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SPECTRAL DISCRETIZATION OF THE NAVIER–STOKES EQUATIONS COUPLED WITH THE HEAT EQUATION

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Abstract. We consider the spectral discretization of the Navier–Stokes equations coupled with the heat equation where the viscosity depends on the temperature, with boundary conditions which involve the velocity and the temperature. This problem admits a variational formulation with three independent unknowns, the velocity, the pressure and the temperature. We prove optimal error estimates and present some numerical experiments which confirm the validity of the discretization.

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1. Introduction

Let Ω be a connected bounded open set in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz-continuous boundary $\partial\Omega$. The following system models the stationary flow of a viscous incompressible fluid, in the case where the viscosity of the fluid depends on the temperature

$$\begin{cases} -\mathrm{div}(\nu(T)\nabla \boldsymbol{u}) + (\boldsymbol{u}\,.\,\nabla)\boldsymbol{u} + \mathbf{grad}\,p = \boldsymbol{f} & \text{in} \quad \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in} \quad \Omega, \\ -\alpha\,\Delta T + (\boldsymbol{u}\,.\,\nabla)T = g & \text{in} \quad \Omega, \\ \boldsymbol{u} = \boldsymbol{u}_0 & \text{on}\ \partial\Omega, \\ T = T_0 & \text{on}\ \partial\Omega. \end{cases} \tag{1.1}$$

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The unknowns are the velocity $\boldsymbol{u}$, the pressure p, and the temperature T of the fluid, while the data are the distributions $\boldsymbol{f}$ and g. The function ν is positive and bounded, while the coefficient α is a positive constant. A similar but slightly more complex model has been derived and analyzed in [3, 6].

Equations (1.1) are a very realistic model for a number of incompressible fluids when the temperature presents high variations, for instance induced by the boundary condition. The solution $(\boldsymbol{u}, p)$ of the first two equations of the system behaves like the viscosity solution of Euler's equations. We refer to [5] for the first study of such a simplification.

We propose a spectral discretization in the basic situation where the domain is a square or a cube. More complex geometries can be treated using the arguments in [11], however we prefer to avoid them for simplicity. The numerical analysis of the nonlinear discrete problem makes use of the approach of Brezzi *et al.* [4], the main difficulty being the lack of compactness of the nonlinear term linked to the viscosity. Nevertheless, we prove the existence of a solution for the discrete problem. We establish *a priori* error estimates for this discretization.

In a final step, we describe the Newton type iterative algorithm that is used to solve the nonlinear discrete problem. Relying once more on the arguments in [4], we verify its convergence. We conclude with some numerical experiments where the viscosity of the fluid ν is a constant, or a function dependent of the space variable or a function that depends on the temperature T. All of these confirm the optimality of the discretization and justify the choice of this formulation.

An outline of the paper is as follows:

- In Section 2, we prove the existence of a solution for problem (1.1).
- The discrete problem is described in Section 3, and we prove optimal *a priori* error estimates for the error.
- The Newton algorithm is described in Section 4 and some numerical experiments are presented.

2. The continuous problem

In order to write a variational formulation of system (1.1), we first make precise the assumptions on the function ν: it belongs to $L^\infty(\mathbb{R})$ and satisfies, for two positive constants ν_1 and ν_2,

$$\text{for a.e. } \tau \in \mathbb{R}, \quad \nu_1 \le \nu(\tau) \le \nu_2. \tag{2.1}$$

Note that these assumptions are not at all restrictive.

The treatment of a nonzero value $\boldsymbol{u}_0$ requires an appropriate lifting of this value in order to treat the nonlinear term, *i.e.* the technical results linked to the Hopf lemma, (see [9], Chap. IV, Lem. 2.3). In order to avoid this further difficulty for the analysis of the problem, we assume that $\boldsymbol{u}_0 = \boldsymbol{0}$.

To go further, for any subset O of Ω with a Lipschitz-continuous boundary ∂O, we consider the full scale of Sobolev spaces $H^s(O)$, $s \in \mathbb{R}$, and also the analogous spaces $H^s(\partial O)$ on its boundary. We need the spaces $W^{m,p}(O)$, for any nonnegative integer m and $1 < p < +\infty$, equipped with the norm $\|.\|_{W^{m,p}(O)}$ and seminorm $|\ .\ |_{W^{m,p}(O)}$. We denote by $W_0^{m,p}(O)$ the closure in $W^{m,p}(O)$ of the space $\mathscr{D}(O)$ of infinitely differentiable functions with a compact support in O, by $W^{-m,p'}(O)$ its dual space (with $\frac{1}{p} + \frac{1}{p'} = 1$), and by $W^{m-\frac{1}{p},p}(\partial O)$ the space of traces of functions in $W^{m,p}(O)$ on ∂O. We also introduce the space

$$L_0^2(O) = \big\{q \in L^2(O);\ \int_O q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0\big\}. \tag{2.2}$$

We thus consider the variational problem:

Find $(\boldsymbol{u}, p, T)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$ such that

$$T = T_0 \quad \text{on } \partial\Omega, \tag{2.3}$$

and that,

$$\begin{aligned}
\forall \boldsymbol{v} \in H_0^1(\Omega)^d, \quad & \int_\Omega \nu(T)(\boldsymbol{x})(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega ((\boldsymbol{u}\,.\,\nabla)\boldsymbol{u})(\boldsymbol{x})\,.\,\boldsymbol{v}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})\,p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \;=\; \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega, \\
\forall q \in L_0^2(\Omega), \quad & -\int_\Omega (\mathrm{div}\,\boldsymbol{u})(\boldsymbol{x})\,q(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0, \\
\forall S \in H_0^1(\Omega), \quad & \alpha \int_\Omega (\mathbf{grad}\,T)(\boldsymbol{x})\,.\,(\mathbf{grad}\,S)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega ((\boldsymbol{u}\,.\,\nabla)T)(\boldsymbol{x})\,S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \;=\; \langle g, S\rangle_\Omega,
\end{aligned} \tag{2.4}$$

where $\langle .,.\rangle_\Omega$ denotes the duality pairing between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ and also between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$.

Standard arguments relying on the density of $\mathscr{D}(\Omega)$ in $H_0^1(\Omega)$ lead to the following result.

Proposition 2.1. *Problem* (1.1) *with* $\boldsymbol{u}_0 = \mathbf{0}$ *and problem* (2.3)–(2.4) *are equivalent: Any triple* $(\boldsymbol{u}, p, T)$ *in* $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$ *is a solution of* (1.1) (*in the distribution sense*) *if and only if it is a solution of* (2.3)–(2.4).

The existence of a solution can be established owing to a fixed-point theorem. Its proof requires the kernel

$$\mathbb{V} = \big\{\boldsymbol{v} \in H_0^1(\Omega)^d;\ \mathrm{div}\,\boldsymbol{v} = 0 \text{ in } \Omega\big\}. \tag{2.5}$$

Theorem 2.2. *For any data* $(\boldsymbol{f}, g)$ *in* $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$ *and* T_0 *in* $H^{\frac{1}{2}}(\partial\Omega)$*, problem* (2.3)–(2.4) *admits at least a solution* $(\boldsymbol{u}, p, T)$ *in* $H_0^1(\Omega)^d \times L_0^2(\Omega) \times H^1(\Omega)$*. Moreover, this solution satisfies, for a constant c only depending on* ν_1 *and* α,

$$\|\,\boldsymbol{u}\,\|_{H^1(\Omega)^d} + \|\,T\,\|_{H^1(\Omega)} \leqslant\; c\Big(\|\,\boldsymbol{f}\,\|_{H^{-1}(\Omega)^d} + \|\,g\,\|_{H^{-1}(\Omega)} + \|\,T_0\,\|_{H^{\frac{1}{2}}(\partial\Omega)}\Big). \tag{2.6}$$

Proof. It is performed in several steps.

1) We refer to the Hopf lemma (see [9], Chap. IV, Lem. 2.3), for the following result: For any $\varepsilon > 0$, there exists a lifting $\overline{T}_0$ of T_0 which satisfies

$$\|\,\overline{T}_0\,\|_{H^1(\Omega)} \leqslant\; c\,\|\,T_0\,\|_{H^{\frac{1}{2}}(\partial\Omega)} \quad \text{and} \quad \|\,\overline{T}_0\,\|_{L^4(\Omega)} \leqslant\; \varepsilon\,\|\,T_0\,\|_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{2.7}$$

where the constant c is independent of ε.

2) Setting $U = (\boldsymbol{u}, T)$ and $V = (\boldsymbol{v}, S)$, we define the mapping Φ from $\mathbb{V} \times H_0^1(\Omega)$ into its dual space by

$$\begin{aligned}
\langle \Phi(U), V\rangle = & \int_\Omega \nu(T + \overline{T}_0)(\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{v})(\boldsymbol{x})\mathrm{d}\boldsymbol{x} + \int_\Omega ((\boldsymbol{u}\,.\,\nabla)\boldsymbol{u})(\boldsymbol{x})\,.\,\boldsymbol{v}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\
& + \alpha \int_\Omega \mathbf{grad}\,(T + \overline{T}_0)(\boldsymbol{x})\,.\,(\mathbf{grad}\,S)(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \\
& + \int_\Omega ((\boldsymbol{u}\,.\,\nabla)(T + \overline{T}_0))(\boldsymbol{x})\,S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega - \langle g, S\rangle_\Omega.
\end{aligned}$$

It follows from (2.1) and the imbedding of $H^1(\Omega)$ into $L^6(\Omega)$ that Φ is continuous on $\mathbb{V} \times H_0^1(\Omega)$. Moreover, it follows from (2.1), (2.5) and (2.7) and the antisymmetry property

$$\int_\Omega \big((\boldsymbol{u}\,.\,\nabla)\overline{T}_0\big)(\boldsymbol{x})\,S(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \big((\boldsymbol{u}\,.\,\nabla)S\big)(\boldsymbol{x})\,\overline{T}_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \tag{2.8}$$

that

$$\langle \Phi(U),\, U\rangle \geqslant \nu_1 \mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} + \alpha \mid T \mid^2_{H^1(\Omega)} - \alpha \mid \overline{T}_0 \mid_{H^1(\Omega)} \mid T \mid_{H^1(\Omega)}$$
$$- \frac{c\,\varepsilon}{2} \parallel T_0 \parallel_{H^{\frac{1}{2}}(\partial\Omega)} (\mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} + \mid T \mid^2_{H^1(\Omega)})$$
$$- \parallel \boldsymbol{f} \parallel_{H^{-1}(\Omega)^d} \mid \boldsymbol{u} \mid_{H^1(\Omega)^d} - \parallel g \parallel_{H^{-1}(\Omega)} \mid T \mid_{H^1(\Omega)} .$$

We now take ε such that

$$c\varepsilon \parallel T_0 \parallel_{H^{\frac{1}{2}}(\partial\Omega)} \leqslant \min\{\nu_1,\, \alpha\}.$$

Thus, we deduce from the previous inequality that

$$\langle \Phi(U),\, U\rangle \geqslant \frac{\min\{\nu_1,\, \alpha\}}{2} (\mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} + \mid T \mid^2_{H^1(\Omega)})$$
$$- \Big(\alpha c \parallel T_0 \parallel_{H^{\frac{1}{2}}(\partial\Omega)} + (\parallel \boldsymbol{f} \parallel^2_{H^{-1}(\Omega)^d} + \parallel g \parallel^2_{H^{-1}(\Omega)})^{\frac{1}{2}}\Big) \Big(\mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} + \mid T \mid^2_{H^1(\Omega)}\Big)^{\frac{1}{2}}.$$

