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FINITE ELEMENT EXTERIOR CALCULUS FOR PARABOLIC PROBLEMS *, **

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Abstract. In this paper, we consider the extension of the finite element exterior calculus from elliptic problems, in which the Hodge Laplacian is an appropriate model problem, to parabolic problems, for which we take the Hodge heat equation as our model problem. The numerical method we study is a Galerkin method based on a mixed variational formulation and using as subspaces the same spaces of finite element differential forms that are used for elliptic problems. We analyze both the semidiscrete and a fully-discrete numerical scheme.

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1. Introduction

In this paper we consider the numerical solution of the *Hodge heat equation*, the parabolic equation associated to the Hodge Laplacian on differential k-forms. The initial-boundary value problem we study is

$$u_t + (d\delta + \delta d)u = f \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\operatorname{tr}(\star u) = 0,\ \operatorname{tr}(\star \mathrm{d}u) = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{1.2}$$

$$u(\,\cdot\,, 0) = u_0 \quad \text{in } \Omega. \tag{1.3}$$

Here the domain $\Omega \subset \mathbb{R}^n$ has a piecewise smooth, Lipschitz boundary, the unknown u is a time dependent differential k-form on Ω, u_t denotes its partial derivative with respect to time, and d, δ, $\star$, and tr denote the exterior derivative, coderivative, Hodge star, and trace operators, respectively.

The numerical methods we consider are mixed finite element methods, which introduce the variable $\sigma = \delta u$, a differential $(k-1)$-form. The mixed method is based on the mixed weak formulation: find

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$(\sigma, u) : [0,T] \to H\Lambda^{k-1} \times H\Lambda^k$, satisfying the equations

$$\langle \sigma, \tau \rangle - \langle \mathrm{d}\tau, u \rangle = 0, \quad \tau \in H\Lambda^{k-1}, \ t \in (0,T], \tag{1.4}$$

$$\langle u_t, v \rangle + \langle \mathrm{d}\sigma, v \rangle + \langle \mathrm{d}u, \mathrm{d}v \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k, \ t \in (0,T], \tag{1.5}$$

together with the initial condition (1.3). Here angular brackets are used to denote the L^2 inner product of differential forms. The notations are explained more fully in the following section. The well-posedness of this mixed problem is established in a precise sense in Theorem 4.4.

In this paper, we restrict ourselves to studying the model problem (1.1)–(1.3), with the corresponding weak formulation (1.4)–(1.5). However, both the numerical method and its analysis can be extended in various ways. In particular, essential boundary conditions and variable coefficients, which we now discuss briefly, are both easily handled. For a fuller discussion of these in the elliptic case (see [3], Sect. 6.2).

The boundary conditions (1.2) are natural in this mixed formulation and so do not appear explicitly in the weak formulation. Essential boundary conditions

$$\operatorname{tr} u = 0, \quad \operatorname{tr} \delta u = 0, \tag{1.6}$$

are treated simply by replacing the spaces $H\Lambda^{k-1}$ and $H\Lambda^k$ by their trace-free subspaces $\mathring{H}\Lambda^{k-1}$ and $\mathring{H}\Lambda^k$ in the weak formulation.

Variable coefficients may be included in the problem by replacing the inner products in the three Hilbert spaces $L^2\Lambda^{k-1}$, $L^2\Lambda^k$, and $L^2\Lambda^{k+1}$ by equivalent weighted inner products, but otherwise retaining the same structure. The new inner product on $L^2\Lambda^k$ may be written in terms of the ordinary L^2 inner product as $\langle \theta\,\cdot\,,\,\cdot\,\rangle_{L^2\Lambda^{k-1}}$ where the coefficient $\theta = \theta(x)$ is a symmetric and uniformly positive definite operator on $\mathrm{Alt}^k\,\mathbb{R}^n$. Similarly, we may introduce coefficients α and γ for the inner products on $(k-1)$- and $(k+1)$-forms. Thus the weak form of the problem with variable coefficients becomes

$$\langle \alpha\sigma, \tau \rangle - \langle \theta\mathrm{d}\tau, u \rangle = 0, \quad \tau \in H\Lambda^{k-1}, \ t \in (0,T],$$
$$\langle \theta u_t, v \rangle + \langle \theta\mathrm{d}\sigma, v \rangle + \langle \gamma\mathrm{d}u, \mathrm{d}v \rangle = \langle f, v \rangle, \quad v \in H\Lambda^k, \ t \in (0,T].$$

The strong equation in this case is

$$\theta u_t + \theta d[\alpha^{-1}\delta(\theta u)] + \delta(\gamma du) = f, \tag{1.7}$$

which reduces to (1.1) if the coefficients are all chosen to be the identity at every point.

Now we interpret the model problem (1.1)–(1.3) in specific cases. In the simplest case of 0-forms ($k = 0$), the differential equation (1.1) is simply the heat equation, $u_t - \Delta u = f$, and the natural boundary condition (1.2) is the Neumann boundary condition, $\partial u/\partial n = 0$. The essential boundary condition, (1.6), is, of course the Dirichlet condition. Moreover, in this case the space $H\Lambda^{k-1}$ vanishes, and the weak formulation (1.4)–(1.5) is the usual (unmixed) one: $u : [0,T] \to H^1(\Omega)$ satisfies

$$\langle u_t, v \rangle + \langle \operatorname{grad} u, \operatorname{grad} v \rangle = \langle f, v \rangle, \quad v \in H^1(\Omega), \ t \in (0,T].$$

In this case, the numerical methods and convergence results obtained in this paper reduce to ones long known [11, 22].

In the case of n-forms, the differential equation is again the heat equation, although the natural boundary condition is now the Dirichlet condition and the essential boundary condition (1.6) is the Neumann condition. For n-forms, the weak formulation seeks $\sigma \in H(\mathrm{div})$, $u \in L^2$ such that

$$\langle \sigma, \tau \rangle - \langle \operatorname{div} \tau, u \rangle = 0, \ \tau \in H(\mathrm{div}), \quad \langle u_t, v \rangle - \langle \operatorname{div} \sigma, v \rangle = \langle f, v \rangle, \ v \in L^2, \quad t \in (0,T].$$

This mixed method for the heat equation was studied in a fundamental paper by Johnson and Thomée [17]. See also the treatment in ([21], Chap. 17) and the analysis in [4], which shows that the conditions required of the mixed finite element spaces are more stringent than the classical Brezzi conditions needed for elliptic problems [6]. Extension to quasilinear parabolic equations has been carried out by Garcia [12, 13] and by Chou and Li [9], max norm estimates were studied by Scholz [19], and superconvergence by Squeff [20] and others [8], directions which are not pursued here. Recently, Holst *et al.* [14] have studied this mixed method for the heat equation in n-dimensions using a finite element exterior calculus framework (in their work they consider hyperbolic problems as well), and Holst and Tiee have extended the results of that work and of the present paper to the case where the domain is a Riemannian hypersurface [16].

For $k = 1$ or 2 in $n = 3$ dimensions, the differential equation (1.1) is the vectorial heat equation,

$$u_t + \operatorname{curl}\operatorname{curl} u - \operatorname{grad}\operatorname{div} u = f.$$

The weak formulations (1.4)–(1.5) for $k = 1$ and 2 correspond to two different mixed formulations of this equation, the former using the scalar field $\sigma = -\operatorname{div} u$ as the second unknown, the latter using the vector field $\sigma = \operatorname{curl} u$. For $k = 1$, the boundary conditions (1.2), which are natural in the mixed formulation, become $u \times n = 0$, $\operatorname{curl} u \times n = 0$, while for $k = 2$ these natural boundary conditions are $u \times n = 0$, $\operatorname{div} u = 0$. For essential boundary conditions, these are reversed.

As an example application of these equations, we consider the eddy current approximation of Maxwell's equations. See, *e.g.*, [1] for more information. The eddy current approximation on a conducting domain $\Omega \subset \mathbb{R}^3$ may be viewed as the limit of the full Maxwell's equations in the case of small electric permitivity ([1], Chap. 2.2). Written in terms of the electric field $E : \Omega \times [0, T] \to \mathbb{R}^3$ and the magnetic induction $B : \Omega \times [0, T] \to \mathbb{R}^3$, this gives the equations

$$\operatorname{curl}(\mu^{-1}B) = j + \eta E, \quad B_t + \operatorname{curl} E = 0, \quad \text{on } \Omega \times [0, T], \tag{1.8}$$

where μ is the magnetic permeability, η is the conductivity, and j is the applied current density. The permeability and conductivity can be positive scalars or positive definite matrices, and they might vary in space. We assume that the applied current density is divergence-free, so the first equation implies that $\operatorname{div}(\eta E) = 0$. The eddy current problem consists of these differential equations, together with the initial condition $E = E_0$ at $t = 0$ and suitable boundary conditions, such as the electric boundary conditions $E \times n = 0$.

Differentiating the first equation of (1.8) in time and combining with the second equation gives the eddy current equation in terms of the electric field alone:

$$\eta E_t + \operatorname{curl}\mu^{-1}\operatorname{curl} E = -j_t.$$

Since $\operatorname{div}(\eta E) = 0$, we have

$$\eta E_t - \eta\operatorname{grad}[\rho\operatorname{div}(\eta E)] + \operatorname{curl}\mu^{-1}\operatorname{curl} E = -j_t,$$

for any convenient choice of coefficient tensor ρ. If we take the coefficients μ, η, and ρ all to be unity, this is precisely the Hodge heat equation for 1-forms. In the general case, it recovers the variable coefficient Hodge heat equation (1.7) with $\alpha = \rho^{-1}$, $\theta = \eta$, and $\gamma = \mu^{-1}$.

Besides the eddy current model, the vectorial heat equation arises in other applications, often as part of more complicated equations. Examples are the Ginzburg–Landau equations for superconductivity [15], and some formulations of incompressible fluid [23].

