotic behavior of solutions and eigenelements to a 2-dimensional and 3-dimensional boundary value problem for the Laplace equation in a domain perforated along part of the boundary. On the boundary of holes we set the homogeneous Dirichlet boundary condition and the Steklov spectral condition on the mentioned part of the outer boundary of the domain. Assuming that the boundary microstructure is periodic, we construct the limit problem and prove the homogenization theorem.

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1. Introduction

The Steklov spectral problem is well known and it has been studying for several decades (see, for instance [1–16]). In the paper [3] the author constructed leading terms of the asymptotic expansion of solutions in the case of spectral Steklov-type problem in a thin domain with a nonsmooth boundary. In the paper [6] the authors prove the connection of the first eigenvalue to a Steklov-type problem in the domain with micro perforation and the constant in the Sobolev inequality for traces. In [7] it was studied the asymptotic behavior of the eigenvalues and respective eigenfunctions to the Steklov-type spectral problem in micro inhomogeneous plane domain. The paper [8] concerns the error in the final elements method for the Steklov-type spectral problem. In [9] the author considers the Steklov-type problem for the p-Laplacian. In [10] the authors studied the connection between the Neumann problem for the Hénon equation and the eigenvalues to one Steklov-type

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problem. The paper [11] is devoted to the influence of boundary conditions on the positiveness of the inverse operator to the biharmonic operator in the case of Steklov-type boundary conditions. In [12] the authors investigate the Steklov spectral problem in a domain with a peak (degenerated corner point) on the boundary. The elliptic problem with critical growth and the Steklov-type spectral conditions in bounded domain was considered in [13]. In the paper [14] the author investigates the homogenization problem with rapidly alternating boundary conditions (the Dirichlet and the Steklov conditions) in the case of dominating of the Steklov condition in the limit. The papers [15, 16] concern the degenerating case of the Steklov-type problem in domains with rapidly alternating boundary conditions.

There are numerous papers dealing with homogenization of problems in domains perforated along the boundary (*e.g.*, see [17–34]). In [17] (see also the short communication [18]), problems in domains were considered under the assumption that the diameter of holes is much less than the distance between them. In particular, the homogenization theorem was proved for a boundary value problem with the Neumann boundary conditions on the exterior boundary and the Dirichlet condition on the boundary of the cavities, the first term of perturbation theory was constructed, and the deviation of the leading terms of the asymptotic expansion from the solution of the original problem was estimated. In addition, the spectral properties of such boundary value problem were analyzed, and the closeness of eigenelements of the original and homogenized problems was estimated. A degenerate quasilinear Dirichlet problem was considered in [25] in a domain with nonperiodic cavities near a boundary. In particular, conditions for the existence of a limit problem were derived, and the weak convergence in L_2 was proved. A problem in a domain perforated along a closed curve was considered in ([26], Chap. I, Sect. 3). It was shown that the solutions of the original problem converge uniformly to the solutions of the limit problem in compact subdomains that do not contain that curve. The paper [27,28] deals with the Dirichlet problem in a domain with nonperiodic cavities, of which perforation along the boundary can be viewed as a special case. The weak convergence of solutions in L_2 was proved in terms of the convergence of the harmonic capacity of cavities. The asymptotic behavior of the solution in a domain perforated randomly along the boundary was analyzed in [29] (see also [30]). The weak convergence of the solution of the original problem with a random structure to the solution of the nonrandom problem with homogenized boundary conditions on the boundary of the domain was proved in the Sobolev space H^1. Solutions of boundary value problems in a domain divided into two parts by a perforated surface of variable thickness were considered in [31, 34]. In particular, the weak convergence of solutions of the original problem to solutions of two independent problems in domains divided by this surface was proved in L_2. The asymptotic behavior of solutions of the boundary value problem in a domain perforated along a manifold with various boundary conditions on the boundaries of cavities was analyzed in [32]. The case in which the perforation makes no contribution in the limit was considered. A problem with a random perforation formed by the union of randomly placed balls of fixed radius multiplied by a small parameter was studied in [33]. The case of perforation along a curve was also considered. The convergence of the eigenvalues was proved for the case in which the perforation vanishes in the limit.

In the papers [19–24] the authors consider problems in 2-dimensional domains periodically perforated along a part of the boundary under the assumption that the Dirichlet boundary condition is set on the boundary of the cavities. Unlike the results of [17, 18], these papers are devoted to the case, when the diameter of the cavities and the distance between them are of the same order of smallness. In [19, 20] the authors obtain the limit (homogenized) problem and prove the strong convergence in H^1 of solutions of the original problem to the solutions of the limit problem. For this case in [21] the authors construct two-terms asymptotics of a solution to a boundary value problem. In [22, 23] it were constructed the leading two terms of the asymptotic expansion of eigenvalues converging to a simple and multiple eigenvalues, respectively, of the limit problem. In the paper [24] the authors study boundary value problem in the case, when the diameter of the cavities and the distance between them are of the same order of smallness as well as in the case when the ratio between the diameter of the cavities and the distance between them tends to zero.

In this paper we study spectral problems with Steklov-type boundary condition in 2D and 3D domains periodically perforated along part of the boundary. Under the assumption that the ratio between the diameter

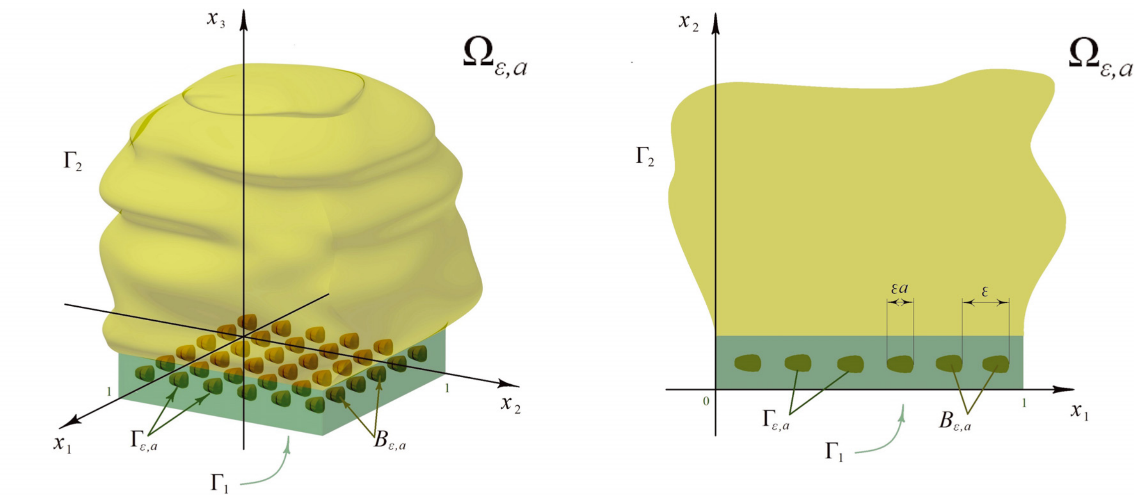

FIGURE 1. Structure of the domain $\Omega_{\varepsilon,a}$.

of the cavities and the distance between them tends to zero, we present different cases of the limiting behavior of eigen-pairs.

The structure of the paper is the following. Section 2 is devoted to the settings and statements of main results. Next Section 3 is devoted to the auxiliary propositions. In Section 4 we construct a model function in semistrip and semi-infinite parallelepiped, which we use in the proof of Theorem 2.2. The proof is in Section 5. Section 6 is devoted to the proof of Theorem 2.3. In Section 7 we prove variational estimate for semistrip and for semi-infinite parallelepiped with the small hole, which we use in the proof of Theorems 2.5 and 2.6. In Section 8 one can find these proofs.

2. SETTING OF THE PROBLEM AND MAIN RESULTS

Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, situated in the semi-plane $x_2 > 0$ for $d = 2$ and in the semi-space $x_3 > 0$ for $d = 3$. Its boundary Γ consists of two parts: $\Gamma = \Gamma_1 \cup \Gamma_2$, where Γ_1 is the segment $[0, 1]$ in the axis $x_2 = 0$ for $d = 2$ and square $[0, 1]^2$ in the plane $x_3 = 0$ for $d = 3$. For $d = 2$, the part Γ_2 is infinitely differentiable and in a neighborhood of the points $(0, 0)$ and $(1, 0)$ coincides with lines $x_1 = 0$ and $x_1 = 1$, respectively. For $d = 3$, the part Γ_2 coincides with the lateral faces of the cube $[0, 1]^3$ in a small neighborhood of the plane $x_3 = 0$ and in addition it is infinitely differentiable everywhere except the vertical edges.

Then assume that B is an arbitrary bounded domain with Lipschitz boundary. Denote $B_a = \{x : \big(a^{-1}(x_1 - b_1), a^{-1}(x_2 - c)\big) \in B\}$ for $d = 2$, $B_a = \{x : \big(a^{-1}(x_1 - b_1), a^{-1}(x_2 - b_2), a^{-1}(x_3 - c)\big) \in B\}$, $j = 1, 2$, for $d = 3$, where $0 < b_j < 1$, $c > 0$ are arbitrary fixed numbers, a is sufficiently small positive parameter, such that $\overline{B_a}$ lies in the semi-strip $\Pi = (0, 1) \times (0, \infty)$ for $d = 2$ and in the semi-infinite parallelepiped $\Pi = (0, 1)^2 \times (0, \infty)$ for $d = 3$.

Denote $B^{\mathbf{k}}_{\varepsilon,a} = \{x : (\varepsilon^{-1}x_1 - \mathbf{k}, \varepsilon^{-1}x_2) \in B_a\}$, $\mathbf{k} \in \mathbb{Z}$ for $d = 2$, $B^{\mathbf{k}}_{\varepsilon,a} = \{x : (\varepsilon^{-1}x_1 - k_1, \varepsilon^{-1}x_2 - k_2, \varepsilon^{-1}x_3) \in B_a\}$, $\mathbf{k} = (k_1, k_2)$, $k_j \in \mathbb{Z}$ for $d = 3$, $B_{\varepsilon,a} = \bigcup\limits_{\mathbf{k}} B^{\mathbf{k}}_{\varepsilon,a}$, $\Gamma_{\varepsilon,a} = \partial B_{\varepsilon,a}$. Hereafter ε is a small positive parameter, $\varepsilon = \varepsilon_N = \frac{1}{N}$, where $N \gg 1$ is a natural number. Define the domain $\Omega_{\varepsilon,a}$ as $\Omega \setminus \overline{B_{\varepsilon,a}}$ (see Fig. 1).

Remark 2.1. In fact we have three scales in the geometry. The magnitude of the diameters of the cavities has the order $\mathcal{O}(a\varepsilon)$, the distance between consecutive holes is equal to ε and the diameter of the domain is of order $\mathcal{O}(1)$.

Solutions of boundary value problems

$$
\begin{aligned}
&-\Delta U_{\varepsilon,a}=0 \quad \text{in } \Omega_{\varepsilon,a}, \qquad && U_{\varepsilon,a}=0 \quad \text{on } \Gamma_{\varepsilon,a},\\
&\frac{\partial U_{\varepsilon,a}}{\partial\nu}=\lambda U_{\varepsilon,a}+f \quad \text{on } \Gamma_1, && \frac{\partial U_{\varepsilon,a}}{\partial\nu}=0 \quad \text{on } \Gamma_2,
\end{aligned}
\tag{2.1}
$$

$$
-\Delta U_0=0 \text{ in } \Omega, \quad \frac{\partial U_0}{\partial\nu}+\sigma_d C_d(B)AU_0=\lambda U_0+f \text{ on } \Gamma_1, \quad \frac{\partial U_0}{\partial\nu}=0 \text{ on } \Gamma_2, \tag{2.2}
$$

where ν is an outer normal, $\lambda\in\mathbb{R}$, and eigenfunctions of spectral problems

$$
\begin{aligned}
&-\Delta u_{\varepsilon,a}=0 \quad \text{in } \Omega_{\varepsilon,a}, \qquad && u_{\varepsilon,a}=0 \quad \text{on } \Gamma_{\varepsilon,a},\\
&\frac{\partial u_{\varepsilon,a}}{\partial\nu}=\lambda_{\varepsilon,a} u_{\varepsilon,a} \quad \text{on } \Gamma_1, && \frac{\partial u_{\varepsilon,a}}{\partial\nu}=0 \quad \text{on } \Gamma_2,
\end{aligned}
\tag{2.3}
$$

$$
-\Delta u_0=0 \text{ in } \Omega, \quad \frac{\partial u_0}{\partial\nu}+\sigma_d C_d(B)Au_0=\lambda_0 u_0 \text{ on } \Gamma_1, \quad \frac{\partial u_0}{\partial\nu}=0 \text{ on } \Gamma_2 \tag{2.4}
$$

are weak (see, for instance, [35], Chap. IV and the next section). Note that $\lambda_{\varepsilon,a}$ and λ_0 are real.