All this yields that $\langle \Phi(U),\, U\rangle$ is nonnegative on the sphere of $\mathbb{V} \times H^1_0(\Omega)$ with radius

$$\mu = \frac{2}{\min\{\nu_1,\, \alpha\}} \Big(\alpha c \parallel T_0 \parallel_{H^{\frac{1}{2}}(\partial\Omega)} + (\parallel \boldsymbol{f} \parallel^2_{H^{-1}(\Omega)^d} + \parallel g \parallel^2_{H^{-1}(\Omega)})^{\frac{1}{2}}\Big). \tag{2.9}$$

3) We recall from ([9], Chap. I, Cor. 2.5), that $\mathscr{D}(\Omega)^d \cap \mathbb{V}$ is dense in $\mathbb{V}$. Thus, there exist an increasing sequence $(\mathbb{V}_n)_n$ of finite-dimensional subspaces of $\mathbb{V}$ and an increasing sequence $(\mathbb{W}_n)_n$ of finite-dimensional subspaces of $H^1_0(\Omega)$ such that $\bigcup_{n\in\mathbb{N}} (\mathbb{V}_n \times \mathbb{W}_n)$ is dense in $\mathbb{V} \times H^1_0(\Omega)$. Moreover, the properties of the function Φ established above still hold with $\mathbb{V} \times H^1_0(\Omega)$ replaced by $\mathbb{V}_n \times \mathbb{W}_n$. Thus, applying Brouwer's fixed-point theorem (see [9], Chap. IV, Cor. 1.1, for instance) yields that, for each n, there exists a $U_n = (\boldsymbol{u}_n,\, T_n)$ satisfying

$$\forall\, V_n \in \mathbb{V}_n \times \mathbb{W}_n, \quad \langle \Phi(U_n),\, V_n\rangle = 0 \quad \text{and} \quad (\mid \boldsymbol{u}_n \mid^2_{H^1(\Omega)^d} + \mid T_n \mid^2_{H^1(\Omega)})^{\frac{1}{2}} \leqslant \mu. \tag{2.10}$$

4) Since the norms of $\boldsymbol{u}_n$ in $H^1(\Omega)^d$ and T_n in $H^1(\Omega)$ are bounded by a constant c (due to the Poincaré–Friedrichs inequality on Ω) and owing to the compactness of the imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, there exists a subsequence, still denoted by $(\boldsymbol{u}_n,\, T_n)_n$ for simplicity, which converges to a pair $(\boldsymbol{u},\, \widetilde{T})$ of $H^1_0(\Omega)^d \times H^1_0(\Omega)$ weakly in $H^1(\Omega)^d \times H^1(\Omega)$ and strongly in $L^4(\Omega)^d \times L^4(\Omega)$. Next, we observe that, for $m \leqslant n$, these $(\boldsymbol{u}_n,\, T_n)$ satisfy

$$\forall\, V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(U_n),\, V_m\rangle = 0.$$

Passing to the limit on n is obvious for the linear terms and follows from the strong convergence in $L^4(\Omega)^d \times L^4(\Omega)$ for the terms $(\boldsymbol{u}_n \,.\, \nabla)\boldsymbol{u}_n$ and $(\boldsymbol{u}_n \,.\, \nabla)T_n$. On the other hand, due to this strong convergence, the sequence $(\nu(T_n + \overline{T}_0)\, \mathbf{grad}\, \boldsymbol{v}_m)_n$ converges to $(\nu(\widetilde{T} + \overline{T}_0)\, \mathbf{grad}\, \boldsymbol{v}_m)$ a.e. in Ω and its norm is bounded by $\nu_2 \parallel \mathbf{grad}\, \boldsymbol{v}_m \parallel_{L^2(\Omega)^{d\times d}}$, so that using the Lebesgue dominated convergence theorem yields the convergence of $(\nu(T_n + \overline{T}_0)\, \mathbf{grad}\, \boldsymbol{v}_m)_n$ to $\nu(\widetilde{T} + \overline{T}_0)\, \mathbf{grad}\, \boldsymbol{v}_m$ in $L^2(\Omega)^{d\times d}$. All this leads to

$$\forall\, V_m \in \mathbb{V}_m \times \mathbb{W}_m, \quad \langle \Phi(\boldsymbol{u}, \widetilde{T}),\, V_m\rangle = 0,$$

and passing to the limit on m is now easy. Thus, we derive that the pair $(\boldsymbol{u}, T = \widetilde{T} + \overline{T}_0)$ satisfies the second and third equation in (2.4) and also

$$\forall \boldsymbol{v} \in \mathbb{V}, \quad \int_\Omega \nu(T)(\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \big((\boldsymbol{u} \,.\, \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \,.\, \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \;=\; \langle \boldsymbol{f}, \boldsymbol{v}\rangle_\Omega. \tag{2.11}$$

5) We recall from ([9], Chap. I, Cor. 2.4), the following inf-sup condition for a positive constant β

$$\forall q \in L^2_0(\Omega), \qquad \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{\int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x}) q(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}}{\| \boldsymbol{v} \|_{H^1(\Omega)^d}} \geqslant \beta \| q \|_{L^2(\Omega)} . \tag{2.12}$$

Thus, owing to equation (2.11), there exists a p in $L^2_0(\Omega)$ (see [9], Chap. I, Lem. 4.1), such that

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad \int_\Omega \nu(T)(\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x}) : (\mathbf{grad}\, \boldsymbol{v})(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ + \int_\Omega \big((\boldsymbol{u} \,.\, \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \,.\, \boldsymbol{v}(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} - \langle \boldsymbol{f}, \boldsymbol{v} \rangle_\Omega = \int_\Omega (\operatorname{div} \boldsymbol{v})(\boldsymbol{x})\, p(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}.$$

Then the triple $(\boldsymbol{u}, p, T)$ is a solution of problem (2.3)–(2.4), and estimate (2.6) is easily derived from (2.7) and (2.10), see (2.9). □

Proposition 2.3. *Assume that the function ν is Lipschitz-continuous, with Lipschitz constant ν^*. There exist two positive constants $c_\sharp$ and $c_\flat$ such that*

(i) *if the data $(\boldsymbol{f}, g)$ in $H^{-1}(\Omega)^d \times H^{-1}(\Omega)$ and T_0 in $H^{\frac{1}{2}}(\partial\Omega)$ satisfy*

$$c_\sharp \left(\| \boldsymbol{f} \|_{H^{-1}(\Omega)^d} + \| g \|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \right) < 1, \tag{2.13}$$

(ii) *if problem (2.3)–(2.4) admits a solution $(\boldsymbol{u}, p, T)$ such that $\boldsymbol{u}$ belongs to $W^{1,q}(\Omega)^d$ with $q > 2$ in dimension $d = 2$ and $q \geqslant 3$ in dimension $d = 3$, and satisfies*

$$c_\flat\, \nu^* \,|\, \boldsymbol{u} \,|_{W^{1,q}(\Omega)^d} < 1, \tag{2.14}$$

then this solution is unique.

Proof. For brevity, we set:

$$c_1 = c \left(\| \boldsymbol{f} \|_{H^{-1}(\Omega)^d} + \| g \|_{H^{-1}(\Omega)} + \|T_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \right),$$

where c is the constant in (2.6). Let $(\boldsymbol{u}_1, p_1, T_1)$ and $(\boldsymbol{u}_2, p_2, T_2)$ be two solutions of problem (2.3)–(2.4), with $\boldsymbol{u}_1$ in $W^{1,q}(\Omega)^d$ satisfying (2.14).
Setting for a while $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2, p = p_1 - p_2$ and $T = T_1 - T_2$, we proceed in three steps.

1) It follows from the third equation in (2.4) that, since T belongs to $H^1_0(\Omega)$,

$$\alpha \,|\, T \,|^2_{H^1(\Omega)} = - \int_\Omega \big((\boldsymbol{u}_1 \,.\, \nabla)T_1 - (\boldsymbol{u}_2 \,.\, \nabla)T_2\big)(\boldsymbol{x})\, T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\ = - \int_\Omega \big((\boldsymbol{u} \,.\, \nabla)T_1\big)(\boldsymbol{x})\, T(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$$

whence

$$\alpha \,|\, T \,|_{H^1(\Omega)} \leqslant c_1\, c_2\, |\boldsymbol{u}|_{H^1(\Omega)^d}, \tag{2.15}$$

where c_2 is the square of the norm of the imbedding of $H^1_0(\Omega)$ into $L^4(\Omega)$.

2) Similarly, we derive from the first equation in (2.4) that

$$\int_\Omega \nu(T_2)(\boldsymbol{x}) \mid \mathbf{grad}\,\boldsymbol{u} \mid^2 (\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = -\int_\Omega \big((\boldsymbol{u}\,.\,\nabla)\boldsymbol{u}_1\big)(\boldsymbol{x})\,.\,\boldsymbol{u}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - \int_\Omega \big(\nu(T_1)-\nu(T_2)\big)(\boldsymbol{x})(\mathbf{grad}\,\boldsymbol{u}_1)(\boldsymbol{x}) : (\mathbf{grad}\,\boldsymbol{u})(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Using appropriate Hölder's inequalities thus yields

$$\nu_1 \mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} \leqslant c_1\,c_2 \mid \boldsymbol{u} \mid^2_{H^1(\Omega)^d} + \nu^*\,c_3 \mid \boldsymbol{u}_1 \mid_{W^{1,q}(\Omega)^d} \mid T \mid_{H^1(\Omega)} \mid \boldsymbol{u} \mid_{H^1(\Omega)^d},$$

where c_3 stands for the norm of the imbedding of $H^1_0(\Omega)$ into $L^{q^*}(\Omega)$, with $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$. By combining this with (2.15) and choosing $c_\sharp$ and $c_\flat$ such that

$$c_1\,c_2\,\nu_1^{-1}\left(1+\nu^*\,c_3\,\alpha^{-1} \mid \boldsymbol{u}_1 \mid_{W^{1,q}(\Omega)^d}\right) < 1,$$

we obtain that $\boldsymbol{u}$ is zero, so that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are equal.

3) It then follows from (2.15) that T_1 and T_2 are equal. Finally, the function p satisfies

$$\forall \boldsymbol{v} \in H^1_0(\Omega)^d, \quad -\int_\Omega (\mathrm{div}\,\boldsymbol{v})(\boldsymbol{x})\,p(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0,$$

so that it is zero (see [9], Chap. I, Sect. 2, for instance). Thus, p_1 and p_2 coincide.

This concludes the proof. □

Assumptions (2.13) and (2.14) are clearly very restrictive and will not be used in what follows. We conclude with a regularity result.