To discretize (1.4), (1.5), we utilize the two main families of finite element differential forms, the $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ spaces. Between them they include lots of the best known families of finite elements on simplicial meshes ([3], Sect. 5). We give both semidiscrete and fully discrete schemes, and the corresponding convergence analysis. Convergence rates under different norms are shown in our final results (see Thms. 5.4 and 6.3 below). These achieve the optimal rates allowed by the finite element spaces provided some regularity assumptions are satisfied. These results also reveal the relation between convergence rates under different norms and the regularity of the exact solution.

The numerical discretization and analysis of the Hodge heat equation presented here has much in common with the numerical analysis of mixed methods for the scalar heat equation developed in [17] (which can be viewed as the special case of n-forms). In particular, the analysis relies on estimates for an appropriate elliptic projection. However, there are significant differences between the case $k = n$ and the general case. One is that the spatial discretization is no longer of the saddle point type considered by Brezzi [6] and so the needed stability properties for the elliptic projection must be gotten from the finite element exterior calculus instead of the Brezzi theory. Particularly significant is the role of harmonic forms, which do not arise in the case of n-forms. It is interesting that, unlike in the elliptic case, harmonic forms do not enter the weak formulation of the Galerkin method for the parabolic problem. However they must be accounted for in the definition of the elliptic projection and the subsequent analysis. This accounts for some technical complications below.

The outline of the remainder of the paper is as follows. In Section 2, we review basic notations from finite element exterior calculus, including the two main families of finite element differential forms, the $\mathcal{P}_r^-\Lambda^k$ and $\mathcal{P}_r\Lambda^k$ families, and some of their properties. In Section 3, we apply the elliptic theory to define an elliptic projection that will be crucial to the error analysis of the time-dependent problem, and to obtain error estimates for it. In Section 4, we turn to the Hodge heat equation at the continuous level and establish well-posedness of the mixed formulation. We then give a convergence analysis for the semidiscrete and fully discrete schemes in Sections 5 and 6, respectively. Finally, we present some numerical examples confirming the results.

2. Preliminaries

We briefly review here some basic notions of finite element exterior calculus for the Hodge Laplacian. Details can be found in ([2], Sect. 2) and ([3], Sects. 3 and 4) and in numerous references given there.

For Ω a domain in $\mathbb{R}^n$ and k an integer, let $L^2\Lambda^k = L^2\Lambda^k(\Omega)$ denote the Hilbert space of differential k-forms on Ω with coefficients in L^2. This is the space of L^2 functions on Ω with values in $\mathrm{Alt}^k\,\mathbb{R}^n$, a finite dimensional Hilbert space of dimension $\binom{n}{k}$ (understood to be 0 if $k < 0$ or $k > n$). We may similarly define Lebesgue spaces $L^p\Lambda^k$ and Sobolev spaces $W_p^m\Lambda^k$ and $H^m\Lambda^k = W_2^m\Lambda^k$. The *Hodge star* operator $\star$ is an isometry of $\mathrm{Alt}^k\,\mathbb{R}^n$ and $\mathrm{Alt}^{n-k}\,\mathbb{R}^n$, and so induces an isometry of $L^2\Lambda^k$ onto $L^2\Lambda^{n-k}$. The inner product in $L^2\Lambda^k$ may be written $\langle u, v\rangle = \int_\Omega u\wedge\star v$, with the corresponding norm denoted $\|u\|$. We view the exterior derivative $d = d^k$ as a unbounded operator from $L^2\Lambda^k$ to $L^2\Lambda^{k+1}$. Its domain, which we denote $H\Lambda^k(\Omega)$, consists of forms $u \in L^2\Lambda^k$ for which the distributional exterior derivative du belongs to $L^2\Lambda^{k+1}$. Assuming, as we shall, that Ω has Lipschitz boundary, the *trace operator* $\mathrm{tr} = \mathrm{tr}_{\partial\Omega}$ maps $H\Lambda^k(\Omega)$ boundedly into an appropriate Sobolev space on $\partial\Omega$ (namely $H^{-1/2}\Lambda^k(\partial\Omega)$). The *coderivative* δ is defined as $\pm\star d\star : H^*\Lambda^k \to H^*\Lambda^{k-1}$, where $H^*\Lambda^k := \star H\Lambda^{n-k}$ and the sign is $-$ if n is even and $(-1)^k$ if n is odd. The adjoint $d^* = d_k^*$ of d^{k-1} is the unbounded operator $L^2\Lambda^k \to L^2\Lambda^{k-1}$ given by restricting δ to the domain of d^*,

$$D(d^*) = \mathring{H}^*\Lambda^k := \{\, u \in H^*\Lambda^k \mid \mathrm{tr}\star u = 0 \,\}.$$

We denote by $\mathfrak{Z}^k$ and $\mathfrak{Z}_k^*$ the null spaces of d^k and d_k^*, respectively. Their orthogonal complements in $L^2\Lambda^k$ are $\mathfrak{B}_k^*$ and $\mathfrak{B}^k$, the ranges of d_{k+1}^* and d^{k-1}, respectively. The orthogonal complement of $\mathfrak{B}^k$ inside $\mathfrak{Z}^k$ is the space of *harmonic forms*

$$\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{Z}_k^* = \{\, \omega \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \,|\, \mathrm{d}\omega = 0, d^*\omega = 0 \,\}.$$

The dimension of $\mathfrak{H}^k$ is equal to the kth Betti number of Ω, so $\mathfrak{H}^k = 0$ for $k \neq 0$ if Ω is contractible. The *Hodge decomposition* of $L^2\Lambda^k$ and of $H\Lambda^k$ follow immediately:

$$L^2\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*, \tag{2.1}$$

$$H\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}, \tag{2.2}$$

where $\mathfrak{Z}^{k\perp} = H\Lambda^k \cap \mathfrak{B}_k^*$ denotes the orthogonal complement of $\mathfrak{Z}^k$ in $H\Lambda^k$. Let $P_{\mathfrak{B}} : L^2\Lambda^k \to \mathfrak{B}^k$ denote the L^2-projection, and similarly for other spaces.

The *Hodge Laplacian* is the unbounded operator $\mathcal{L} = dd^* + d^*d : D(\mathcal{L}) \subset L^2\Lambda^k \to L^2\Lambda^k$ with the domain

$$D(\mathcal{L}) = \{\, v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k \;\mid\; d^*v \in H\Lambda^{k-1}, \mathrm{d}v \in \mathring{H}^*\Lambda^{k+1} \,\}.$$

The null space of $\mathcal{L}$ consists precisely of the harmonic forms $\mathfrak{H}^k$.

For any $f \in L^2\Lambda^k$, there exists a unique solution $u = Kf \in D(\mathcal{L})$ satisfying

$$\mathcal{L}u = f \pmod{\mathfrak{H}}, \quad u \perp \mathfrak{H}^k,$$

(see [3], Thm. 3.1). The solution u satisfies the Hodge Laplacian boundary value problem

$$(d\delta + \delta d)u = f - P_{\mathfrak{H}}f \text{ in } \Omega, \quad \mathrm{tr}\star u = 0,\ \mathrm{tr}\star \mathrm{d}u = 0 \text{ on } \partial\Omega,$$

together with side condition $u \perp \mathfrak{H}^k$ required for uniqueness. The solution operator K is a compact operator $L^2\Lambda^k \to H\Lambda^k \cap \mathring{H}^*\Lambda^k$ and *a fortiori*, is compact as an operator from $L^2\Lambda^k$ to itself.

Now we consider the mixed finite element discretization of the Hodge Laplacian boundary value problem, following [3]. This is based on the mixed weak formulation, which seeks $\sigma \in H\Lambda^{k-1}$, $u \in H\Lambda^k$, and $p \in \mathfrak{H}^k$ such that

$$\begin{aligned}\langle \sigma, \tau\rangle - \langle \mathrm{d}\tau, u\rangle &= 0, \quad \tau \in H\Lambda^{k-1},\\ \langle \mathrm{d}\sigma, v\rangle + \langle \mathrm{d}u, \mathrm{d}v\rangle + \langle p, v\rangle &= \langle f, v\rangle, \quad v \in H\Lambda^k,\\ \langle u, q\rangle &= 0, \quad q \in \mathfrak{H}^k.\end{aligned}$$

It admits a unique solution given by $u = Kf$, $\sigma = d^*u$, $p = P_{\mathfrak{H}}f$. We discretize the mixed formulation using Galerkin's method. For this, let Λ_h^{k-1} and Λ_h^k be finite dimensional subspaces of $H\Lambda^{k-1}$ and $H\Lambda^k$, respectively, satisfying $d\Lambda_h^{k-1} \subset \Lambda_h^k$. We define the space of *discrete harmonic forms* $\mathfrak{H}_h^k$ as the orthogonal complement of $\mathfrak{B}_h^k := d\Lambda_h^{k-1}$ inside $\mathfrak{Z}_h^k := \mathfrak{Z} \cap \Lambda_h^k$. This immediately gives the *discrete Hodge decomposition*

$$\Lambda_h^k = \mathfrak{B}_h^k \oplus \mathfrak{H}_h^k \oplus \mathfrak{Z}_h^{k\perp},$$

where $\mathfrak{Z}_h^{k\perp}$ is the orthogonal complement of $\mathfrak{Z}_h^k$ inside Λ_h^k.