Hereafter $\sigma_2=2\pi$, $\sigma_3=4\pi$, $C_2(B)=1$ and $C_3(B)>0$ is the capacity (see [36] and Rem. 4.6 below) of the domain B. In particular, $C_3(B)=1$, if B is a unit ball.

The main goal of the paper is to prove the following statements.

Theorem 2.2. *Suppose that*

$$
-\frac{1}{\varepsilon\ln a}\longrightarrow A\neq\infty \quad \textit{for } d=2, \qquad \frac{a}{\varepsilon}\longrightarrow A\neq\infty \quad \textit{for } d=3, \tag{2.5}
$$

$f\in L_2(\Gamma_1)$ and λ is not an eigenvalue of the problem (2.4).

Then:

1) *boundary value problem (2.1) has a unique solution in $W_2^1(\Omega_{\varepsilon,a})$ for any sufficiently small ε, and moreover the following uniform in ε estimate:*

$$
\|U_{\varepsilon,a}\|_{W_2^1(\Omega)}\leqslant C\|f\|_{L_2(\Gamma_1)} \tag{2.6}
$$

holds true, where the function $U_{\varepsilon,a}$ is extended in $\overline{B_{\varepsilon,a}}$ by zero;

2) *for the solution of problem (2.1) the following strong convergence*

$$
U_{\varepsilon,a}\underset{\varepsilon\to0}{\to}U_0 \qquad \textit{in } W_2^1(\Omega) \tag{2.7}
$$

takes place, if $A=0$, and the weak convergence

$$
U_{\varepsilon,a}\underset{\varepsilon\to0}{\rightharpoonup}U_0 \qquad \textit{in } W_2^1(\Omega) \tag{2.8}
$$

holds true, if $A\neq0$, where U_0 is a solution of the homogenized (limit) problem (2.2).

Theorem 2.3.

I. *Suppose that the condition (2.5) holds, and the multiplicity of the eigenvalue λ_0 to the problem (2.4) equals to n. Then there exist n eigenvalues $\lambda_{\varepsilon,a}^{(l)}$ of problem (2.3), $l=1,\dots,n$ (with respect to their multiplicities) converging to λ_0 as $\varepsilon\to0$.*

II. *Let* $u^{(l)}_{\varepsilon,a}$ *be orthonormalized in* $L_2(\varGamma_1)$ *eigenfunctions of problem* (2.3)*, corresponding to* $\lambda^{(l)}_{\varepsilon,a}$*. Then from the sequence* $\left\{\varepsilon_k = \frac{1}{k}\right\}_{k=1}^{\infty}$ *and any sequence* $\{a_k\}_{k=1}^{\infty}$ *(for which the convergence* (2.5) *takes place) one can choose subsequences* $\{\varepsilon_{k'}\}$*,* $\{a_{k'}\}$ *such that the convergence*

$$u^{(l)}_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} u^{(l)}_{*} \qquad \text{in } W_2^1(\varOmega) \tag{2.9}$$

holds, if $A = 0$*, and weak convergence*

$$u^{(l)}_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} u^{(l)}_{*} \qquad \text{in } W_2^1(\varOmega) \tag{2.10}$$

holds, if $A \neq 0$*, where the functions* $u^{(l)}_{\varepsilon,a}$ *are extended in* $\overline{B_{\varepsilon,a}}$ *by zero, and* $u^{(l)}_{*}$ *are orthonormalized in* $L_2(\varGamma_1)$ *eigenfunctions of problem* (2.4)*, corresponding to* λ_0 *(they depend on the choice of the sequence* $\{a_k\}_{k=1}^{\infty}$ *and the subsequence).*

Remark 2.4. From Theorem 2.3 it follows that if λ_0 is a simple eigenvalue of the homogenized problem (2.4), u_0 is the respective normalized in $L_2(\varGamma_1)$ eigenfunction, the condition (2.5) holds, then $\lambda_{\varepsilon,a}$ is the unique eigenvalue of problem (2.3), converging to λ_0, and for corresponding eigenfunction $u_{\varepsilon,a}$ (normalized in $L_2(\varGamma_1)$ and extended by zero in $B_{\varepsilon,a}$) up to the sign of $u_{\varepsilon,a}$ (*i.e.* with right choice of the sign of $u_{\varepsilon,a}$) the convergence

$$\|u_{\varepsilon,a} - u_0\|_{W_2^1(\varOmega)} \underset{\varepsilon\to 0}{\longrightarrow} 0$$

holds for $A = 0$, and

$$u_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} u_0 \quad \text{â } W_2^1(\varOmega),$$

holds for $A \neq 0$.

Theorem 2.5. *Assume that*

$$\varepsilon \ln a \longrightarrow 0 \quad \text{for } d = 2, \qquad \frac{\varepsilon}{a} \longrightarrow 0 \quad \text{for } d = 3. \tag{2.11}$$

Then the minimal eigenvalue $\lambda_{\varepsilon,a}$ *of problem* (2.3) *converge to* $+\infty$ *as* $\varepsilon \to 0$*.*

Theorem 2.6. *Assume that the condition* (2.11) *holds,* $f \in L_2(\varGamma_1)$ *and* λ *is fixed.*

Then for the solution of problem (2.1) *the strong convergence*

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\to} 0 \qquad \text{in } W_2^1(\varOmega) \tag{2.12}$$

holds, where the function $U_{\varepsilon,a}$ *is extended in* $\overline{B_{\varepsilon,a}}$ *by zero.*

Remark 2.7. The assumption (2.5) (in Thms. 2.2 and 2.3) means that the cavities are distributed sufficiently "rarely" along the part of the boundary. This assumption leads to Neumann or Fourier (Robin) type boundary conditions on $\varGamma_1$ in the homogenized problem.

The asymptotics (2.11) means that the cavities are distributed sufficiently "frequently" along the part of the boundary. In this case the homogeneous Dirichlet condition is the limit (homogenized) boundary condition on $\varGamma_1$. Note that the asymptotics (2.11) differs for $d = 2$ and $d = 3$. For instance, in two-dimensional case the cavities can be very small, *i.e.* $a = \mathcal{O}(\varepsilon^{\varrho})$ for any $\varrho \geqslant 0$, unlike the three-dimensional case, where $a = \mathcal{O}(\varepsilon^{\varrho})$ for $0 \leqslant \varrho < 1$.

The same asymptotics as in Theorems 2.2–2.6 has been studied for similar problems (see [17, 18, 34, 37, 38]).

3. Preliminaries and auxiliary propositions

Let us remind the definition of weak solutions. Assume that $f \in L_2(\varGamma_1)$. The function $U_0 \in W_2^1(\varOmega)$ is called a weak solution of problem (2.2), if for any $v \in W_2^1(\varOmega)$ the integral identity

$$\int\limits_{\varOmega} (\nabla U_0, \nabla v)\mathrm{d}x + \sigma_d C_d(B) A \int\limits_{\varGamma_1} U_0 v \mathrm{d}s = \lambda \int\limits_{\varGamma_1} U_0 v \mathrm{d}s + \int\limits_{\varGamma_1} f v \mathrm{d}s \tag{3.1}$$

holds true.

Denote by $W_2^1(\varOmega_{\varepsilon,a}; \varGamma_{\varepsilon,a})$ (by $W_2^1(\varOmega; \varGamma_1)$) the subset of functions belonging to $W_2^1(\varOmega_{\varepsilon,a})$ (to $W_2^1(\varOmega)$) and vanishing on $\varGamma_{\varepsilon,a}$ (on $\varGamma_1$).

And finally the function $U_{\varepsilon,a} \in W_2^1(\varOmega_{\varepsilon,a}; \varGamma_{\varepsilon,a})$ is called a weak solution of problem (2.1), if for any $v \in W_2^1(\varOmega_{\varepsilon,a}; \varGamma_{\varepsilon,a})$ the integral identity

$$\int\limits_{\varOmega_{\varepsilon,a}} (\nabla U_{\varepsilon,a}, \nabla v)\mathrm{d}x = \lambda \int\limits_{\varGamma_1} U_{\varepsilon,a} v \, \mathrm{d}x + \int\limits_{\varGamma_1} f v \mathrm{d}x \tag{3.2}$$

takes place.

Naturally a nontrivial weak solution u_0 of problem (2.4) is called an eigenfunction of problem (2.4), and the number λ_0 is called an eigenvalue of problem (2.4).

Analogously, a nontrivial weak solution $u_{\varepsilon,a}$ of problem (2.3) is called an eigenfunction of problem (2.3), and the number $\lambda_{\varepsilon,a}$ is called an eigenvalue of this problem.

Remark 3.1. Obviously, the function $W_2^1(\varOmega_{\varepsilon,a}; \varGamma_{\varepsilon,a})$, extended in $\overline{B_{\varepsilon,a}}$ by zero, belongs to $W_2^1(\varOmega)$. Then in what follows we consider functions from $W_2^1(\varOmega_{\varepsilon,a}; \varGamma_{\varepsilon,a})$ as functions from $W_2^1(\varOmega)$, keeping for them the same notation. Bearing in mind this fact, one can rewrite the integral identity (3.2) in the form

$$\int\limits_{\varOmega} \nabla U_{\varepsilon,a} \nabla v \mathrm{d}x = \lambda \int\limits_{\varGamma_1} U_{\varepsilon,a} v \mathrm{d}s + \int\limits_{\varGamma_1} f v \mathrm{d}s, \tag{3.3}$$

respectively.

Remark 3.2. The standard norm in $W_2^1(\varOmega)$ is equivalent to the norm in $\|u\|_{H^1(\varOmega)}$, generated by the following scalar product:

$$(u, v)_{H^1(\varOmega)} = \int\limits_{\varOmega} (\nabla u, \nabla v)\mathrm{d}x + \int\limits_{\varGamma_1} u v \mathrm{d}s$$

(see, for instance, [35], Chap. III, Sect. 5.6).

Note that for any fixed $\lambda \in \mathbb{R}$ solution of problem (2.1) satisfies the uniform in ε and a *a priori* estimate

$$\|U_{\varepsilon,a}\|_{H^1(\varOmega)} \leqslant C \left(\|U_{\varepsilon,a}\|_{L_2(\varGamma_1)} + \|f\|_{L_2(\varGamma_1)} \right). \tag{3.4}$$

In fact, substituting in the integral identity (3.3) $v = U_{\varepsilon,a}$ as a test-function and adding $\|U_{\varepsilon,a}\|^2_{L_2(\varGamma_1)}$ for both parts of the identity and using the Cauchy–Bunjakovski–Schwartz and the Poincaré inequalities and also the trace theorem, we get

$$\begin{aligned} \|U_{\varepsilon,a}\|^2_{H^1(\varOmega)} &\leqslant C \left(\|f\|_{L_2(\varGamma_1)} \|U_{\varepsilon,a}\|_{L_2(\varGamma_1)} + \|U_{\varepsilon,a}\|^2_{L_2(\varGamma_1)} \right) \\ &\leqslant C \left(\|f\|_{L_2(\varGamma_1)} + \|U_{\varepsilon,a}\|_{L_2(\varGamma_1)} \right) \|U_{\varepsilon,a}\|_{H^1(\varOmega)}. \end{aligned}$$

The estimate (3.4) follows from this.

Lemma 3.3. *Assume that* $\{m(k)\}_{k=1}^{\infty}$ *is an increasing sequence of natural numbers, the sequence* $\{a_k\}_{k=1}^{\infty}$ *is converging,* $U_{\varepsilon_{m(k)},a_k}$ *is a solution of problem* (2.1) *with* $f = f_k$,

$$\|U_{\varepsilon_{m(k)},a_k}\|_{L_2(\Gamma_1)} = 1 \tag{3.5}$$

and the inequality

$$\|U_{\varepsilon_{m(k)},a_k}\|_{H^1(\Omega)} > k\|f_k\|_{L_2(\Gamma_1)} \tag{3.6}$$

holds.

Then

$$\|f_k\|_{L_2(\Gamma_1)} \underset{k\to\infty}{\longrightarrow} 0 \tag{3.7}$$

and there exist a subsequence of indexes $\{k'\}$ *and a function* $U^* \in W_2^1(\Omega)$ *such that the following weak convergence:*

$$U_{\varepsilon_{m(k')},a_{k'}} \rightharpoonup U^* \quad in\ H^1(\Omega) \tag{3.8}$$

holds as $k' \to \infty$ *and the strong convergence*

$$U_{\varepsilon_{m(k')},a_{k'}} \to U^* \quad in\ L_2(\Omega) \tag{3.9}$$

holds, wherein

$$\|U^*\|_{L_2(\Gamma_1)} = 1. \tag{3.10}$$

Proof. Due to (3.4), (3.6) and (3.5) it follows that

$$\|U_{\varepsilon_{m(k)},a_k}\|_{H^1(\Omega)} \leqslant C_1$$

as $k \to \infty$. From this estimate and (3.6) the estimate (3.7) follows. From this estimate, weak compactness of bounded sets in Hilbert spaces and compact imbedding $W_2^1(\Omega)$ in $L_2(\Omega)$ (see, for instance, [35], Chap. II, Sect. 8) one can prove the remaining statements of the lemma. □

4. Construction of model functions in semi-strip and in semi-infinite parallelepiped

In accordance with the strategy suggested in [38] for the problem with rapidly changing type of boundary conditions (see also [24] for the problem in domain perforated along part of the boundary) the following statement is crucial for the proof of Theorems 2.2.