Proposition 2.4. *There exist a real number $q_0 > 2$ only depending on the geometry of Ω and on the ratio ν_2/ν_1 and a real number $q_1 > 1$ only depending on the geometry of Ω such that, for any q, $2 \leqslant q \leqslant q_0$, and q', $1 \leqslant q' \leqslant q_1$, and for any data $(\boldsymbol{f}, g)$ in the space $W^{-1,q}(\Omega)^d \times L^{q'}(\Omega)$ and T_0 in $W^{2-\frac{1}{q'},q'}(\partial\Omega)$, any solution $(\boldsymbol{u}, p, T)$ of problem (2.3)–(2.4) belongs to $W^{1,q}(\Omega)^d \times L^q(\Omega) \times W^{2,q'}(\Omega)$. Moreover, q_1 is $\geqslant \frac{4}{3}$ for a general domain Ω and $\geqslant 2$ when Ω is convex.*

Proof. Proving the regularity of the velocity follows from the approach in [12]. The regularity of the pressure is a direct consequence of this. Finally, the regularity of the temperature is deduced from the standard properties of the Laplace operator (see [10], Thm. 4.3.2.4, [7], Thm. 2, or [8], Cor. 3.10). □

3. The discrete problem and its *a priori* analysis

We now consider the discretization of problem (2.3)–(2.4) in the case where $\Omega =]-1,1[^d$, $d = 2$ or 3. Let N be an integer $\geqslant 2$. We introduce the space $\mathbb{P}_N(\Omega)$ of polynomials with d variables and degree $\leqslant N$ with respect to each variable and the space $\mathbb{P}^0_N(\Omega)$ of polynomials in $\mathbb{P}_N(\Omega)$ vanishing on the boundary of Ω. Based on these definitions, we introduce the discrete spaces

$$\mathbb{X}_N = \mathbb{P}^0_N(\Omega)^d,\ \mathbb{M}_N = \mathbb{P}_{N-2}(\Omega) \cap L^2_0(\Omega),$$
$$\mathbb{Y}_N = \mathbb{P}_N(\Omega), \quad \mathbb{Y}^0_N = \mathbb{Y}_N \cap H^1_0(\Omega).$$

The reason for the choice of the space $\mathbb{M}_N$ is that it does not contain spurious modes (see [1], Chap. V).

We introduce the space $\mathbb{P}_N(-1,1)$ of restrictions to $[-1,1]$ of polynomials with degree $\leqslant N$. Setting $\xi_0 = -1$ and $\xi_N = 1$, we consider the $N-1$ nodes ξ_j, $1 \leqslant j \leqslant N-1$, and the $N+1$ weights ρ_j, $0 \leqslant j \leqslant N$, of the Gauss–Lobatto quadrature formula. We recall that the following equality holds

$$\forall \phi \in \mathbb{P}_{2N-1}(-1,1), \quad \int_{-1}^{1} \phi(\zeta)\,\mathrm{d}\zeta = \sum_{i=0}^{N} \phi(\xi_i)\,\rho_i. \tag{3.1}$$

We also recall ([2], Chap. IV, Cor. 1.10) the following property, which is useful in what follows

$$\forall \phi_N \in \mathbb{P}_N(-1,1), \quad \|\phi_N\|^2_{L^2(-1,1)} \leqslant \sum_{i=0}^{N} \phi_N^2(\xi_i)\,\rho_i \leqslant 3\,\|\phi_N\|^2_{L^2(-1,1)}. \tag{3.2}$$

Relying on this formula, we introduce the discrete product, defined on continuous functions u and v by

$$(u,v)_N = \begin{cases} \sum_{i=0}^{N} \sum_{j=0}^{N} u(\xi_i,\xi_j)\, v(\xi_i,\xi_j)\, \rho_i\rho_j & \text{if } d=2, \\ \sum_{i=0}^{N} \sum_{j=0}^{N} \sum_{k=0}^{N} u(\xi_i,\xi_j,\xi_k)\, v(\xi_i,\xi_j,\xi_k)\, \rho_i\rho_j\rho_k & \text{if } d=3. \end{cases}$$

It follows from (3.2) that this discrete product is a scalar product on $\mathbb{P}_N(\Omega)$. Let $\mathcal{I}_N$ denote the Lagrange interpolation operator at the nodes of the grid

$$\Sigma_N = \begin{cases} \{\boldsymbol{x} = (\xi_i,\xi_j); \quad 0 \leqslant i,j \leqslant N\} & \text{if } d=2, \\ \{\boldsymbol{x} = (\xi_i,\xi_j,\xi_k); \quad 0 \leqslant i,j,k \leqslant N\} & \text{if } d=3, \end{cases}$$

with values in the space $\mathbb{P}_N(\Omega)$. Finally, let $i_N^{\partial\Omega}$ denote the Lagrange interpolation operator at the nodes of $\Sigma_N \cap \partial\Omega$ with values in the space of traces of $\mathbb{P}_N(\Omega)$.

We now assume that the function T_0 is continuous on $\partial\Omega$ and $\boldsymbol{f}$, g are continuous on $\overline{\Omega}$. Thus the discrete problem is constructed from (2.3)–(2.4) by using the Galerkin method combined with numerical integration. It reads

Find $(\boldsymbol{u}_N, p_N, T_N)$ in $\mathbb{X}_N \times \mathbb{M}_N \times \mathbb{Y}_N$ such that

$$T_N = i_N^{\partial\Omega} T_0 \quad \text{on } \partial\Omega, \tag{3.3}$$

and that

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad \big(\nu(T_N)\,\mathbf{grad}\,\boldsymbol{u}_N, \mathbf{grad}\,\boldsymbol{v}_N\big)_N + \big((\boldsymbol{u}_N \,.\, \nabla)\boldsymbol{u}_N, \boldsymbol{v}_N\big)_N - \big(\mathrm{div}\,\boldsymbol{v}_N, p_N\big)_N = (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\ &\forall q_N \in \mathbb{M}_N, \quad -\big(\mathrm{div}\,\boldsymbol{u}_N,\, q_N\big)_N = 0, \\ &\forall S_N \in \mathbb{Y}_N^0, \quad \alpha\big(\mathbf{grad}\,T_N,\ \mathbf{grad}\,S_N\big)_N + \big((\boldsymbol{u}_N \,.\, \nabla)T_N,\, S_N\big)_N = (g, S_N)_N. \end{aligned} \tag{3.4}$$

The existence of a solution can be derived by the same arguments in Section 2, however we prefer to follow the approach of [4] to obtain directly more precise results.

We recall the existence of a discrete inf-sup condition between the spaces $\mathbb{X}_N$ and $\mathbb{M}_N$ (see [1], Chap. V, Thm. 25.7)

$$\forall\, q_N \in \mathbb{M}_N, \quad \sup_{\boldsymbol{v}_N \in \mathbb{X}_N} \frac{\int_\Omega (\mathrm{div}\,\boldsymbol{v}_N)(\boldsymbol{x})\, q_N(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}{\|\,\boldsymbol{v}_N\,\|_{H^1(\Omega)^d}} \geqslant c\,N^{-(d-1)/2}\,\|\,q_N\,\|_{L^2(\Omega)}\,. \tag{3.5}$$

For any real-valued measurable function τ on Ω, we introduce the modified Stokes operator $\mathcal{S}(\tau)$, which associates with any datum $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$ the part $\boldsymbol{u}$ of the solution $(\boldsymbol{u},p)$ of the generalized Stokes problem

$$\begin{cases} -\mathrm{div}(\nu(\tau)\nabla\boldsymbol{u}) + \mathbf{grad}\,p = \boldsymbol{F} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

We also consider the operator $\widetilde{\mathcal{S}}(\tau)$ which associates with any datum $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$ the part p of the solution $(\boldsymbol{u}, p)$ of this same problem.

We introduce the inverse $\mathcal{L}$ of the Laplace operator which associates with any datum (g, T_0) in $H^{-1}(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ the solution T in $H^1(\Omega)$ of the problem

$$\begin{cases} -\alpha\,\Delta T = g & \text{in } \Omega, \\ T = T_0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Thus it is readily verified that, when setting $U = (\boldsymbol{u}, T)$, problem (2.3)–(2.4) can be written equivalently as

$$\mathcal{F}(U) \;=\; U + \begin{pmatrix} \mathcal{S}(T) & 0 \\ 0 & \mathcal{L} \end{pmatrix} \mathcal{G}(U) \;=\; 0, \quad \text{with} \quad \mathcal{G}(U) \;=\; \begin{pmatrix} (\boldsymbol{u}\,.\,\nabla)\boldsymbol{u} - \boldsymbol{f} \\ ((\boldsymbol{u}\,.\,\nabla)T - g,\, T_0) \end{pmatrix}. \tag{3.8}$$

Similarly, let $\mathcal{S}_N(\tau)$ denote the discrete Stokes operator, *i.e.*, the operator which associates with any data $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$, the part $\boldsymbol{u}_N$ of the solution $(\boldsymbol{u}_N, p_N)$ in $\mathbb{X}_N \times \mathbb{M}_N$ of the Stokes problem

$$\begin{aligned} &\forall\, \boldsymbol{v}_N \in \mathbb{X}_N, \quad \big(\nu(\tau)\nabla\,\boldsymbol{u}_N, \nabla\,\boldsymbol{v}_N\big)_N - \big(\operatorname{div}\boldsymbol{v}_N, p_N\big)_N = \langle \boldsymbol{F}, \boldsymbol{v}_N\rangle_\Omega, \\ &\forall\, q_N \in \mathbb{M}_N, \quad -\big(\operatorname{div}\boldsymbol{u}_N,\, q_N\big)_N \;=\; 0. \end{aligned} \tag{3.9}$$

Let finally $\mathcal{L}_N$ denote the operator which associates with any datum G in $H^{-1}(\Omega)$ and any continuous function R_0 in $H^{\frac{1}{2}}(\partial\Omega)$, the function R_N in $\mathbb{Y}_N$, equal to $i_N^{\partial\Omega} R_0$ on $\partial\Omega$ and which satisfies

$$\forall\, S_N \in \mathbb{Y}_N^0, \quad \alpha\,\big(\mathbf{grad}\,R_N,\, \mathbf{grad}\,S_N\big)_N \;=\; \langle G, S_N\rangle_\Omega. \tag{3.10}$$

With the notation $U_N = (\boldsymbol{u}_N, T_N)$, problem (3.3)–(3.4) can equivalently be written as

$$\mathcal{F}_N(U_N) \;=\; U_N + \begin{pmatrix} \mathcal{S}_N(T_N) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} \mathcal{G}_N(U_N) = 0, \quad \text{with} \quad \mathcal{G}_N(U_N) = \begin{pmatrix} \mathcal{G}_{N1} \\ (\mathcal{G}_{N2}, T_0) \end{pmatrix}. \tag{3.11}$$

The two components $\mathcal{G}_{N1}$ and $\mathcal{G}_{N2}$ are defined in the dual spaces of $\mathbb{X}_N$ and $\mathbb{Y}_N^0$, respectively, by

$$\begin{aligned} &\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad \int_\Omega \mathcal{G}_{N1}(\boldsymbol{x})\,.\,\boldsymbol{v}_N(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \big((\boldsymbol{u}_{\boldsymbol{N}}\,.\,\nabla)\boldsymbol{u}_N - \boldsymbol{f}, \boldsymbol{v}_N\big)_N \\ &\forall S_N \in \mathbb{Y}_N^0, \quad \int_\Omega \mathcal{G}_{N2}(\boldsymbol{x})\; S_N(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = \big((\boldsymbol{u}_{\boldsymbol{N}}\,.\,\nabla)T_N - g, S_N\big)_N. \end{aligned}$$

Lemma 3.1. *There exists a constant $c > 0$ such that the following continuity property holds*

$$\langle \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle \leqslant c\,(\|\,\boldsymbol{u}_N\,\|_{H^1(\Omega)^d} + \|\,\mathcal{I}_N\boldsymbol{f}\,\|_{L^2(\Omega)^d})\,\|\,\boldsymbol{v}_N\,\|_{H^1(\Omega)^d}\,. \tag{3.12}$$

