ESAIM: M2AN 50 (2016) 1615–1630
DOI: 10.1051/m2an/2015095

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

NUMERICAL ANALYSIS OF DARCY PROBLEM ON SURFACES

Alberto Ferroni[1], Luca Formaggia[1] and Alessio Fumagalli[1]

Abstract. Surface problems play a key role in several theoretical and applied fields. In this work the main focus is the presentation of a detailed analysis of the approximation of the classical porous media flow problem: the Darcy equation, where the domain is a regular surface. The formulation considers the mixed form and the numerical approximation adopts a classical pair of finite element spaces: piecewise constant for the scalar fields and Raviart–Thomas for vector fields, both written on the tangential space of the surface. The main result is the proof of the order of convergence where the discretization error, due to the finite element approximation, is coupled with a geometrical error. The latter takes into account the approximation of the real surface with a discretized one. Several examples are presented to show the correctness of the analysis, including surfaces with boundary.

Mathematics Subject Classification. 65N30, 65N15, 76S05.

Received March 11, 2015. Revised November 19, 2015. Accepted December 17, 2015.

1. Introduction

In several application, like biology [5] or geophysics, the domains where some or part of problems have to be solved are surfaces or lines. In this particular framework several works in literature are present, mainly focused on the derivation and approximation of diffusive processes. Normally the resulting mathematical equations considered involve the classical Laplace–Beltrami operator [11]. A numerical approximation of this problems is presented in [1, 3, 10, 12] where standard Lagrangian finite element spaces are considered.

With this choice only the primary unknown field is computed directly, while a possible secondary unknown, like the tangential gradients, should be computed as a post process, often resulting in a poor approximation [4]. In some applications, *e.g.* in geophysics, the most important unknowns are often the secondary ones, which represent the fluxes or a macroscopic velocity, which play as a transport fields for advected quantities. Consequently, we are interested in problems where both the primary and secondary unknowns are computed directly. This is possible by employing a mixed formulation of the differential problem.

An important example of this choice, which is part of the motivations of this paper, are presented in [7, 15, 17, 18, 20]. In this series of papers a reduced model is considered to approximate the flow and pressure fields in fractures. The fractures are represented as object of co-dimension one and the reduced models

Keywords and phrases. PDEs on surfaces, Darcy problem, mixed finite elements.

[1] Politecnico di Milano, Department of Mathematics. Piazza Leonardo da Vinci 32, 20133 Milan, Italy.
alberto.ferroni@polimi.it; luca.formaggia@polimi.it; alessio.fumagalli@polimi.it

considered are Darcy-type equations written in the tangential spaces of each fracture. Assuming that the porous matrix is impervious, like in [15], the resulting problem is only written for the fractures.

In this work we propose and analyse a mathematical approach to the formulation of Darcy problems on surfaces embedded in $\mathbb{R}^3$. The main part of the paper is devoted to the derivation of a proper framework for the numerical approximation of such problems. The finite element spaces considered are the classical piecewise constant for scalar fields and Raviart–Thomas for vector fields, but projected on the approximated surface. Particular attention is devoted to the well posedness of the resulting discrete problem and to prove the order of convergence including also the geometrical error in the estimation. We allow the surface to be closed so no boundary conditions can be imposed and then suitable additional conditions should be taken into consideration, such as zero-mean pressure.

We mention that mixed surface FEM have been recently analyzed in Holst and Stern [19] in a rather abstract setting and using exterior calculus techniques. In this paper we focus on a more standard analysis of mixed finite elements for Darcy flow, with several numerical results to support the theoretical findings.

The paper is organized as follow. Section 2 introduces the notation used in the paper as well as the physical problem with some assumptions on the data. The weak formulation of the physical problem and the correct functional setting is described in Section 3, where also the inf-sup condition is proved. Section 4 introduces, describes and analyses the numerical approximation where the discrete inf-sup condition is presented. An error estimation, from the chosen discretization, is derived in Section 5. In Section 6 a collection of examples highlights the potentiality of the proposed methods and gives a numerical validation of the derived theoretical results. Finally, Section 7 is devoted to the conclusions.

2. The governing equations

We assume that the physical domain $\mathit{\Gamma}$ is a C^3 compact, connected orientable manifold embedded in $\mathbb{R}^3$ described by a signed distance function $d : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\mathit{\Gamma} = \{\boldsymbol{x} \in U \,:\, d(\boldsymbol{x}) = 0\},$$

where U is an open subset of $\mathbb{R}^3$ containing $\mathit{\Gamma}$. If $\mathit{\Gamma}$ is an open manifold the boundary is indicated by $\partial\mathit{\Gamma}$. The outward-pointing normal is defined as $\boldsymbol{n}(\boldsymbol{x}) := \nabla d(\boldsymbol{x}) / |\nabla d(\boldsymbol{x})|$,where $\nabla d(\boldsymbol{x}) \neq 0$ almost everywhere on $\mathit{\Gamma}$. Another quantity that will be useful afterwards is the Hessian matrix H of the distance function d, where $H_{ij} := \frac{\partial^2 d}{\partial x_i \partial x_j}$.

In what follows, given a function $u : \mathit{\Gamma} \to \mathbb{R}$, we will indicate its lifting, on a given open set U containing $\mathit{\Gamma}$, as $\tilde{u}$ such that $\tilde{u}|_{\mathit{\Gamma}} = u$. The tangential gradient of u will be then defined as

$$\nabla_{\mathit{\Gamma}} u := \nabla \tilde{u} - (\nabla \tilde{u} \cdot \boldsymbol{n})\, \boldsymbol{n}. \tag{2.1}$$

Note that $\nabla_{\mathit{\Gamma}} u$ is independent from choice of the extension $\tilde{u}$ as shown in [9]. Introducing $P = I - \boldsymbol{n} \otimes \boldsymbol{n}$, where $\otimes$ is the tensor product $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j$, we can rewrite (2.1) as $\nabla_{\mathit{\Gamma}} u = P \nabla \tilde{u}$. The definition of the tangential divergence is now straightforward, in fact a smooth given vector field $\boldsymbol{u} : \mathit{\Gamma} \to \mathbb{R}^3$ we have $\nabla_{\mathit{\Gamma}} \cdot \boldsymbol{u} := P : \nabla \tilde{\boldsymbol{u}}$.

The problem we are interested to solve is the classical Darcy problem [2] defined on the regular surface $\mathit{\Gamma}$. The two unknowns are the tangential Darcy velocity $\boldsymbol{u}$ and the pressure p. The problem is defined in the following way

$$\begin{cases} \eta \boldsymbol{u} + \nabla_{\mathit{\Gamma}} p = \boldsymbol{g} & \text{in } \mathit{\Gamma} \\ \nabla_{\mathit{\Gamma}} \cdot \boldsymbol{u} = f & \text{in } \mathit{\Gamma} \\ p = \hat{p} & \text{on } \gamma^N \\ \boldsymbol{u} \cdot \boldsymbol{\mu} = b & \text{on } \gamma^D \end{cases}, \tag{2.2}$$

where $\boldsymbol{\mu}$ is the outward unit normal of $\partial\Gamma = \gamma^N \cup \gamma^D$, with $\gamma^N \cap \gamma^D = \emptyset$. The main datum in (2.2) is the inverse of the permeability, defined as

$$\eta \in L^\infty(\Gamma) \quad \text{and} \quad \exists \eta_{\min} \in \mathbb{R}^+ : \eta(\boldsymbol{x}) > \eta_{\min} \geq 0 \quad \forall \boldsymbol{x} \in \Gamma. \tag{2.3}$$

We set $\eta_{\max} = \operatorname{supess}_{\boldsymbol{x}\in\Gamma} \eta(\boldsymbol{x})$. Moreover the scalar source term is defined as $f \in L^2(\Gamma)$ and the boundary conditions are imposed on the Darcy velocity $\boldsymbol{u}$ and on the pressure p though sufficiently smooth functions b and $\hat{p}$, respectively. Finally, the vector field $\boldsymbol{g}$ may represent a gravity term and the scalar field f may be viewed as a source or a sink. We also allow $\partial\Gamma = \emptyset$ in which case to have a well posed problem we require the following condition

$$\int_\Gamma f - \nabla_\Gamma \cdot \eta^{-1}\boldsymbol{g} \mathrm{d}x = \int_\Gamma p \mathrm{d}x = 0.$$

In the case of permeable rock matrix the velocity field is still tangent to the fracture but the source term f includes also the flux from the surrounding medium into the fracture, see for example [20]. In the forthcoming analysis, to ease the presentation, we will assume that some of the aforementioned data are zero.