Lemma 4.1. *There exists a function* $W_{\varepsilon,a}(x)$ *from* $W_2^1(\Omega_{\varepsilon,a};\Gamma_{\varepsilon,a}) \cap W_2^2(\Omega_{\varepsilon,a})$*, such that the relations*

$$\left.\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right|_{\Gamma_1} = -\frac{2\pi}{\varepsilon \ln a} \quad for\ d=2, \qquad \left.\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right|_{\Gamma_1} = \frac{4\pi C_3(B)a}{\varepsilon} \quad for\ d=3, \tag{4.1}$$

$$\|1 - W_{\varepsilon,a}\|_{L_2(\Omega)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.2}$$

$$\|\Delta W_{\varepsilon,a}\|_{L_2(\Omega_{\varepsilon,a})} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.3}$$

$$\|1 - W_{\varepsilon,a}\|_{L_2(\Gamma)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.4}$$

$$\left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|_{L_2(\Gamma_2)} \underset{\varepsilon\to 0}{\longrightarrow} 0, \tag{4.5}$$

hold, if (2.5) *is true.*

In its turn, according to the approach of [24,38] the function $W_{\varepsilon,a}$ is constructed as

$$W_{\varepsilon,a}(x) = W_a\left(\frac{x}{\varepsilon}\right),$$

where $W_a(\xi)$ is 1-periodic in ξ_1 for $d=2$ and 1-periodic in ξ_1, ξ_2 for $d=3$. This section is devoted to the construction of this function W_a by the method of matching of asymptotic expansions [24,39,40].

For $d=2$ we denote by $\mathbb{R}^2_+$ the semi-plane $x_2 \geqslant 0$, and by $x_0^{(\mathbf{k})}$ the points with coordinates $x_1 = b_1 + \mathbf{k}$, $x_2 = c$. Here $\mathbf{k} \in \mathbb{Z}$. For $d=3$ we denote by $\mathbb{R}^3_+$ the semi-space $x_3 \geqslant 0$, and by $x_0^{(\mathbf{k})}$ the points with coordinates $x_j = b_j + k_j$, $j=1,2$, $x_3 = c$. Here $\mathbf{k} = (k_1, k_2)$ and $k_j \in \mathbb{Z}$.

Denote $x_0 := x_0^{(\mathbf{0})}$, $y := x - x_0$ and $\Sigma := \{x : x_1 \in (0,1),\ x_2 = 0\}$ for $d=2$, $\Sigma := \{x : x_1, x_2 \in (0,1),\ x_3 = 0\}$ for $d=3$. Define $G_2(t) := \ln t$, $G_3(t) := -t^{-1}$.

Let us remind that $\Pi = (0,1) \times (0,\infty)$ for $d=2$ and $\Pi = (0,1)^2 \times (0,\infty)$ for $d=3$.

Lemma 4.2. *There exists a 1-periodic in x_1 for $d=2$ and in x_1, x_2 for $d=3$ function $g_d \in C^\infty\left(\mathbb{R}^d_+ \setminus \bigcup\limits_{\mathbf{k}} \left\{x_0^{(\mathbf{k})}\right\}\right)$, which satisfies the problem*

$$\begin{cases} \Delta g_d = 0 & \text{if } x \in \Pi \setminus \{x_0\}, \\ \frac{\partial g_d}{\partial \nu} = \alpha & \text{if } x \in \Sigma \end{cases}$$

for

$$\alpha = \sigma_d, \tag{4.6}$$

has the differentiable asymptotics

$$g_d(x) = O\left(\mathrm{e}^{-2\pi x_d}\right), \quad x_d \to +\infty$$

and in a neighborhood of x_0 has the representation

$$g_d(x) = G_d(|y|) + g_d^{(1)}(x),$$

where $g_d^{(1)}(x)$ is an infinitely smooth function in the neighborhood of this point including this point.

Proof. Let infinitely smooth cut-off function $\chi(t)$ be equal to 1 as $t \leqslant \frac{1}{3}\mathfrak{T}$ and zero as $t \geqslant \frac{2}{3}\mathfrak{T}$, where $\mathfrak{T} = \min\{b_1; 1-b_1; c\}$ for $d=2$ and $\mathfrak{T} = \min\{b_1; 1-b_1; b_2; 1-b_2; c\}$ for $d=3$. We look for the function g in Π in the form

$$g_d(x) = \chi\left(|y|\right) G_d(|y|) + g_d^{(1)}(x). \tag{4.7}$$

Acting to this function by the Laplace operator we get the problem for functions of $g_d^{(1)}(x)$. We have

$$\begin{cases} \Delta g_d^{(1)} = F_d & \text{as } x \in \Pi, \\ \frac{\partial g_d^{(1)}}{\partial x_d} = -\alpha & \text{as } x_d = 0 \end{cases} \tag{4.8}$$

where $F_d \in C_0^\infty(\Pi)$, where $C_0^\infty(\Pi)$ is the set of infinitely smooth functions with compact support in Π. Using the Fourier method of separating variables, it is easy to show that there exists a 1-periodic in x_1 for $d=2$ and x_1, x_2 for $d=3$ function $g_d^{(1)} \in C^\infty(\mathbb{R}^d_+)$ with differentiable asymptotics

$$g_d^{(1)}(x) = O\left(\mathrm{e}^{-2\pi x_d}\right), \quad x_d \to +\infty,$$

which is a solution of boundary value problem (4.8) for some $\alpha \in \mathbb{R}$. Hence, using (4.8) we get all the statements of the Lemma except (4.6).

Denote by S_δ the circle for $d=2$ and the ball for $d=3$ of the radius $\delta \ll 1$ centered in x_0. Integrating by parts the left-hand side of the identity

$$\int\limits_{\Pi\setminus S_\delta} \Delta g_d \mathrm{d}x = 0,$$

by means of the boundary conditions on g_d, we deduce

$$\int\limits_{\partial(\Pi\setminus S_\delta)} \frac{\partial g_d}{\partial \nu}\mathrm{d}s = 0.$$

Therefore due to (4.7) we get

$$\alpha - \sigma_d + O(\delta) = 0.$$

Passing to the limit as $\delta \to 0$, we derive (4.6). □

Corollary 4.3. *The differentiable asymptotics*

$$g_d(x) = G_d(|y|) + c_{\Pi,d} + P_1^{\Pi,d}(y) + O\left(|y|^2\right), \qquad y \to 0, \tag{4.9}$$

holds, where $c_{\Pi,d}$ is a constant and $P_1^{\Pi,d}(y)$ is a homogeneous polynomial of the first order.

Remark 4.4. Note that the existence of a solution to the problem (4.8) can be proved by means of the variational results (see Prop. 2.2 from [41] or Chap. 5 of the book [42]).

Lemma 4.5. *There exist functions $V_0^{(d)}, V_1^{(d)} \in C^\infty\left(\mathbb{R}^d\setminus\overline{B}\right) \cap C\left(\mathbb{R}^d\setminus B\right)$, being solutions of the problems*

$$\Delta V_i^{(d)} = 0, \quad x \in \mathbb{R}^d\setminus\overline{B}, \qquad V_i^{(d)} = 0, \quad x \in \partial B,$$

and having differentiable asymptotics

$$\begin{aligned} &V_0^{(2)}(x) = \ln|x| + c_B + O(|x|^{-1}), &&V_0^{(3)}(x) = 1 - C_3(B)|x|^{-1} + P_1^{B,3}(x)|x|^{-3} + O(|x|^{-3}),\\ &V_1^{(2)}(x) = P_1^{\Pi,2}(x) + \widetilde{c} + O(|x|^{-1}), &&V_1^{(3)}(x) = P_1^{\Pi,3}(x) - C_3(P,B)|x|^{-1} + O(|x|^{-2}) \end{aligned}$$

as $|x| \to \infty$, where $P_1^{B,3}(y)$ is a homogeneous polynomial of the first order.

Proof. It is wellknown (see, for instance, [43], Sect. 5.8), that for $\varphi \in C(\partial B)$ the boundary value problem

$$\Delta v = 0, \quad x \in \mathbb{R}^d\setminus\overline{B}, \qquad v = \varphi, \quad x \in \partial B \tag{4.10}$$

has a solution $v \in C^\infty\left(\mathbb{R}^d\setminus\overline{B}\right) \cap C\left(\mathbb{R}^d\setminus B\right)$ with differentiable asymptotics

$$v(x) = C|x|^{-d+2} + \sum_{j=1}^{d} C_j x_j |x|^{-d} + O\left(|x|^{-d}\right), \quad |x| \to \infty. \tag{4.11}$$

Then, the functions $V_0^{(2)}(x) = \ln|x-x_B| + \widetilde{V}_0^{(2)}(x)$, where x_B is an arbitrary point from B, $V_0^{(3)}(x) = 1 + \widetilde{V}_0^{(3)}(x)$, $V_1^{(d)}(x) = P_1^{\Pi,d}(x) + \widetilde{V}_1^{(d)}(x)$, where $\widetilde{V}_0^{(2)}(x)$, $\widetilde{V}_0^{(3)}(x)$, $\widetilde{V}_1^{(d)}(x)$ are above mentioned solutions of the boundary value problem for $\varphi(x) = -\ln|x-x_B|$, $\varphi(x) = -1$, $\varphi(x) = -P_1^{\Pi,d}(x)$, respectively, satisfy the statement of our Lemma. □

Remark 4.6. Note that if $d=3$ in (4.10) and $\varphi \equiv 1$, then the constant $C = C_3(B)$ in (4.11) is called the capacity (harmonic capacity) of B. If $d=2$ in (4.10) and $\varphi = -\ln|x|$, then the constant $C = c_B$ in (4.11) is called the logarithmic capacity of B.

In analogous way as Lemma 4.2 one can prove the following statement.

Lemma 4.7. *Let $d=3$. Then there exists a 1-periodic in x_1, x_2 function $\widehat{g}_3 \in C^\infty\left(\mathbb{R}^3_+ \setminus \bigcup\limits_{\mathbf{k}} \left\{x_0^{(\mathbf{k})}\right\}\right)$, which satisfies the problem*

$$\begin{cases} \Delta \widehat{g}_3 = 0 & \text{if } x \in \Pi \setminus \{x_0\}, \\ \frac{\partial \widehat{g}_3}{\partial \nu} = 0 & \text{if } x \in \Sigma, \end{cases}$$

has the differentiable asymptotics

$$\widehat{g}_3(x) = O\left(\mathrm{e}^{-2\pi x_3}\right), \quad x_3 \to +\infty$$

and in a neighborhood of x_0 has the representation

$$\widehat{g}_3(x) = P_1^{B,3}(y)|y|^{-3} + \widehat{g}_3^{(1)}(x),$$

where $\widehat{g}_3^{(1)}(x)$ is an infinitely smooth function in the neighborhood of this point including this point.

Corollary 4.8. *The differentiable asymptotics*

$$\widehat{g}_3(x) = P_1^{B,3}(y)|y|^{-3} + \widehat{c}_{B,\Pi} + O\left(|y|\right), \qquad y \to 0, \tag{4.12}$$

holds.

Denote $\Pi_a = \Pi \setminus \overline{B_a}$ (see Fig. 2) and define in $\overline{\Pi_a}$

$$\begin{aligned} W_a(x) :=& \left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1 - \frac{1}{\ln a}\left(g_2(x) + c_B - c_{\Pi,2}\right)\right) \\ & - \frac{1}{\ln a}\chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(2)}\left(\frac{y}{a}\right) + aV_1^{(2)}\left(\frac{y}{a}\right)\right) && \text{for } d=2\ , \\ W_a(x) :=& \left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1 + aC_3(B)\left(g_3(x) - c_{\Pi,3}\right) + a^2\left(C_3(P,B)g_3(x) + \widehat{g}_3(x)\right)\right) \\ & + \chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(3)}\left(\frac{y}{a}\right) + aV_1^{(3)}\left(\frac{y}{a}\right)\right) && \text{for } d=3\ . \end{aligned} \tag{4.13}$$

We assume that $\beta \in (0,1)$ following the method of matching of asymptotic expansions.

