# NUMERICAL ANALYSIS FOR A THREE INTERACTING SPECIES MODEL WITH NONLOCAL AND CROSS DIFFUSION *,** 

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#### Abstract

In this paper, we consider a reaction-diffusion system describing three interacting species in the food chain structure with nonlocal and cross diffusion. We propose a semi-implicit finite volume scheme for this system, we establish existence and uniqueness of the discrete solution, and it is also showed that the discrete solution generated by the given scheme converges to the corresponding weak solution for the model studied. The convergence proof is based on the use of the discrete Sobolev embedding inequalities with general boundary conditions and a space-time $L^{1}$ compactness argument that mimics the compactness lemma due to Kruzhkov. Finally we give some numerical examples.


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## 1. Introduction

It is clear that species do not exist alone in nature; therefore, to study the persistence and extinction of each population in systems of two or more interacting species have more biological significance. The classical ecological models of interacting populations typically have focussed on two species. Two species systems have long played dominating roles in ecology, systems such as predator-prey, plant-herbivore or plant-pest, etc. However, it has been recognized that this kind of ecological systems by two interacting species can account for only a small number of the phenomena that are commonly exhibited in nature. This is particularly true in community studies where the essence of the behaviour of a complex system may only be understood when the interactions among

[^0]a large number of species are incorporated. Of course, the increasing number of differential equations and the increasing dimensionality raise considerable additional problems both for the experimenter and the theoretician. Nonetheless, such models need to be analyzed because certain three-species communities have become the focus of considerable attention.

Mathematical developments also suggest that models which involve only two species as the basic buildings blocks may miss important ecological behavior. Results that are much more complicated than those seen in twospecies models appeared in early theoretical studies of three-species (e.g. [20]), models based on local stability analyses. Rosenzweig began the exploitation in three trophic levels by adding a third species and the trophic level [20]. Hastings and Powell [13] studied the three-species food chain, and they found that there is a "tea-cup" attractor in the system. Yodzis and Innes [25] showed that changes in dynamics are associated with increased resource carrying capacity and gave estimates for resource-consumer body mass ratios that permit robust limit cycles. Klebanoff and Hastings [14] studied the dynamics of the model given by Hastings and Powell in [13] using the co-dimension two bifurcation theory to show the existence of chaotic dynamics. McCann and Yodzis [17] performed numerical simulation for the same system and showed that chaos ocurred in some region of the parameter space.

By the way, first-order differential equations like the system studied by Hastings and Powell in [13] reflects only population changes because of the predation in a situation where predator and prey are not spatially dependent. It does not take into account either the fact that predators and preys naturally develop strategies to survive, nor the fact that population is usually not homogeneously distributed. The two aspects mentioned before involve process of diffusion which can be a little complicated since different concentration levels of preys and predators cause different population movements. These movements can be determined by the concentration of the same species or the other species (diffusion and cross-diffusion, respectively). With this in mind, Shigesada, Kawasaki and Teramoto in [21] proposed a strongly coupled reaction-diffusion model (SKT model) with reaction terms of Lotka-Volterra type to describe spatial segregation of interacting population species in one-dimensional space. Since that time the two species SKT competing system continue being of great interest in mathematical analysis as well as in real-life modelling. Furthermore, three or multi-species system and the SKT model in any space dimension has recently focused a lot of attention due to their more complicated patterns, besides the SKT models with other kind of reaction terms are also proposed and investigated. Considering the above two aspects, Wen and Zhong investigated a strongly coupled reaction-diffusion system of the HP model given by Hastings and Powell in [13], in which the population is not homogeneously distributed caused by the consideration of diffusions. The authors established the existence of non-constant positive steady states of their system through using the Leray-Schauder degree theory [26].

The model that we considered in this paper is based on the HP food chain model given in [13], besides we consider a diffusion terms as Wen and Zhong in their model studied in [26]. We have a reaction-diffusion system in which the population is not homogeneously distributed due to the consideration of nonlocal and cross diffusion terms. Cross-diffusion expresses the population fluxes of one species due to the presence of the other species. The dynamics of interacting population with cross-diffusion are investigated by several researchers. Beginning with Turing [24] in 1952, diffusion and cross-diffusion have been observed as causes of the spontaneous emergence of ordered structures, namely stationary patterns. For the ecological systems with cross-diffusion and LotkaVolterra type reaction terms in [16] is studied the effect of diffusion, self-diffusion and cross-diffusion of the two species SKT competition model. Moreover, in [19] the authors investigated a three species predator-prey model with cross-diffusion and found that the stationary patterns do not emerge from the diffusion of individual species but only appear with the introduction of cross-diffusion. The concept of this phenomena was also studied by Galiano et al. $[11,12]$, Bendahmane et al. $[1,7]$, and many other authors. In this kind of models were noticed that when the cross-diffusion is nonlinear, difficulties increased in the mathematical analysis. Furthermore, there is not general theory available that covers all possible cross-diffusion models. Tian, Lin and Pedersen in [23] studied the reaction-diffusion systems with nonlinear cross-diffusion, the aim of the authors is to study what role the cross-diffusion plays in the process of pattern formation.