3. Weak formulation and functional setting

For simplicity we consider Neumann homogeneous boundary conditions, the result may be extended by using a standard lifting technique to impose the boundary data. We introduce the weak formulation of problem (2.2), defining a suitable functional setting. First we introduce the following functional space defined on the manifold Γ, with its associated norm

$$\boldsymbol{H}_{\mathrm{div}}(\Gamma) := \left\{ \boldsymbol{v} \in \left[L^2(\Gamma)\right]^3, \nabla_\Gamma \cdot \boldsymbol{v} \in L^2(\Gamma) \right\}$$

and

$$\|\boldsymbol{v}\|^2_{\mathrm{div},\Gamma} := \|\boldsymbol{v}\|^2_{0,\Gamma} + \|\nabla_\Gamma \cdot \boldsymbol{v}\|^2_{0,\Gamma},$$

where $\|\cdot\|_{0,A}$ is the L^2 norm on the domain A. In the following it will be useful to introduce the standard scalar product in $L^2(A)$ as $(\cdot,\cdot)_A$. We indicate the functional space and the norm for the velocity with $\boldsymbol{W}$, namely

$$\boldsymbol{W} := \left\{ \boldsymbol{v} \in \boldsymbol{H}_{\mathrm{div}}(\Gamma), \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma, \boldsymbol{v}\cdot\boldsymbol{\mu} = 0 \text{ on } \gamma^D \right\} \quad \text{with} \quad \|\boldsymbol{v}\|_{\boldsymbol{W}} := \|\boldsymbol{v}\|_{\mathrm{div},\Gamma}.$$

For the pressure field we consider the standard L^2 space with its classical norm. We have

$$Q := L^2(\Gamma) \quad \text{with} \quad \|q\|_Q = \|q\|_{0,\Gamma}.$$

If $\left|\gamma^N\right| = 0$ the pressure is uniquely defined in the quotient space $L^2(\Gamma)/\mathbb{R}$. Thus, we set

$$Q := L^2_0(\Gamma) = \left\{ v \in L^2(\Gamma) : (v,1)_\Gamma = 0 \right\} \quad \text{with} \quad \|q\|_Q = \|q\|_{0,\Gamma},$$

to select the representative solution in $L^2(\Gamma)/\mathbb{R}$ with zero mean.

The derivation of the weak formulation of the problem (2.2) is quite standard except the integration by part of the tangential gradient of the pressure. Taking a test function $\boldsymbol{v} \in \boldsymbol{W}$ and considering the boundary conditions, following [12] we have that

$$\int_\Gamma \nabla_\Gamma p \cdot \boldsymbol{v} \mathrm{d}x = -\int_\Gamma p \nabla_\Gamma \cdot \boldsymbol{v} \mathrm{d}x - \int_\Gamma \mathcal{K} p\, \boldsymbol{v}\cdot\boldsymbol{n} \mathrm{d}x + \int_{\partial\Gamma} \hat{p}\boldsymbol{v}\cdot\boldsymbol{\mu} \mathrm{d}\sigma,$$

where $\mathcal{K} = \nabla_\Gamma \cdot \boldsymbol{n}$ is the surface curvature. The term that involves the curvature is zero since we have required that $\boldsymbol{v} \cdot \boldsymbol{n} = 0$. We introduce the bilinear forms $a(\cdot,\cdot) : \boldsymbol{W} \times \boldsymbol{W} \to \mathbb{R}$ and $b(\cdot,\cdot) : \boldsymbol{W} \times Q \to \mathbb{R}$, defined as

$$a(\boldsymbol{u},\boldsymbol{v}) := (\eta \boldsymbol{u}, \boldsymbol{v})_\Gamma, \quad \text{and} \quad b(\boldsymbol{v},q) := -(p, \nabla_\Gamma \cdot \boldsymbol{v})_\Gamma$$

The functionals are $F(\cdot) : Q \to \mathbb{R}$ and $G(\cdot) : \boldsymbol{W} \to \mathbb{R}$, defined as

$$F(q) := -(f,q)_\Gamma. \quad \text{and} \quad G(\boldsymbol{v}) := -(\hat{p}, \boldsymbol{v} \cdot \boldsymbol{\mu})_{\partial\Gamma} + (\boldsymbol{g}, \boldsymbol{v})_\Gamma.$$

The weak formulation of problem (2.2) is

Problem 1 (Weak formulation)**.** Given η as in (2.3), find $(\boldsymbol{u}, p) \in \boldsymbol{W} \times Q$ such that

$$\begin{cases} a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) = G(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{W} \\ b(\boldsymbol{u},q) = F(q) & \forall q \in Q \end{cases}. \tag{3.1}$$

Theorem 3.1 (Well posedness)**.** *Under the given hypotheses on the data, Problem 1 is well posed.*

Proof. To ease the presentation we consider the case where $\hat{p}$ and $\boldsymbol{g}$ are zero and $\gamma^D = \emptyset$. However a similar result can be obtained in more general cases. Since (3.1) is a saddle-point problem we have to prove the inf-sup condition [4,14]. We consider the functional space $\boldsymbol{W}_0 = \{\boldsymbol{v} \in \boldsymbol{W},\, b(\boldsymbol{v},q) = 0\, \forall q \in Q\}$ and we introduce $\boldsymbol{v} \in \boldsymbol{W}_0$. Then we have $\nabla_\Gamma \cdot \boldsymbol{v} = 0$ almost everywhere in $\boldsymbol{W}_0$ and for each function in $\boldsymbol{W}_0$ the relation $\|\boldsymbol{v}\|_{\boldsymbol{W}} = \|\boldsymbol{v}\|_{L^2(\Gamma)}$ holds true. Using this result we can prove the coercivity of $a(\cdot,\cdot)$ on W_0

$$a(\boldsymbol{u},\boldsymbol{u}) = (\eta \boldsymbol{u}, \boldsymbol{u})_\Gamma \geq \eta_{\min} \|\boldsymbol{u}\|^2_{L^2(\Gamma)} = \eta_{\min} \|\boldsymbol{u}\|^2_{\boldsymbol{W}} \quad \forall \boldsymbol{u} \in \boldsymbol{W}_0.$$

Then, thanks to the hypothesis on η and the Schwarz inequality we have the continuity of the bilinear form $a(\cdot,\cdot)$ on W

$$|a(\boldsymbol{u},\boldsymbol{v})| \leq \eta_{\max} \|\boldsymbol{u}\|_{\boldsymbol{W}} \|\boldsymbol{v}\|_{\boldsymbol{W}} \qquad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{W}.$$

Similarly for the bilinear form $b(\cdot,\cdot)$ we obtain its continuity using the Schwarz inequality

$$|b(\boldsymbol{v},q)| \leq \|\nabla_\Gamma \cdot \boldsymbol{v}\|_{L^2(\Gamma)} \|q\|_{L^2(\Gamma)} \leq \|\boldsymbol{v}\|_{\boldsymbol{W}} \|q\|_Q \qquad \forall (\boldsymbol{v},q) \in \boldsymbol{W} \times Q.$$

Finally we need to prove the inf-sup condition, *i.e.* that there exists a positive constant $\beta \in \mathbb{R}^+$ such that

$$\forall q \in Q, \exists \boldsymbol{v} \in \boldsymbol{W} \quad \text{such that} \quad b(\boldsymbol{v},q) \geq \beta \|\boldsymbol{v}\|_{\boldsymbol{W}} \|q\|_Q.$$

Given a function $q \in Q$ we consider the following auxiliary problem

$$\begin{cases} -\nabla_\Gamma \cdot (\nabla_\Gamma \varphi) = q & \text{in } \Gamma \\ \varphi = 0 & \text{on } \partial\Gamma \end{cases}. \tag{3.2}$$

Problem (3.2) admits a unique solution $\varphi \in H^2(\Gamma)$ such that $\|\varphi\|_{H^2(\Gamma)} \leq C\|q\|_{L^2(\Gamma)}$, see [11]. Choosing $\boldsymbol{v} = \nabla_\Gamma \varphi$, from problem (3.2) we have $-\nabla_\Gamma \cdot \boldsymbol{v} = q$. Considering the aforementioned results, the following inequality holds true

$$\begin{aligned} \|\boldsymbol{v}\|^2_{\boldsymbol{W}} &= \|\boldsymbol{v}\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma \cdot \boldsymbol{v}\|^2_{L^2(\Gamma)} = \|\nabla_\Gamma \varphi\|^2_{L^2(\Gamma)} + \|q\|^2_{L^2(\Gamma)} \\ &\leq \|\varphi\|^2_{H^2(\Gamma)} + \|q\|^2_{L^2(\Gamma)} \leq (C+1)\|q\|^2_{L^2(\Gamma)}. \end{aligned}$$

Imposing $C^* = (C+1)^{\frac{1}{2}}$ we finally obtain the inf-sup condition

$$b(\boldsymbol{v},q) = -(q, \nabla_\Gamma \cdot \boldsymbol{v})_\Gamma = \|q\|^2_{L^2(\Gamma)} \geq \frac{1}{C^*} \|\boldsymbol{v}\|_{\boldsymbol{W}} \|q\|_Q,$$

with $\beta = 1/C^*$. Thanks to this results we can conclude that (1) is well posed [4]. □

4. Numerical discretization

To provide a discrete formulation for Problem 1 we have to introduce a suitable approximation of the surface Γ. Following the approach presented in [11], we consider a polyhedral surface Γ_h consisting in the union of non-overlapping triangles K of diameter h_K,with vertices lying on Γ. We denote with h the maximum among the diameters, $h = \max_{K\in\Gamma_h} h_K$. We also require the resulting grid to be conforming. With the aforementioned hypotheses the resulting discrete surface Γ_h is of class $C^{0,1}$. In the case that the surface is not closed, we indicate by $\partial\Gamma_h$ the boundary of Γ_h composed by the boundary edges of Γ_h.