Now, denote $B_a^{\mathbf{k}} = \{x : (x_1 - \mathbf{k}, x_2) \in B_a\}$, $\mathbf{k} \in \mathbb{Z}$ for $d=2$, $B_a^{\mathbf{k}} = \{x : (x_1 - k_1, x_2 - k_2, x_3) \in B_a\}$, $\mathbf{k} = (k_1, k_2)$, $k_j \in \mathbb{Z}$ for $d=3$, $B^a = \bigcup\limits_{\mathbf{k}} B_a^{\mathbf{k}}$ and extend the function $W_a(x)$ 1-periodically in x_1 for $d=2$ and in x_1, x_2 for $d=3$, keeping the same notation $W_a(x)$.

Theorem 4.9. *The function $W_a(x) \in C^\infty\left(\mathbb{R}^d_+ \setminus B^a\right)$ is 1-periodic in x_1 for $d=2$, and in x_1, x_2 for $d=3$, has the differentiable asymptotics*

$$\begin{aligned} W_a(x) =& 1 - \frac{1}{\ln a}\left(c_B - c_{\Pi,2} + O\left(\mathrm{e}^{-2\pi x_2}\right)\right) && \text{as } x_2 \to \infty \text{ for } d=2, \\ W_a(x) =& 1 - a\left(C_3(B)c_{\Pi,3} + O\left(\mathrm{e}^{-2\pi x_3}\right)\right) && \text{as } x_3 \to \infty \text{ for } d=3 \end{aligned} \tag{4.14}$$

uniform in a, and satisfies the problem

$$\begin{cases} \Delta W_a = F_a & \text{if } x \in \Pi_a, \\ \frac{\partial W_a}{\partial \nu} = -\frac{2\pi}{\ln a} \quad \text{for } d=2, & \frac{\partial W_a}{\partial \nu} = a4\pi C_3(B) \quad \text{for } d=3, \quad \text{if } x \in \Sigma, \\ W_a = 0 & \text{if } x \in \partial B_a, \end{cases}$$

where $F_a \in C_0^\infty(\Pi_a)$.

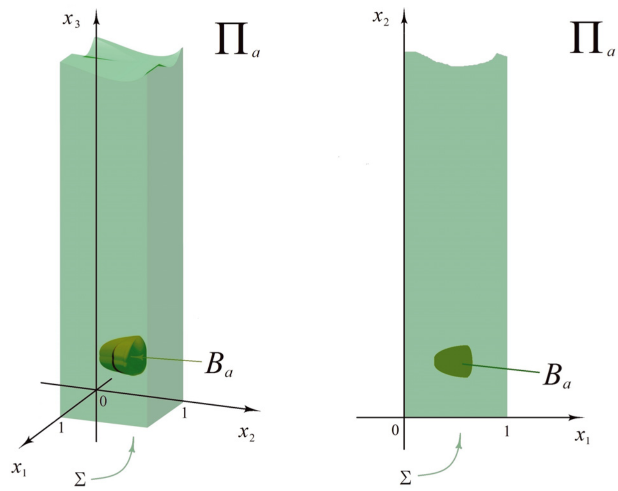

FIGURE 2. The cell of periodicity.

Moreover, as $a \to 0$,

$$\|1 - W_a\|_{L_2(\Sigma)} = O\left(\frac{1}{|\ln a|}\right) \quad \text{for } d = 2, \qquad \|1 - W_a\|_{L_2(\Sigma)} = O(a) \quad \text{for } d = 3, \tag{4.15}$$

$$\begin{aligned} \left\|1 - \left(W_a + \frac{1}{\ln a}(c_B - c_{\Pi,2})\right)\right\|_{L_2(\Pi_a)} &= O\left(\frac{1}{|\ln a|}\right) && \text{for } d = 2, \\ \|1 - (W_a + aC_3(B)c_{\Pi,3})\|_{L_2(\Pi_a)} &= O\left(a + a^{\frac{3\beta}{2}}\right) && \text{for } d = 3, \end{aligned} \tag{4.16}$$

$$\|F_a\|_{L_2(\Pi_a)} = O\left(\frac{1}{|\ln a|}\left(a^\beta + a^{1-2\beta}\right)\right) \quad \text{for } d = 2, \tag{4.17}$$

$$\|F_a\|_{L_2(\Pi_a)} = O\left(a^{1+\frac{3}{2}\beta} + a^{2-\frac{1}{2}\beta} + a^{3-\frac{7}{2}\beta}\right) \quad \text{for } d = 3. \tag{4.18}$$

Proof. All the statements of the Theorem except (4.16) and (4.17) follow from the definition (4.13) of the function W_a and Lemmas 4.2 and 4.5.

Let us prove (4.16). Suppose that

$$\begin{aligned} W_a^{(1)}(x) &:= -\frac{1}{\ln a}\left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right) g_2(x), \\ W_a^{(2)}(x) &:= -\frac{1}{\ln a}\chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(2)}\left(\frac{y}{a}\right) + aV_1^{(2)}\left(\frac{y}{a}\right) + \ln a - c_B + c_{\Pi,2}\right) \quad \text{for } d = 2, \end{aligned}$$

and

$$\begin{aligned} W_a^{(1)}(x) &=: a\left(1 - \chi\left(\frac{|y|}{a^\beta}\right)\right)\left(C_3(B)g_3(x) + a\left(C_3(P,B)g_3(x) + \widehat{g}_3(x)\right)\right), \\ W_a^{(2)}(x) &:= \chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(3)}\left(\frac{y}{a}\right) + aV_1^{(3)}\left(\frac{y}{a}\right) - 1 + aC_3(B)c_{\Pi,3}\right) \quad \text{for } d = 3. \end{aligned}$$

Then

$$\begin{aligned} W_a(x) + \frac{1}{\ln a}\left(c_B - c_{\Pi,2}\right) - 1 =& W_a^{(1)}(x) + W_a^{(2)}(x) \quad \text{for } d = 2, \\ W_a(x) + aC_3(B)c_{\Pi,3} - 1 =& W_a^{(1)}(x) + W_a^{(2)}(x) \quad \text{for } d = 3. \end{aligned} \tag{4.19}$$

Due to Lemma 4.2 we get

$$\|W_a^{(1)}\|_{L_2(\Pi_a)} = O\left(\frac{1}{|\ln a|}\right) \quad \text{for } d = 2, \qquad \|W_a^{(1)}\|_{L_2(\Pi_a)} = O\left(a\right) \quad \text{for } d = 3. \tag{4.20}$$

Denote

$$\widetilde{T}_a := \left\{x:\ |y| \leqslant \frac{2a^\beta}{3}\mathfrak{T}\right\}.$$

Since $\operatorname{supp}\chi\left(\frac{|y|}{a^\beta}\right) \subset \widetilde{T}_a$, then

$$\operatorname{supp} W_a^{(2)}(x) \subset \widetilde{T}_a.$$

For $x \in \widetilde{T}_a$, due to Lemma 4.5 we have that

$$-\frac{1}{\ln a}\left(V_0^{(2)}\left(\frac{y}{a}\right) + aV_1^{(2)}\left(\frac{y}{a}\right) + \ln a - c_B + c_{\Pi,2}\right) = O(1) \quad \text{as } d = 2,$$

and

$$\left(V_0^{(3)}\left(\frac{y}{a}\right) + aV_1^{(3)}\left(\frac{y}{a}\right) - 1 + aC_3(B)c_{\Pi,3}\right) = O(1) \quad \text{as } d = 3.$$

Hence, due to the definition of $W_a^{(2)}(x)$ we have

$$\|W_a^{(2)}\|_{L_2(\Pi_a)} = O\left(a^\beta\right). \tag{4.21}$$

From (4.19)–(4.21) we derive (4.16).

Since $F_a = \Delta W_a$, then the definition of W_a leads to

$$F_a = F_a^{(1)} + F_a^{(2)}, \tag{4.22}$$

where

$$\begin{aligned} F_a^{(1)}(x) =& -\left(1 - \frac{1}{\ln a}\left(g_2(x) + c_B - c_{\Pi,2} - V_0^{(2)}\left(\frac{y}{a}\right) - aV_1^{(2)}\left(\frac{y}{a}\right)\right)\right)\Delta\chi\left(\frac{|y|}{a^\beta}\right), \\ F_a^{(2)}(x) =& -2\nabla\left(1 - \frac{1}{\ln a}\left(g_2(x) + c_B - c_{\Pi,2} - V_0^{(2)}\left(\frac{y}{a}\right) - aV_1^{(2)}\left(\frac{y}{a}\right)\right)\right)\nabla\chi\left(\frac{|y|}{a^\beta}\right) \quad \text{for } d = 2. \end{aligned}$$

and

$$\begin{aligned} F_a^{(1)}(x) =& -\left(1 + aC_3(B)\left(g_3(x) - c_{\Pi,3}\right) + a^2\left(C_3(P,B)g_3(x) + \widehat{g}_3(x)\right) - V_0^{(3)}\left(\frac{y}{a}\right) - aV_1^{(3)}\left(\frac{y}{a}\right)\right)\Delta\chi\left(\frac{|y|}{a^\beta}\right), \\ F_a^{(2)}(x) =& -2\nabla\left(1 + aC_3(B)\left(g_3(x) - c_{\Pi,3} + a^2\left(C_3(P,B)g_3(x) + \widehat{g}_3(x)\right)\right) - V_0^{(3)}\left(\frac{y}{a}\right) - aV_1^{(3)}\left(\frac{y}{a}\right)\right)\nabla\chi\left(\frac{|y|}{a^\beta}\right) \end{aligned}$$

for $d = 3$. Keeping in mind

$$\operatorname{supp}\Delta\chi\left(\frac{|y|}{a^\beta}\right), \operatorname{supp}\nabla\chi\left(\frac{|y|}{a^\beta}\right) \subset T_a := \left\{x:\ \frac{a^\beta}{3}\mathfrak{T} \leqslant |y| \leqslant \frac{2a^\beta}{3}\mathfrak{T}\right\},$$

we conclude, that

$$\operatorname{supp} F_a,\ \operatorname{supp} F_a^{(i)} \subset T_a \tag{4.23}$$

also. Bearing in mind (4.9), (4.12) and Lemma 4.5 we deduce

$$-\left(1-\frac{1}{\ln a}\left(g_2(x)+c_B-c_{\Pi,2}-V_0^{(2)}\left(\frac{y}{a}\right)-aV_1^{(2)}\left(\frac{y}{a}\right)\right)\right)=O\left(\frac{1}{\ln a}\left(a^{2\beta}+a^{1-\beta}\right)\right),$$

$$-2\nabla\left(1-\frac{1}{\ln a}\left(g_2(x)+c_B-c_{\Pi,2}-V_0^{(2)}\left(\frac{y}{a}\right)-aV_1^{(2)}\left(\frac{y}{a}\right)\right)\right)=O\left(\frac{1}{\ln a}\left(a^{\beta}+a^{1-2\beta}\right)\right)\quad \text{for } x\in T_a,\ d=2$$

and

$$\begin{aligned}
-\Big(1+aC_3(B)\left(g_3(x)-c_{\Pi,3}\right)+a^2\left(C_3(P,B)g_3(x)+\widehat{g}_3(x)\right)-V_0^{(3)}\left(\frac{y}{a}\right)-aV_1^{(3)}\left(\frac{y}{a}\right)\Big)&\\
=O\left(a^{1+2\beta}+a^2+a^{3(1-\beta)}\right)&,\\
-2\nabla\Big(1+aC_3(B)\left(g_3(x)-c_{\Pi,3}\right)+a^2\left(C_3(P,B)g_3(x)+\widehat{g}_3(x)\right)-V_0^{(3)}\left(\frac{y}{a}\right)-aV_1^{(3)}\left(\frac{y}{a}\right)\Big)&\\
=O\left(a^{1+\beta}+a^{2-\beta}+a^{3-4\beta}\right)&
\end{aligned}$$

for $x\in T_a$, $d=3$. Then, since

$$\Delta\chi\left(\frac{|y|}{a^\beta}\right)=O(a^{-2\beta}),\qquad \nabla\chi\left(\frac{|y|}{a^\beta}\right)=O(a^{-\beta}),$$

we derive

$$\begin{aligned}
F_a^{(i)}(x)=&O\left(\frac{1}{\ln a}\left(1+a^{1-3\beta}\right)\right) && \text{for } x\in T_a,\ d=2,\\
F_a^{(i)}(x)=&O\left(a+a^{2(1-\beta)}+a^{3-5\beta}\right) && \text{for } x\in T_a,\ d=3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|F_a^{(i)}\|_{L_2(T_a)}=&O\left(\frac{1}{|\ln a|}\left(a^{\beta}+a^{1-2\beta}\right)\right)\quad \text{for } d=2,\\
\|F_a^{(i)}\|_{L_2(T_a)}=&O\left(a^{1+\frac{3}{2}\beta}+a^{2-\frac{1}{2}\beta}+a^{3-\frac{7}{2}\beta)}\right)\quad \text{for } d=3.
\end{aligned}$$

From this relation, (4.22) and (4.23) we obtain (4.17). □

5. Proof of Theorem 2.2

Before proving Theorem 2.2 we prove two auxiliary propositions.