The Galerkin method seeks $\sigma_h \in \Lambda_h^{k-1}$, $u_h \in \Lambda_h^k$, $p_h \in \mathfrak{H}_h^k$ such that

$$\begin{aligned}\langle \sigma_h, \tau\rangle - \langle \mathrm{d}\tau, u_h\rangle &= 0, \quad \tau \in \Lambda_h^{k-1},\\ \langle \mathrm{d}\sigma_h, v\rangle + \langle \mathrm{d}u_h, \mathrm{d}v\rangle + \langle p_h, v\rangle &= \langle f, v\rangle, \quad v \in \Lambda_h^k,\\ \langle u_h, q\rangle &= 0, \quad q \in \mathfrak{H}_h^k.\end{aligned} \tag{2.3}$$

For the analysis of this discretization, we require the existence of a third space $\Lambda_h^{k+1} \subset H\Lambda^{k+1}$ which contains $d\Lambda_h^k$, so that $\Lambda_h^{k-1} \xrightarrow{d} \Lambda_h^k \xrightarrow{d} \Lambda_h^{k+1}$ is a subcomplex of the segment $H\Lambda^{k-1} \xrightarrow{d} H\Lambda^k \xrightarrow{d} H\Lambda^{k+1}$ of the de Rham complex. Further we require that there exists a *bounded cochain projection*, *i.e.*, bounded linear projection maps $\pi_h^j : H\Lambda^j \to \Lambda_h^j$, $j = k-1, k, k+1$, such that the diagram

$$\begin{array}{ccccc} H\Lambda^{k-1} & \xrightarrow{d} & H\Lambda^k & \xrightarrow{d} & H\Lambda^{k+1} \\ \pi_h^{k-1}\downarrow & & \pi_h^k\downarrow & & \pi_h^{k+1}\downarrow \\ \Lambda_h^{k-1} & \xrightarrow{d} & \Lambda_h^k & \xrightarrow{d} & \Lambda_h^{k+1} \end{array} \tag{2.4}$$

commutes. A key result of the finite element exterior calculus is that, under these assumptions, the Galerkin equations (2.3) admit a unique solution and provide a stable discretization.

Another important aspect of the finite element exterior calculus is the construction of finite element spaces Λ_h^k which satisfy these hypotheses, *i.e.*, which combine to form de Rham subcomplexes with bounded cochain projections. Let there be given a shape regular family of meshes $\mathcal{T}_h$ with mesh size h tending to 0. For each $r \geq 1$, we define two finite element subspaces of $H\Lambda^k$, denoted $\mathcal{P}_r\Lambda^k(\mathcal{T}_h)$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$. For $k=0$, these two spaces coincide and equal the degree r Lagrange finite element subspace of $H^1(\Omega)$. For $k=n$, $\mathcal{P}_r^-\Lambda^n(\mathcal{T}_h)$ coincides with $\mathcal{P}_{r-1}\Lambda^n(\mathcal{T}_h)$, which may be viewed as the space of all piecewise polynomials of degree at most $r-1$, without inter-element continuity constraints. However, for $0<k<n$,

$$\mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \subsetneq \mathcal{P}_r\Lambda^k(\mathcal{T}_h).$$

For stable mixed finite elements for the Hodge Laplacian, we have four possibilities (which reduce to just one for $k=0$ and to two for $k=1$ or n):

$$\Lambda_h^{k-1} = \left\{ \begin{matrix} \mathcal{P}_r\Lambda^{k-1}(\mathcal{T}_h) \\ \text{or} \\ \mathcal{P}_r^-\Lambda^{k-1}(\mathcal{T}_h) \end{matrix} \right\}, \quad \Lambda_h^k = \left\{ \begin{matrix} \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h) \\ \text{or} \\ \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h) \ (\text{if } r>1) \end{matrix} \right\}. \tag{2.5}$$

Concerning the auxiliary space Λ_h^{k+1} in (2.4), if $\Lambda_h^k = \mathcal{P}_r^-\Lambda^k(\mathcal{T}_h)$, we take $\Lambda_h^{k+1} = \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}_h)$, while if $\Lambda_h^k = \mathcal{P}_{r-1}\Lambda^k(\mathcal{T}_h)$, we take $\Lambda_h^{k+1} = \mathcal{P}_{r-1}^-\Lambda^{k+1}(\mathcal{T}_h)$.

For this choice of spaces, it is known ([2], Sect. 5.4, [3], Sect. 5.5, [10]) that there exist cochain projections as in (2.4) for which $\pi_h^j : L^2\Lambda^j \to \Lambda_h^j$ is bounded in $L^2\Lambda^j$ uniformly with respect to h. In particular, this implies that there is a constant C independent of h such that

$$\|u - \pi_h^j u\| \leq C \inf_{v\in\Lambda_h^j} \|u - v\|, \quad u \in L^2\Lambda^j. \tag{2.6}$$

Moreover, we have the approximation estimates

$$\|u - \pi_h^j u\| \leq Ch^s\|u\|_s, \quad 0 \leq s \leq \begin{cases} r, & \Lambda_h^j = \mathcal{P}_r^-\Lambda^j(\mathcal{T}_h), \\ r+1, & \Lambda_h^j = \mathcal{P}_r\Lambda^j(\mathcal{T}_h). \end{cases} \tag{2.7}$$

Note that we use $\|u\|_s$ as a notation for the Sobolev norm $\|u\|_{H^s\Lambda^j}$.

3. Elliptic projection of the exact solution

As usual, we shall obtain error estimates for the finite element approximation to the evolution equation by comparing it to an appropriate elliptic projection of the exact solution into the finite element space. In this section we define the elliptic projection and establish error estimates for it.

Given any $u \in D(\mathcal{L})$, the elliptic projection of u is defined as $(\hat\sigma_h, \hat u_h, \hat p_h) \in \Lambda_h^{k-1} \times \Lambda_h^k \times \mathfrak{H}_h^k$, such that

$$\langle \hat\sigma_h, \tau\rangle - \langle \mathrm{d}\tau, \hat u_h\rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \tag{3.1}$$

$$\langle \mathrm{d}\hat\sigma_h, v\rangle + \langle \mathrm{d}\hat u_h, \mathrm{d}v\rangle + \langle \hat p_h, v\rangle = \langle \mathcal{L}u, v\rangle, \quad v \in \Lambda_h^k, \tag{3.2}$$

$$\langle \hat u_h, q\rangle = \langle u, q\rangle, \quad q \in \mathfrak{H}_h^k. \tag{3.3}$$

By Theorem 3.8 of [3] there exists a unique solution to (3.1)–(3.3). Now we follow the approach of [3] to derive error estimates. To this end, we introduce some notation. First, let $P_{\mathfrak{H}_h} : L^2\Lambda^k \to \mathfrak{H}_h^k$ denote the L^2-projection. From (3.3), $P_{\mathfrak{H}_h}\hat u_h = P_{\mathfrak{H}_h} u$. Moreover, from ([3], Sect. 3.4),

$$\hat p_h = P_{\mathfrak{H}_h}(\mathcal{L}u) = P_{\mathfrak{H}_h}(\mathrm{d}\sigma),$$

where $\sigma = d^*u$, the last equality holding because $d^*du \in \mathfrak{B}_k^* \perp \mathfrak{Z}^k$, but $\mathfrak{H}_h^k \subset \mathfrak{Z}_h^k \subset \mathfrak{Z}^k$.

The following norms often appear in the analysis, and we follow ([3], Sect. 3.5) and define $\beta = \beta_h^k$, $\mu = \mu_h^k$, and $\eta = \eta_h^k$ by

$$\beta = \|(I-\pi_h)K\|_{\mathcal{L}(L^2\Lambda^k, L^2\Lambda^k)},\ \mu = \|(I-\pi_h)P_{\mathfrak{H}^k}\|_{\mathcal{L}(L^2\Lambda^k, L^2\Lambda^k)},$$
$$\eta = \max_{j=0,1} \max[\|(I-\pi_h)dK\|_{\mathcal{L}(L^2\Lambda^{k-j}, L^2\Lambda^{k-j+1})}, \|(I-\pi_h)d^*K\|_{\mathcal{L}(L^2\Lambda^{k+j}, L^2\Lambda^{k+j-1})}].$$

These take into account the approximation properties of the finite element spaces, and the regularity of the solution operator K. From (2.6) and the compactness of $K : L^2\Lambda^k \to H\Lambda^k \cap \mathring{H}^*\Lambda^k$, we conclude that $\eta, \beta, \mu \to 0$ as $h \to 0$. Assuming H^2 regularity for the Hodge Laplacian (by which we mean both that $\|Kf\|_2 \le C\|f\|_0$ for all $f \in L^2\Lambda^k$ and that $\mathfrak{H}^k \subset H^2\Lambda^k$), then we have

$$\eta = O(h), \quad \beta, \mu = O(h^{\min(2,r+1)}) \tag{3.4}$$

for any of the choices of spaces in (2.5) where r denotes the largest degree of complete polynomials in the space Λ_h^k. Note that $\beta = O(h^2)$ except in the case $\Lambda_h^k = \mathcal{P}_1^-\Lambda^k$.

Finally, we denote the best approximation error in the L^2 norm by

$$E(w) = \inf_{v \in \Lambda_h^k} \|w - v\|, \quad w \in L^2\Lambda^k,\ k = 0, \ldots, n.$$

We are now ready to give the error estimates for the elliptic projection.