Proof. By definition, we have

$$\begin{aligned} \langle \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle &= \sum_{i,j=1}^d \left(\boldsymbol{u}_{Nj}\frac{\partial \boldsymbol{u}_{Ni}}{\partial x_j} - \boldsymbol{f}, \boldsymbol{v}_{Ni}\right)_N \\ &= \sum_{i,j=1}^d \left(\left(\boldsymbol{u}_{Nj}\boldsymbol{v}_{Ni}, \frac{\partial \boldsymbol{u}_{Ni}}{\partial x_j}\right)_N - (\boldsymbol{f}, \boldsymbol{v}_{Ni})_N\right) \\ &= \sum_{i,j=1}^d \left(\left(\mathcal{I}_N(\boldsymbol{u}_{Nj}\boldsymbol{v}_{Ni}), \frac{\partial \boldsymbol{u}_{Ni}}{\partial x_j}\right)_N - (\mathcal{I}_N\boldsymbol{f}, \boldsymbol{v}_{Ni})_N\right) \end{aligned}$$

whence, by using (3.2), we obtain

$$\langle \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N \rangle \leqslant 3^d \sum_{i,j=1}^{d} \Big(\| \mathcal{I}_N(\boldsymbol{u}_{Nj}\boldsymbol{v}_{Ni}) \|_{L^2(\Omega)^d} \| \frac{\partial \boldsymbol{u}_{Ni}}{\partial x_j} \|_{L^2(\Omega)^d} + \| \mathcal{I}_N \boldsymbol{f} \|_{L^2(\Omega)^d} \| \boldsymbol{v}_{Ni} \|_{L^2(\Omega)^d} \Big).$$

We recall from ([1], Rem. 13.5), that,

$$\forall \varphi_M \in \mathbb{P}_M(\Omega), \quad \| \mathcal{I}_N \varphi_M \|_{L^2(\Omega)} \leqslant c \left(1 + \frac{M}{N}\right)^d \| \varphi_M \|_{L^2(\Omega)} . \tag{3.13}$$

By taking $M = 2N$, we derive

$$\langle \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N \rangle \leqslant c \sum_{i,j=1}^{d} \Big(\| \boldsymbol{u}_{Nj} \|_{L^4(\Omega)^d} \| \boldsymbol{v}_{Ni} \|_{L^4(\Omega)^d} \| \boldsymbol{u}_{Ni} \|_{H^1(\Omega)^d} + \| \mathcal{I}_N \boldsymbol{f} \|_{L^2(\Omega)^d} \| \boldsymbol{v}_{Ni} \|_{L^2(\Omega)^d} \Big).$$

We conclude by noting that $H^1(\Omega)$ is embedded in $L^4(\Omega)$. □

We recall the basic properties of the discrete operators $\mathcal{S}_N(\tau)$ and $\mathcal{L}_N$. The operator $\mathcal{S}_N(\tau)$ satisfies the following properties: For any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$,

$$\| \mathcal{S}_N(\tau)\, \boldsymbol{F} \|_{H^1(\Omega)^d} \leqslant c \| \boldsymbol{F} \|_{H^{-1}(\Omega)^d}, \tag{3.14}$$

and, if moreover $\widetilde{\mathcal{S}}(\tau)\boldsymbol{F}$ belongs to $H^{s-1}(\Omega)$ and $\mathcal{S}(\tau)\boldsymbol{F}$ to $H^s(\Omega)^d$ for a real number s, $s \geqslant 1$,

$$\| \big(\mathcal{S}(\tau) - \mathcal{S}_N(\tau)\big)\, \boldsymbol{F} \|_{H^1(\Omega)^d} \leqslant c\, N^{1-s} \Big(\| \mathcal{S}(\tau)\boldsymbol{F} \|_{H^s(\Omega)^d} + \| \widetilde{\mathcal{S}}(\tau)\boldsymbol{F} \|_{H^{s-1}(\Omega)} \Big). \tag{3.15}$$

The analogous properties concerning the operator $\mathcal{L}_N$ read: For any G in $H^{-1}(\Omega)$,

$$\| \mathcal{L}_N(G, 0) \|_{H^1(\Omega)} \leqslant c \| G \|_{H^{-1}(\Omega)} . \tag{3.16}$$

and, if moreover $\mathcal{L}G$ belongs to $H^s(\Omega)^d$, $s \geqslant 1$, and R_0 belongs to $H^\sigma(\partial\Omega)$, for a real number σ, $\sigma > \frac{d-1}{2}$,

$$\| (\mathcal{L} - \mathcal{L}_N)(G, R_0) \|_{H^1(\Omega)} \leqslant c\, N^{1-s} \| \mathcal{L}G \|_{H^s(\Omega)} + N^{\frac{1}{2}-\sigma} \| R_0 \|_{H^\sigma(\partial\Omega)} . \tag{3.17}$$

Note that these properties yield the following convergence result, for any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$ and any G in $H^{-1}(\Omega)$,

$$\lim_{N \to +\infty} \| \big(\mathcal{S}(\tau) - \mathcal{S}_N(\tau)\big)\boldsymbol{F} \|_{H^1(\Omega)^d} = 0, \quad \lim_{N \to +\infty} \| (\mathcal{L} - \mathcal{L}_N)(G, 0) \|_{H^1(\Omega)} = 0. \tag{3.18}$$

From now on, we denote by

$$\mathcal{X}(\Omega) = H_0^1(\Omega)^d \times H^1(\Omega), \quad \mathcal{X}_N = \mathbb{X}_N \times \mathbb{Y}_N.$$

Assumption 3.2. The solution $(\boldsymbol{u}, p, T)$ of problem (2.3)–(2.4) satisfies:

(i) the velocity $\boldsymbol{u}$ belongs to $H^\rho(\Omega)^d$ and the temperature T belongs to $H^\rho(\Omega)$, for some $\rho > 1$;
(ii) the pair $U = (\boldsymbol{u}, T)$ is such that $D\mathcal{F}(U)$ is an isomorphism of $\mathcal{X}(\Omega)$.

Note that these assumptions are not restrictive, compared with the hypotheses of Proposition 2.3 for the uniqueness of the solution.

We are thus in a position to prove the preliminary results which we need for applying the theorem of Brezzi *et al.* [4]. This requires an approximation $U_N^\diamond = (\boldsymbol{u}_N^\diamond, T_N^\diamond)$ of U in $\mathbb{X}_N \times \mathbb{Y}_N$ which satisfies (see [2], Chap. III, Thm. 2.4) for the real number ρ of Assumption 3.2 and $0 \leqslant t \leqslant \rho$,

$$\|\boldsymbol{u} - \boldsymbol{u}_N^\diamond\|_{H^t(\Omega)^d} \leqslant c\, N^{t-\rho} \parallel \boldsymbol{u} \parallel_{H^\rho(\Omega)^d}, \quad \|T - T_N^\diamond\|_{H^{t+1}(\Omega)} \leqslant c\, N^{t-\rho} \parallel T \parallel_{H^{\rho+1}(\Omega)^d}. \tag{3.19}$$

Lemma 3.3. *If that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d, \sigma > \frac{d}{2}$, the following result holds*

$$\langle \mathcal{G}_1(\boldsymbol{u}_N) - \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle_\Omega \leqslant c\,\left(N^{-\frac{1}{2}} \parallel \boldsymbol{u}_N \parallel_{H^1(\Omega)^d} + N^{-\sigma} \parallel \boldsymbol{f} \parallel_{H^\sigma(\Omega)^d}\right) \parallel \boldsymbol{v}_N \parallel_{H^1(\Omega)^d} \tag{3.20}$$

with

$$\langle \mathcal{G}_1(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle_\Omega = \int_\Omega ((\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N - \boldsymbol{f})(\boldsymbol{x}) \cdot \boldsymbol{v}_N(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}.$$

Proof. Denoting for brevity the scalar product of $L^2(\Omega)$ by $(\cdot,\cdot)$, we have,

$$\langle \mathcal{G}_1(\boldsymbol{u}_N) - \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle_\Omega = ((\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N - \boldsymbol{f}, \boldsymbol{v}_N) - ((\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N - \boldsymbol{f}, \boldsymbol{v}_N)_N.$$

If N' stands for the integer part of $\frac{N-1}{2}$, we introduce an approximation $\boldsymbol{u}_{N'}$ of $\boldsymbol{u}_N$ in $\mathbb{P}_{N'}(\Omega)$ and we note the identity

$$((\boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_{N'}, \boldsymbol{v}_N) = ((\boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_{N'}, \boldsymbol{v}_N)_N.$$

Inserting it, we obtain

$$\begin{aligned}\langle \mathcal{G}_1(\boldsymbol{u}_N) - \mathcal{G}_{N1}(\boldsymbol{u}_N), \boldsymbol{v}_N\rangle_\Omega &= \big((\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N - (\boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_{N'}, \boldsymbol{v}_N\big)\\ &\quad + \big((\boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_{N'} - (\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N), \boldsymbol{v}_N\big)_N\\ &\quad - \big((\boldsymbol{f}, \boldsymbol{v}_N) - (\mathcal{I}_N \boldsymbol{f}, \boldsymbol{v}_N)_N\big).\end{aligned}$$

The arguments for evaluating the first two quantities are the same, so we only consider the first one. We have

$$\big((\boldsymbol{u}_N\ .\ \nabla)\boldsymbol{u}_N - (\boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_{N'}, \boldsymbol{v}_N\big) = ((\boldsymbol{u}_N - \boldsymbol{u}_{N'})\ .\ \nabla \boldsymbol{u}_N, \boldsymbol{v}_N) + ((\boldsymbol{u}_{N'}\ .\ \nabla)(\boldsymbol{u}_N - \boldsymbol{u}_{N'}), \boldsymbol{v}_N).$$

Writing

$$((\boldsymbol{u}_N - \boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_N, \boldsymbol{v}_N) = \int_\Omega \sum_{i,j=1}^d (\boldsymbol{u}_{Ni} - \boldsymbol{u}_{N'i})(\boldsymbol{x})\, \boldsymbol{v}_{Nj}\,(\boldsymbol{x}) \nabla \boldsymbol{u}_{Ni}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x},$$

we obtain

$$\begin{aligned}((\boldsymbol{u}_N - \boldsymbol{u}_{N'}\ .\ \nabla)\boldsymbol{u}_N, \boldsymbol{v}_N) &\leqslant \|\boldsymbol{u}_N - \boldsymbol{u}_{N'}\|_{L^3(\Omega)^d}\, \|\boldsymbol{v}_N\|_{L^6(\Omega)^d}\, \|\nabla \boldsymbol{u}_N\|_{L^2(\Omega)^d}\\ &\leqslant \|\boldsymbol{u}_N - \boldsymbol{u}_{N'}\|_{H^{\frac{1}{2}}(\Omega)^d}\, \|\boldsymbol{v}_N\|_{H^1(\Omega)^d} |\,\boldsymbol{u}_N|_{H^1(\Omega)^d},\end{aligned}$$

we conclude by using (3.19) for $t = \frac{1}{2}$ and $\rho = 1$.