Proof of Lemma 4.1. Then from the definition of $W_{\varepsilon,a}(x)$ and Theorem 4.9 one gets (4.1). Due to (4.16) we have

$$\begin{aligned}
\|1-W_{\varepsilon,a}\|^2_{L_2(\Omega_{\varepsilon,a})}\leqslant&\left\|1-\left(W_{\varepsilon,a}+\frac{1}{\ln a}\left(c_B-c_{\Pi,2}\right)\right)\right\|^2_{L_2(\Omega_{\varepsilon,a})}\\
&+\left\|\frac{1}{\ln a}\left(c_B-c_{\Pi,2}\right)\right\|^2_{L_2(\Omega_{\varepsilon,a})}=O\left(\frac{\varepsilon}{|\ln a|^2}+\frac{1}{|\ln a|^2}\right)\qquad \text{for } d=2,\\
\|1-W_{\varepsilon,a}\|^2_{L_2(\Omega_{\varepsilon,a})}\leqslant&\left\|1-\left(W_{\varepsilon,a}+aC_3(B)c_{\Pi,3}\right)\right\|^2_{L_2(\Omega_{\varepsilon,a})}\\
&+\|aC_3(B)c_{\Pi,3}\|^2_{L_2(\Omega_{\varepsilon,a})}=O\left(\varepsilon\big(a+a^{\frac{3\beta}{2}}\big)^2+a^2\right)\qquad \text{for } d=3,
\end{aligned}\tag{5.1}$$

Now (4.2) follows from (5.1) for any $\beta \in (0,1)$. Using (4.15), we get

$$\|1 - W_{\varepsilon,a}\|_{L_2(\Gamma)} = O\left(\frac{1}{|\ln a|}\right) \quad \text{for } d = 2, \qquad \|1 - W_{\varepsilon,a}\|_{L_2(\Gamma)} = O\left(a\right) \quad \text{for } d = 3, \tag{5.2}$$

which leads to (4.4).

Assume

$$\Gamma_{2,\varepsilon} := \{x:\, x_1 = 0,\, 0 < x_2 < \varepsilon^{\frac{1}{2}}\} \cup \{x:\, x_1 = 1,\, 0 < x_2 < \varepsilon^{\frac{1}{2}}\} \quad \text{for } d = 2,$$

and

$$\Gamma_{2,\varepsilon} := \left(\partial(0,1)^2\right) \times \left(0, \varepsilon^{\frac{1}{2}}\right) \quad \text{for } d = 3.$$

Since $g_d(x) \in C^\infty\left(\overline{\Pi}\backslash\{x_0\}\right)$, then from the definition of $W_{\varepsilon,a}(x)$ and (4.14) we obtain

$$\begin{aligned}
&\left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|_{L_2(\Gamma_2\backslash\Gamma_{2,\varepsilon})} = O\left(\frac{\mathrm{e}^{-2\pi\varepsilon^{-\frac{1}{2}}}}{\varepsilon|\ln a|}\right), \qquad \left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|^2_{L_2(\Gamma_{2,\varepsilon})} = O\left(\frac{\varepsilon^{\frac{1}{2}}}{\varepsilon^2|\ln a|^2}\right) && \text{for } d = 2,\\
&\left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|_{L_2(\Gamma_2\backslash\Gamma_{2,\varepsilon})} = O\left(\frac{a\mathrm{e}^{-2\pi\varepsilon^{-\frac{1}{2}}}}{\varepsilon}\right), \qquad \left\|\frac{\partial W_{\varepsilon,a}}{\partial \nu}\right\|^2_{L_2(\Gamma_{2,\varepsilon})} = O\left(\frac{a^2\varepsilon^{\frac{1}{2}}}{\varepsilon^2}\right) && \text{for } d = 3.
\end{aligned}$$

Due to the condition (2.5) we obtain the convergence (4.5).

From the definition of $W_{\varepsilon,a}(x)$ and (4.17) we have

$$\begin{aligned}
\|\Delta W_{\varepsilon,a}\|_{L_2(\Omega_{\varepsilon,a})} &= O\left(\frac{1}{\varepsilon^{\frac{3}{2}}|\ln a|}\left(a^\beta + a^{1-2\beta}\right)\right) && \text{for } d = 2,\\
\|\Delta W_{\varepsilon,a}\|_{L_2(\Omega_{\varepsilon,a})} &= O\left(\left(\frac{a}{\varepsilon}\right)^{\frac{3}{2}}\left(a^{-\frac{1}{2}+\frac{3}{2}\beta} + a^{\frac{1}{2}-\frac{1}{2}\beta} + a^{\frac{3}{2}-\frac{7}{2}\beta)}\right)\right) && \text{for } d = 3.
\end{aligned} \tag{5.3}$$

One can see that due to (2.5) the relations (5.3) imply (4.3) for any $\frac{1}{3} < \beta < \frac{3}{7}$. □

Lemma 5.1. *Let condition* (2.5) *hold. Assume also that the function* $U_{\varepsilon,a} \in W_2^1\left(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a}\right)$ *converges weakly*

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U^* \quad \text{in } W_2^1(\Omega). \tag{5.4}$$

Then for any functions $v \in C^\infty(\overline{\Omega})$ *the convergences*

$$\int\limits_{\Omega_{\varepsilon,a}} U_{\varepsilon,a}(vW_{\varepsilon,a})\mathrm{d}x \to \int\limits_{\Omega} U^* v\,\mathrm{d}x, \tag{5.5}$$

$$\int\limits_{\Omega_{\varepsilon,a}} \left(\nabla U_{\varepsilon,a}, \nabla(vW_{\varepsilon,a})\right)\mathrm{d}x \to \int\limits_{\Omega} (\nabla U^*, \nabla v)\mathrm{d}x + \sigma_d C_d(B) A \int\limits_{\Gamma_1} U^* v \mathrm{d}x_1 \tag{5.6}$$

take place.

Proof. Since $v \in C^\infty(\overline{\Omega})$, the Lemma 4.1 implies

$$vW_{\varepsilon,a} \in W_2^1\left(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a}\right) \cap W_2^2\left(\Omega_{\varepsilon,a}\right).$$

Due to (4.2) we have

$$v(1 - W_{\varepsilon,a}) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } L_2(\Omega).$$

The convergence (5.5) follows from this and (4.4).

Let us show (5.6). Integrating by parts, we get

$$\begin{aligned}\int\limits_{\Omega_{\varepsilon,a}} \big(\nabla U_{\varepsilon,a}, \nabla(vW_{\varepsilon,a})\big)\mathrm{d}x = &-\int\limits_{\Omega_{\varepsilon,a}} \Delta v W_{\varepsilon,a} U_{\varepsilon,a}\mathrm{d}x - \int\limits_{\Omega_{\varepsilon,a}} 2(\nabla v, \nabla W_{\varepsilon,a})U_{\varepsilon,a}\mathrm{d}x \\ &- \int\limits_{\Omega_{\varepsilon,a}} v\Delta W_{\varepsilon,a}U_{\varepsilon,a}\mathrm{d}x + \int\limits_{\Gamma} \frac{\partial v}{\partial \nu} W_{\varepsilon,a}U_{\varepsilon,a}\mathrm{d}s + \int\limits_{\Gamma} \frac{\partial W_{\varepsilon,a}}{\partial \nu} vU_{\varepsilon,a}\mathrm{d}s.\end{aligned} \tag{5.7}$$

Since

$$\Delta v\,(1 - W_{\varepsilon,a}) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } L_2(\Omega), \qquad \frac{\partial v}{\partial \nu}\,(1 - W_{\varepsilon,a}) \underset{\varepsilon\to 0}{\longrightarrow} 0 \quad \text{in } L_2(\Gamma)$$

due to (4.2) and (4.4), then from (5.4) we get

$$-\int\limits_{\Omega_{\varepsilon,a}} \Delta v W_{\varepsilon,a} U_{\varepsilon,a}\mathrm{d}x + \int\limits_{\Gamma} \frac{\partial v}{\partial \nu} W_{\varepsilon,a}U_{\varepsilon,a}\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} -\int\limits_{\Omega} \Delta v U^*\mathrm{d}x + \int\limits_{\Gamma} \frac{\partial v}{\partial \nu} U^*\mathrm{d}s = \int\limits_{\Omega} (\nabla U^*, \nabla v)\mathrm{d}x. \tag{5.8}$$

By means of (4.3) and (5.4) we have

$$\int\limits_{\Omega_{\varepsilon,a}} v\Delta W_{\varepsilon,a}U_{\varepsilon,a}\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.9}$$

Bearing in mind that $\nabla(W_{\varepsilon,a} - 1) = \nabla W_{\varepsilon,a}$ and integrating by parts, we derive

$$\begin{aligned}\int\limits_{\Omega_{\varepsilon,a}} (\nabla v, \nabla W_{\varepsilon,a})U_{\varepsilon,a}\mathrm{d}x &= \int\limits_{\Omega_{\varepsilon,a}} \big(\nabla v, \nabla(W_{\varepsilon,a} - 1)\big)U_{\varepsilon,a}\mathrm{d}x \\ &= -\int\limits_{\Omega_{\varepsilon,a}} (W_{\varepsilon,a} - 1)\big((\nabla U_{\varepsilon,a}, \nabla v) + U_{\varepsilon,a}\Delta v\big)\mathrm{d}x \\ &\quad + \int\limits_{\Gamma} (W_{\varepsilon,a} - 1)U_{\varepsilon,a}\frac{\partial v}{\partial \nu}\mathrm{d}s.\end{aligned}$$

Using this identity, (4.2), (4.4) and (5.4) we deduce

$$\int\limits_{\Omega_{\varepsilon,a}} (\nabla v, \nabla W_{\varepsilon,a})U_{\varepsilon,a}\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.10}$$

Due to (4.1) and (5.4) we obtain

$$\int\limits_{\Gamma_1} \frac{\partial W_{\varepsilon,a}}{\partial \nu} vU_{\varepsilon,a}\mathrm{d}s = -\frac{\sigma_d C_d(B)}{\varepsilon G_d(a)} \int\limits_{\Gamma_1} vU_{\varepsilon,a}\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} \sigma_d C_d(B) A \int\limits_{\Gamma_1} vU^*\mathrm{d}s. \tag{5.11}$$

Finally, (4.5) and (5.4) lead to the convergence

$$\int\limits_{\Gamma_2} \frac{\partial W_{\varepsilon,a}}{\partial \nu} vU_{\varepsilon,a}\mathrm{d}s \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{5.12}$$

The convergence (5.6) follows immediately from (5.7)–(5.12). □

Proof of Theorem 2.2. Since the Fredholm alternative for problem (2.1) holds (see, for instance, [35], Chap. II, Sect. 3), then it is sufficient to show the estimate (2.6) to prove *1)*. If $U_{\varepsilon,a} = 0$ on Γ_1, then estimate (2.6) follows from (3.4) due to Remark 3.2.

Otherwise, due to the linearity of problem (2.1) it is sufficient to prove this estimate for normalized in $L_2(\Gamma_1)$ functions $U_{\varepsilon,a}$. In this case the proof is based on contradiction. Let the estimate (2.6) be wrong. Then, due to Remark 3.2 there exists a sequence of natural numbers, such that the conditions of Lemma 3.3 are fulfilled. Due to the lemma the convergence (3.7) takes place and there exist a subsequence of indexes $\{k'\}$ and a function $U^* \in W_2^1(\Omega)$, such that the convergences (3.8) and (3.9) and the identity (3.10) hold as $k' \to \infty$.

Hence, using Lemma 5.1 and the convergence (3.7), we get the identity

$$\int\limits_{\Omega} (\nabla U^*, \nabla v)\,\mathrm{d}x + \sigma_d C_d(B) A \int\limits_{\Gamma_1} U^* v \mathrm{d}s = \lambda \int\limits_{\Gamma_1} U^* v\,\mathrm{d}s$$

for any $v \in C^\infty(\overline{\Omega})$. From the embedding of $C^\infty(\overline{\Omega})$ in $W_2^1(\Omega)$ it follows that this identity holds for any $v \in W_2^1(\Omega)$. On the one hand $U^* \neq 0$ due to (3.10), and on the other hand, λ is not an eigenvalue of the limit problem (2.4), hence we have a contradiction. This contradiction proves the estimate (2.6).