Theorem 3.1. *Let $u \in D(\mathcal{L})$ and let $(\hat{\sigma}_h, \hat{u}_h)$ be defined by (3.1)–(3.3). Then we have*

$$\|d(\sigma - \hat{\sigma}_h)\| \le CE(\mathrm{d}\sigma), \tag{3.5}$$
$$\|\sigma - \hat{\sigma}_h\| \le C(E(\sigma) + \eta E(\mathrm{d}\sigma)), \tag{3.6}$$
$$\|\hat{p}_h\| \le C\mu E(\mathrm{d}\sigma), \tag{3.7}$$
$$\|d(u - \hat{u}_h)\| \le C(E(\mathrm{d}u) + \eta E(\mathrm{d}\sigma)), \tag{3.8}$$
$$\|u - \hat{u}_h\| \le C(E(u) + E(P_{\mathfrak{H}}u) + \eta[E(\mathrm{d}u) + E(\sigma)] + (\eta^2 + \beta)E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}}u)). \tag{3.9}$$

Proof. This is essentially proven in [3], except that there it is assumed that $u \perp \mathfrak{H}$ and $\hat{u}_h \perp \mathfrak{H}_h$. To account for this difference, let $\tilde{u} = u - P_{\mathfrak{H}}u$ and $\tilde{u}_h = \hat{u}_h - P_{\mathfrak{H}_h}\hat{u}_h$. Then (3.1) and (3.2) continue to hold with u and u_h replaced by $\tilde{u}$ and $\tilde{u}_h$, respectively, and, in place of (3.3), we have

$$\langle \tilde{u}_h, q \rangle = 0, \quad q \in \mathfrak{H}_h^k.$$

Application of Theorem 3.11 of [3] (with $f = \mathcal{L}u$ and $p = 0$) then gives the (3.5)–(3.8), and, instead of (3.9), we get

$$\begin{aligned}\|\tilde{u} - \tilde{u}_h\| &\le C(E(\tilde{u}) + \eta[E(\mathrm{d}u) + E(\sigma)] + (\eta^2 + \beta)E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}}u)) \\ &\le C(E(u) + E(P_{\mathfrak{H}}u) + \eta[E(\mathrm{d}u) + E(\sigma)] + (\eta^2 + \beta)E(\mathrm{d}\sigma) + \mu E(P_{\mathfrak{B}}u)).\end{aligned}$$

Thus $\|\tilde{u} - \tilde{u}_h\|$ is bounded by the right-hand side of (3.9), and, to complete the proof, it suffices bound $P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}\hat{u}_h$ by same quantity. Now

$$P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}\hat{u}_h = P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}u = (I - P_{\mathfrak{H}_h})P_{\mathfrak{H}}u - P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}}u),$$

For the first term on the right-hand side, we use ([3], Thm. 3.5) and (2.6) to get

$$\|(I - P_{\mathfrak{H}_h})P_{\mathfrak{H}}u\| \le \|(I - \pi_h)P_{\mathfrak{H}}u\| \le CE(P_{\mathfrak{H}}u).$$

To estimate the second term, we use the Hodge decomposition (2.2) to write $u - P_{\mathfrak{H}} u = u_b + u_\perp$ with $u_b \in \mathfrak{B}^k, u_\perp \in \mathfrak{Z}^{k\perp}$. Since $\mathfrak{H}_h^k \subset \mathfrak{Z}^k$, $P_{\mathfrak{H}_h} u_\perp = 0$, and since $\pi_h u_b \in \mathfrak{B}_h^k$, $P_{\mathfrak{H}_h} \pi_h u_b = 0$. Hence $P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}} u) = P_{\mathfrak{H}_h}(I - \pi_h) u_b$. We normalize this quantity by setting

$$q = P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}} u)/\|P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}} u)\| \in \mathfrak{H}_h^k.$$

Then $P_{\mathfrak{H}} q \in \mathfrak{H}$, and, by ([3], Thm. 3.5), $\|q - P_{\mathfrak{H}} q\| \le \|(I - \pi_h) P_{\mathfrak{H}} q\| \le \mu$. Therefore,

$$\|P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}} u)\| = (P_{\mathfrak{H}_h}(u - P_{\mathfrak{H}} u), q) = (P_{\mathfrak{H}_h}(I - \pi_h) u_b, q) = ((I - \pi_h) u_b, q).$$

Now $(I - \pi_h) u_b \in \mathfrak{B}^k$, and so is orthogonal to $\mathfrak{H}$. Thus

$$((I - \pi_h) u_b, q) = ((I - \pi_h) u_b, q - P_{\mathfrak{H}} q) \le \|(I - \pi_h) u_b\| \|q - P_{\mathfrak{H}} q\| \le C\mu E(P_{\mathfrak{B}} u),$$

by (2.6). Combining these results, we get

$$\|P_{\mathfrak{H}} u - P_{\mathfrak{H}_h} \hat{u}_h\| \le C[E(P_{\mathfrak{H}} u) + \mu E(P_{\mathfrak{B}} u)],$$

completing the proof of the theorem. □

Assuming sufficient regularity of u and $\sigma = d^* u$, we can combine the estimates of the theorem with the approximation results of (2.7) to obtain rates of convergence for the elliptic projection. The precise powers of h and Sobolev norms that arise depend on the particular choice of spaces in (2.5). For example, if we take $\Lambda_h^{k-1} = \mathcal{P}_r \Lambda^{k-1}(\mathcal{T}_h)$, then we can show the optimal estimate $\|\sigma - \sigma_h\| \le C h^{r+1} \|\sigma\|_{r+1}$, but, if $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$, then clearly we can only have $\|\sigma - \sigma_h\| = O(h^r)$. Rather than give a complicated statement of the results, covering all the possible cases, in the following theorem and below we restrict to a particular choice of spaces from among the possibilities in (2.5). Moreover, we assume $r > 1$, since the case $r = 1$ is slightly different. However, very similar results can be obtained for any of the choices of spaces permitted in (2.5), including for $r = 1$, in the same way. Finally, we introduce the space

$$\bar{H}^r = \{\, u \in H^r \Lambda^k \mid P_{\mathfrak{H}} u \in H^r \Lambda^k,\ P_{\mathfrak{B}} u \in H^{r-2} \Lambda^k \,\},$$

with the associated norm

$$\|u\|_{\bar{H}^r} = \|u\|_r + \|P_{\mathfrak{H}} u\|_r + \|P_{\mathfrak{B}} u\|_{r-2},$$

since it will arise frequently below.

Theorem 3.2. *Assume H^2 regularity for the Hodge Laplacian, so (3.4) holds and suppose that we use the finite element spaces $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$ and $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ (so that the auxilliary space is $\Lambda_h^{k+1} = \mathcal{P}_r^- \Lambda^{k+1}(\mathcal{T}_h)$), for some $r > 1$. Then we have the following convergence rates for the elliptic projection:*

$$\begin{gathered}
\|d(\sigma - \hat{\sigma}_h)\| \le C h^r \|\mathrm{d}\sigma\|_r, \\
\|\sigma - \hat{\sigma}_h\| \le C h^r \|\sigma\|_r, \\
\|\hat{p}_h\| \le C h^r \|\mathrm{d}\sigma\|_{r-2}, \\
\|d(u - \hat{u}_h)\| \le C h^r (\|\mathrm{d}u\|_r + \|\mathrm{d}\sigma\|_{r-1}), \\
\|u - \hat{u}_h\| \le C h^r \|u\|_{\bar{H}^r}.
\end{gathered}$$

Remark 3.3. If we set r in the statement of Theorem 3.2 to be equal to 1, then, in view of (3.4), we have $\beta, \mu = O(h)$. Employing Theorem 3.1, we find that the first, second, and fourth bound asserted by the theorem still hold, while the third and fifth bound are replaced by

$$\|\hat{p}_h\| \le C h^2 \|\mathrm{d}\sigma\|_1, \quad \|u - \hat{u}_h\| \le C h (\|u\|_1 + \|P_{\mathfrak{H}} u\|_1 + \|\mathrm{d}\sigma\| + \|P_{\mathfrak{B}} u\|).$$

4. Well-posedness of the parabolic problem

We now turn to the Hodge heat equation. In this section we demonstrate well-posedness of the initial-boundary value problem (1.4), (1.5). The key tool is the Hille–Yosida–Phillips theory as presented, for example, in [5] and [7].

We begin by showing that the Hodge Laplacian is maximal monotone (equivalently, in the terminology of [7], that its negative is m-dissipative). This is the key hypotheses needed to apply the Hille–Yosida–Phillips theory to the problem (1.1)–(1.3).

Theorem 4.1. *The Hodge Laplacian $\mathcal{L}$ is maximal monotone. That is, it satisfies*

$$\langle \mathcal{L}v, v\rangle \geq 0, \quad \forall v \in D(\mathcal{L}),$$

and, for any $f \in L^2\Lambda^k$, there exists $u \in D(\mathcal{L})$ such that $u + \mathcal{L}u = f$.

Proof. For any $v \in D(\mathcal{L})$, $\langle \mathcal{L}v, v\rangle = \langle \mathrm{d}v, \mathrm{d}v\rangle + \langle d^*v, d^*v\rangle$, so the monotonicity inequality is obvious. Now, for any $f \in L^2\Lambda^k$, the Riesz representation theorem furnishes a unique $u \in H\Lambda^k \cap \mathring{H}^*\Lambda^k$ such that

$$\langle \mathrm{d}u, \mathrm{d}v\rangle + \langle d^*u, d^*v\rangle + \langle u, v\rangle = \langle f, v\rangle, \quad v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k. \tag{4.1}$$

We shall show that this u belongs to $D(\mathcal{L})$, from which it follows immediately that $u + \mathcal{L}u = f$.