Unlike the classical finite elements method, the discrete domain Γ_h will not in general be included in Γ, thus adding to the approximation error a component that accounts for the error introduced by the discretization of the geometry. To ensure a sufficiently good approximation of the surface Γ we assume $\Gamma_h \subset U$, where U is a strip of width $\delta > 0$ in which the decomposition

$$\boldsymbol{x} = \boldsymbol{a}(\boldsymbol{x}) + d(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x}) \quad \boldsymbol{x} \in U, \tag{4.1}$$

is unique, being $\boldsymbol{a} : U \to \Gamma$ a projection function, d the distance of $\boldsymbol{x}$ from Γ and $\boldsymbol{n}$ its unit normal. We suppose also that the map $\boldsymbol{a}$ is bijective up to the boundary leading to $\partial\Gamma = \boldsymbol{a}(\partial\Gamma_h)$, see [10]. Thanks to the regularity of the surface there exists a δ such that (4.1) holds.

We set the finite element spaces accordingly with the previous section. We start by defining the space $\boldsymbol{H}_{\mathrm{div}}(\Gamma_h)$ as

$$\boldsymbol{H}_{\mathrm{div}}(\Gamma_h) := \{\boldsymbol{v}_h \in [L^2(\Gamma_h)]^3,\, \nabla_{\Gamma_h}\cdot \boldsymbol{v}_h \in L^2(\Gamma_h)\}.$$

Then, the finite spaces for velocity and pressure are

$$\boldsymbol{W}_h := \{\boldsymbol{v}_h \in \boldsymbol{H}_{\mathrm{div}}(\Gamma_h),\, \boldsymbol{v}_h\cdot\boldsymbol{n}_h = 0 \text{ on } \Gamma_h,\, \boldsymbol{v}_h\cdot\boldsymbol{\mu}_h = 0 \text{ on } \gamma_h^N,\, \boldsymbol{v}_h|_K \in \mathbb{RT}^0(K)\}$$
$$Q_h := \{q_h \in L^2(K) : q_h|_K \in \mathbb{P}^0(K)\},$$

where $\mathbb{RT}^0$ is the Raviart–Thomas finite elements space of lowest order degree. If $\left|\gamma^N\right| = 0$ to recover the uniqueness of the discrete solution, we consider the following discrete space for the pressure field

$$Q_h := \left\{q_h \in L^2(K) : q_h|_K \in \mathbb{P}^0(K)\right\} \cap L^2_0(\Gamma_h).$$

We introduce $a_h(\cdot,\cdot) : \boldsymbol{W}_h \times \boldsymbol{W}_h \to \mathbb{R}$ and $b_h(\cdot,\cdot) : \boldsymbol{W}_h \times Q_h \to \mathbb{R}$, defined as

$$a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) := (\eta_h\boldsymbol{u}_h,\boldsymbol{v}_h)_{\Gamma_h} \quad \text{and} \quad b_h(\boldsymbol{v}_h,q_h) := -(q_h,\nabla_{\Gamma_h}\cdot\boldsymbol{v}_h)_{\Gamma_h},$$

and the linear functionals $F_h(\cdot) : Q_h \to \mathbb{R}$ and $G_h(\cdot) : \boldsymbol{W}_h \to \mathbb{R}$, given by

$$F_h(q_h) := -(f_h,q_h)_{\Gamma_h} \quad \text{and} \quad G_h(\boldsymbol{v}_h) := -\left(\hat{p}_h,\boldsymbol{v}_h\cdot\boldsymbol{\mu}_h\right)_{\partial\Gamma_h} + \left(\boldsymbol{g}_h,\boldsymbol{v}_h\right)_{\Gamma_h}.$$

Where η_h, f_h, $\hat{p}_h$ and $\boldsymbol{g}_h$ are a suitable approximation of the data problem on Γ_h and $\partial\Gamma_h$. We will see in the next section how to choose this approximation. Given the previous definitions, the discrete problem is

Problem 2 (Discrete weak formulation)**.** Find $(\boldsymbol{u}_h,p_h) \in \boldsymbol{W}_h \times Q_h$ such that

$$\begin{cases} a_h(\boldsymbol{u}_h,\boldsymbol{v}_h) + b_h(\boldsymbol{v}_h,p_h) = G_h(\boldsymbol{v}_h) & \forall \boldsymbol{v}_h \in \boldsymbol{W}_h \\ b_h(\boldsymbol{u}_h,q_h) = F_h(q_h) & \forall q_h \in Q_h \end{cases}. \tag{4.2}$$

It can be proved that for Problem 2 all results presented in the previous section for the continuous problem are still valid, using in the proof that $|\varphi_h|_{H^1(\Gamma_h)} \le c\|q_h\|_{L^2(\Gamma_h)}$ (see [11]).

To compare the exact solution defined on Γ with the discrete one defined on Γ_h, we consider the projection of the latter on Γ. As concerns scalar functions we adopt the choice presented in [11], *i.e.* to lift the functions $q_h \in Q_h$ as $\widetilde{q}_h(\boldsymbol{a}(\boldsymbol{x})) = q_h(\boldsymbol{x})$. This kind of lifting, however, does not work properly for the velocity field, in fact it does not map a function in $\boldsymbol{H}_{\mathrm{div}}(\Gamma_h)$ in a function of $\boldsymbol{H}_{\mathrm{div}}(\Gamma)$. In order to preserve this feature, we have used the so called Piola transformation, refer to [22] for a more detailed presentation.

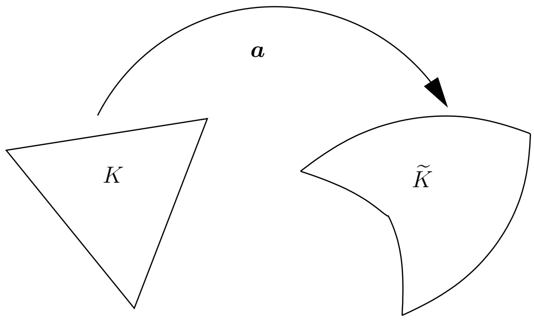


Figure 1. Graphical representation of the map $\boldsymbol{a}$ which transform a triangle K defined in Γ_h to a curved triangle $\widetilde{K}$ defined in Γ.

Definition 4.1 (Piola transformation)**.** Consider $\Omega_0 \subset \mathbb{R}^n$ and let F be a non-degenerate map from Ω_0 to $\Omega \subset \mathbb{R}^n$. Let also be $J = DF(\boldsymbol{X})$, with $J_{ij} = \partial F_i/\partial X_j$, and $\boldsymbol{\Psi} \in [L^2(\Omega_0)]^n$. The Piola transformation $\mathcal{F}$ is then defined as

$$\mathcal{F}(\boldsymbol{\Psi}) := \frac{1}{|\mathrm{det} J|} J\boldsymbol{\Psi} \circ F^{-1}.$$

We now consider a triangle $K \in \Gamma_h$ and its projection on the surface Γ given by the curved triangle $\widetilde{K} = \{\boldsymbol{a}(\boldsymbol{x}) \in \Gamma : \boldsymbol{x} \in K\}$. We use a coordinate system local to the triangle K, so that a generic point $\hat{\boldsymbol{x}} \in K$ has coordinates $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, 0)$.

We now extend $\boldsymbol{a}$ to $\mathbb{R}^3$ introducing a new map $\boldsymbol{\Psi} : \mathbb{R}^3 \to \mathbb{R}^3$, defined as

$$\boldsymbol{\Psi}(\hat{\boldsymbol{x}}) := \boldsymbol{a}(\hat{\boldsymbol{x}}) + \hat{x}_3 \boldsymbol{n}(\hat{\boldsymbol{x}}). \tag{4.3}$$

The map $\boldsymbol{\Psi}$ is the one we consider for the construction of the Piola transformation.

The lifting of a scalar function $q_h : K \to \mathbb{R}$ to $\widetilde{q}_h : \widetilde{K} \to \mathbb{R}$ is therefore given by

$$\widetilde{q}_h(\boldsymbol{\Psi}(\hat{\boldsymbol{x}})) = q_h(\hat{\boldsymbol{x}}) \qquad \hat{\boldsymbol{x}} \in K, \tag{4.4}$$

while, given $F := \nabla\boldsymbol{\Psi}$, the lifting of a vectorial function $\boldsymbol{w}_h : K \to \mathbb{R}^3$ to $\widetilde{\boldsymbol{w}}_h : \widetilde{K} \to \mathbb{R}^3$ is defined as

$$\widetilde{\boldsymbol{w}}_h(\boldsymbol{\Psi}(\hat{\boldsymbol{x}})) = \frac{1}{|\mathrm{det} F|} F\boldsymbol{w}_h(\hat{\boldsymbol{x}}) \qquad \hat{\boldsymbol{x}} \in K. \tag{4.5}$$

The tensor F has the following structure

$$F = [\boldsymbol{t}_1 \quad \boldsymbol{t}_2 \quad \boldsymbol{n}], \tag{4.6}$$

where, following [12], $\boldsymbol{t}_i = \partial\boldsymbol{a}/\partial x_i$, $i = 1, 2$ has components $\boldsymbol{t}_{i,j} = \delta_{ji} - n_j n_i - dH_{ji}$ for $j = 1, 2, 3$ with H the Hessian of the distance function d.

Remark 4.2. It is immediate to show that $\boldsymbol{t}_i \cdot \boldsymbol{n} = 0$ for $i = 1, 2$.