Let us now prove the statement *2)*. Assume that $\{a_k\}_{k=1}^\infty$, is a sequence, such that for $\varepsilon = \varepsilon_k$, $a = a_k$ as $k \to \infty$ the convergence (2.5) holds, $\{k'\}$ is an arbitrary subsequence of natural numbers. Then using estimate (2.6), we conclude that there exist $U^* \in W_2^1(\Omega)$ and a subsequence of this sequence, such that the convergence (5.4) takes place on this subsequence. Writing down the integral identity (3.2) with the test function equals to $vW_{\varepsilon,a}$ with arbitrary function $v \in C^\infty(\overline{\Omega})$, we obtain

$$\int\limits_{\Omega_{\varepsilon,a}} \big(\nabla U_{\varepsilon,a}, \nabla\big(vW_{\varepsilon,a}\big)\big)\,\mathrm{d}x = \lambda \int\limits_{\Gamma_1} U_{\varepsilon,a}\big(vW_{\varepsilon,a}\big)\,\mathrm{d}s + \int\limits_{\Gamma_1} f\big(vW_{\varepsilon,a}\big)\,\mathrm{d}s.$$

Then due to Lemma 5.1 the identity

$$\int\limits_{\Omega} (\nabla U^*, \nabla v)\,\mathrm{d}x + \sigma_d C_d(B) A \int\limits_{\Gamma_1} U^* v\,\mathrm{d}s = \lambda \int\limits_{\Gamma_1} U^* v\,\mathrm{d}s + \int\limits_{\Gamma_1} f v\,\mathrm{d}s$$

holds for any function $v \in C^\infty(\overline{\Omega})$. The embedding of $C^\infty(\overline{\Omega})$ in $W_2^1(\Omega)$ and the uniqueness of the solution of the limit problem (2.2) gives $U^* = U_0$. This fact, the arbitrariness of the choice of the subsequence $\{k'\}$ and the convergence (5.4) lead to the weak convergence (2.8).

It remains to prove for $A = 0$ the strong convergence (2.7). Using the integral identity (3.1), weak convergence (2.8) and the compactness of the embedding of $W_2^1(\Omega)$ in $L_2(\Omega)$ and in $L_2(\Gamma_1)$, we derive

$$\begin{aligned}
\|U_{\varepsilon,a} - U_0\|^2_{W_2^1(\Omega)} &= \|\nabla(U_{\varepsilon,a} - U_0)\|^2_{L_2(\Omega)} + \|U_{\varepsilon,a} - U_0\|^2_{L_2(\Omega)} \\
&= \|U_{\varepsilon,a} - U_0\|^2_{L_2(\Omega)} + \|\nabla U_{\varepsilon,a}\|^2_{L_2(\Omega)} + \|\nabla U_0\|^2_{L_2(\Omega)} \\
&\quad - \int\limits_{\Omega} \big(\nabla U_{\varepsilon,a}, \nabla U_0\big)\,\mathrm{d}x - \int\limits_{\Omega} \big(\nabla U_0, \nabla U_{\varepsilon,a}\big)\,\mathrm{d}x \\
&= \|U_{\varepsilon,a} - U_0\|^2_{L_2(\Omega)} + \|\nabla U_{\varepsilon,a}\|^2_{L_2(\Omega)} + \|\nabla U_0\|^2_{L_2(\Omega)} - 2\int\limits_{\Omega} \big(\nabla U_0, \nabla U_{\varepsilon,a}\big)\,\mathrm{d}x \\
&= \|U_{\varepsilon,a} - U_0\|^2_{L_2(\Omega)} + \int\limits_{\Gamma_1} (f + \lambda U_0) U_0\,\mathrm{d}s + \int\limits_{\Gamma_1} (f + \lambda U_{\varepsilon,a}) U_{\varepsilon,a}\,\mathrm{d}s \\
&\quad - 2\int\limits_{\Gamma_1} (f + \lambda U_0) U_{\varepsilon,a}\,\mathrm{d}s \underset{\varepsilon \to 0}{\longrightarrow} 0.
\end{aligned}$$

Theorem is proved. ∎

From the proof of statement 2) of Theorem 2.2 one can derive a stronger assertion.

Lemma 5.2. *Let the condition (2.5) hold, assume also that λ is not an eigenvalue of problem (2.4), $U_{\varepsilon,a}$ is the solution of problem (2.1) for $f = f_{\varepsilon,a}$, U_0 is the solution of problem (2.2) for $f = f_0$, the weak convergence*

$$f_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} f_0 \qquad \text{in } L_2(\Gamma_1) \tag{5.13}$$

holds.

Then the convergence (2.7) and (2.8) take place.

Obviously the following proposition holds true.

Lemma 5.3. *Suppose that the condition (2.5) holds and λ is not an eigenvalue of the problem (2.4), $U_{0,\varepsilon,a}$ is the solution of problem (2.2) for $f = f_{\varepsilon,a}$, U_0 is the solution of problem (2.2) for $f = f_0$ and the weak convergence (5.13) holds.*

Then the weak convergence

$$U_{0,\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U_0 \qquad \text{in } W_2^1(\Omega)$$

takes place.

Remark 5.4. We use these two statements to prove the convergence of eigenpairs of the problem (2.3). More precisely, we introduce operators mapping the right-hand side of the equation to the trace of the solution and prove the operator convergence of them.

6. Proof of Theorem 2.3

Denote by $\mathcal{P}_{\varepsilon,a}$ and $\mathcal{P}_0$ operators $\mathcal{P}_{\varepsilon,a}, \mathcal{P}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$, mapping f to the traces on Γ_1 of solutions to boundary value problems (2.1) and (2.2), respectively, for $\lambda = -1$. For such lambda these operators are compact, selfadjoint and positive.

Lemmas 5.2 and 5.3 lead to the following statement.

Lemma 6.1. *If condition (2.5) and weak convergence (5.13) hold, then*

$$\mathcal{P}_{\varepsilon,a} f_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{P}_0 f_0, \quad \mathcal{P}_0 f_{\varepsilon,a} \underset{\varepsilon\to 0}{\longrightarrow} \mathcal{P}_0 f_0 \qquad \text{in} \quad L_2(\Gamma_1)$$

strongly.

Lemma 6.2. *If condition (2.5) holds, then*

$$\|\mathcal{P}_{\varepsilon,a} - \mathcal{P}_0\| \underset{\varepsilon\to 0}{\longrightarrow} 0$$

in the operator norm.

Proof. Assume the contrary. Then, without loss of generality one can say that there exists a number $\delta > 0$ and normalized in $L_2(\Gamma_1)$ functions $f_{\varepsilon,a}$, for which the following estimate:

$$\|\mathcal{P}_{\varepsilon,a} f_{\varepsilon,a} - \mathcal{P}_0 f_{\varepsilon,a}\|_{L_2(\Gamma_1)} \geqslant \delta \quad \text{as} \quad \varepsilon \to 0 \tag{6.1}$$

holds. Due to the weak compactness of the Hilbert space $L_2(\Gamma_1)$ we conclude that for some subsequence of indexes ε, a the weak convergence (5.13) takes place. Then using Lemma 6.1 we have

$$\|\mathcal{P}_{\varepsilon,a} f_{\varepsilon,a} - \mathcal{P}_0 f_{\varepsilon,a}\|_{L_2(\Gamma_1)} \leqslant \|\mathcal{P}_{\varepsilon,a} f_{\varepsilon,a} - \mathcal{P}_0 f_0\|_{L_2(\Gamma_1)} + \|\mathcal{P}_0 f_0 - \mathcal{P}_0 f_{\varepsilon,a}\|_{L_2(\Gamma_1)} \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

which contradicts (6.1). Lemma is proved. ∎

Denote by $\mathcal{L}_{\varepsilon,a}$, $\mathcal{L}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$ the operators inverse to $\mathcal{P}_{\varepsilon,a}$, $\mathcal{P}_0 : L_2(\Gamma_1) \to L_2(\Gamma_1)$. From Lemma 6.2 and ([44], Chap. 9, Sect. 4) (see also [45], Chap. IV, Thms. 2.25 and 3.16) we easily derive the following proposition.

Lemma 6.3. *Suppose that the condition* (2.5) *holds, and the multiplicity of the eigenvalue* Λ_0 *to the operator* $\mathcal{L}_0$ *equals to* n*. Then there exist exactly* n *eigenvalues* $\Lambda^{(l)}_{\varepsilon,a}$ *of the operator* $\mathcal{L}_{\varepsilon,a}$, $l = 1, \ldots, n$ (*with respect to their multiplicities*) *converging to* Λ_0 *as* $\varepsilon \to 0$.

Since obviously $\Lambda^{(l)}_{\varepsilon,a} = \lambda^{(l)}_{\varepsilon,a} + 1$, $\Lambda_0 = \lambda_0 + 1$, then the next assertion follows.

Lemma 6.4. *Suppose that the condition* (2.5) *holds, and the multiplicity of the eigenvalue* λ_0 *to the problem* (2.4) *equals to* n*. Then there exist exactly* n *eigenvalues* $\lambda^{(l)}_{\varepsilon,a}$ *of problem* (2.3), $l = 1, \ldots, n$ (*with respect to their multiplicities*) *converging to* λ_0 *as* $\varepsilon \to 0$.

Proof of Theorem 2.3. The statement I follows from Lemma 6.4.

The integral identity of problem (2.3) has the form

$$\int\limits_{\Omega} (\nabla u^{(l)}_{\varepsilon,a}, \nabla v) \mathrm{d}x = \lambda^{(l)}_{\varepsilon,a} \int\limits_{\Gamma_1} u^{(l)}_{\varepsilon,a} v \mathrm{d}x.$$

Bearing in mind $\lambda^{(l)}_{\varepsilon,a} \to \lambda_0$ and $\|u^{(l)}_{\varepsilon,a}\|_{L_2(\Gamma_1)} = 1$, substituting in this identity $v = u^{(l)}_{\varepsilon,a}$ and bearing in mind Remark 3.2, we get

$$\|u^{(l)}_{\varepsilon,a}\|_{W^1_2(\Omega)} \leqslant C. \tag{6.2}$$

Using this estimate (analogous to the estimate (2.6)) we complete the proof of the statement II, repeating the proof of 2) of Theorem 2.2. □

7. Variational estimates for the semi-strip and for the semi-infinite parallelepiped with small hole

Denote $\Pi(t) := \Sigma \times (0, t)$. Here and throughout $t > 0$ and a are such numbers, that $\Pi(t) \cap \overline{B_a} = \overline{B_a}$. Denote $\Pi_a(t) := \Pi(t) \backslash \overline{B_a}$. Define the space $H^1(\Pi_a(t); \partial B_a)$ is a completion by the norm

$$\|w\|_{H^1(\Pi_a(t))} := \left(\int\limits_{\Pi_a(t)} |\nabla w|^2 \, \mathrm{d}x + \int\limits_{\Sigma} w^2 \mathrm{d}s \right)^{1/2} \tag{7.1}$$

of functions from $C^\infty(\overline{\Pi_a(t)})$, vanishing on ∂B_a.

The aim of this Section is to prove the next statement.

Theorem 7.1. *There exists a constant* $C > 0$*, such that the uniform in* a *estimate*

$$\begin{aligned} &\inf_{\substack{w \in H^1(\Pi_a(t); \partial B_a) \\ w \neq 0}} \frac{\int\limits_{\Pi_a(t)} |\nabla w|^2 \, \mathrm{d}x}{\int\limits_{\Sigma} w^2 \mathrm{d}x_1} \geqslant C \frac{1}{|\ln a|} \qquad \text{for } d = 2, \\ &\inf_{\substack{w \in H^1(\Pi_a(t); \partial B_a) \\ w \neq 0}} \frac{\int\limits_{\Pi_a(t)} |\nabla w|^2 \, \mathrm{d}x}{\int\limits_{\Sigma} w^2 \mathrm{d}x_1 \mathrm{d}x_2} \geqslant C a \qquad \text{for } d = 3 \end{aligned} \tag{7.2}$$

holds.

We emphasize that it is not necessary to have $a \to 0$.

We use this estimate in the proofs of Lemma 8.1 and Corollary 8.2, which are crucial in proofs of Theorems 2.5 and 2.6.

The Steklov type spectral problem

$$\begin{cases} -\Delta w_a = 0 \quad \text{as } x \in \Pi_a(t), \\ w_a = 0 \quad \text{as } x \in \partial B_a, \\ \frac{\partial w_a}{\partial \nu} = 0 \quad \text{as } x \in \partial \Pi(t) \backslash \overline{\Sigma}, \\ \frac{\partial w_a}{\partial \nu} = \Lambda(a) w_a \quad \text{as } x \in \Sigma, \end{cases} \tag{7.3}$$

we consider in a weak sense. That is, a nontrivial element w_a from $H^1(\Pi_a(t); \partial B_a)$ is called a weak eigenfunction of problem (7.3), and $\Lambda(a)$ is an eigenvalue, if for any $v \in H^1(\Pi_a(t); \partial B_a)$ the identity

$$\int\limits_{\Pi_a(t)} (\nabla w_a, \nabla v) \,\mathrm{d}x = \Lambda(a) \int\limits_{\Sigma} w_a v \mathrm{d}s \tag{7.4}$$

holds true.