To show that $u \in D(\mathcal{L})$, we must show that $\mathrm{d}u \in \mathring{H}^*\Lambda^{k+1}$ and $d^*u \in H\Lambda^{k-1}$. From (4.1), $f - u$ is orthogonal to $\mathfrak{H}^k$, so, using the Hodge decomposition of $L^2\Lambda^k$, we may write $f - u = df_1 + d^*f_2$ with $f_1 \in H\Lambda^{k-1} \cap \mathfrak{B}^*_{k-1}$ and $f_2 \in \mathring{H}^*\Lambda^{k+1} \cap \mathfrak{B}^{k+1}$. Then

$$\langle f - u, v\rangle = \langle df_1 + d^*f_2, v\rangle = \langle f_1, d^*v\rangle + \langle f_2, \mathrm{d}v\rangle, \quad v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k.$$

Combining with (4.1), we get

$$\langle \mathrm{d}u - f_2, \mathrm{d}v\rangle + \langle d^*u - f_1, d^*v\rangle = 0, \quad v \in H\Lambda^k \cap \mathring{H}^*\Lambda^k. \tag{4.2}$$

Now $\mathrm{d}u, f_2 \in \mathfrak{B}^{k+1}$, so there exists $v \in \mathfrak{Z}^{k\perp} = H\Lambda^k \cap \mathfrak{B}^*_k$ such that $\mathrm{d}v = \mathrm{d}u - f_2$. Choosing this v in (4.2), we find $\mathrm{d}u = f_2 \in \mathring{H}^*\Lambda^{k+1}$, as desired. Similarly $d^*u = f_1 \in H\Lambda^{k-1}$. □

Since $\mathcal{L}$ is maximal monotone and self-adjoint, we obtain the following existence theorem. This is proved in [7] in Theorems 3.1.1 and 3.2.1 for $f = 0$ and $u_0 \in L^2\Lambda^k$, and in Proposition 4.1.6 for general f and u_0 in $D(\mathcal{L})$. Combining the two results by superposition, gives the theorem.

Theorem 4.2. *Suppose that $u_0 \in L^2\Lambda^k$ and $f \in C([0,T]; L^2\Lambda^k)$ are given and that either $f \in L^1((0,T); D(\mathcal{L}))$ or $f \in W^1_1((0,T); L^2\Lambda^k)$. Then there exists a unique $u \in C([0,T]; L^2\Lambda^k) \cap C((0,T]; D(\mathcal{L})) \cap C^1((0,T]; L^2\Lambda^k)$, such that*

$$u_t + \mathcal{L}u = f \text{ on } \Omega \times (0,T], \quad u(0) = u_0.$$

If further, $u_0 \in D(\mathcal{L})$, then $u \in C([0,T]; D(\mathcal{L})) \cap C^1([0,T]; L^2\Lambda^k)$.

We denote by $S(t) : L^2\Lambda^k \to L^2\Lambda^k$ the solution operator for the homogeneous problem ($f \equiv 0$), so $u(t) = S(t)u_0$ solves $u_t + \mathcal{L}u = 0$, $u(0) = u_0$. Then $S(t)$ is a contraction in $L^2\Lambda^k$ for all $t \in [0,T]$, *i.e.*, $\|S(t)\| \leq 1$, and $S(t)$ commutes with $\mathcal{L}$ on $D(\mathcal{L})$ (Thm. 3.1.1 of [7]).

We can measure the regularity of the solution (for general f) by using the iterated domains defined by $D(\mathcal{L}^l) = \{\, u \in D(\mathcal{L}^{l-1}) \,|\, \mathcal{L}^{l-1}u \in D(\mathcal{L}) \,\}$, $l \geq 2$. The next theorem shows that if f is more regular, then the solution is also more regular.

Theorem 4.3. *Suppose that in addition to the hypotheses of Theorem 4.2, we have that f belongs to $C((0,T]; D(\mathcal{L})) \cap L^1((0,T); D(\mathcal{L}^2))$. Then*

$$u \in C^1((0,T]; D(\mathcal{L})). \tag{4.3}$$

Proof. If $f = 0$, then ([5], Thm. 7.7) implies that

$$u \in C^k((0,T]; D(\mathcal{L}^l)),$$

for all $k, l \geq 0$. Therefore, it is sufficient to treat the case $u_0 = 0$, which we do using Duhamel's principle. By Proposition 4.1.6 of [7], the solution is given by

$$u(t) = \int_0^t S(t-s)f(s)\mathrm{d}s$$

in this case, and, assuming that f satisfies the hypotheses of Theorem 4.2,

$$u \in C([0,T]; D(\mathcal{L})) \cap C^1([0,T]; L^2\varLambda^k).$$

Now $f \in L^1((0,T); D(\mathcal{L}^2))$, so

$$\mathcal{L}^2 u(t) = \int_0^t S(t-s)\mathcal{L}^2 f(s)\,\mathrm{d}s,$$

by the commutativity of $S(t-s)$ and $\mathcal{L}$. Since $S(t-s)$ is a contraction in $L^2\varLambda^k$, this implies that $u \in C([0,T]; D(\mathcal{L}^2))$ and so $\mathcal{L}u \in C([0,T]; D(\mathcal{L}))$. Since we also assume that $f \in C((0,T]; D(\mathcal{L}))$, (4.3) follows immediately from the equation $u_t = f - \mathcal{L}u$. □

Next we show that the solution u guaranteed by Theorem 4.2, together with $\sigma = d^*u$, is a solution of the mixed problem (1.4), (1.5). Since $u \in C((0,T]; D(\mathcal{L}))$, $\sigma = d^*u \in C((0,T]; H\varLambda^{k-1})$ and (1.4) holds. Clearly

$$\langle u_t, v\rangle + \langle \mathcal{L}u, v\rangle = \langle f, v\rangle, \quad v \in L^2\varLambda^k, \ t \in (0,T].$$

Since $u \in C((0,T]; D(\mathcal{L}))$, we have

$$\langle \mathcal{L}u, v\rangle = \langle dd^*u, v\rangle + \langle d^*\mathrm{d}u, v\rangle = \langle \mathrm{d}\sigma, v\rangle + \langle \mathrm{d}u, \mathrm{d}v\rangle, \quad v \in H\varLambda^k, \ t \in (0,T].$$

Combining the last two equations gives (1.5).

We are now ready to state the main result for this section.

Theorem 4.4. *Suppose that* $u_0 \in L^2\varLambda^k$ *and* $f \in C([0,T]; L^2\varLambda^k)$ *are given and that either* $f \in L^1((0,T); D(\mathcal{L}))$ *or* $f \in W_1^1((0,T); L^2\varLambda^k)$*. Then there exist unique*

$$\sigma \in C((0,T]; H\varLambda^{k-1}), \quad u \in C([0,T]; L^2\varLambda^k) \cap C((0,T]; D(\mathcal{L})) \cap C^1((0,T]; L^2\varLambda^k),$$

satisfying the mixed problem (1.4), (1.5) *and the initial condition* $u(0) = u_0$*. If, moreover, the hypotheses of Theorem* 4.3 *are satisfied, then* (4.3) *holds.*

Proof. We have already established existence. For uniqueness, we assume $f = 0$ and take $\tau = \sigma$ in (1.4) and $v = u$ in (1.5), to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 = -\|\sigma\|^2 - \|\mathrm{d}u\|^2 \leq 0.$$

Therefore $\|u\|^2$ is decreasing in time, so if $u(0) = 0$, then $u \equiv 0$. Finally, (1.4) then implies that $\sigma \equiv 0$. □

5. The semidiscrete finite element method

The semidiscrete finite element method for the Hodge heat equation is Galerkin's method applied to the mixed variational formulation (1.4), (1.5). That is, we choose finite element spaces Λ_h^{k-1} and Λ_h^k as in (2.5) for some value of $r \geq 1$, and seek $(\sigma_h, u_h) \in C([0,T]; \Lambda_h^{k-1}) \times C^1([0,T]; \Lambda_h^k)$, such that $u_h(0) = u_h^0$, a given initial value in Λ_h^k, and

$$\langle \sigma_h, \tau \rangle - \langle \mathrm{d}\tau, u_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \ t \in (0,T], \tag{5.1}$$

$$\langle u_{h,t}, v \rangle + \langle \mathrm{d}\sigma_h, v \rangle + \langle \mathrm{d}u_h, \mathrm{d}v \rangle = \langle f, v \rangle, \quad v \in \Lambda_h^k, \ t \in (0,T]. \tag{5.2}$$

In this section we shall establish convergence estimates for this scheme.

We may interpret the semidiscrete solution in terms of two operators, $d_h^* : \Lambda_h^k \to \Lambda_h^{k-1}$ and $\mathcal{L}_h : \Lambda_h^k \to \Lambda_h^k$, which are discrete analogues of d^* and $\mathcal{L}$, respectively. For $v \in \Lambda_h^k$, $d_h^* v \in \Lambda_h^{k-1}$ is defined by the equation

$$\langle d_h^* v, \tau \rangle = \langle v, \mathrm{d}\tau \rangle, \quad \tau \in \Lambda_h^{k-1},$$

and the discrete Hodge Laplacian $\mathcal{L}_h : \Lambda_h^k \to \Lambda_h^k$ is given by $\mathcal{L}_h = d_h^* d + d d_h^*$. The following characterization is then a direct consequence of the definitions.

Lemma 5.1. *The pair $(\sigma_h, u_h) \in C([0,T]; \Lambda_h^{k-1}) \times C^1([0,T]; \Lambda_h^k)$ solves (5.1) and (5.2) if and only if $u_h(t) \in C^1([0,T]; \Lambda_h^k)$ solves*

$$u_{h,t} + \mathcal{L}_h u_h = P_h f, \quad 0 \leq t \leq T, \tag{5.3}$$

where P_h is L^2 projection of f onto Λ_h^k, and $\sigma_h = d_h^* u_h$.

From the theory of ordinary differential equations, there exists a unique solution $u_h \in C^1([0,T]; \Lambda_h^k)$ solving the ODE (5.3) and taking a given initial value. Letting $\sigma_h = d_h^* u_h$, we obtain a unique solution to the semidiscrete finite element scheme (5.1), (5.2).

Remark 5.2. The formulation (5.3) is useful for theoretical purposes, but is typically not implemented directly, rather only implicitly *via* the mixed method. This is because the operator d_h^* is not local. Even if the finite element function v is supported in just a few elements, $d_h^* v$ will generally have global support.

Next, we turn to the convergence analysis. In Proposition 5.3 we shall give error estimates for the difference between the semidiscrete finite element solution and the elliptic projection of the exact solution of the evolution equations. Combining these estimates with the estimates from Section 3 for the elliptic projection gives error estimates for the semidiscrete finite element method, which we present in Theorem 5.4.