Since $\mathrm{d}\sigma = |\boldsymbol{t}_1 \wedge \boldsymbol{t}_2| \mathrm{d}\sigma_h$ and $\mathrm{det} F = (\boldsymbol{t}_1 \wedge \boldsymbol{t}_2) \cdot \boldsymbol{n}$ we have $\mathrm{d}\sigma = \xi_h \mathrm{d}\sigma_h$ where $\xi_h = |\mathrm{det} F|$.

Remark 4.3. The matrix F is defined element wise. If we glue together all the local F, we find a global matrix that, to ease the notation, we will still indicate as F. In the following it will be clear from the domain of integration if we are referring to the local map or to the global one.

We recall a useful lemma about the properties of the considered geometry. For a complete proof refer to [12].

Lemma 4.4. *Assume Γ and Γ_h defined as above. Then,*

$$\|d\|_{L^\infty(\Gamma_h)} \leq ch^2.$$

Moreover, the quotient $\xi_h = d\sigma/d\sigma_h$ previously defined satisfies

$$\|1 - \xi_h\|_{L^\infty(\Gamma_h)} \leq ch^2.$$

We now deduce an important relationship between functions defined on K and their lifting on $\widetilde{K}$. We consider a couple of functions $(\boldsymbol{w}_h, q_h) \in \boldsymbol{W}_h \times Q_h$ and the corresponding lifting $(\widetilde{\boldsymbol{w}}_h, \widetilde{q}_h) \in \widetilde{\boldsymbol{W}}_h \times \widetilde{Q}_h$, where $\widetilde{\boldsymbol{W}}_h$ and $\widetilde{Q}_h$ are defined as

$$\widetilde{\boldsymbol{W}}_h := \left\{ \widetilde{\boldsymbol{w}}_h(\boldsymbol{x}) = \frac{1}{\xi_h} F \boldsymbol{w}_h \circ \boldsymbol{\Psi}^{-1}(\boldsymbol{x}),\, \boldsymbol{w}_h \in \boldsymbol{W}_h,\, \boldsymbol{x} \in \Gamma \right\},$$
$$\widetilde{Q}_h := \left\{ \widetilde{q}_h(\boldsymbol{x}) = q_h \circ \boldsymbol{\Psi}^{-1}(\boldsymbol{x}),\, q_h \in Q_h,\, \boldsymbol{x} \in \Gamma \right\}.$$

From definitions (4.4) and (4.5) we have that $\widetilde{\boldsymbol{W}}_h \subset \boldsymbol{W}$ and $\widetilde{Q}_h \subset Q$. For such functions the following relation holds

$$\begin{aligned}(\nabla_\Gamma \cdot \widetilde{\boldsymbol{w}}_h, \widetilde{q}_h)_{\widetilde{K}} &= -(\widetilde{\boldsymbol{w}}_h, \nabla_\Gamma \widetilde{q}_h)_{\widetilde{K}} + (\widetilde{\boldsymbol{w}}_h \cdot \widetilde{\boldsymbol{\mu}}_K, \widetilde{q}_h)_{\partial \widetilde{K}} \\ &= -(\widetilde{\boldsymbol{w}}_h, (I - \boldsymbol{n} \otimes \boldsymbol{n}) \nabla \widetilde{q}_h)_{\widetilde{K}} + (\widetilde{\boldsymbol{w}}_h \cdot \widetilde{\boldsymbol{\mu}}_K, \widetilde{q}_h)_{\partial \widetilde{K}}.\end{aligned}$$

For the boundary term we use the relation $\widetilde{\boldsymbol{\mu}}_K \widetilde{\mathrm{d}e} = \xi_h F^{-T} \boldsymbol{\mu}_K \,\mathrm{d}e$ (see [6], Thm. 1.7-1), where $\widetilde{\mathrm{d}e}$ ($\mathrm{d}e$ resp.) is the infinitesimal boundary element of $\widetilde{K}$ (K resp.) and $\widetilde{\boldsymbol{\mu}}_K$ ($\boldsymbol{\mu}_K$ resp.) is the outward unit normal to $\partial \widetilde{K}$ (∂K resp.). Therefore we obtain

$$\begin{aligned}(\nabla_\Gamma \cdot \widetilde{\boldsymbol{w}}_h, \widetilde{q}_h)_{\widetilde{K}} &= -\left(F \boldsymbol{w}_h, (I - \boldsymbol{n} \otimes \boldsymbol{n}) F^{-\top} \nabla q_h\right)_K + \left(\frac{1}{\xi_h} F \boldsymbol{w}_h \cdot \xi_h F^{-\top} \boldsymbol{\mu}_K, q_h\right)_{\partial K} \\ &= -\left(\boldsymbol{w}_h, F^\top (I - \boldsymbol{n} \otimes \boldsymbol{n}) F^{-\top} \nabla q_h\right)_K + (\boldsymbol{w}_h \cdot \boldsymbol{\mu}_K, q_h)_{\partial K}.\end{aligned}$$

In addition we have that

$$F^\top \boldsymbol{n} \otimes \boldsymbol{n} F^{-\top} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{e}_3 \otimes \boldsymbol{e}_3.$$

In the coordinate system local to K, $\boldsymbol{e}_3$ coincides with the normal $\boldsymbol{n}_h$ and so we have

$$F^\top \boldsymbol{n} \otimes \boldsymbol{n} \, F^{-\top} = \boldsymbol{n}_h \otimes \boldsymbol{n}_h.$$

Using this last relation we write that

$$\begin{aligned}(\nabla_\Gamma \cdot \widetilde{\boldsymbol{w}}_h, \widetilde{q}_h)_{\widetilde{K}} &= -(\boldsymbol{w}_h, (I - \boldsymbol{n}_h \otimes \boldsymbol{n}_h) \nabla q_h)_K + (\boldsymbol{w}_h \cdot \boldsymbol{\mu}_K, q_h)_{\partial K} \\ &= (\nabla_{\Gamma_h} \cdot \boldsymbol{w}_h, q_h)_K = \left(\frac{1}{\xi_h} \nabla_{\Gamma_h} \cdot \boldsymbol{w}_h, \widetilde{q}_h\right)_{\widetilde{K}}.\end{aligned} \tag{4.7}$$

Now we are able to prove the following

Lemma 4.5. *Given* $(\boldsymbol{w}_h, q_h) \in \boldsymbol{W}_h \times Q_h$ *and the corresponding lifting onto* Γ $(\widetilde{\boldsymbol{w}}_h, \widetilde{q}_h) \in \widetilde{\boldsymbol{W}}_h \times \widetilde{Q}_h$, *there exists some positive constants* C_1, C_2 *and* C_3 *such that the following inequalities hold*

$$\frac{1}{C_1}\|q_h\|_{L^2(K)} \leq \|\widetilde{q}_h\|_{L^2(\widetilde{K})} \leq C_1\|q_h\|_{L^2(K)}$$
$$\frac{1}{C_2}\|\boldsymbol{w}_h\|_{L^2(K)} \leq \|\widetilde{\boldsymbol{w}}_h\|_{L^2(\widetilde{K})} \leq C_2\|\boldsymbol{w}_h\|_{L^2(K)}$$
$$\frac{1}{C_3}\|\nabla_{\Gamma_h}\cdot\boldsymbol{w}_h\|_{L^2(K)} \leq \|\nabla_{\Gamma}\cdot\widetilde{\boldsymbol{w}}_h\|_{L^2(\widetilde{K})} \leq C_3\|\nabla_{\Gamma_h}\cdot\boldsymbol{w}_h\|_{L^2(K)}$$

Proof. The first inequality is proved in [11]. For the second inequality we have by the definition of the L^2-norm

$$\|\widetilde{\boldsymbol{w}}_h\|^2_{L^2(\widetilde{K})} = (\widetilde{\boldsymbol{w}}_h, \widetilde{\boldsymbol{w}}_h)_{\widetilde{K}} = \left(\frac{1}{\xi_h}F^\top F\boldsymbol{w}_h, \boldsymbol{w}_h\right)_K.$$

Matrix $F^\top F$ is given by

$$F^\top F = \begin{bmatrix} |\boldsymbol{t}_1|^2 & \boldsymbol{t}_1\cdot\boldsymbol{t}_2 & 0 \\ \boldsymbol{t}_1\cdot\boldsymbol{t}_2 & |\boldsymbol{t}_2|^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From the definition of $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ it is straightforward that

$$|\boldsymbol{t}_i|^2 = 1 + \mathcal{O}(h^2) \quad \text{and} \quad \boldsymbol{t}_1\cdot\boldsymbol{t}_2 \approx -n_1 n_2 + \mathcal{O}(h^2).$$

Following [12], we have that exists $c \in \mathbb{R}^+$ such that $\|n_i\|_{L^\infty(K)} \leq ch$, $i = 1, 2$, then

$$\|\boldsymbol{t}_1\cdot\boldsymbol{t}_2\|_{L^\infty(K)} \leq ch^2.$$

In conclusion, the following inequality holds

$$\left\|I - F^\top F\right\|_{L^\infty(K)} \leq ch^2.$$

Thanks to this last relation and Lemma 4.4 the second estimate immediately follows. The boundary term in (4.7) is second order in h and can be neglected, then the last inequality of the lemma can be written as

$$\begin{aligned}\|\nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h\|^2_{L^2(\widetilde{K})} = (\nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h, \nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h)_{\widetilde{K}} &\leq \left(\frac{1}{\xi_h}\nabla_{\Gamma_h}\cdot\boldsymbol{w}_h, \nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h\right)_{\widetilde{K}} \\ &\leq \left\|\xi_h^{-1}\right\|_{L^\infty(\widetilde{K})}\|\nabla_{\Gamma_h}\cdot\boldsymbol{w}_h\|_{L^2(\widetilde{K})}\|\nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h\|_{L^2(\widetilde{K})}.\end{aligned}$$