To evaluate the third term, we have for any $\boldsymbol{f}_{N-1}$ in $\mathbb{P}_{N-1}(\Omega)^d$

$$\begin{aligned}(\boldsymbol{f}, \boldsymbol{v}_N) - (\boldsymbol{f}, \boldsymbol{v}_N)_N &= \int_\Omega \boldsymbol{f}(\boldsymbol{x}).\, \boldsymbol{v}_N(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N)_N\\ &= \int_\Omega (\boldsymbol{f} - \boldsymbol{f}_{N-1})(\boldsymbol{x}).\, \boldsymbol{v}_N(\boldsymbol{x})\mathrm{d}\boldsymbol{x} - (\boldsymbol{f} - \boldsymbol{f}_{N-1}, \boldsymbol{v}_N)_N,\end{aligned}$$

whence

$$(\boldsymbol{f},\boldsymbol{v}_N)-(\boldsymbol{f},\boldsymbol{v}_N)_N \leqslant \big(\parallel \boldsymbol{f}-\boldsymbol{f}_{N-1}\parallel_{L^2(\Omega)^d}+3^d\parallel \mathcal{I}_N\boldsymbol{f}-\boldsymbol{f}_{N-1}\parallel_{L^2(\Omega)^d}\big)\parallel \boldsymbol{v}_N\parallel_{L^2(\Omega)^d},$$
$$(\boldsymbol{f},\boldsymbol{v}_N)-(\boldsymbol{f},\boldsymbol{v}_N)_N \leqslant c\big(\parallel \boldsymbol{f}-\mathcal{I}_N\boldsymbol{f}\parallel_{L^2(\Omega)^d}+\inf_{f_{N-1}\in\mathbb{P}_{N-1}(\Omega)}\parallel \boldsymbol{f}-\boldsymbol{f}_{N-1}\parallel_{L^2(\Omega)^d}\big)\parallel \boldsymbol{v}_N\parallel_{H^1(\Omega)^d}.$$

By taking $\boldsymbol{f}_{N-1}$ equal to the image of $\boldsymbol{f}$ by the $L^2(\Omega)$ orthogonal projection operator (see [2], Chap. III), using ([2], Chap. IV, Thm 2.6), and ([2], Chap. III, Thm 2.4), we obtain

$$(\boldsymbol{f},\boldsymbol{v}_N)-(\boldsymbol{f},\boldsymbol{v}_N)_N \leqslant cN^{-\sigma}\parallel \boldsymbol{f}\parallel_{H^\sigma(\Omega)^d}\parallel \boldsymbol{v}_N\parallel_{H^1(\Omega)^d}.$$

We are now in a position to prove the following lemmas, we denote by $\mathcal{E}$ the space of endomorphisms of $\mathcal{X}(\Omega)$. Here, D stands for the differential operator. □

Lemma 3.4. *Assume that ν is of class $\mathscr{C}^2$ on $\mathbb{R}$, with bounded derivatives, and Assumption 3.2 holds. There exists a positive integer N_0 such that, for all $N\geqslant N_0$, the operator $D\mathcal{F}_N(U_N^\diamond)$ is an isomorphism of $\mathbb{X}_N\times\mathbb{Y}_N$ with the norm of its inverse bounded independently of N.*

Proof. We write the expansion

$$\begin{aligned}D\mathcal{F}_N(U_N^\diamond)=D\mathcal{F}(U)&-\begin{pmatrix}(\mathcal{S}-\mathcal{S}_N)(T) & 0\\ 0 & \mathcal{L}-\mathcal{L}_N\end{pmatrix}D\mathcal{G}(U)-\begin{pmatrix}\mathcal{S}_N & 0\\ 0 & \mathcal{L}_N\end{pmatrix}(D\mathcal{G}(U)-D\mathcal{G}(U_N^\diamond))\\ &-\begin{pmatrix}\mathcal{S}_N & 0\\ 0 & \mathcal{L}_N\end{pmatrix}(D\mathcal{G}(U_N^\diamond)-D\mathcal{G}_N(U_N^\diamond))-\begin{pmatrix}D(\mathcal{S}-\mathcal{S}_N)(T) & 0\\ 0 & 0\end{pmatrix}\mathcal{G}(U)\\ &-\begin{pmatrix}D\mathcal{S}_N(T) & 0\\ 0 & 0\end{pmatrix}(\mathcal{G}(U)-\mathcal{G}(U_N^\diamond))-\begin{pmatrix}D\mathcal{S}_N(T) & 0\\ 0 & 0\end{pmatrix}(\mathcal{G}(U_N^\diamond)-\mathcal{G}_N(U_N^\diamond)).\end{aligned}$$

Due to part (ii) of Assumption 3.2, it suffices to check that the last six terms in the right-hand side tend to zero when N tends to $+\infty$ in the norm of the space $\mathcal{E}$. Let $W_N=(\boldsymbol{w}_N,R_N)$ be any element in the unit sphere of $\mathcal{X}_N$.

1) We observe that

$$D\mathcal{G}(U).W_N=\begin{pmatrix}(\boldsymbol{u}\,.\,\nabla)\boldsymbol{w}_N+(\boldsymbol{w}_N\,.\,\nabla)\boldsymbol{u}\\ ((\boldsymbol{u}\,.\,\nabla)R_N+(\boldsymbol{w}_N\,.\,\nabla)T,0)\end{pmatrix}.$$

Thus, the compactness of the imbedding of $H^1(\Omega)$ into $L^q(\Omega)$, with $q<\infty$ in dimension $d=2$ and $q<6$ in dimension $d=3$, combined with the regularity of $\boldsymbol{u}$ yields that both terms $(\boldsymbol{u}\,.\,\nabla)\boldsymbol{w}_N+(\boldsymbol{w}_N\,.\,\nabla)\boldsymbol{u}$ and $(\boldsymbol{u}\,.\,\nabla)R_N+(\boldsymbol{w}_N\,.\,\nabla)T$ belong to a compact subset of $H^{-1}(\Omega)^d$ and $H^{-1}(\Omega)$ respectively. Combining all this with (3.18) leads to

$$\lim_{N\to+\infty}\left\|\begin{pmatrix}(\mathcal{S}-\mathcal{S}_N)(T) & 0\\ 0 & \mathcal{L}-\mathcal{L}_N\end{pmatrix}D\mathcal{G}(U)\right\|_{\mathcal{E}}=0. \tag{3.21}$$

2) Due to the definition of $D\mathcal{G}$, we must now investigate the convergence of the two terms

$$((\boldsymbol{u}-\boldsymbol{u}_N^\diamond)\,.\nabla)\boldsymbol{w}_N+(\boldsymbol{w}_N\,.\,\nabla)(\boldsymbol{u}-\boldsymbol{u}_N^\diamond),\quad ((\boldsymbol{u}-\boldsymbol{u}_N^\diamond)\,.\nabla)R_N+(\boldsymbol{w}_N\,.\,\nabla)(T-T_N^\diamond).$$

By applying (3.19) with a $t<\rho$ such that $H^t(\Omega)$ is imbedded in $L^r(\Omega)$, with $r>2$ in dimension $d=2$ and $r=3$ in dimension $d=3$, and combining it with (3.14) and (3.16), we derive

$$\lim_{N\to+\infty}\left\|\begin{pmatrix}\mathcal{S}_N & 0\\ 0 & \mathcal{L}_N\end{pmatrix}(D\mathcal{G}(U)-D\mathcal{G}(U_N^\diamond))\right\|_{\mathcal{E}}=0. \tag{3.22}$$

3) Similarly, using an extension of Lemma 3.3 obviously yields

$$\lim_{N\to+\infty}\left\| \begin{pmatrix} \mathcal{S}_N & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (D\mathcal{G}(U_N^\diamond) - D\mathcal{G}_N(U_N^\diamond)) \right\|_{\mathcal{E}} = 0. \tag{3.23}$$

4) On the other hand, we note that, for any $\boldsymbol{F}$ in $H^{-1}(\Omega)^d$,

$$\begin{aligned}(D\mathcal{S}(T)R_N)\boldsymbol{F} &= \mathcal{S}(T)\big(-\operatorname{div}(\partial_\tau\nu(T)\,R_N\,\nabla\mathcal{S}(T)\boldsymbol{F})\big),\\ (D\mathcal{S}_N(T)R_N)\boldsymbol{F} &= \mathcal{S}_N(T)\big(-\operatorname{div}(\partial_\tau\nu(T)\,R_N\,\nabla\mathcal{S}_N(T)\boldsymbol{F})\big).\end{aligned} \tag{3.24}$$

By subtracting the second line from the first one, we derive

$$\begin{aligned}(D(\mathcal{S}-\mathcal{S}_N)(T)\,R_N)\boldsymbol{F} = {}& (\mathcal{S}-\mathcal{S}_N)(T)\big(-\operatorname{div}(\partial_\tau\nu(T)\,R_N\,\nabla\mathcal{S}(T)\boldsymbol{F})\big)\\ &+\mathcal{S}_N(T)\big(-\operatorname{div}(\partial_\tau\nu(T)\,R_N\,\nabla(\mathcal{S}-\mathcal{S}_N)\,(T)\boldsymbol{F})\big).\end{aligned}$$

Denoting by $\boldsymbol{F}$ the first component of $\mathcal{G}(U)$, we see that $\mathcal{S}(T)\boldsymbol{F}$ is equal to $-\boldsymbol{u}$, see (3.8). First, using the compactness of the imbedding of $H^1(\Omega)$ into $L^r(\Omega)$ for any $r<\infty$ in dimension $d=2$ and $r<6$ in dimension $d=3$, we deduce from the regularity assumption on $\boldsymbol{u}$ that, when W_N runs through the unit sphere of $\mathcal{X}_N$, the quantity $-\operatorname{div}(\partial_\tau\nu(T)\,R_N\,\nabla\mathcal{S}(T)\boldsymbol{F})$ belongs to a compact subset of $H^{-1}(\Omega)^d$. Thus, the convergence of the first term to zero follows from (3.18).

To handle the second term, we observe from (3.14) that it suffices to prove the convergence of the quantity $\|\,\nabla(\mathcal{S}-\mathcal{S}_N)\,(T)\boldsymbol{F}\,\|_{L^{q^*}(\Omega)^{d\times d}}$, with $\frac{1}{q}+\frac{1}{q^*}=\frac{1}{2}$ for the q introduced in the beginning of the proof. Since $\mathcal{S}\,(T)\boldsymbol{F}$ coincides with $-\boldsymbol{u}$, by using the injection of $H^1(\Omega)$ into $W^{1,q^*}(\Omega)$, we obtain

$$\|\,\nabla(\mathcal{S}-\mathcal{S}_N)\,(T)\boldsymbol{F}\,\|_{L^{q^*}(\Omega)^{d\times d}} \leqslant c\left(N^{1-s}\,\big(\,\|\,\boldsymbol{u}\,\|_{H^s(\Omega)^d}+\|\,p\,\|_{H^{s-1}(\Omega)}\,\big)\right).$$

Hence, we derive

$$\lim_{N\to+\infty}\left\| \begin{pmatrix} D(\mathcal{S}-\mathcal{S}_N)(T) & 0 \\ 0 & 0 \end{pmatrix} \mathcal{G}(U) \right\|_{\mathcal{E}} = 0. \tag{3.25}$$

5) The convergence of the fifth term is deduced from (3.19), (3.12) and the stability of $D\mathcal{S}_N(\tau)$ and the convergence of the last term is obtained with the same arguments as for Lemma 3.3.

This concludes the proof. □

Lemma 3.5. *If the function ν belongs to $W^{2,\infty}(\mathbb{R})$, with Lipschitz-continuous derivatives, there exist a neighbourhood of $U_N^\diamond$ in $\mathcal{X}_N$ and a constant $c>0$ such that the operator $D\mathcal{F}_N$ satisfies the following Lipschitz property, for all U_N^* in this neighbourhood,*

$$\|\,D\mathcal{F}_N(U_N^\diamond)-D\mathcal{F}_N(U_N^*)\,\|_{\mathcal{E}} \leqslant c\,\mu(N)\,\|\,U_N^\diamond-U_N^*\,\|_{\mathcal{X}(\Omega)}, \tag{3.26}$$

with $\mu(N)$ equal to $|\log N|^{\frac{1}{2}}$ in dimension $d=2$ and to N in dimension $d=3$.