Assume the conditions of Theorem 4.3 hold, so the exact solution

$$u \in C([0,T]; L^2\Lambda^k) \cap C^1((0,T]; D(\mathcal{L})).$$

For each $t > 0$, we can then define the elliptic projection of $u(t)$ and of $u_t(t)$; see (3.1)–(3.3). Writing $(\hat{\sigma}_h(t), \hat{u}_h(t), \hat{p}_h(t))$ for the former, it is easy to see that its time-derivative, $(\hat{\sigma}_{h,t}, \hat{u}_{h,t}, \hat{p}_{h,t})$, is the elliptic projection of u_t. From Theorems 3.1 and 3.2 we obtain error estimates, such as

$$\|u_t - \hat{u}_{h,t}\| \leq C(E(u_t) + E(P_{\mathfrak{H}} u_t) + \eta[E(\mathrm{d}u_t) + E(\sigma_t)] + (\eta^2 + \beta)E(\mathrm{d}\sigma_t) + \mu E(P_{\mathfrak{B}} u_t)) \leq Ch^r \|u_t\|_{\bar{H}^r}, \tag{5.4}$$

with the last inequality holding for the choice of spaces made in Theorem 3.2 (and similar results holding for the other allowable choices of spaces). Now, from (3.1),

$$\langle \hat{\sigma}_h, \tau \rangle - \langle \mathrm{d}\tau, \hat{u}_h \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \ t \in (0,T], \tag{5.5}$$

and, substituting $\mathcal{L}u = -u_t + f$ into (3.2),

$$\langle \hat{u}_{h,t}, v \rangle + \langle \mathrm{d}\hat{\sigma}_h, v \rangle + \langle \mathrm{d}\hat{u}_h, dv \rangle = \langle \hat{u}_{h,t} - u_t, v \rangle + \langle f, v \rangle - \langle \hat{p}_h, v \rangle. \tag{5.6}$$

Define

$$\varSigma_h = \hat{\sigma}_h - \sigma_h, \quad U_h = \hat{u}_h - u_h,$$

the difference between the elliptic projection and the finite element solution. Subtracting (5.1) and (5.2) from (5.5) and (5.6), respectively, gives

$$\langle \varSigma_h, \tau \rangle - \langle \mathrm{d}\tau, U_h \rangle = 0, \quad \tau \in \varLambda_h^{k-1},\ 0 < t \le T, \tag{5.7}$$

$$\langle U_{h,t}, v \rangle + \langle d\varSigma_h, v \rangle + \langle dU_h, dv \rangle = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, v \rangle,\ v \in \varLambda_h^k,\ 0 < t \le T. \tag{5.8}$$

We shall now use these equations to derive bounds on $\varSigma_h$ and U_h in terms of $\hat{u}_{h,t} - u_t$ and $\hat{p}_h$, for which we derived bounds in Section 3. In the remainder of the paper, we adopt the notation $\|\cdot\|_{L^\infty(L^2)}$ for the norm in $L_\infty(0,T;L^2\varLambda^k(\varOmega))$ and similarly for other norms.

Proposition 5.3. *Assume $u_0 \in D(L)$. Then*

$$\|U_h\|_{L^\infty(L^2)} + \|\varSigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} \le C(\|U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}),$$
$$\|\varSigma_h\|_{L^\infty(L^2)} + \|d\varSigma_h\|_{L^2(L^2)} \le C(\|d_h^* U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^2(L^2)}).$$

Proof. By Theorem 4.2, $u \in C([0,T];D(L)) \cap C^1([0,T];L^2\varLambda^k)$. For each $t \in (0,T]$, take $\tau = \varSigma_h(t) \in \varLambda_h^{k-1}$ in (5.7) and $v = U_h(t) \in \varLambda_h^k$ in (5.8), and add to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U_h\|^2 + \|\varSigma_h\|^2 + \|dU_h\|^2 = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, U_h \rangle, \tag{5.9}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\|U_h\|^2 \le 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\|\|U_h\|.$$

Taking $t^* \in [0,T]$ such that $\|U_h\|_{L^\infty(L^2)} = \|U_h(t^*)\|$, and integrating this inequality from 0 to t^* gives

$$\|U_h(t^*)\|^2 \le \|U_h(0)\|^2 + 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}\|U_h\|_{L^\infty(L^2)},$$

whence

$$\|U_h\|_{L^\infty(L^2)} \le \|U_h(0)\| + 2\|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}, \tag{5.10}$$

which gives the desired bound on U_h. To get the bound on $\varSigma_h$ and $\mathrm{d}U_h$, integrate (5.9) over $t \in [0,T]$. This gives

$$\|\varSigma_h\|^2_{L^2(L^2)} + \|dU_h\|^2_{L^2(L^2)} \le \frac{1}{2}\|U_h(0)\|^2 + \|U_h\|_{L^\infty(L^2)}\|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)},$$

and so, by (5.10),

$$\|\varSigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} \le C(\|U_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^1(L^2)}),$$

which completes the proof of the first inequality.

To prove the second inequality, we differentiate (5.7) in time and take $\tau = \varSigma_h \in \varLambda_h^{k-1}$, and then add to (5.8) with $v = d\varSigma_h \in \varLambda_h^k$ (here we use the subcomplex property $d\varLambda_h^{k-1} \subset \varLambda_h^k$). This gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varSigma_h\|^2 + \|d\varSigma_h\|^2 = \langle \hat{u}_{h,t} - u_t - \hat{p}_h, d\varSigma_h \rangle.$$

By integrating in time, first over $[0,t^*]$ with $t^* \in [0,T]$ chosen so that $\|\varSigma_h\|_{L^\infty(L^2)} = \|\varSigma_h(t^*)\|$, and then over all of $[0,T]$, we deduce that

$$\|\varSigma_h\|_{L^\infty(L^2)} + \|d\varSigma_h\|_{L^2(L^2)} \le C(\|\varSigma_h(0)\| + \|\hat{u}_{h,t} - u_t - \hat{p}_h\|_{L^2(L^2)}).$$

Finally, we note from (5.1) and (3.1) that $\varSigma_h = d_h^* U_h$, and so complete the proof. □

Now suppose, for simplicity, that we choose the initial data u_h^0 to equal the elliptic projection of u_0. Then $U_h(0) = 0$ and the right-hand sides of the inequalities in Proposition 5.3 simplify. Bounding them using Theorem 3.2 and (5.4) we get, for the choice of spaces indicated in the theorem,

$$\|U_h\|_{L^\infty(L^2)} + \|\Sigma_h\|_{L^2(L^2)} + \|dU_h\|_{L^2(L^2)} \le Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|dd^*u\|_{L^1(H^{r-2})}),$$
$$\|\Sigma_h\|_{L^\infty(L^2)} + \|d\Sigma_h\|_{L^2(L^2)} \le Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|dd^*u\|_{L^2(H^{r-2})}).$$

Combining these estimates with the estimates in Theorem 3.2 for the elliptic projection, we obtain the main result of the section.

Theorem 5.4. *Suppose that, in addition to the hypotheses of Theorem 3.2 and 4.3, $u_0 \in D(\mathcal{L})$. Let (σ, u) be the solution of (1.4), (1.5) satisfying (1.3), and (σ_h, u_h) the solution of (5.1), (5.2) with the spaces selected as in Theorem 3.2 and $u_h(0)$ chosen to be equal to the elliptic projection of u_0. Then, we have the following error estimates for the semidiscrete finite element method:*

$$\|\sigma - \sigma_h\|_{L^2(L^2)} \le Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|d^*u\|_{L^2(H^r)}),$$
$$\|\sigma - \sigma_h\|_{L^\infty(L^2)} \le Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}),$$
$$\|d(\sigma - \sigma_h)\|_{L^2(L^2)} \le Ch^r(\|u_t\|_{L^2(\bar{H}^r)} + \|dd^*u\|_{L^2(H^r)}),$$
$$\|u - u_h\|_{L^\infty(L^2)} \le Ch^r(\|u\|_{L^\infty(\bar{H}^r)} + \|u_t\|_{L^1(\bar{H}^r)}),$$
$$\|d(u - u_h)\|_{L^2(L^2)} \le Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|\mathrm{d}u\|_{L^2(H^r)} + \|dd^*u\|_{L^2(H^{r-1})}).$$

6. The fully discrete finite element method

If we combine the semidiscrete finite element method with a standard time-stepping scheme to solve the resulting system of ordinary differential equations, we obtain a fully discrete finite element method for the Hodge heat equations (1.4) and (1.5). For simplicity, we use backward Euler's method with constant time step $\Delta t = T/M$. We may choose any of the pairs of finite element spaces indicated in (2.5) for any value of $r \ge 1$, but, as above, for simplicity we restrict ourselves to the choice $\Lambda_h^{k-1} = \mathcal{P}_r^- \Lambda^{k-1}(\mathcal{T}_h)$ and $\Lambda_h^k = \mathcal{P}_r^- \Lambda^k(\mathcal{T}_h)$ with $r > 1$, the results for the other cases being simple variants. The fully discrete method seeks $\sigma_h^n \in \Lambda_h^{k-1}, u_h^n \in \Lambda_h^k$, satisfying the equations

$$\langle \sigma_h^n, \tau \rangle - \langle \mathrm{d}\tau, u_h^n \rangle = 0, \quad \tau \in \Lambda_h^{k-1}, \tag{6.1}$$
$$\left\langle \frac{u_h^n - u_h^{n-1}}{\Delta t}, v \right\rangle + \langle \mathrm{d}\sigma_h^n, v \rangle + \langle \mathrm{d}u_h^n, \mathrm{d}v \rangle = \langle f(t^n), v \rangle, \quad v \in \Lambda_h^k. \tag{6.2}$$

for $1 \le n \le M$. It is easy to see that this linear system for u_h^n, σ_h^n is invertible at each time step. We initialize by choosing $u_h^0 \in \Lambda_h^k$. We also define $\sigma_h^0 \in \Lambda^{k-1}$ so that (6.1) holds for $n = 0$.