Using the first inequality of the lemma and the estimate for ξ_h we can obtain the last inequality. □

5. Error analysis

In this section, to ease the analysis and the presentation of the forthcoming results, we consider fully homogeneous boundary conditions and zero vector source term. Moreover we suppose that η is constant so that $\eta_h = \eta$. Finally, from (4.7) we can obtain the following useful relation

$$\int_\Gamma \nabla_\Gamma\cdot\widetilde{\boldsymbol{w}}_h\,\widetilde{q}_h\,\mathrm{d}x = \int_{\Gamma_h}\nabla_{\Gamma_h}\cdot\boldsymbol{w}_h\,q_h\,\mathrm{d}x_h. \tag{5.1}$$

Therefore the approximation of the bilinear form $b(\cdot,\cdot)$ with $b_h(\cdot,\cdot)$ will not bring any additional error due to the discretization of the geometry. The additional term is only linked to the approximation of the bilinear form $a(\cdot,\cdot)$, in particular from

$$\int_K \eta \boldsymbol{u}_h \cdot \boldsymbol{v}_h \,\mathrm{d}x_h = \int_{\widetilde{K}} \eta \xi_h \left(F^{-\top} F^{-1} \widetilde{\boldsymbol{u}}_h \right) \cdot \widetilde{\boldsymbol{v}}_h \,\mathrm{d}x. \tag{5.2}$$

If we define $B_h := \xi_h F^{-\top} F^{-1}$ we can rewrite the discrete Problem 2 as

$$\begin{cases} (\eta B_h \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma - (\widetilde{p}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{v}}_h)_\Gamma = 0 & \forall \widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h \\ (\widetilde{q}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{u}}_h)_\Gamma = (f, q_h)_\Gamma & \forall \widetilde{q}_h \in \widetilde{Q}_h \end{cases}. \tag{5.3}$$

In this case we have chosen $f_h(\hat{\boldsymbol{x}}) = \xi_h f(\boldsymbol{\Psi}(\hat{\boldsymbol{x}}))$ in order to have $F_h = F$ on Γ.

Remark 5.1. In practice is often simpler to compute the source term as $f_h(\hat{\boldsymbol{x}}) = f(\boldsymbol{\Psi}(\hat{\boldsymbol{x}}))$, thus adding an extra term of order $\mathcal{O}(h^2)$ to the error given by the difference between f and f_h.

Lemma 5.2. *If* $(\boldsymbol{u}_h, p_h)$ *is solution of* (4.2)*, then its correspondent lift to* Γ*, indicated with* $(\widetilde{\boldsymbol{u}}_h, \widetilde{p}_h)$*, is solution of* (5.3) *and* vice versa*.*

Proof. Thanks to relations (5.1) and (5.2) we immediately get the equivalence between problems (4.2) and (5.3). □

To provide an error estimate for our problem we need to recall some results on saddle-points problems, see [4, 21] for a detailed analyses. Introducing the following discrete functional space $\widetilde{\boldsymbol{W}}_h^f := \{\widetilde{\boldsymbol{w}}_h \in \widetilde{\boldsymbol{W}}_h : (\widetilde{q}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{w}}_h - f)_\Gamma = 0 \ \forall \widetilde{q}_h \in \widetilde{Q}_h\}$, Lemma 5.3 holds true [21].

Lemma 5.3. *If the spaces* $\widetilde{\boldsymbol{W}}_h$ *and* $\widetilde{Q}_h$ *satisfy the inf-sup condition then for each* $f \in L^2(\Gamma)$ *there exist a unique* $\widetilde{\boldsymbol{w}}_h^f \in (\widetilde{\boldsymbol{W}}_h^0)^\perp$ *such that:*

$$(\widetilde{q}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{w}}_h^f - f)_\Gamma = 0 \quad \forall \widetilde{q}_h \in \widetilde{Q}_h \tag{5.4}$$

and

$$\left\| \widetilde{\boldsymbol{w}}_h^f \right\|_{\boldsymbol{W}} \leq \frac{1}{\beta} \sup_{\widetilde{q}_h \in \widetilde{Q}_h, \widetilde{q}_h \neq 0} \frac{(f, \widetilde{q}_h)_\Gamma}{\|\widetilde{q}_h\|_Q} \cdot \tag{5.5}$$

Furthermore if $\widetilde{\boldsymbol{u}}_h \in \widetilde{\boldsymbol{W}}_h$ *satisfies*

$$(\eta B_h \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma = 0 \quad \forall \widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h^0,$$

then there exists a unique $\widetilde{p}_h \in \widetilde{Q}_h$ *such that*

$$-(\widetilde{p}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{v}}_h)_\Gamma + (\eta B_h \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma = 0 \quad \forall \widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h \tag{5.6}$$

and

$$\|\widetilde{p}_h\|_{L^2} \leq \frac{1}{\beta} \sup_{\widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h, \widetilde{\boldsymbol{v}}_h \neq 0} \frac{(\eta B_h \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma}{\|\widetilde{\boldsymbol{v}}_h\|_{\boldsymbol{W}}} \cdot \tag{5.7}$$

By setting $\widetilde{\boldsymbol{u}}_h = \widetilde{\boldsymbol{u}}_h^0 + \widetilde{\boldsymbol{w}}_h^f$, with $\widetilde{\boldsymbol{u}}_h^0 \in \widetilde{\boldsymbol{W}}_h^0$ and $\widetilde{\boldsymbol{w}}_h^f \in \widetilde{\boldsymbol{W}}_h^f$, we can rewrite (5.3) as: find $\widetilde{u}_h^0 \in \widetilde{\boldsymbol{W}}_h^0$ such that:

$$\left(\eta B_h \widetilde{\boldsymbol{u}}_h^0, \widetilde{\boldsymbol{v}}_h\right)_\Gamma = -\left(\eta B_h \widetilde{\boldsymbol{w}}_h^f, \widetilde{\boldsymbol{v}}_h\right)_\Gamma \quad \forall \widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h^0. \tag{5.8}$$

Thanks to the Lax–Milgram lemma, it exists a unique solution $\widetilde{\boldsymbol{u}}_h^0 \in \widetilde{\boldsymbol{W}}_h^0$ of (5.8) that satisfies $\|\widetilde{\boldsymbol{u}}_h^0\|_{\boldsymbol{W}} \le C\|\widetilde{\boldsymbol{w}}_h^f\|_{L^2}$, for a $C > 0$. Then, from (5.5) and (5.7) we have:

$$\|\widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} \le \frac{C}{\beta}\|f\|_{L^2} \quad \text{and} \quad \|\widetilde{p}_h\|_{L^2} \le \frac{C}{\beta}\|\widetilde{\boldsymbol{u}}_h\|_{L^2}. \tag{5.9}$$

Remark 5.4. From equality (5.1) and from the well-posedness of Problem 2, we have that $\forall \widetilde{q}_h \in Q_h$, $\exists \boldsymbol{v} \in \widetilde{\boldsymbol{W}}_h$ such that

$$b(\widetilde{\boldsymbol{v}}_h, \widetilde{q}_h) \ge \beta \|\boldsymbol{v}_h\|_{\boldsymbol{H}_{\mathrm{div}}(\Gamma_h)} \|q_h\|_{L^2(\Gamma_h)}, \tag{5.10}$$

where β is a positive constant. Then, from the inequalities of Lemma (4.5), we can conclude that the spaces $\widetilde{\boldsymbol{W}}_h$ and $\widetilde{Q}_h$ satisfy an inf-sup condition.

Lemma 5.5. *Let* $(\boldsymbol{u}, p) \in \boldsymbol{W} \times Q$ *be the solution of the continuous problem* (3.1)*,* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{W}_h \times Q_h$ *the solution of the discrete problem* (4.2) *and* $(\widetilde{\boldsymbol{u}}_h, \widetilde{p}_h) \in \widetilde{\boldsymbol{W}}_h \times \widetilde{Q}_h$ *its corresponding lift to* Γ*, then the following inequality holds*

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} + \|p - \widetilde{p}_h\|_Q \le C\left(\|I - B_h\|_{L^\infty}\|f\|_{L^2} + \inf_{\widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h} \|\boldsymbol{u} - \widetilde{\boldsymbol{v}}_h\|_{\boldsymbol{W}} + \inf_{\widetilde{q}_h \in \widetilde{Q}_h} \|p - \widetilde{q}_h\|_Q\right). \tag{5.11}$$

Proof. To find an estimate for the discretization error we write (5.3) in the following form, which highlights the classical saddle point structure:

$$\begin{cases} (\eta \widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma - (\widetilde{p}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{v}}_h)_\Gamma = (\eta(I - B_h)\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma & \forall \widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h \\ (\widetilde{q}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{u}}_h)_\Gamma = (f, q_h)_\Gamma & \forall \widetilde{q}_h \in \widetilde{Q}_h \end{cases}. \tag{5.12}$$