Proof. Let us introduce the matrix operators

$$\mathcal{M}(\xi)=\begin{pmatrix}\mathcal{S}(\xi) & 0\\ 0 & \mathcal{L}\end{pmatrix},\qquad \mathcal{M}_N(\xi)=\begin{pmatrix}\mathcal{S}_N(\xi) & 0\\ 0 & \mathcal{L}_N\end{pmatrix}.$$

Setting $U_N^*=(\boldsymbol{u}_N^*,T_N^*)$, we have

$$\begin{aligned}D\mathcal{F}_N(U_N^\diamond)-D\mathcal{F}_N(U_N^*) = {}&(\mathcal{M}_N(T_N^\diamond)-\mathcal{M}_N(T_N^*))D\mathcal{G}_N(U_N^\diamond)+(D\mathcal{M}_N(T_N^\diamond)-D\mathcal{M}_N(T_N^*))\mathcal{G}_N(U_N^\diamond)\\ &+\mathcal{M}_N(T_N^*)(D\mathcal{G}_N(U_N^\diamond)-D\mathcal{G}_N(U_N^*))+D\mathcal{M}_N(T_N^*)(\mathcal{G}_N(U_N^\diamond)-\mathcal{G}_N(U_N^*)).\end{aligned}$$

We have to evaluate these quantities, for any $W_N = (\boldsymbol{w}_N, R_N)$ in the unit sphere of $\mathcal{X}_N$ and $\boldsymbol{\psi}_N$ in the unit sphere of $\mathbb{X}_N$. Since evaluating the last two terms follows from Lemma 3.3 and an extension of it, we only consider the first two terms. All constants c in what follows only depend on the norms $\| U_N^\diamond \|_{\mathcal{X}(\Omega)}, \| U_N^* \|_{\mathcal{X}(\Omega)}$ and $\| \nu \|_{W^{2,\infty}(\mathbb{R})}$.

1) We have

$$(\mathcal{M}_N(T_N^\diamond) - \mathcal{M}_N(T_N^*))D\mathcal{G}_N(U_N^\diamond)W_N = \mathcal{M}_N(T_N^\diamond)\begin{pmatrix} A \\ 0 \end{pmatrix},$$

with

$$A = \mathrm{div}\big((\nu(T_N^\diamond) - \nu(T_N^*))\nabla\mathcal{S}_N(T_N^*)((\boldsymbol{u}_N^\diamond \,.\, \nabla)\boldsymbol{w}_N + (\boldsymbol{w}_N \,.\, \nabla)\boldsymbol{u}_N^\diamond, \boldsymbol{\psi}_N)_N\big).$$

There exists a constant c only depending on the Lipschitz property of ν such that,

$$\| (\mathcal{M}_N(T_N^\diamond) - \mathcal{M}_N(T_N))D\mathcal{G}_N(U_N^\diamond)W_N \|_{\mathcal{X}(\Omega)} \leqslant c \| T_N^\diamond - T_N^* \|_{L^2(\Omega)} \| \boldsymbol{w}_N \|_{L^\infty(\Omega)} .$$

We conclude by applying the inverse inequality [13], valid for any polynomial φ_N in $\mathbb{P}_N(\Omega)$,

$$\|\varphi_N\|_{L^\infty(\Omega)} \leqslant cN^{\frac{2d}{\delta}} \|\varphi_N\|_{L^\delta(\Omega)},$$

and noting that
- in dimension $d = 3$, $H^1(\Omega)$ is embedded in $L^6(\Omega)$.
- in dimension $d = 2$, $H^1(\Omega)$ is embedded in any $L^\delta(\Omega)$ with the norm of the imbedding smaller than $c\sqrt{\delta}$ (see [14]), (we thus take δ equal to $\log N$).

2) On the other hand, combining the second part of (3.24) with a further triangle inequality

$$\| ((D\mathcal{M}_N(T_N^\diamond) - D\mathcal{M}_N(T_N^*)R_N)\mathcal{G}_N(U_N^\diamond) \|_{\mathcal{X}(\Omega)} \leqslant c \| T_N^\diamond - T_N^* \|_{L^2(\Omega)} \| R_N \|_{L^\infty(\Omega)} .$$

The same arguments as in part 1) yields the desired result. □

Lemma 3.6. *Assume that ν is of class $\mathscr{C}^2$ on $\mathbb{R}$ and that the solution $(\boldsymbol{u}, p, T)$ of problem (2.3)–(2.4) belongs to $H^s(\Omega)^d \times H^{s-1}(\Omega) \times H^s(\Omega)$ for a real number s, $s \geqslant 1$, and the data $\boldsymbol{f}$ belongs to $H^\sigma(\Omega)^d$ for a real number σ, $\sigma > \frac{d}{2}$. Then, the following estimate is satisfied*

$$\| \mathcal{F}_N(U_N^\diamond) \|_{\mathcal{X}(\Omega)} \leqslant \; c\, N^{1-s} \left(\| \boldsymbol{u} \|_{H^s(\Omega)^d} + \| p \|_{H^{s-1}(\Omega)} + \| T \|_{H^s(\Omega)}\right) + c' N^{-\sigma} \; \| \boldsymbol{f} \|_{H^\sigma(\Omega)^d} .$$

Proof. Since $\mathcal{F}(U)$ is zero, we have

$$\| \mathcal{F}_N(U_N^\diamond) \|_{\mathcal{X}(\Omega)} \leqslant \| U - U_N^\diamond \|_{\mathcal{X}(\Omega)} + \left\| \begin{pmatrix} (\mathcal{S} - \mathcal{S}_N)(T) & 0 \\ 0 & \mathcal{L} - \mathcal{L}_N \end{pmatrix} \mathcal{G}(U) \right\|_{\mathcal{X}(\Omega)}$$
$$+ \left\| \begin{pmatrix} \mathcal{S}_N(T) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (\mathcal{G}(U) - \mathcal{G}(U_N^\diamond)) \right\|_{\mathcal{X}(\Omega)} + \left\| \begin{pmatrix} \mathcal{S}_N(T) & 0 \\ 0 & \mathcal{L}_N \end{pmatrix} (\mathcal{G}(U_N^\diamond) - \mathcal{G}_N(U_N^\diamond)) \right\|_{\mathcal{X}(\Omega)}.$$

The first term is bounded in (3.19). Evaluating the second term follows from (3.15) and (3.17) by noting that, if $\mathbf{F}$ denotes the first component of $\mathcal{G}(U)$, $\mathcal{S}\mathbf{F}$ is equal to $-\boldsymbol{u}$ and $\mathbf{F}$ is equal to $\boldsymbol{f} + \mathrm{div}(\nu(T)\nabla\boldsymbol{u}) - \mathbf{grad}\, p$. To bound the third term, we apply (3.14), triangle inequalities and estimate (3.19). Finally, proving the estimate for the fourth term is obtained by using the standard arguments for the error issued from numerical integration combined with the same arguments as in the proof of Lemma 3.3.

Thanks to Lemmas 3.4 to 3.6, we are now in a position to prove the main result of this section. □

Theorem 3.7. *Let $(\boldsymbol{u}, p, T)$ be a solution of problem (2.3)–(2.4) which satisfies Assumption 3.2 for $\rho > d-1$ and belongs to $H^s(\Omega)^d \times H^{s-1}(\Omega) \times H^s(\Omega)$, $s > 1$. We further assume that the function ν is of class $\mathscr{C}^2$ on $\mathbb{R}$ with Lipschitz-continuous derivatives and that the data $\boldsymbol{f}$ belong to $H^\sigma(\Omega)^d$ for a real number σ, $\sigma > \frac{d}{2}$. Then, there exist a positive number $N_\diamond$ and a constant $c_\diamond$ such that, for all $N \geqslant N_\diamond$, problem (3.3)–(3.4) has a unique solution $(\boldsymbol{u}_N, p_N, T_N)$ such that $(\boldsymbol{u}_N, T_N)$ belongs to the ball of $\mathcal{X}(\Omega)$ with center $(\boldsymbol{u}, T)$ and radius $c_\diamond \mu_N^{-1}$ for the constant μ_N introduced in Lemma 3.5. Moreover, this solution satisfies*

$$\begin{aligned}
&\| \boldsymbol{u} - \boldsymbol{u}_N \|_{H^1(\Omega)^d} + \| T - T_N \|_{H^1(\Omega)} + N^{-(d-1)/2} \| p - p_N \|_{L^2(\Omega)} \\
&\leqslant c\, N^{1-s} (\| \boldsymbol{u} \|_{H^s(\Omega)^d} + \| p \|_{H^{s-1}(\Omega)} + \| T \|_{H^s(\Omega)}) + c' N^{-\sigma} \| \boldsymbol{f} \|_{H^\sigma(\Omega)^d} \, . \qquad (3.27)
\end{aligned}$$

Proof. Combining Lemmas 3.4 to 3.6 with Brezzi–Rappaz–Raviart's theorem [4] (see also [9], Chap. IV, Thm 3.1) yields for N sufficiently large, the existence of a solution (u_N, T_N), its local uniqueness and the desired estimates for $\| \boldsymbol{u} - \boldsymbol{u}_N \|_{H^1(\Omega)^d}$ and $\| T - T_N \|_{H^1(\Omega)}$.

Moreover, thanks to the discrete inf-sup condition (3.5), there exists a unique p_N in $\mathbb{M}_N$ such that:

$$\int_\Omega (\mathrm{div} v_N)(\boldsymbol{x}) p_N(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \int_\Omega \nu(T_N)(\mathbf{grad}\boldsymbol{u}_N)(\boldsymbol{x}) : (\mathbf{grad}\boldsymbol{v}_N)(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \int_\Omega \big((\boldsymbol{u}_N \,.\, \nabla)\boldsymbol{u}_N\big)(\boldsymbol{x}) \,.\, \boldsymbol{v}_N(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} - (\boldsymbol{f}, \boldsymbol{v}_N),$$

whence, for any q_N in $\mathbb{M}_N$,

$$\begin{aligned}
\int_\Omega (\mathrm{div}\, \boldsymbol{v}_N)(\boldsymbol{x})\, (p_N - q_N)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} &= \int_\Omega \big(\nu(T_N)(\mathbf{grad}\, \boldsymbol{u}_N)(\boldsymbol{x}) - \nu(T)(\mathbf{grad}\, \boldsymbol{u})(\boldsymbol{x})\big) : (\mathbf{grad}\, \boldsymbol{v}_N)(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&+ \int_\Omega \big((\boldsymbol{u}_N \,.\, \nabla)\boldsymbol{u}_N - (\boldsymbol{u} \,.\, \nabla)\boldsymbol{u}\big)(\boldsymbol{x}) \,.\, \boldsymbol{v}_N(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} + \int_\Omega (\mathrm{div}\boldsymbol{v}_N)(\boldsymbol{x})(p - q_N)(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}.
\end{aligned}$$

So by using successively (3.5), triangle inequalities and the error estimates on $\boldsymbol{u}$ and $\boldsymbol{T}$, we derive the estimate for $\| p - p_N \|_{L^2(\Omega)}$.

