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PREFACE

Discretization methods of Partial Differential Equations (PDEs) that support polygonal or polyhedral (in short, polytopal) grids have been receiving extensive interest in recent years. Polytopal grids offer, for instance, a natural framework to handle hanging nodes, different cell shapes within the same mesh, and non-matching interfaces resulting from local mesh adaptation or from multi-domain mesh gluing. Polytopal grids often lead to enhanced computational efficiency owing to the flexibility provided in the geometrical approximation of the domain and of the problem. Interesting natural applications include basin and reservoir simulation, problems with moving boundaries (as encountered in fluid-structure interaction, crack propagation and contact problems), materials with complex inclusions, topology optimization, image processing, and many others.

Traditional discretization methods turn out to have limitations in the range of applicable meshes. For instance, the popular Finite Element method uses specific (geometry-related) polynomials to reconstruct shape functions within cells from degrees of freedom while ensuring some continuity (or conformity) property across cell boundaries. Instead, Finite Volume methods typically handle piecewise constant functions, but the traditional Two-Point Flux Approximation to discretize a second-order differential operator hinges on stringent orthogonality constraints for the underlying grid. The above limitations have been overcome in the last decade or so, and at the same time, several discretization methods that can cope with polytopal grids appeared in the literature. Just to give a short, and by no means exhaustive, list of examples, we mention the Mimetic Discretization method (see [8], the book [5], and the links to mixed and hybrid finite volumes in [18]), the polygonal Finite Element Method [24,25], the Virtual Element Method [3,4], the Hybrid High-Order method [16,17], the Hybridizable Discontinuous Galerkin method [12], and the Weak Galerkin Method [26]. Unified analysis frameworks also appeared recently for the lowest-order case, such as Gradient Schemes [19] and Compatible Discrete Operator schemes [7]. Furthermore, discontinuous Galerkin methods offer a natural setting supporting polytopal meshes; see the book [15] and references therein, as well as the recent developments in [9] to include meshes with face degeneration.

Owing to these encouraging results, the devising and analyzing of polytopal discretization methods has grown into a topic of large interest in the mathematical and engineering communities. The present Special Issue is motivated by such developments. While the field is quite vast and is growing in different interesting directions, the manuscripts collected herein provide a relatively broad, yet non-exhaustive, sampling of emerging or consolidating research topics. The contents can be loosely classified into three groups. Some contributions aim at uncovering links between existing methods or at providing unifying analysis frameworks. This is the case for [13] which bridges the Hybrid High-Order method to the Hybridizable Discontinuous Galerkin method and to the recent nonconforming Virtual Element Method [2] also analyzed in this Issue. Unifying frameworks are devised in [20] which further elaborates the framework of Gradient Schemes, and in [11] which further develops the notion of Finite Element Systems. Another group of contributions brings new developments to existing techniques. Salient examples are the *hp*-analysis of polytopal discontinuous Galerkin methods [10], the interpolation error analysis for harmonic coordinates [21], and the extension to elliptic PDEs in mixed form of the Virtual Element method [6] and of the Mimetic Discretization method [22]. A third group of contributions explores challenging applications of polytopal discretization methods, such as the Plane Wave Virtual Element method for the Helmholtz equation [23], the Mimetic Finite Difference approximation of flows in fractured porous media [1], and the geometrically-defined vector basis functions for computational electromagnetics [14].

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