By subtracting (3.1) and (5.12) and adding and subtracting to the result a vector $\widetilde{\boldsymbol{w}}_h^* \in \widetilde{\boldsymbol{W}}_h^f$, we obtain

$$(\eta(\widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{w}}_h^*), \widetilde{\boldsymbol{v}}_h)_\Gamma + (p - \widetilde{p}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{v}}_h)_\Gamma = (\eta(\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*), \widetilde{\boldsymbol{v}}_h)_\Gamma + (\eta(I - B_h)\widetilde{\boldsymbol{u}}_h, \widetilde{\boldsymbol{v}}_h)_\Gamma. \tag{5.13}$$

By choosing $\widetilde{\boldsymbol{v}}_h = \widetilde{\boldsymbol{u}}_h - \widetilde{\boldsymbol{w}}_h^*$, with $\widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h^0$, and using (5.9), we get:

$$\|\widetilde{\boldsymbol{v}}_h\|_{\boldsymbol{W}} = \|\widetilde{\boldsymbol{v}}_h\|_{L^2} \le C\left(\|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*\|_{L^2} + \|I - B_h\|_{L^\infty}\|f\|_{L^2}\right),$$

from which it follows that

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} \le C\left(\inf_{\widetilde{\boldsymbol{w}}_h^* \in \widetilde{\boldsymbol{W}}_h^f} \|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*\|_{\boldsymbol{W}} + \|I - B_h\|_{L^\infty}\|f\|_{L^2}\right). \tag{5.14}$$

Now we want to show that

$$\inf_{\widetilde{\boldsymbol{w}}_h^* \in \widetilde{\boldsymbol{W}}_h^f} \|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*\|_{\boldsymbol{W}} \le C \inf_{\widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h} \|\boldsymbol{u} - \widetilde{\boldsymbol{v}}_h\|_{\boldsymbol{W}}. \tag{5.15}$$

From Lemma 5.3, for all $\widetilde{\boldsymbol{v}}_h \in \widetilde{\boldsymbol{W}}_h$, there exists a unique $\widetilde{\boldsymbol{z}}_h \in (\widetilde{\boldsymbol{W}}_h^0)^\perp$ such that

$$(\widetilde{q}_h, \nabla_\Gamma \cdot \widetilde{\boldsymbol{z}}_h)_\Gamma = (\widetilde{q}_h, \nabla_\Gamma \cdot (\boldsymbol{u} - \widetilde{\boldsymbol{v}}_h))_\Gamma \quad \forall \widetilde{q}_h \in Q_h,$$

and $\|\widetilde{\boldsymbol{z}}_h\|_{\boldsymbol{W}} \leq C\|\nabla_\Gamma \cdot (\boldsymbol{u} - \widetilde{\boldsymbol{v}}_h)\|_{L^2}$. Setting $\widetilde{\boldsymbol{w}}_h^* = \widetilde{\boldsymbol{z}}_h + \widetilde{\boldsymbol{v}}_h$, we have $\widetilde{\boldsymbol{w}}_h^* \in \widetilde{\boldsymbol{W}}_h^f$ and we obtain $\|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*\|_{\boldsymbol{W}} \leq \|\boldsymbol{u} - \widetilde{\boldsymbol{v}}_h\|_{\boldsymbol{W}}$, from which (5.15) follows. We consider now the term which involves the pressure. From (5.13) by adding and subtracting the term $(\widetilde{q}_h, \nabla_\Gamma \widetilde{\boldsymbol{v}}_h)_\Gamma$, where $\widetilde{q}_h \in \widetilde{Q}_h$, and using (5.10) we obtain

$$\|\widetilde{q}_h - \widetilde{p}_h\|_{L^2} \leq C\left(\|I - B_h\|_{L^\infty}\|f\|_{L^2} + \|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} + \|p - \widetilde{q}_h\|_{L^2}\right).$$

By using relation (5.14) and the generality of $\widetilde{q}_h$ we have

$$\|p - \widetilde{p}_h\|_Q \leq C\left(\inf_{\widetilde{\boldsymbol{w}}_h^* \in \widetilde{\boldsymbol{W}}_h^f} \|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h^*\|_{\boldsymbol{W}} + \|I - B_h\|_{L^\infty}\|f\|_{L^2} + \inf_{\widetilde{q}_h \in \widetilde{Q}_h} \|p - \widetilde{q}_h\|_{L^2}\right). \tag{5.16}$$

Considering the bounds (5.14) and (5.16) we obtain (5.11). □

In (5.11) we observe that, as expected, the error is composed by two different terms, the first related to the finite element discretization and the second related to the approximation of the geometry of the problem. In particular, as seen in the previous section for $F^\top F$, we can immediately conclude that

$$\|I - B_h\|_{L^\infty(\Gamma)} \leq ch^2.$$

Thus the contribution of the geometric error in the Darcy problem is of the second order with respect to the grid size h. We prove the main result of this section.

Theorem 5.6 (Order of convergence)**.** *Let $(\boldsymbol{u}, p) \in \boldsymbol{W} \times Q$ be the solution of the continuous problem (3.1), $(\boldsymbol{u}_h, p_h) \in \boldsymbol{W}_h \times Q_h$ the solution of the discrete problem (4.2) and $(\widetilde{\boldsymbol{u}}_h, \widetilde{p}_h) \in \widetilde{\boldsymbol{W}}_h \times \widetilde{Q}_h$ its corresponding lift to Γ. Assuming that the solution is regular enough and that $\xi_h \in H^1(K)$, then the following inequality holds*

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} + \|p - \widetilde{p}_h\|_Q \leq Ch\left(\|\nabla_\Gamma \cdot \boldsymbol{u}\|_{H^1(\Gamma)} + \|\boldsymbol{u}\|_{H^1(\Gamma)} + |p|_{H^1(\Gamma)}\right).$$

Proof. If we neglect in (5.11) the geometric contribution to the error, we have

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{u}}_h\|_{\boldsymbol{W}} + \|p - \widetilde{p}_h\|_Q \leq \left(\inf_{\boldsymbol{w}_h \in \widetilde{\boldsymbol{W}}_h} \|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h\|_{\boldsymbol{W}} + \inf_{\widetilde{q}_h \in \widetilde{Q}_h} \|p - \widetilde{q}_h\|_Q\right).$$

We start considering the estimate for the velocity field and we introduce the function $\hat{\boldsymbol{u}} : \Gamma_h \to \mathbb{R}^3$, defined as follows

$$\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}) := \xi_h\, F^{-1}\boldsymbol{u}(\Psi(\hat{\boldsymbol{x}})) \quad \text{with} \quad \hat{\boldsymbol{x}} \in \Gamma_h.$$

So $\hat{\boldsymbol{u}}$ is the projection of the exact solution on discrete surface. Thanks to lemma (4.5) we have

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}_{\mathrm{div}}(\widetilde{K})} \leq \|\hat{\boldsymbol{u}} - \boldsymbol{w}_h\|_{\boldsymbol{H}_{\mathrm{div}}(K)}.$$

This relation, together with standard results for $\boldsymbol{H}_{\mathrm{div}}$, gives us

$$\|\boldsymbol{u} - \widetilde{\boldsymbol{w}}_h\|_{\boldsymbol{H}_{\mathrm{div}}(\widetilde{K})} \leq Ch_k\left(|\nabla_{\Gamma_h} \cdot \hat{\boldsymbol{u}}|_{H^1(K)} + |\hat{\boldsymbol{u}}|_{\boldsymbol{H}_{\mathrm{div}}(K)}\right).$$

We see now how to estimate the right hand side of the inequality. From the definition of the H^1 semi-norm it follows that

$$|\nabla_{\Gamma_h} \cdot \hat{\boldsymbol{u}}|^2_{H^1(K)} = \|\nabla_{\Gamma_h}(\nabla_{\Gamma_h} \cdot \hat{\boldsymbol{u}})\|^2_{L^2(K)} = (\nabla_{\Gamma_h}(\nabla_{\Gamma_h} \cdot \hat{\boldsymbol{u}}), \nabla_{\Gamma_h}(\nabla_{\Gamma_h} \cdot \hat{\boldsymbol{u}}))_K.$$

From ([13], Sect. 4) and (4.7) we obtain $\nabla_{\Gamma_h}(\nabla_{\Gamma_h}\cdot\hat{\boldsymbol{u}}) = P_h(I-dH)\nabla_\Gamma(\xi_h\nabla_\Gamma\cdot\boldsymbol{u})$, with $P_h = I - \boldsymbol{n}_h\otimes\boldsymbol{n}_h$, which inserted in the semi-norm definition gives us

$$|\nabla_{\Gamma_h}\cdot\hat{\boldsymbol{u}}|^2_{H^1(K)} = (A_h\nabla_\Gamma(\xi_h\nabla_\Gamma\cdot\boldsymbol{u}),\nabla_\Gamma(\xi_h\nabla_\Gamma\cdot\boldsymbol{u}))_K,$$

where A_h is defined as $A_h := P(I-dH)P_h(I-dH)P/\xi_h$. We know that ξ_h^{-1} is bounded and moreover, from ([11], Sect. 5), we have

$$P(I-dH)P_h(I-dH)P = PP_hP + \mathcal{O}(h^2).$$

Then,

$$PP_hP = P - (\boldsymbol{n}_h - (\boldsymbol{n}_h\cdot\boldsymbol{n})\boldsymbol{n})(\boldsymbol{n}_h - (\boldsymbol{n}_h\cdot\boldsymbol{n})\boldsymbol{n})^\top\mathcal{O}(h^2).$$