Next, we turn to the convergence analysis. We first obtain error estimates for the difference between the fully discrete finite element solution and the elliptic projection of the exact solution of the evolution equations. These are stated in (6.10) and (6.11). Combining these estimates with the estimates from Section 3 for the elliptic projection, we obtain the error estimates for the fully discrete finite element method presented in Theorem 6.3.

The analysis is similar to that for the semidiscrete finite element method, but with some extra complications arising from the time discretization. Let $(\hat{\sigma}_h^n, \hat{u}_h^n, \hat{p}_h^n)$ be the elliptic projection of $u^n = u(t^n)$.

Now, from (3.1)

$$\langle \hat{\sigma}_h^n, \tau \rangle - \langle \mathrm{d}\tau, \hat{u}_h^n \rangle = 0, \quad \tau \in \Lambda_h^{k-1},\ 0 \le n \le M, \tag{6.3}$$

and, from (3.2) and the equation $u_t + Lu = f$,

$$\left\langle \frac{\hat u_h^n - \hat u_h^{n-1}}{\Delta t}, v\right\rangle + \langle \mathrm{d}\hat\sigma_h^n, v\rangle + \langle \mathrm{d}\hat u_h^n, dv\rangle = \left\langle \frac{\hat u_h^n - \hat u_h^{n-1}}{\Delta t} - u_t^n, v\right\rangle + \langle f^n, v\rangle - \langle \hat p_h^n, v\rangle$$
$$= \left\langle \frac{(\hat u_h^n - u^n) - (\hat u_h^{n-1} - u^{n-1})}{\Delta t}, v\right\rangle + \left\langle \frac{u^n - u^{n-1}}{\Delta t} - u_t^n, v\right\rangle + \langle f^n, v\rangle - \langle \hat p_h^n, v\rangle,$$
$$v \in \Lambda_h^k,\ 1 \le n \le M. \quad (6.4)$$

Set

$$\Sigma_h^n = \hat\sigma_h^n - \sigma_h^n, \quad U_h^n = \hat u_h^n - u_h^n,$$

the difference between the elliptic projection and the finite element solution at each time step. Subtracting (6.1) and (6.2) from (6.3) and (6.4), respectively, gives

$$\langle \Sigma_h^n, \tau\rangle - \langle \mathrm{d}\tau, U_h^n\rangle = 0, \quad \tau \in \Lambda_h^{k-1},\ 0 \le n \le M, \quad (6.5)$$

and

$$\left\langle \frac{U_h^n - U_h^{n-1}}{\Delta t}, v\right\rangle + \langle d\Sigma_h^n, v\rangle + \langle dU_h^n, dv\rangle$$
$$= \left\langle \frac{(\hat u_h^n - u^n) - (\hat u_h^{n-1} - u^{n-1})}{\Delta t}, v\right\rangle + \left\langle \frac{u^n - u^{n-1}}{\Delta t} - u_t^n, v\right\rangle - \langle \hat p_h^n, v\rangle \quad (6.6)$$
$$= \langle z^n, v\rangle, \quad v \in \Lambda_h^k,\ 1 \le n \le M,$$

where $z^n \in L^2\Lambda^k$ is defined by the last equation. We easily see that

$$\|z^n\| \le \frac{1}{\Delta t}\int_{t^{n-1}}^{t^n} \|(\hat u_{h,t} - u_t)(s)\|\mathrm{d}s + \frac{\Delta t}{2}\|u_{tt}\|_{L^\infty(L^2)} + \|\hat p_h^n\|.$$

By Theorem 3.2, the last term on the right hand side is bounded by $Ch^r\|dd^*u\|_{L^\infty(H^{r-2})}$, and, by (5.4), the first term on the right hand side by

$$\frac{Ch^r}{\Delta t}\|u_t\|_{L^1([t^{n-1},t^n],\bar H^r)}.$$

Thus we have proved:

Proposition 6.1.

$$\|z^n\| \le \frac{\Delta t}{2}\|u_{tt}\|_{L^\infty(L^2)} + Ch^r\left(\|dd^*u\|_{L^\infty(H^{r-2})} + \frac{1}{\Delta t}\|u_t\|_{L^1([t^{n-1},t^n],\bar H^r)}\right).$$

We shall now use equations (6.5) and (6.6) to derive bounds on Σ_h and U_h in terms of z. Toward this end we adopt the notation

$$\|f\|_{l^\infty(X)} = \max_{1\le n\le M}\|f^n\|_X,\ \|f\|_{l^2(X)} = \left(\Delta t\sum_{n=1}^M \|f^n\|_X^2\right)^{1/2},\ \|f\|_{l^1(X)} = \Delta t\sum_{n=1}^M\|f^n\|_X.$$

Proposition 6.2. *Assume $u_0 \in D(L)$. Then*

$$\|U_h\|_{l^\infty(L^2)} + \|\Sigma_h\|_{l^2(L^2)} + \|dU_h\|_{l^2(L^2)} \le C(\|U_h(0)\| + \|z\|_{l^1(L^2)}),$$
$$\|\Sigma_h\|_{l^\infty(L^2)} + \|d\Sigma_h\|_{l^2(L^2)} \le C(\|d_h^*U_h(0)\| + \|z\|_{l^2(L^2)}).$$

Proof. Take $\tau = \varSigma_h^n \in \varLambda_h^{k-1}$ in (6.5), $v = U_h^n \in \varLambda_h^k$ in (6.6), and add to obtain

$$\Delta t(\|\varSigma_h^n\|^2 + \|dU_h^n\|^2) + \|U_h^n\|^2 = (U_h^{n-1} + \Delta t z^n, U_h^n), \tag{6.7}$$

which implies

$$\|U_h^n\| \le \|U_h^{n-1}\| + \Delta t\|z^n\|.$$

By iteration,

$$\|U_h\|_{l^\infty(L^2)} \le \|U_h^0\| + \Delta t \sum_{n=1}^{M} \|z^n\|, \tag{6.8}$$

which is the desired bound on U_h. To get the bound on $\varSigma_h$ and dU_h, we derive from (6.7) that

$$\frac{1}{2}\|U_h^n\|^2 - \frac{1}{2}\|U_h^{n-1}\|^2 + \Delta t(\|\varSigma_h^n\|^2 + \|dU_h^n\|^2) \le (\Delta t z^n, U_h^n) \le \Delta t\|z^n\|\|U_h\|_{l^\infty(L^2)}.$$

Summing then gives

$$\frac{1}{2}\|U_h^M\|^2 - \frac{1}{2}\|U_h^0\|^2 + \|\varSigma_h\|^2_{l^2(L^2)} + \|dU_h\|^2_{l^2(L^2)} \le \|U_h\|_{l^\infty(L^2)}\|\|z\|_{l^1(L^2)},$$

and so, by (6.8),

$$\|\varSigma_h\|^2_{l^2(L^2)} + \|dU_h\|^2_{l^2(L^2)} \le \|U_h^0\|^2 + \frac{3}{2}\|z\|^2_{l^1(L^2)},$$

which completes the proof of the first inequality.

To prove the second inequality, we take $\tau = \varSigma_h^n \in \varLambda_h^{k-1}$ in (6.5) at both time level $n-1$ and level n. This gives

$$(\varSigma_h^{n-1}, \varSigma_h^n) = (d\varSigma_h^n, U_h^{n-1}), \quad (\varSigma_h^n, \varSigma_h^n) = (d\varSigma_h^n, U_h^n), \quad 1 \le n \le M. \tag{6.9}$$

Next take $v = d\varSigma_h^n \in \varLambda_h^k$ in (6.6) and substitute (6.9) to get

$$(\varSigma_h^n - \varSigma_h^{n-1}, \varSigma_h^n) + \Delta t\|d\varSigma_h^n\|^2 = \Delta t(z^n, d\varSigma_h^n),\ 1 \le n \le M,$$

whence

$$\|\varSigma_h^n\|^2 - \|\varSigma_h^{n-1}\|^2 + \Delta t\|d\varSigma_h^n\|^2 \le \Delta t\|z^n\|^2.$$

Again we get a telescoping sum, so

$$\|\varSigma_h\|^2_{l^\infty(L^2)} + \|d\varSigma_h\|^2_{l^2(L^2)} \le C\left(\|\varSigma_h^0\|^2 + \|z\|^2_{l^2(L^2)}\right).$$

This implies the second inequality and so completes the proof of the proposition. □

As in Section 5, we choose the initial data u_h^0 to equal the elliptic projection of u_0 for simplicity. Then $U_h(0) = 0$ and the right-hand sides of the inequalities in Proposition 6.2 simplify. Bounding them *via* Proposition 6.1 we get for the first

$$\|U_h\|_{l^\infty(L^2)} + \|\varSigma_h\|_{l^2(L^2)} + \|dU_h\|_{l^2(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|dd^*u\|_{L^\infty(H^{r-2})} + \|u_t\|_{L^1(\bar{H}^r)}). \tag{6.10}$$

For the second, we bound the $L^1([t^{n-1}, t^n])$ norm in Proposition 6.1 by Δt times the L^∞ norm, and substitute the resulting bound for z in the second estimate of Proposition 6.2, obtaining

$$\|\varSigma_h\|_{l^\infty(L^2)} + \|d\varSigma_h\|_{l^2(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|dd^*u\|_{L^\infty(H^{r-2})} + \|u_t\|_{L^\infty(\bar{H}^r)}). \tag{6.11}$$

Combining (6.10), (6.11) with the estimates in Theorem 3.2 for the elliptic projection, we obtain the main result of the section.