Lemma 7.2. *The minimal eigenvalue Λ_a of problem (7.3) satisfies the relation*

$$\Lambda_a = \inf_{\substack{w \in H^1(\Pi_a(t); \partial B_a) \\ w \neq 0}} \frac{\int\limits_{\Pi_a(t)} |\nabla w|^2 \,\mathrm{d}x}{\int\limits_{\Sigma} w^2 \mathrm{d}s} \cdot \tag{7.5}$$

Proof. Define the space $H^1(\Pi(t))$ as a completion by the norm

$$\|w\|_{H^1(\Pi(t))} := \left(\int\limits_{\Pi(t)} |\nabla w|^2 \,\mathrm{d}x + \int\limits_{\Sigma} w^2 \mathrm{d}s \right)^{1/2} \tag{7.6}$$

of the set of functions from $C^\infty(\overline{\Pi(t)})$. Denote by $(u, v)_1$ and $(u, v)_0$ the scalar products in $H^1(\Pi(t))$ and $L_2(\Sigma)$, respectively. Considering functions from $H^1(\Pi_a(t); \partial B_a)$ extended in $\overline{B_a}$ by zero, we rewrite the identity (7.4) in the form

$$(w_a, v)_1 = (\Lambda_a + 1)\,(w_a, v)_0,$$

which due to the Riesz Theorem (see, for instance, [35], Chap. II, Sect. 3), can be regarded as

$$(w_a, v)_1 = (\Lambda_a + 1)\,(\mathcal{A}_a w_a, v)_1,$$

where the operator

$$\mathcal{A}_a : H^1(\Pi_a(t); \partial B_a) \to H^1(\Pi_a(t); \partial B_a)$$

is defined by the formula

$$(\mathcal{A}_a w_a, v)_1 = (w_a, v)_0 \tag{7.7}$$

for any $v \in H^1(\Pi_a(t); \partial B_a)$. Thus, the minimization problem for the first eigenvalue of problem (7.3) leads to the problem for the minimal characteristic value μ_a for the operator $\mathcal{A}_a$ in $H^1(\Pi_a(d); \partial B_a)$:

$$w_a = \mu_a \mathcal{A}_a w_a,$$

where

$$\mu_a = \Lambda_a + 1. \tag{7.8}$$

It is easy to see, that the operator $\mathcal{A}_a$ defined in (7.7), is selfadjoint nonnegative linear and bounded, and moreover,

$$\begin{aligned}\|\mathcal{A}_a w - \mathcal{A}_a z\|^2_{H^1(\Pi(t))} &= (\mathcal{A}_a w - \mathcal{A}_a z, \mathcal{A}_a w - \mathcal{A}_a z)_1 = (w - z, \mathcal{A}_a w - \mathcal{A}_a z)_0 \\ &\leqslant \|w - z\|_{L_2(\Sigma)} \|\mathcal{A}_a w - \mathcal{A}_a z\|_{L_2(\Sigma)} \leqslant \|w - z\|_{L_2(\Sigma)} \|\mathcal{A}_a w - \mathcal{A}_a z\|_{H^1(\Pi(t))}.\end{aligned}$$

Hence,

$$\|\mathcal{A}_a w - \mathcal{A}_a z\|_{H^1(\Pi(t))} \leqslant \|w - z\|_{L_2(\Sigma)}.$$

Using this estimate, the boundedness of the operator $\mathcal{A}_a$, the equivalence of the norm (7.6) and the standard norm in $W_2^1(\Pi(t))$, and the compactness in $L_2(\Sigma)$ of the traces of functions bounded in $W_2^1(\Pi(t))$ (see, for instance, [35], Chap. III, Sect. 5, Thms. 5 and 4), we conclude that the operator $\mathcal{A}_a$ is completely continuous. Then (see, for instance, [35], Chap. 2, Sect. 5 Thm. 1) keeping in mind (7.7) we get

$$\mu_a = \inf_{\substack{w \in H^1(\Pi_a(t); \partial B_a) \\ w \neq 0}} \frac{\|w_a\|^2_{H^1(\Pi(t))}}{(\mathcal{A}_a w_a, w_a)_1} = \inf_{\substack{w \in H^1(\Pi_a(t); \partial B_a) \\ w \neq 0}} \frac{\int\limits_{\Pi_a} |\nabla w_a|^2 \, \mathrm{d}x}{\int\limits_{\Sigma} w_a^2 \mathrm{d}s} + 1.$$

Formula (7.5) follows from this and (7.8). □

Due to Lemma 7.2 the minimal eigenvalue λ_a of problem (7.3) is a monotonically increasing in a function (not only for small a). Hence, to prove Theorem 7.1 it is sufficient to prove the next theorem.

Theorem 7.3. *If $a \to 0$, then*

$$\Lambda_a = -\frac{2\pi}{\ln a}(1 + o(1)) \quad \textit{for } d = 2, \qquad \Lambda_a = 4\pi C_3(B) a (1 + o(1)) \quad \textit{for } d = 3. \tag{7.9}$$

The rest of the section is devoted to the proof of this Theorem.

Lemma 7.4. *If $a \to 0$, then the minimal eigenvalue Λ_a of problem (7.3) goes to zero.*

Proof. In analogous way as it was done in [38] it is easy to show, that for any function $v \in H^1(\Pi(t))$ there exist functions $v_a \in H^1(\Pi_a; \partial B_a)$, such that $v_a \to v$ in $H^1(\Pi(t))$ as $a \to 0$. Taking $v \equiv 1$, we have

$$\frac{\int\limits_{\Pi_a(t)} |\nabla v_a|^2 \, \mathrm{d}x}{\int\limits_{\Sigma} v_a^2 \mathrm{d}s} \underset{a \to 0}{\longrightarrow} 0.$$

Lemma follows from this and (7.5). □

Lemma 7.5. *Assume that $\Lambda(a)$ is an eigenvalue of problem (7.3), converging to zero as $a \to 0$, w_a is the respective normalized in $L_2(\Sigma)$ eigenfunction. Then from any sequence $a_k \underset{k \to \infty}{\longrightarrow} 0$ as $k \to \infty$ it is possible to choose a subsequence $\{a_{k'}\}$, such that the strong convergence*

$$w_a \underset{a \to 0}{\longrightarrow} w^* \qquad \textit{in } L_2(\Sigma) \tag{7.10}$$

holds on this subsequence, where

$$w^* = 1 \quad \textit{or} \quad w^* = -1. \tag{7.11}$$

Proof. Substituting $v = w_a$ as a test function in (7.4), we get

$$\|w_a\|_{H^1(\Pi(t))} \leqslant C.$$

Keeping in mind the equivalence of the norm (7.6) and the standard norm in $W_2^1(\Pi(t))$, we get

$$\|w_a\|_{W_2^1(\Pi(t))} \leqslant \widetilde{C}.$$

The weak compactness of bounded sequence in a Hilbert space, the compactness in $L_2(\Sigma)$ of traces of functions bounded in $W_2^1(\Pi(t))$ lead to the following: from any sequence $a_k \underset{k\to\infty}{\longrightarrow} 0$ as $k \to \infty$ one can choose a subsequence $\{a_{k'}\}$, such that the strong convergence (7.10) holds on this subsequence and the weak convergence

$$w_a \underset{a_{k'}\to 0}{\rightharpoonup} w^* \qquad \text{in } H^1(\Pi(t)) \tag{7.12}$$

holds.

It remains to prove formulae (7.11). Suppose that v is an arbitrary function from $H^1(\Pi(t))$, and functions $v_a \in H^1(\Pi_a; \partial B_a)$ extended by zero in B_a, satisfy $v_a \to v$ in $H^1(\Pi(t))$ as $a \to 0$. The possibility of construction of such a sequence follows from the description of the micro inhomogeneous geometry of the domain Π_a. Then, substituting v_a as a test function in (7.4) and passing to the limit as $a \to 0$ bearing in mind (7.10) and (7.12), we obtain

$$\int\limits_{\Pi(t)} (\nabla w^*, \nabla v)\,\mathrm{d}x = 0.$$

Due to the arbitrariness of the choice of v we have $w^* = \text{const}$. Finally, this fact, the convergence (7.10), and $\|w_a\|_{L_2(\Sigma)} = 1$ prove (7.11). □

Next Corollary follows from Lemmas 7.4 and 7.5.

Corollary 7.6. *The unique eigenvalue of problem* (7.3)*, converging to zero as* $a \to 0$*, is the minimal simple eigenvalue* Λ_a*, and the respective eigenfunction* w_a *normalized in* $L_2(\Sigma)$*, satisfies Lemma* 7.5*.*

Proof. Assume the contrary, *i.e.* in addition to the minimum eigenvalue $\Lambda_a \to 0$, there exists another eigenvalue $\widetilde{\Lambda}_a \neq \Lambda_a$, $\widetilde{\Lambda}_a \to 0$ and suppose that $\widetilde{w}_a$ is the respective eigenfunction normalized $L_2(\Sigma)$. Then from (7.4) it follows that

$$\int\limits_{\Pi_a(t)} (\nabla w_a, \nabla \widetilde{w}_a)\,\mathrm{d}x = \Lambda(a) \int\limits_{\Sigma} w_a \widetilde{w}_a \mathrm{d}s, \qquad \int\limits_{\Pi_a(t)} (\nabla \widetilde{w}_a, \nabla w_a)\,\mathrm{d}x = \widetilde{\Lambda}(a) \int\limits_{\Sigma} \widetilde{w}_a w_a \mathrm{d}s$$

and hence, w_a and $\widetilde{w}_a$ are orthogonal in $L_2(\Sigma)$, which contradicts with (7.10) and (7.11). Then the statement of Corollary follows from Lemma 6.3. □

The proof of the next Lemma is completely analogous to the proof of Lemma 4.2.

Lemma 7.7. *There exists a function* $\widetilde{g}_d \in C^\infty\left(\overline{\Pi(t)} \setminus \{x_0\}\right)$*, which satisfies the problem*

$$\begin{cases} \Delta \widetilde{g}_d = 0 & \text{as } x \in \Pi(t) \setminus \{x_0\}, \\ \frac{\partial \widetilde{g}_d}{\partial \nu} = 0 & \text{as } x \in \partial\Pi(t) \backslash \overline{\Sigma}, \\ \frac{\partial \widetilde{g}_d}{\partial \nu} = \sigma_d & \text{as } x \in \Sigma \end{cases}$$

and represents in a neighborhood of the point x_0*, in the form*

$$\widetilde{g}_d(x) = G_d(|y|) + \widetilde{g}_d^{(1)}(x)$$

where $\widetilde{g}_d^{(1)}$ *is infinitely differentiable function in the neighborhood of this point.*

Corollary 7.8. *The differentiable asymptotics*

$$\widetilde{g}_d(x) = G_d(|y|) + c_{\Pi,d} + P_1^{\Pi,d}(y) + O\left(|y|^2\right), \qquad y \to 0,$$

holds, where $c_{\Pi,d}$ is a constant and $P_1^{\Pi,d}(y)$ is a homogeneous polynomial of the first order.