Because in the reference system local to the triangle K, we have $\boldsymbol{n}_h = \boldsymbol{e}_3$, then

$$|\boldsymbol{n}_h - (\boldsymbol{n}_h\cdot\boldsymbol{n})\boldsymbol{n}| = |\boldsymbol{e}_3 - n_3\boldsymbol{n}| = \sqrt{1-n_3^2} = \sqrt{n_1^2+n_2^2} \approx \mathcal{O}(h).$$

Therefore for matrix A_h holds the relation $A_h \approx P + \mathcal{O}(h^2)$, Moreover, thanks to the regularity of the surface P is bounded and so it is A_h. Then, we can obtain

$$|\nabla_{\Gamma_h}\cdot\hat{\boldsymbol{u}}|_{H^1(K)} \le \|A_h\|^{\frac{1}{2}}_{L^\infty(\widetilde{K})}\|\nabla_\Gamma(\xi_h\,\nabla_\Gamma\cdot\boldsymbol{u})\|_{L^2(\widetilde{K})}.$$

Applying triangular and Schwarz's inequalities

$$\|\nabla_\Gamma(\xi_h\,\nabla_\Gamma\cdot\boldsymbol{u})\|_{L^2(\widetilde{K})} \le \|\nabla_\Gamma\xi_h\|_{L^2(\widetilde{K})}\|\nabla_\Gamma\cdot\boldsymbol{u}\|_{L^2(\widetilde{K})} + \|\xi_h\|_{L^2(\widetilde{K})}\|\nabla_\Gamma(\nabla_\Gamma\cdot\boldsymbol{u})\|_{L^2(\widetilde{K})} \le C^*\|\nabla_\Gamma\cdot\boldsymbol{u}\|_{H^1(\widetilde{K})}, \tag{5.17}$$

where $C^* = \max\left\{\|\nabla_\Gamma\xi_h\,\|_{L^2(\widetilde{K})}, \|\xi_h\|_{L^2(\widetilde{K})}\right\}$. Defining $C_1 = C^*\|A_h\|^{\frac{1}{2}}_{L^\infty(\widetilde{K})}$, we have proved that

$$|\nabla_{\Gamma_h}\cdot\hat{\boldsymbol{u}}|_{H^1(K)} \le C_1\|\nabla_\Gamma\cdot\boldsymbol{u}\|_{H^1(\widetilde{K})}. \tag{5.18}$$

In analogous way we can show the following inequality for the semi-norm

$$|\hat{\boldsymbol{u}}|_{H^1(K)} \le C_2\|\boldsymbol{u}\|_{H^1(\widetilde{K})}. \tag{5.19}$$

Summing (5.18) and (5.19) over all triangles we obtain the velocity estimate. We consider now the estimate for pressure and, similarly to what we have done for velocity, we introduce the lift of the exact solution p to Γ_h as

$$\hat{p}(\hat{\boldsymbol{x}}) := p(\Psi(\hat{\boldsymbol{x}})) \quad \text{with} \quad \hat{\boldsymbol{x}}\in\Gamma_h.$$

From Lemma 4.5 and standard interpolation results we have

$$\|p - \widetilde{q}_h\|_{L^2(\widetilde{K})} \le C\|\hat{p} - q_h\|_{L^2(K)} \le C_3h_K\,|\hat{p}|_{H^1(K)}\,.$$

Finally we exploit the results of ([11], Lem. 3) and we obtain

$$\|p - \widetilde{q}_h\|_{L^2(\widetilde{K})} \le CC_3h_K\,|p|_{H^1(\widetilde{K})}\,.$$

Considering the contribution of all the elements we have the desired estimation for the pressure. □

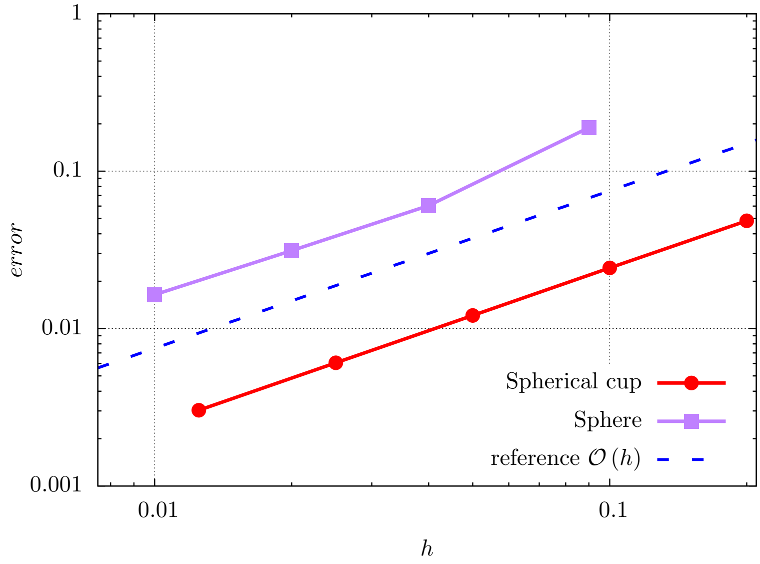


FIGURE 2. Error decay for the sphere and the spherical cup compared with a reference curve of $\mathcal{O}(h)$.

6. Applicative examples

We present in the following sub-sections some examples to show the goodness of the proposed approximation. In particular we show the error convergence for two different geometries: a sphere and a toroid. The choice is driven by the analytical solutions proposed in the literature for these geometries. The results are in good agreement with the theory. The simulations we propose in this paper are based on a code developed inside the library for finite elements LifeV [16] developed by École Polytechnique Fédérale de Lausanne (CMCS), Politecnico di Milano (MOX), INRIA under the projects REO and ESTIME and Emory University (Math&CS). Finally to ensure that the geometrical error is small enough and to increase the accurateness of the numerical solution, we have used the software presented in [8] to increase grid quality.

6.1. Example 1

We consider problem (2.2) solved on two different domains Γ_1 and Γ_2, where the former is a unit sphere while the latter a spherical cup limited by $\theta \in [-\pi/2, \pi/2]$ and $\phi \in [0, 2\pi]$. A unit permeability is considered and the scalar source term is taken as $f(\theta,\phi) = 2(2\cos^2\theta - \sin^2\theta)$ such that the exact solution is $p(\theta,\phi) = \cos^2\theta$. For Γ_1 the problem does not require boundary conditions, hence to have a well-posed problem we impose the solution in one point. While for Γ_2 we consider Neumann boundary conditions equal to the exact solution. The advantage of using a spherical domain, in addition to the use of spherical coordinate in finding the exact solution, is that we explicitly know the distance function $d(\boldsymbol{x}) = |\boldsymbol{x}| - 1$.

In Figure 2 we present the error history, for the two problems, decreasing the mesh size. It is clear that in both cases the error obtained scales at least as $\mathcal{O}(h)$, confirming the theoretical result presented in Theorem 5.6.

Observing the solutions reported in Figure 3, for the sphere, and in Figure 4, for the spherical cup, we can notice that the velocity field obtained is tangent to the surface and flows in the opposite direction of the pressure gradient, as we expect from the Darcy's law.

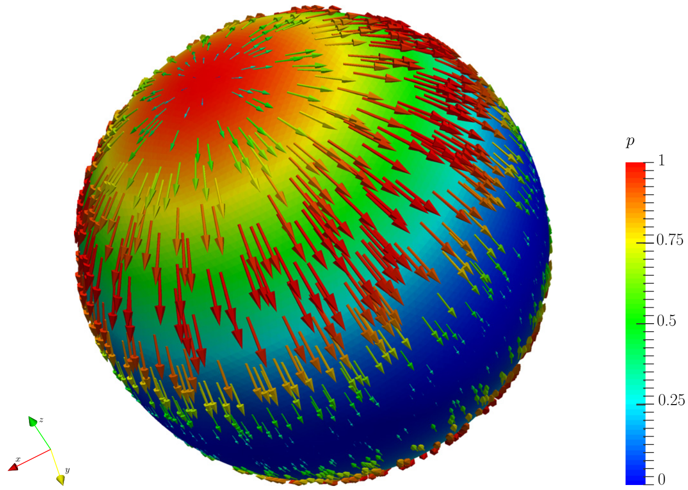
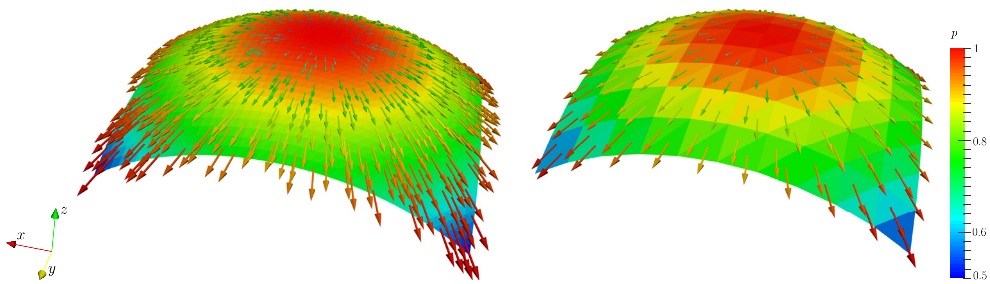


FIGURE 3. Numerical solution on the unit sphere. Both pressure and velocity are represented. The arrows for the latter are coloured and sized as the velocity magnitude. We can notice that in the two poles of the sphere and in its equator the pressure change slowly so does the velocity.