Estimate (3.27) is fully optimal. Moreover the regularity assumptions on the solution $(\boldsymbol{u}, p, T)$ are not at all restrictive in dimension $d = 2$. □

4. Numerical experiments

The numerical experiments have been performed in the two-dimensional case, on the square $\Omega =]-1, 1[^2$. Problem (3.3)–(3.4) is solved via the following iterative algorithm. We set:

$$\mathcal{H}(U_N) = U_N + \mathcal{M}_N(T_N)\mathcal{G}_N(U_N).$$

Applying Newton's method consists in solving iteratively the equation

$$U_N^m = U_N^{m-1} - D\mathcal{H}(U_N^{m-1})^{-1}\mathcal{H}(U_N^{m-1}),$$

which can equivalently be written as: $U_N^m = U_N^{m-1} - W_N^{m-1}$, where $W_N^{m-1} = (\boldsymbol{z}_N^{m-1}, \chi_N^{m-1})$ is a solution of the problem:

$$D\mathcal{H}(U_N^{m-1}) W_N^{m-1} = \mathcal{H}(U_N^{m-1}).$$

But Newton's algorithm can also be applied to one unknown, for simplicity. In fact, we iteratively solve the following problems, for $m \geqslant 1$. They read

Find $(\boldsymbol{z}_N^{m-1}, \varphi_N^{m-1}, \chi_N^{m-1})$ in $\mathbb{X}_N \times \mathbb{M}_N \times \mathbb{Y}_N$ such that

$$\begin{aligned}
\forall \boldsymbol{v}_N \in \mathbb{X}_N, \quad & (\nu(T_N^{m-1})\nabla \boldsymbol{z}_N^{m-1}, \nabla \boldsymbol{v}_N)_N + ((\boldsymbol{u}_N^{m-1}\,.\,\nabla)\boldsymbol{z}_N^{m-1} + (\boldsymbol{z}_N^{m-1}\,.\,\nabla)\boldsymbol{u}_N^{m-1}, \boldsymbol{v}_N)_N - (\operatorname{div} v_N, \varphi_N^{m-1})_N \\
& = (\nu(T_N^{m-1})\nabla \boldsymbol{u}_N^{m-1}, \nabla \boldsymbol{v}_N)_N + ((\boldsymbol{u}_N^{m-1}\,.\,\nabla)\boldsymbol{u}_N^{m-1}, \boldsymbol{v}_N)_N \\
& \quad - (\operatorname{div} \boldsymbol{v}_N, p_N^{m-1})_N - (\boldsymbol{f}, \boldsymbol{v}_N)_N, \\
\forall q_N \in \mathbb{M}_N, \quad & \quad - (\operatorname{div} \boldsymbol{z}_N^{m-1}, q_N)_N = 0, \\
\forall S_N \in \mathbb{Y}_N^0, \quad & \alpha(\nabla \chi_N^{m-1}, \nabla S_N)_N + ((\boldsymbol{z}_N^{m-1}\,.\,\nabla)T_N^{m-1} r + (\boldsymbol{u}_N^{m-1}\,.\,\nabla)\chi_N^{m-1}, S_N)_N \\
& = \alpha(\nabla T_N^{m-1}, \nabla S_N)_N + ((\boldsymbol{u}_N^{m-1}\,.\,\nabla)T_N^{m-1}, S_N)_N - (g, S_N)_N.
\end{aligned} \tag{4.1}$$

The convergence of this method can be easily derived from [4] (see [9], Chap. IV, Thm. 6.5) owing to Lemmas 3.4 to 3.6. We only skip its proof for brevity. It can also be noted that the matrix must be reassembled at each iteration (due to the dependency of the viscosity with respect to T_N^{m-1}), which make its use expensive. However, very few iterations are needed before the solution stops moving. On the other hand and as well-known, the key point for the Newton's method is the initial guess: to do this, by taking for a while the viscosity constant, it is very easy to uncouple the unknowns, more precisely to solve first the Navier–Stokes (or even Stokes) equations, next the heat equation; but we take it equal to zero when possible.

The numerical experiments that we present in the sequel are performed by a MATLAB code and the global system is solved by a GMRES algorithm.

To start, we take in all the calculations $\boldsymbol{u}_N^0, T_N^0, p_N^0$ zeros on internal nodes, knowing that whatever the choice of these solutions the algorithm converges. In all tests the number of iterations needed for better convergence of the Newton algorithm, varies between 5 and 10. We work with ν as a constant, as a function dependent of the space variable and finally as a function which depends on the temperature T.

- Case where $\nu(T)$ is a constant equal to 10^{-2}.
  In the first experiment, the exact solution is given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} y(1-x^2)^{\frac{11}{2}}(1-y^2)^{\frac{9}{2}} \\ -x(1-x^2)^{\frac{9}{2}}(1-y^2)^{\frac{11}{2}} \end{pmatrix}, \quad p(x,y) = x^2 + y - \frac{1}{3}, \quad T(x,y) = (x^2+y^2)^{\frac{7}{2}}. \tag{4.2}$$

  The errors of solution computed with $N = 24$ are presented in Figure 1, for the two components of the velocity on the top, the pressure and the temperature on the bottom.
- Case where $\nu(T)$ is a function equal to $x + y + 1$.
  (i) In the first case, we work with regular functions where the boundary conditions are non-homogeneous, and we attain an error of 10^{-13} from $N = 6$.
  (ii) In the second experiment, we work with the solution given in (4.2). In Figure 2, we present the convergence of the relative errors in $\boldsymbol{u}$, p and T in the $L^2(\Omega)^2, H^1(\Omega)^2, L^2(\Omega)$ or $H^1(\Omega)$ norm in logarithmic scales, as a function of N, for N varying from 8 to 30.
- Case where $\nu(T)$ depend on T.
  (i) We first consider a smooth solution in the case where $\nu(T)$ equal to $3\sqrt{T^2+1}+2$,

$$\boldsymbol{u}(x,y) = \begin{pmatrix} y^2 \\ x^2 \end{pmatrix}, \quad p(x,y) = \sin(x+y), \quad T(x,y) = \cos(xy). \tag{4.3}$$

  In Figure 3 we present the error of solution obtained from (4.3) computed with $N = 20$.
  (ii) We next consider the solution in the case where $\nu(T)$ is given by $T + 1$,

$$\boldsymbol{u}(x,y) = \begin{pmatrix} x\sin(\pi xy) \\ -y\sin(\pi xy) \end{pmatrix}, \quad p(x,y) = x + y, \quad T(x,y) = xy. \tag{4.4}$$

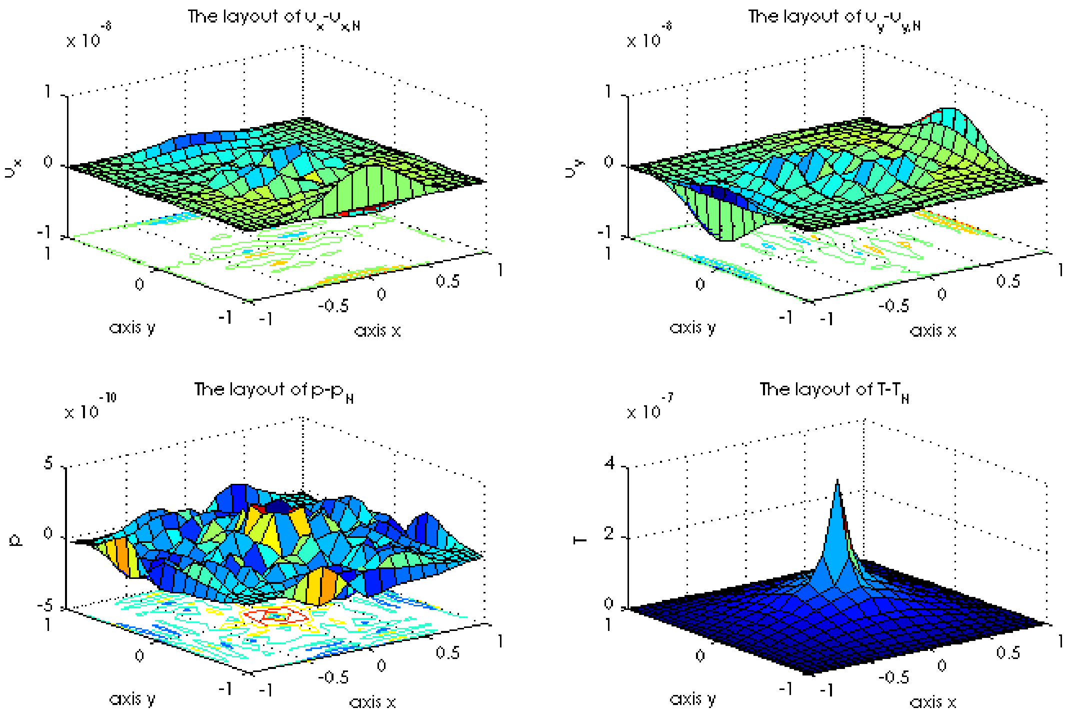

FIGURE 1. The errors of solution obtained from (4.2).

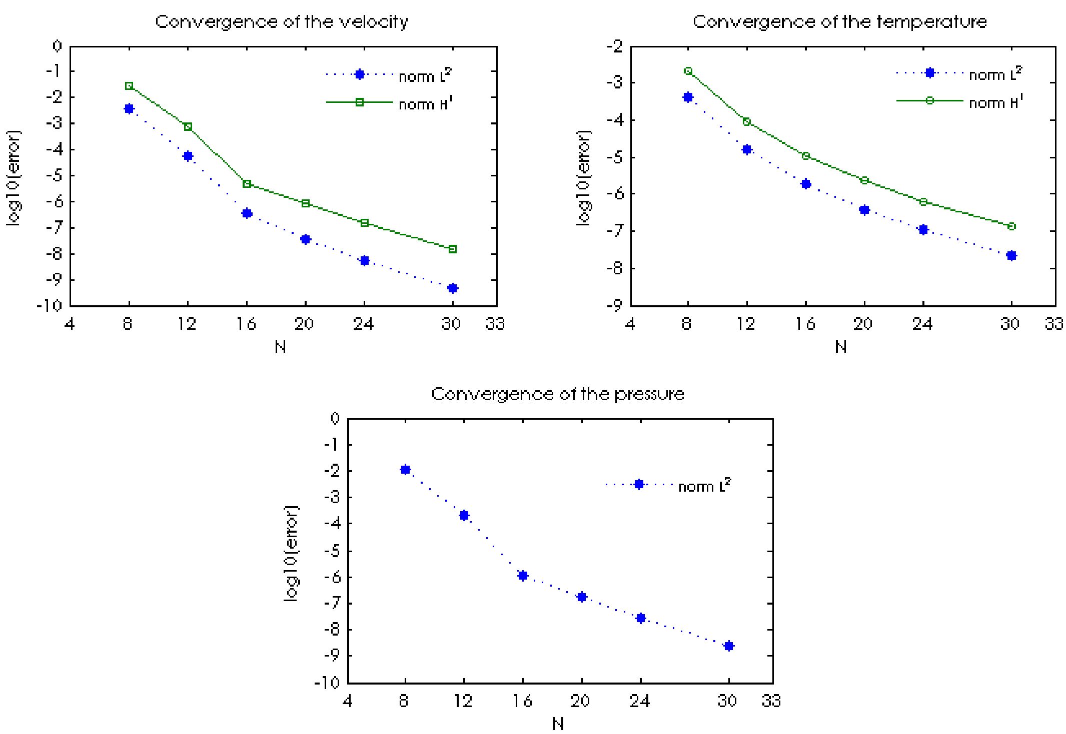

FIGURE 2. The estimations of errors of the solution obtained from (4.2).