Theorem 6.3. *Under the same assumptions as Theorem 5.4, Let (σ, u) be the solution of (1.4), (1.5) satisfying (1.3), and (σ_h^n, u_h^n) the solution of (6.1), (6.2) with u_h^0 equal to the elliptic projection of u_0. Then, we have the following error estimates for the fully discrete finite element method:*

$$\|\sigma - \sigma_h\|_{l^2(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}),$$
$$\|\sigma - \sigma_h\|_{l^\infty(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|u_t\|_{L^\infty(\bar{H}^r)} + \|d^*u\|_{L^\infty(H^r)}),$$
$$\|d(\sigma - \sigma_h)\|_{l^2(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|u_t\|_{L^\infty(\bar{H}^r)} + \|dd^*u\|_{L^\infty(H^r)}),$$
$$\|u - u_h\|_{l^\infty(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|u\|_{L^\infty(\bar{H}^r)} + \|u_t\|_{L^1(\bar{H}^r)})),$$
$$\|d(u - u_h)\|_{l^2(L^2)} \le C\Delta t\|u_{tt}\|_{L^\infty(L^2)} + Ch^r(\|u_t\|_{L^1(\bar{H}^r)} + \|\mathrm{d}u\|_{L^\infty(H^r)} + \|dd^*u\|_{L^\infty(H^{r-1})}).$$

The error estimates are analogous to those of Theorem 5.4 for the semidiscrete solution, with each containing an additional $O(\Delta t)$ term coming from the time discretization. For each quantity, the error is of order $O(\Delta t + h^r)$.

7. Numerical examples

In this section we present results obtained by implementing the numerical methods described above using the FEniCS finite element software library [18].

First we compute a two-dimensional example for the 1-form Hodge heat equation. Using vector proxies, we may write the parabolic equations (1.1)–(1.3) as

$$u_t + (\operatorname{curl}\operatorname{rot} - \nabla \operatorname{div})u = f \text{ in } \Omega \times [0, T],$$
$$u \cdot n = \operatorname{rot} u = 0 \text{ on } \partial\Omega \times [0, T], \quad u(\,\cdot\,, 0) = u_0 \text{ in } \Omega,$$

where

$$\operatorname{rot} u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \operatorname{curl} u = \left(\frac{\partial u}{\partial x_2}, -\frac{\partial u}{\partial x_1}\right).$$

We choose Ω to be a square annulus $[0, 1] \times [0, 1]\backslash[0.25, 0.75] \times [0.25, 0.75]$ and take the exact solution as

$$u = \begin{pmatrix} 100x(x-1)(x-0.25)(x-0.75)t \\ 100y(y-1)(y-0.25)(y-0.75)t \end{pmatrix}.$$

Note that this function is not orthogonal to 1-harmonic forms on Ω. We use the finite element spaces $\mathcal{P}_r\Lambda^0(\mathcal{T}_h)$ (Lagrange elements of degree r) for $\sigma = -\operatorname{div} u$ and $\mathcal{P}_r^-\Lambda^1(\mathcal{T}_h)$ (Raviart–Thomas elements) for u, starting with an initial unstructured mesh, and then refining it uniformly. We take $\Delta t = 0.0001$ and compute the error at time $T = 0.01$ (after 100 time steps). Tables 1 and 2 show the results for $r = 1$ and 2 respectively. The rates of convergence are just as predicted by the theory.

For the second example, we let Ω be the unit cube $[0, 1] \times [0, 1] \times [0, 1]$ in $\mathbb{R}^3$, and again solve the 1-form Hodge heat equation. Using vector proxies, the initial–boundary value problem becomes

$$u_t + (\operatorname{curl}\operatorname{curl} - \nabla \operatorname{div})u = f \text{ in } \Omega \times [0, T]$$
$$u \cdot n = 0,\ \operatorname{curl} u \times n = 0 \text{ on } \partial\Omega \times [0, T], \quad u(\,\cdot\,, 0) = u_0 \text{ in } \Omega.$$

Table 1. Computation with $P_1\Lambda^0 \times P_1^-\Lambda^1$ in two dimensions.

Mesh size	$\|\sigma - \sigma_h\|$	Rate	$\|\nabla(\sigma - \sigma_h)\|$	Rate	$\|u - u_h\|$	Rate
h	0.0008490	1.99	0.1026276	1.01	0.0010586	0.96
$h/2$	0.0002132	1.99	0.0512846	1.00	0.0005341	0.99
$h/4$	0.0000534	2.00	0.0256528	1.00	0.0002678	1.00
$h/8$	0.0000133	2.00	0.0128295	1.00	0.0001340	1.00

TABLE 2. Computation with $P_2\Lambda^0 \times P_2^-\Lambda^1$ in two dimensions.

Mesh size	$\|\sigma - \sigma_h\|$	Rate	$\|\nabla(\sigma - \sigma_h)\|$	Rate	$\|u - u_h\|$	Rate
h	0.0000093	3.03	0.0016510	2.03	0.0000705	1.97
$h/2$	0.0000012	3.00	0.0004119	2.00	0.0000178	1.99
$h/4$	0.0000001	3.00	0.0001031	2.00	0.0000045	1.99
$h/8$	0.0000000	3.04	0.0000258	2.00	0.0000011	2.00

TABLE 3. Computation with $P_1\Lambda^0 \times P_1^-\Lambda^1$ in three dimensions.

Mesh size	$\|\sigma - \sigma_h\|$	Rate	$\|\nabla(\sigma - \sigma_h)\|$	Rate	$\|u - u_h\|$	Rate
h	0.0023326	2.06	0.0260155	1.02	0.0026024	1.00
$h/2$	0.0005735	2.02	0.0134836	0.95	0.0013499	0.95
$h/4$	0.0001429	2.01	0.0068169	0.98	0.0006879	0.97

We take the exact solution to be

$$u = \begin{pmatrix} \sin(\pi x_1)t \\ \sin(\pi x_2)t \\ \sin(\pi x_3)t \end{pmatrix}.$$

Table 3 shows the errors and rates of convergence for linear elements on a sequence of uniform meshes, again at time $T = 0.01$ after 100 time steps. Once again, the rates of convergence are just as predicted by the theory.

REFERENCES

[1] A. Alonso Rodríguez and A. Valli, Eddy current approximation of Maxwell equations. Theory, algorithms and applications. Vol. 4 of *MS&A Modeling, Simulation and Applications*. Springer-Verlag Italia, Milan (2010).

[2] D.N. Arnold, R.S. Falk and R. Winther, Finite element exterior calculus, homological techniques, and applications. *Acta Numer.* **15** (2006) 1–155.

[3] D.N. Arnold, R.S. Falk and R. Winther, Finite element exterior calculus: from Hodge theory to numerical stability. *Bull. Amer. Math. Soc.* (N.S.) **47** (2010) 281–354.

[4] D. Boffi and L. Gastaldi, Analysis of finite element approximation of evolution problems in mixed form. *SIAM J. Numer. Anal.* **42** (2004) 1502–1526.

[5] H. Brezis, Functional analysis, Sobolev spaces and partial differential equations. *Universitext*. Springer, New York (2011).

[6] F. Brezzi, On the existence, uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers. *Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge* **8** (1974), 129–151.

[7] Th. Cazenave and A. Haraux, An introduction to semilinear evolution equations. Vol. 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York (1998).

[8] H. Chen, R. Ewing and R. Lazarov, Superconvergence of mixed finite element methods for parabolic problems with nonsmooth initial data. *Numer. Math.* **78** (1998) 495–521.

[9] S.-H. Chou and Q. Li, Error estimates for mixed finite element methods for nonlinear parabolic problems. *Numer. Methods Partial Differ. Eq.* **8** (1992) 395–404.

[10] S.H. Christiansen and R. Winther, Smoothed projections in finite element exterior calculus. *Math. Comput.* **77** (2008) 813–829.

[11] J. Douglas, Jr. and T. Dupont, Galerkin methods for parabolic equations. *SIAM J. Numer. Anal.* **7** (1970) 575–626.

[12] S.M.F. Garcia, Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: the continuous-time case. *Numer. Methods Partial Differ. Eq.* **10** (1994) 129–147.

[13] S.M.F. Garcia, Improved error estimates for mixed finite-element approximations for nonlinear parabolic equations: the discrete-time case. *Numer. Methods Partial Differ. Eq.* **10** (1994) 149–169.

[14] A. Gillette, M. Holst and Y. Zhu, Finite element exterior calculus for evolution problems. Preprint arXiv:1202.1573v3 (2015).

[15] L. Gor'kov and G. Éliashberg, Generalization of the Ginzburg–Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities. *Soviet Phys. J. Exp. Theor. Phys.* **27** (1968) 328–334.

[16] M. Holst and Ch. Tiee, Finite element exterior calculus for parabolic evolution problems on riemannian hypersurfaces. Preprint arXiv:1509.05524 (2015).

[17] C. Johnson and V. Thomée, Error estimates for some mixed finite element methods for parabolic type problems. *RAIRO Anal. Numér.* **15** (1981) 41–78.

[18] A. Logg, K.-A. Mardal, G.N. Wells, et al., *Automated solution of differential equations by the finite element method.* Springer (2012).

[19] R. Scholz, Optimal L_∞-estimates for a mixed finite element method for second order elliptic and parabolic problems. *Calcolo* **20** (1983) 355–377.

[20] M.C.J. Squeff, Superconvergence of mixed finite element methods for parabolic equations. *RAIRO Modél. Math. Anal. Numér.* **21** (1987) 327–352.

[21] V. Thomée, Galerkin finite element methods for parabolic problems, 2nd edition. Vol. 25 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin (2006).

[22] M.F. Wheeler, A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.* **10** (1973) 723–759.

[23] S.O. Wilson, Differential forms, fluids, and finite models. *Proc. Amer. Math. Soc.* **139** (2011) 2597–2604.