(A) Solution for a fine mesh (B) Solution for a coarse mesh

FIGURE 4. Numerical solution on a spherical cup of a unit sphere. Both pressure and velocity are represented. The arrows for the latter are coloured and sized as the velocity magnitude. We can see that the solution is smoother when refining the mesh.

6.2. Example 2

In this second example we consider as surface a torus defined by

$$\Gamma = \left\{ (x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2}-1\right)^2 + z^2 - 0.6^2 = 0 \right\}.$$

The exact solution for the pressure, expressed in toroidal coordinates, is given by

$$p(\phi,\theta) = \sin(3\phi)\cos(3\theta+\phi),$$

and the correspondent source term is equal to

$$f(\phi,\theta) = \frac{1}{r^2}(9\sin(3\phi)\cos(3\theta+\phi)) - \frac{(-10\sin(3\phi)\cos(3\theta+\phi) - 6\cos(3\phi)\sin(3\theta+\phi))}{(R-r\cos(\theta))^2}$$
$$- \frac{1}{r(R-r\cos(\theta))}(3\sin(\phi)\sin(3\phi)\sin(3\theta+\phi)),$$

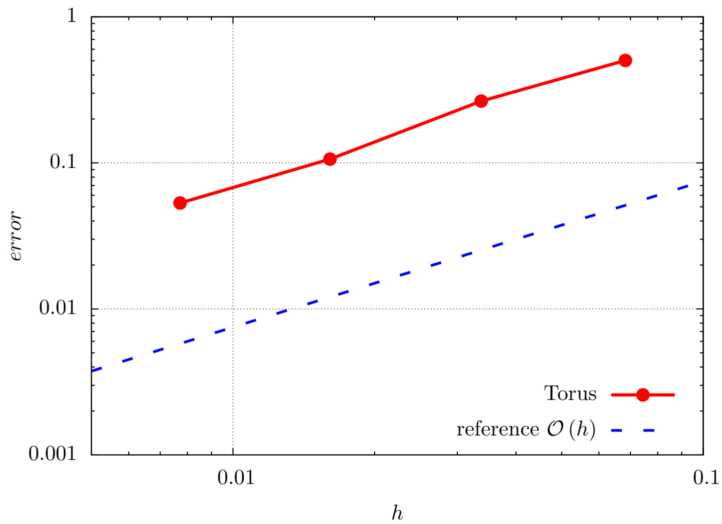


FIGURE 5. Error decay for the torus compared with a reference curve of $\mathcal{O}(h)$.

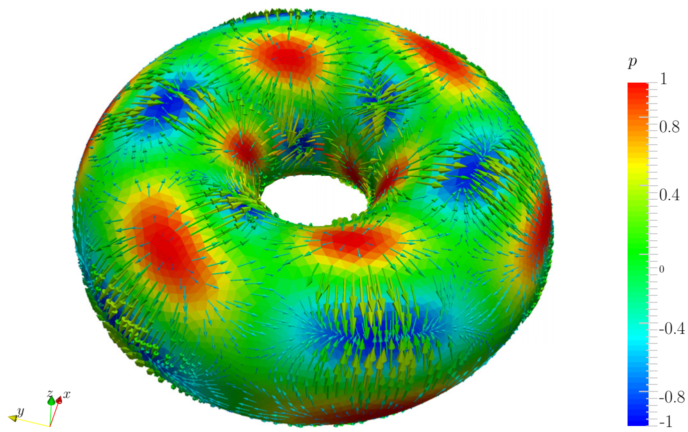


FIGURE 6. Numerical solution on the torus. Both pressure and velocity are represented. The arrows for the latter are coloured and sized as the velocity magnitude.

where $r = 0.6$ and $R = 1$. As in the previous case a unique solution is obtained by imposing the exact solution in one point. In Figure 5 we can observe that, also in this example, the decay of the error confirms the results presented in the theory. Figure 6 shows the obtained solution.

7. CONCLUSIONS

In this work we have presented a framework to solve Darcy problems on regular manifolds. The numerical discretization chosen is the classical pair of piecewise constant, for the pressure, and lowest order tangential Raviart–Thomas, for the Darcy velocity, finite element spaces. In this context we have provided an analysis of the relations between the quantities defined on the real surface and the ones defined on its discretization. Then we have used this properties in order to prove some results for the convergence of the approximation error. The numerical experiments proposed have confirmed the estimate presented in the theory.

A possible development of the work, could be the application of the obtained results to more realistic cases, for example in solving the Darcy problem defined in a whole basin. In such a case we should introduce suitable coupling conditions between the domain and the reduced model of the fracture and, in the case of a network of fractures, we should provide models for the flow along the intersecting curves.

Other possible topics for future extensions include consideration of higher-order surface approximations, more general classes of finite element spaces, more general differential operators, and/or superconvergent postprocessing techniques.

Acknowledgements. The authors wish to thank Antonio Cervone, Franco Dassi, Guido Iori, Anna Scotti and Marco Verani for many fruitful discussions. The authors also thank the anonymous reviewers for their helpful suggestions.

References

[1] P.F. Antonietti, A. Dedner, P. Madhavan, S. Stangalino, B. Stinner and M. Verani, High order discontinuous Galerkin methods on surfaces. *SIAM J. Numer. Anal.* **53** (2015) 1145–1171.

[2] J. Bear, *Dynamics of Fluids in Porous Media*. American Elsevier (1972).

[3] M. Bertalmío, L.-T. Cheng, S. Osher and G. Sapiro, Variational problems and partial differential equations on implicit surfaces. *J. Comput. Phys.* **174** (2001) 759–780.

[4] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Vol. 15 of *Comput. Math*. Springer Verlag, Berlin (1991).

[5] C.D. Cantwell, S. Yakovlev, R.M. Kirby, N.S. Peters and S.J. Sherwin, High-order spectral/*hp* element discretisation for reaction-diffusion problems on surfaces: Application to cardiac electrophysiology. *J. Comput. Phys.* **257** (2014) 813–829.

[6] P.G. Ciarlet, Mathematical Elasticity Volume I: Three-Dimensional Elasticity. Vol. 20 of *Stud. Math. Appl*. Elsevier (1988).

[7] C. D'Angelo and A. Scotti, A mixed finite element method for Darcy flow in fractured porous media with non-matching grids. *Math. Model. Numer. Anal.* **46** (2012) 465–489.

[8] F. Dassi, S. Perotto, L. Formaggia and P. Ruffo, Efficient geometric reconstruction of complex geological structures. *Math. Comput. Simul.* **106** (2014) 163–184.

[9] M.C. Delfour and J.-P. Zolésio, *Shapes and Geometries*, 2nd edition. Society for Industrial and Applied Mathematics (2011).

[10] A. Demlow and G. Dziuk, An adaptive finite element method for the Laplace-Beltrami operator on implicitly defined surfaces. *SIAM J. Numer. Anal.* **45** (2007) 421–442.

[11] G. Dziuk, Finite elements for the Beltrami operator on arbitrary surfaces. In *Partial Differential Equations and Calculus of Variations*. Edited by S. Hildebrandt and R. Leis. Vol. 1357 of *Lect. Notes Math.* Springer, Berlin, Heidelberg (1988) 142–155.

[12] G. Dziuk and C.M Elliott, Finite elements on evolving surfaces. *IMA J. Numer. Anal.* **27** (2007) 262–292.

[13] G. Dziuk and C.M Elliott, Surface finite elements for parabolic equations. *J. Comput. Math.* **25** (2007).

[14] A. Ern and J.-L. Guermond, Theory and Practice of Finite Elements. *Applied Mathematical Sciences*. Springer (2004).

[15] L. Formaggia, A. Fumagalli, A. Scotti and P. Ruffo, A reduced model for Darcy's problem in networks of fractures. *ESAIM: M2AN* **48** (2014) 1089–1116.

[16] G. Fourestey and S. Deparis, Lifev user manual. Available at http://lifev.org November (2010).

[17] A. Fumagalli and A. Scotti, A numerical method for two-phase flow in fractured porous media with non-matching grids. Computational Methods in Geologic CO2 Sequestration. *Adv. Water Resour.* **62** (2013) 454–464.

[18] A. Fumagalli and A. Scotti, An efficient XFEM approximation of Darcy flow in arbitrarly fractured porous media. *Oil Gas Sci. Technol. – Revue d'IFP Energies Nouvelles* **69** (2014) 555–564.

[19] M. Holst and A. Stern, Geometric variational crimes: Hilbert complexes, finite element exterior calculus, and problems on hypersurfaces. *Found. Comput. Math.* **12** (2012) 263–293.

[20] V. Martin, J. Jaffré and J.E. Roberts, Modeling fractures and barriers as interfaces for flow in porous media. *SIAM J. Sci. Comput.* **26** (2005) 1667–1691.

[21] A. Quarteroni and A. Valli, Numerical Approximation of Partial Differential Equations. Vol. 23 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin (1994).

[22] M.E. Rognes, R.C Kirby and A. Logg, Efficient assembly of h(div) and h(curl) conforming finite elements. *SIAM J. Sci. Comput.* **31** (2009) 4130–4151.