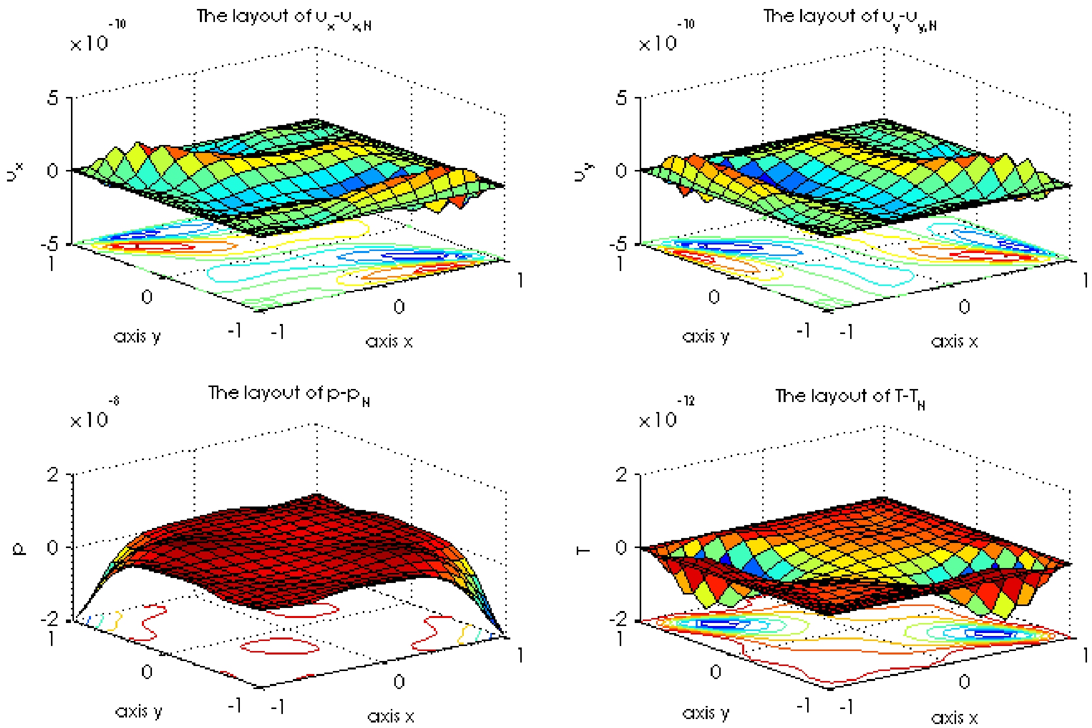

FIGURE 3. The errors of the solution corresponding to (4.3).

In Figure 4 we present the quantities:

$$\log_{10}\|\boldsymbol{u}-\boldsymbol{u}_N\|_{L^2(\Omega)^2},\ \log_{10}\|\boldsymbol{u}-\boldsymbol{u}_N\|_{H^1(\Omega)^2},\ \log_{10}\|p-p_N\|_{L^2(\Omega)},$$
$$\log_{10}\|T-T_N\|_{L^2(\Omega)} \quad \text{and} \quad \log_{10}\|T-T_N\|_{H^1(\Omega)}$$

as functions of N, for N varying from 8 to 24. We observe the good convergence for $N = 24$.

(iii) We now present numerical experiments in the case when $\nu(T)$ is taken to be equal to T^2+T, the datum $\boldsymbol{f}$ is equal to zero, and the datum g is 1.
The boundary velocity $\boldsymbol{h} = (h_x, h_y)$ is given by

$$h_x(x,y) = \begin{cases} 0 & \text{if} \quad y = \pm 1, \\ \sin(\pi x), & \text{otherwise,} \end{cases} \qquad h_y(x,y) = \begin{cases} 0 & \text{if} \quad x = \pm 1, \\ \sin(\pi y), & \text{otherwise.} \end{cases} \tag{4.5}$$

And the boundary condition is replaced by

$$\boldsymbol{u} = \boldsymbol{h} \quad \text{on } \partial\Omega, \quad T_0 = 0.$$

Note that the data satisfy the usual compatibility condition $\int_{\partial\Omega} \boldsymbol{h}(\tau)\cdot\boldsymbol{n}(\tau)\mathrm{d}\tau = 0$. We present in Figure 5 the isovalues of the two components of the velocity, the pressure and the temperature obtained from (4.5) computed with $N = 26$.

(iv) For the last numerical experiment we work with the solution given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} 2\sin(\pi x)^2\sin(\pi y)\cos(\pi y) \\ -2\sin(\pi y)^2\sin(\pi x)\cos(\pi x) \end{pmatrix}, \quad p(x,y) = x^2 - y^2, \quad T(x,y) = xy. \tag{4.6}$$

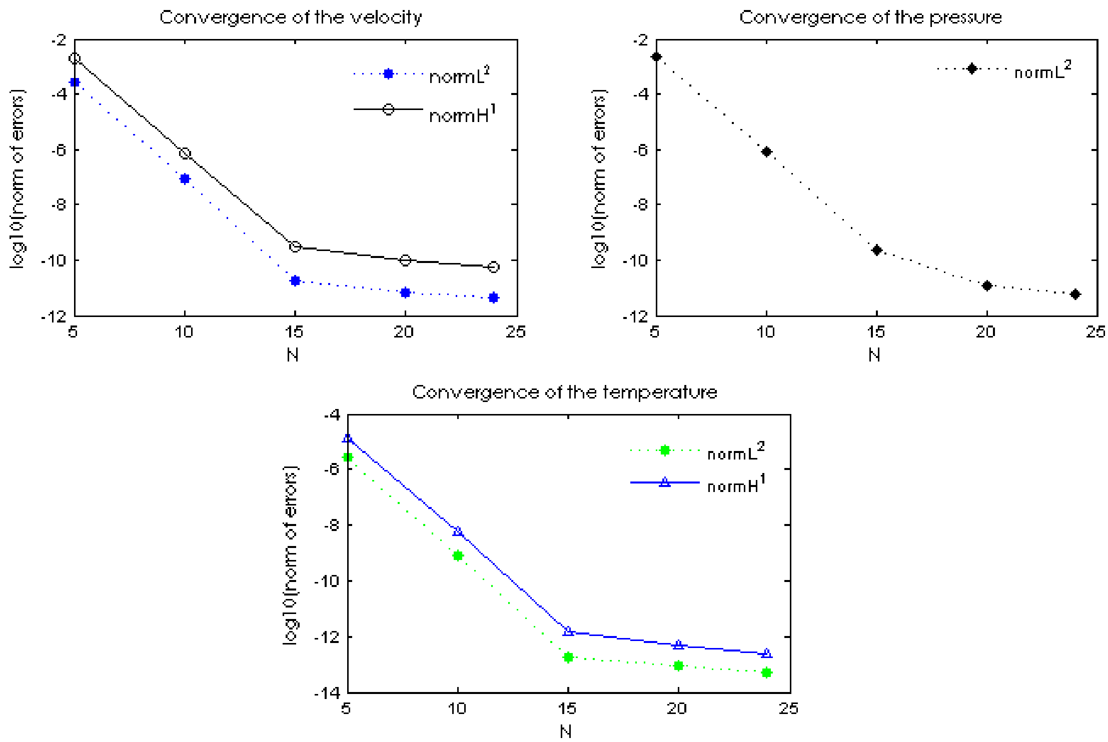

FIGURE 4. The estimations of error of the solution obtained from (4.4).

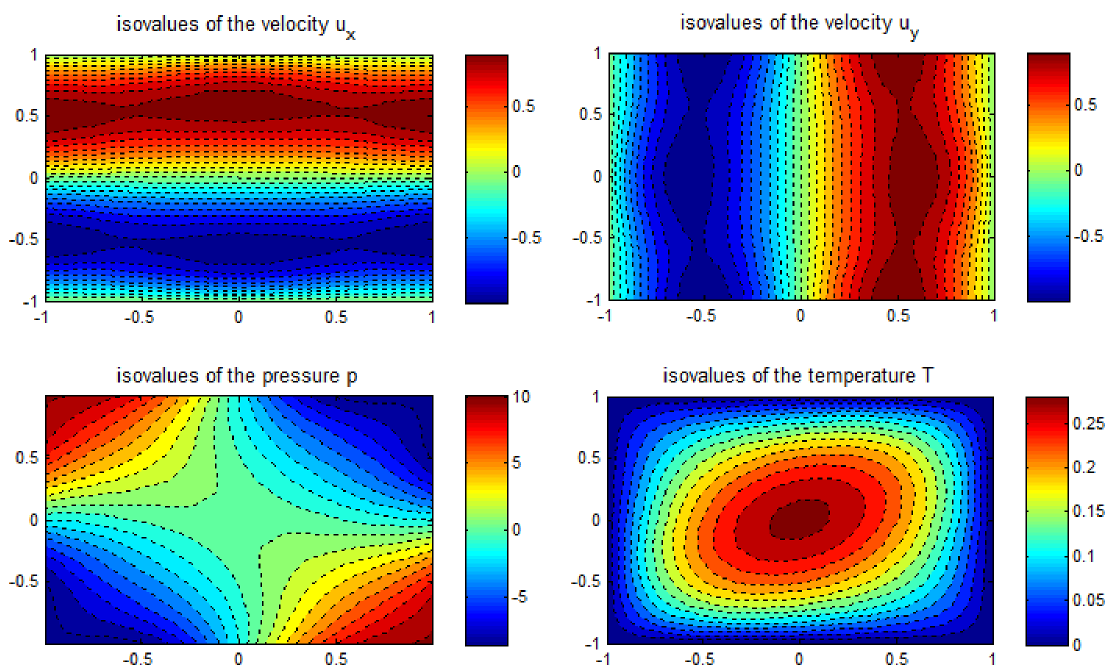

FIGURE 5. Isovalues of the discrete solution corresponding to (4.5).

TABLE 1. Convergence of the solution (4.6) as a function of $\nu(T)$.

$\nu(T)$	$T+1$	$\frac{1}{25}(T+1)$	$\frac{1}{50}(T+1)$	$\frac{1}{75}(T+1)$	$\frac{1}{100}(T+1)$
$\|\|\boldsymbol{u}-\boldsymbol{u}_N\|\|_{L^2(\Omega)^2}$	7.64e-9	1.48e-8	4.16e-8	5.50e-8	9.62e-8
$\|\|\boldsymbol{u}-\boldsymbol{u}_N\|\|_{H^1(\Omega)^2}$	8.76e-8	2.76e-7	4.23e-7	7.67e-7	9.33e-7
$\|\|p-p_N\|\|_{L^2(\Omega)}$	1.35e-8	4.16e-8	6.16e-8	9.02e-8	4.73e-7
$\|\|T-T_N\|\|_{L^2(\Omega)}$	4.23e-11	3.03e-10	4.06e-10	8.39e-10	1.58e-9
$\|\|T-T_N\|\|_{H^1(\Omega)}$	5.68e-10	9.18e-10	2.28e-9	5.21e-9	6.93e-9

In Table 1 we present different values of $\nu(T)$. We observe the stability of the algorithm and the variation of the errors of the velocity, pressure and temperature computed with $N = 16$.

In Table 1 we note that if $\nu(T) = T + 1$ is replaced by $\nu(T) = \dfrac{T+1}{M}$ then as M increases the errors grows. So in order to restore the accuracy one needs to increase the discretization parameter N: for example we obtain the same precision of the solution in the following two cases $N = 16$, $\nu(T) = T + 1$ and $N = 18$, $\nu(T) = \frac{1}{100}(T+1)$.

Developing the software for the three-dimensional case is in progress. It just started so we have not face difficulties yet, as it require a lot of time due to the large size of the matrices to invert.

5. Conclusions

We believe that the model that we propose has many applications: Think of a fluid with exothermic chemical reactions or the flow of water in a partially heated channel. In simple geometries, the spectral method seems appropriate to solve it due to its high accuracy and also to its simplicity of implementation. When coupling it with the Newton's algorithm, we obtain an efficient way of solving this model in a large number of cases.

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