Proof of Theorem 7.3. Denote

$$\begin{aligned}
\widetilde{W}_a(x) := &\left(1-\chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1-\frac{1}{\ln a}\left(\widetilde{g}_2(x)+c_B-c_{\Pi,2}\right)\right)\\
&-\frac{1}{\ln a}\chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(2)}\left(\frac{y}{a}\right)+aV_1^{(2)}\left(\frac{y}{a}\right)\right) \qquad \text{for } d=2\ ,\\
\widetilde{W}_a(x) := &\left(1-\chi\left(\frac{|y|}{a^\beta}\right)\right)\left(1+aC_3(B)\left(\widetilde{g}_3(x)-c_{\Pi,3}\right)\right)\\
&+\chi\left(\frac{|y|}{a^\beta}\right)\left(V_0^{(3)}\left(\frac{y}{a}\right)+aV_1^{(3)}\left(\frac{y}{a}\right)\right) \qquad \text{for } d=3\ .
\end{aligned}$$

Obviously it belongs to $W_2^2(\Pi_a(t))$. Using Lemmas 7.7 and 4.5, we obtain in analogous way as in the proof of Theorem 4.2, that the function $\widetilde{W}_a$ satisfies the problem

$$\begin{cases}
-\Delta\widetilde{W}_a = F_a \quad \text{as } x\in\Pi_a(t),\\
\widetilde{W}_a = 0 \quad \text{as } x\in\partial B_a,\\
\frac{\partial\widetilde{W}_a}{\partial\nu} = 0 \quad \text{as } x\in\partial\Pi(t)\backslash\overline{\Sigma},\\
\frac{\partial\widetilde{W}_a}{\partial\nu} = -\frac{2\pi}{\ln a} \quad \text{for } d=2, \qquad \frac{\partial\widetilde{W}_a}{\partial\nu} = a4\pi C_3(B) \quad \text{for } d=3, \quad \text{if } x\in\Sigma,,
\end{cases} \tag{7.13}$$

where

$$\|1-\widetilde{W}_a\|_{L_2(\Sigma)} = O\left(\frac{1}{\ln a}\right) \quad \text{for } d=2, \qquad \|1-\widetilde{W}_a\|_{L_2(\Sigma)} = O\left(a\right) \quad \text{for } d=3,$$

$$\begin{aligned}
\|F_a\|_{L_2(\Pi_a(t))} &= O\left(\frac{1}{\ln a}\left(a^\beta + a^{1-2\beta}\right)\right) \qquad \text{for } d=2,\\
\|F_a\|_{L_2(\Pi_a(t))} &= O\left(a^{1+\frac{3}{2}\beta} + a^{2-\frac{5}{2}\beta}\right) \qquad \text{for } d=3.
\end{aligned}$$

Suppose that $0<\beta<\frac{1}{2}$. Then

$$\|F_a\|_{L_2(\Pi_a(t))} \underset{a\to 0}{\longrightarrow} 0, \qquad \|\widetilde{W}_a - 1\|_{L_2(\Sigma)} \underset{a\to 0}{\to} 0. \tag{7.14}$$

Substituting $v=\widetilde{W}_a$ and $\Lambda(a)=\Lambda_a$ in (7.4), we get

$$\int\limits_{\Pi_a(t)} (\nabla w_a, \nabla\widetilde{W}_a)\,\mathrm{d}x = \Lambda_a \int\limits_{\Sigma} w_a\widetilde{W}_a \mathrm{d}s.$$

Multiplying the equation in (7.13) by w_a and integrating this equation by parts over $\Pi_a(t)$, we derive

$$\int\limits_{\Pi_a(t)} (\nabla\widetilde{W}_a, \nabla w_a)\,\mathrm{d}x = -\frac{\sigma_d C_d(B)}{G_d(a)}\int\limits_{\Sigma} w_a\widetilde{W}_a\,\mathrm{d}s + \int\limits_{\Pi_a(t)} F_a w_a \mathrm{d}x.$$

From these two identities we deduce

$$\left(\frac{\sigma_d C_d(B)}{G_d(a)} + \Lambda_a\right) \int\limits_{\Sigma} w_a \widetilde{W}_a \,\mathrm{d}s = \int\limits_{\Pi_a(t)} F_a w_a \mathrm{d}x.$$

The representation (7.9) follows from this, as $a \to 0$, and also from (7.10), (7.11) and (7.14). □

8. Proof of Theorems 2.5 and 2.6

Before proving Theorems 2.5 and 2.6 we give one auxiliary proposition.

Lemma 8.1. *If the condition* (2.11) *holds, then there exists a constant* $C > 0$*, such that for functions* $v \in W_2^1(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a})$ *the estimate*

$$\begin{aligned} \|v\|^2_{L_2(\Gamma_1)} &\leqslant C\varepsilon\,|\ln a|\,\|v\|^2_{W_2^1(\Omega_{\varepsilon,a})} && \text{for } d=2\,, \\ \|v\|^2_{L_2(\Gamma_1)} &\leqslant C\varepsilon a^{-1}\|v\|^2_{W_2^1(\Omega_{\varepsilon,a})} && \text{for } d=3 \end{aligned} \tag{8.1}$$

holds.

Proof. For any functions $w \in H^1(\Pi_a(t); \partial B_a)$ due to (7.2) we have

$$\begin{aligned} \|w\|^2_{L_2(\Sigma)} &\leqslant C\,|\ln a|\,\|\nabla w\|^2_{L_2(\Pi(t))} && \text{for } d=2\,, \\ \|w\|^2_{L_2(\Sigma)} &\leqslant C a^{-1}\|\nabla w\|^2_{L_2(\Pi(t))} && \text{for } d=3. \end{aligned}$$

Denote

$$\begin{aligned} \Gamma^{(j)}_{1,\varepsilon} &:= \{x:\ j\varepsilon < x_1 < (j+1)\varepsilon,\ x_2 = 0\} && \text{for } d=2\,, \\ \Gamma^{(j,i)}_{1,\varepsilon} &:= \{x:\ j\varepsilon < x_1 < (j+1)\varepsilon,\ i\varepsilon < x_2 < (i+1)\varepsilon,\ x_3 = 0\} && \text{for } d=3. \end{aligned}$$

From these inequalities we derive for functions v belonging to $W_2^1(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a})$ the following estimates

$$\varepsilon^{-1}\|v\|^2_{L_2\left(\Gamma^{(j)}_{1,\varepsilon}\right)} \leqslant C\,|\ln a|\,\|\nabla v\|^2_{L_2\left(\Gamma^{(j)}_{1,\varepsilon}\times(0,t\varepsilon)\right)},$$
$$\varepsilon^{-1}\|v\|^2_{L_2(\Gamma_1)} \leqslant C\,|\ln a|\,\|\nabla v\|^2_{L_2(\Pi(t\varepsilon))} \leqslant C\,|\ln a|\,\|v\|^2_{W_2^1(\Omega_{\varepsilon,a})}$$

for $d = 2$, and

$$\varepsilon^{-1}\|v\|^2_{L_2\left(\Gamma^{(j,i)}_{1,\varepsilon}\right)} \leqslant C a^{-1}\|\nabla v\|^2_{L_2\left(\Gamma^{(j,i)}_{1,\varepsilon}\times(0,t\varepsilon)\right)},$$
$$\varepsilon^{-1}\|v\|^2_{L_2(\Gamma_1)} \leqslant C a^{-1}\|\nabla v\|^2_{L_2(\Pi(t\varepsilon))} \leqslant C a^{-1}\|v\|^2_{W_2^1(\Omega_{\varepsilon,a})}$$

for $d = 3$.

Finally, the estimate (8.1) follows immediately. □

Corollary 8.2. *Suppose that* (2.11) *is true and the weak convergence*

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} U^* \quad \text{in } W_2^1(\Omega) \tag{8.2}$$

takes place. Then $U^* \in W_2^1(\Omega; \Gamma_1)$.

Proof of Theorem 2.5. Suppose that the statement of Theorem is not true. Then without loss of generality we assume that $\lambda_{\varepsilon,a} \to \lambda_0$ as $\varepsilon \to 0$. Due to the integral identity

$$\int\limits_{\Omega} (\nabla u_{\varepsilon,a}, \nabla v)\,\mathrm{d}x = \lambda_{\varepsilon,a} \int\limits_{\Gamma_1} u_{\varepsilon,a} v \mathrm{d}s \qquad \forall\ v \in W_2^1(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a}), \tag{8.3}$$

of problem (2.3) we derive that

$$\|\nabla u_{\varepsilon,a}\|^2_{L_2(\Omega)} = \lambda_{\varepsilon,a} \|u_{\varepsilon,a}\|^2_{L_2(\Gamma_1)} = \lambda_{\varepsilon,a} \underset{\varepsilon \to 0}{\to} \lambda_0,$$

where $u_{\varepsilon,a}$ is a normalized in $L_2(\Gamma_1)$ eigenfunction of problem (2.3). Hence (see Rem. 3.2)

$$\|u_{\varepsilon,a}\|_{W_2^1(\Omega)} \leqslant C.$$

Then there exists such a subsequence of indexes that the following weak convergence

$$u_{\varepsilon,a} \underset{\varepsilon \to 0}{\rightharpoonup} u_* \not\equiv 0 \qquad \text{in } \ W_2^1(\Omega) \tag{8.4}$$

holds on it. Denote by $C_0^\infty(\overline{\Omega}; \Gamma_1)$ the subset of functions from $C^\infty(\overline{\Omega})$, vanishing in a neighborhood of Γ_1. Assume that $v \in C_0^\infty(\overline{\Omega}; \Gamma_1)$. It is easy to see, that $v \in W_2^1(\Omega_{\varepsilon,a}; \Gamma_{\varepsilon,a})$ for sufficiently small ε. Substituting an arbitrary $v \in C_0^\infty(\overline{\Omega}; \Gamma_1)$ in (8.3), passing to the limit as $\varepsilon \to 0$ and bearing in mind the convergence (8.4), we obtain the identity

$$\int\limits_{\Omega} (\nabla u^*, \nabla v)\ \mathrm{d}x = 0. \tag{8.5}$$

The dense imbedding of $C_0^\infty(\overline{\Omega}; \Gamma_1)$ into $W_2^1(\Omega; \Gamma_1)$ leads to the validity of the identity for any functions $v \in W_2^1(\Omega; \Gamma_1)$. Since $u^* \in W_2^1(\Omega; \Gamma_1)$ by means of the convergence (8.4) and Corollary 8.2, substituting $v = u^*$ in (8.5), we derive that $u^* \equiv 0$, which contradicts (8.4). Theorem is proved. □

Proof of Theorem 2.6. To prove the theorem we need the estimate (2.6) for the solution $U_{\varepsilon,a}$ of the problem (2.1) also in the case (2.11). Due to the linearity of problem (2.1) it is sufficient to prove this estimate for normalized in $L_2(\Gamma_1)$ functions $U_{\varepsilon,a}$. In this case the proof is based on the contradiction. Let estimate (2.6) be wrong. Then due to Remark 3.2 there exists a sequence of natural numbers, such that the conditions of Lemma 3.3 are fulfilled. Due to Lemma 3.3 the convergence (3.7) takes place and there exist a subsequence of indexes $\{k'\}$ and a function $U^* \in W_2^1(\Omega)$, such that the convergences (3.8) and (3.9) and the identity (3.10) hold as $k' \to \infty$. And moreover, $U^* \in W_2^1(\Omega; \Gamma_1)$ due to Corollary 8.2. Substituting an arbitrary $v \in C_0^\infty(\overline{\Omega}; \Gamma_1)$ in (3.2) (as $\varepsilon = \varepsilon_{m(k')}$ and $a = a_{k'}$), passing to the limit as $k' \to \infty$ and bearing in mind the convergence (3.7), we obtain the identity

$$\int\limits_{\Omega} (\nabla U^*, \nabla v)\ \mathrm{d}x = 0. \tag{8.6}$$

The dense imbedding of $C_0^\infty(\overline{\Omega}; \Gamma_1)$ into $W_2^1(\Omega; \Gamma_1)$ leads to the validity of the identity for any functions $v \in W_2^1(\Omega; \Gamma_1)$. On the one hand $U^* \neq 0$ due to (3.10), and on the other hand from (8.6) we get $U^* \equiv const$ which vanishes on Γ_1, consequently $U^* \equiv 0$. Hence, we obtain the contradiction. Therefore, the estimate (2.6) holds.

Assume that $\{a_k\}_{k=1}^\infty$ is a sequence, such that for $\varepsilon = \varepsilon_k$, $a = a_k$ as $k \to \infty$ the convergence (2.11) holds, $\{k'\}$ is an arbitrary subsequence of natural numbers. Then, due to estimate (2.6) there exist $U^* \in W_2^1(\Omega)$ and a subsequence of this subsequence, such that the convergence (8.2) is valid on this subsequence, and moreover, $U^* \in W_2^1(\Omega; \Gamma_1)$ because of Corollary 8.2.

Substituting an arbitrary $v \in C_0^\infty(\overline{\Omega};\Gamma_1)$ in (3.2) (as $\varepsilon = \varepsilon_{k'}$ and $a = a_{k'}$), passing to the limit as $k' \to \infty$ and keeping in mind the convergence (8.2), we get (8.6). The embedding of $C_0^\infty(\overline{\Omega};\Gamma_1)$ into $W_2^1(\Omega;\Gamma_1)$ leads to the validity of the identity (8.6) for any functions $v \in W_2^1(\Omega;\Gamma_1)$. Substituting $v = U^*$ in (8.6), we conclude that $U^* \equiv 0$. This fact, the arbitrariness of the choice of the subsequence $\{k'\}$ and the convergence (8.2) give the weak convergence

$$U_{\varepsilon,a} \underset{\varepsilon\to 0}{\rightharpoonup} 0 \qquad \text{in } W_2^1(\Omega). \tag{8.7}$$

Using the integral identity (3.2), weak convergence (8.7) and the compactness of embedding $W_2^1(\Omega)$ into $L_2(\Omega)$, we derive

$$\begin{aligned}\|U_{\varepsilon,a}\|^2_{W_2^1(\Omega)} =& \|\nabla U_{\varepsilon,a}\|^2_{L_2(\Omega)} + \|U_{\varepsilon,a}\|^2_{L_2(\Omega)} \\ =& \|U_{\varepsilon,a}\|^2_{L_2(\Omega)} - \int\limits_\Omega f U_{\varepsilon,a}\,\mathrm{d}x \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

Hence we have the strong convergence (2.12). Theorem is proved. □

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