Let $\Omega \subset \mathbb{R}^{\ell}(\ell=2,3)$ be a bounded open domain with smooth boundary $\partial \Omega$, over a time span $(0, T), T>0$ and $\Omega_{T}:=\Omega \times(0, T)$, we have the following model:

$$
\begin{cases}\partial_{t} u_{1}-\operatorname{div}\left(d_{1}\left(\int_{\Omega} u_{1} \mathrm{~d} x\right) \nabla u_{1}\right)-\operatorname{div}\left[\left(\alpha_{1} u_{1}+u_{2}\right) \nabla u_{1}+u_{1} \nabla u_{2}\right]=F\left(u_{1}, u_{2}, u_{3}\right), & \text { in } \quad \Omega_{T},  \tag{1.1}\\ \partial_{t} u_{2}-\operatorname{div}\left(d_{2}\left(\int_{\Omega} u_{2} \mathrm{~d} x\right) \nabla u_{2}\right) \\ -\operatorname{div}\left[u_{2} \nabla u_{1}+\left(u_{1}+\alpha_{2} u_{2}+u_{3}\right) \nabla u_{2}+u_{2} \nabla u_{3}\right]=G\left(u_{1}, u_{2}, u_{3}\right), & \text { in } \quad \Omega_{T}, \\ \partial_{t} u_{3}-\operatorname{div}\left(d_{3}\left(\int_{\Omega} u_{3} \mathrm{~d} x\right) \nabla u_{3}\right)-\operatorname{div}\left[u_{3} \nabla u_{2}+\left(u_{2}+\alpha_{3} u_{3}\right) \nabla u_{3}\right]=H\left(u_{1}, u_{2}, u_{3}\right), & \text { in } \quad \Omega_{T}, \\ u_{i}(x, 0)=u_{i, 0}(x)>0, \quad \text { in } \Omega, \text { for } \quad i=1,2,3,\end{cases}
$$

we complete the system (1.1) with Neumann boundary conditions:

$$
\begin{equation*}
\frac{\partial u_{1}}{\partial \eta}=\frac{\partial u_{2}}{\partial \eta}=\frac{\partial u_{3}}{\partial \eta}=0, \quad \text { on } \quad \Sigma_{T}:=\partial \Omega \times(0, T) \tag{1.2}
\end{equation*}
$$

where $\eta$ is the unit outward normal to $\partial \Omega$. Herein $\alpha_{i}>0$ is known as self-diffusion rate for $i=1,2,3$. The cross-diffusion rate is assumed to be equal to 1 . The nonlinearities $F, G$, and $H$ take the form:

$$
\begin{align*}
& F\left(u_{1}, u_{2}, u_{3}\right)=\left(1-\frac{u_{1}}{k}\right) u_{1}-\frac{L_{2} M_{2} u_{1} u_{2}}{R_{0}+u_{1}} \\
& G\left(u_{1}, u_{2}, u_{3}\right)=-L_{2} u_{2}+\frac{L_{2} M_{2} u_{1} u_{2}}{R_{0}+u_{1}}-\frac{L_{3} M_{32} u_{2} u_{3}}{C_{0}+u_{2}} \\
& H\left(u_{1}, u_{2}, u_{3}\right)=-L_{3} u_{3}+\frac{L_{3} M_{32} u_{2} u_{3}}{C_{0}+u_{2}} \tag{1.3}
\end{align*}
$$

In our model, $u_{1}(x, t)$ represents population density of the species at the lowest level of the food chain (preys), $u_{2}(x, t)$ represents population density of the species that preys upon $u_{1}$ (predator), and $u_{3}(x, t)$ represents population density of the species that preys upon $u_{2}$ (superpredator). The constant $k$ is the carrying capacity of $u_{1}$ species. $R_{0}$ and $C_{0}$ are the half saturation densities of $u_{1}$ and $u_{2}$, respectively. Moreover $L_{2}$ and $L_{3}$ are the mass-specific metabolic rates of $u_{2}$ and $u_{3}$, respectively. $M_{2}$ is a measure of ingestion rate per unit metabolic rate of $u_{2}$, and $M_{32}$ denotes the ingestion rate for $u_{3}$ on prey term $u_{2}$. Notice that all the parameters are positive.

In this work, the diffusion rates $d_{i}>0, i=1,2,3$ are supposed to depend on the whole of each population in the domain rather than on the local density, i.e. the diffusion of individuals is guided by the global state of the population in the medium. For instance, if we want to model species having the tendency to leave crowded zones, a natural assumption would be to assume that $d_{i}$ is an increasing function of its argument. Otherwise, if we are dealing with species attracted by the growing population, one will suppose that the nonlocal diffusion $d_{i}$ decreases. We assume that each $d_{i}: \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function satisfying the following: there exist constants $a_{i}, C>0$ such that

$$
\begin{equation*}
a_{i} \leq d_{i} \quad \text { and } \quad\left|d_{i}\left(I_{1}\right)-d_{i}\left(I_{2}\right)\right| \leq C\left|I_{1}-I_{2}\right| \quad \text { for all } \quad I_{1}, I_{2} \in \mathbb{R}, \quad \text { for } \quad i=1,2,3 \tag{1.4}
\end{equation*}
$$

Moreover, for technical reasons to prove the existence of weak solution, we shall assume that the coefficients $\alpha_{i}, i=1,2,3$ satisfy

$$
\begin{equation*}
\alpha_{1}>\frac{1}{2}, \alpha_{2}>1 \text { and } \alpha_{3}>\frac{1}{2} \tag{1.5}
\end{equation*}
$$

Observe that we can rewrite (1.1) as follows

$$
\left\{\begin{array}{l}
\partial_{t} u_{1}-\operatorname{div}\left(d_{1}\left(\int_{\Omega} u_{1} \mathrm{~d} x\right) \nabla u_{1}\right)-\operatorname{div}\left(\mathcal{A}_{11} \nabla u_{1}+\mathcal{A}_{12} \nabla u_{2}\right)=F\left(u_{1}, u_{2}, u_{3}\right),  \tag{1.6}\\
\partial_{t} u_{2}-\operatorname{div}\left(d_{2}\left(\int_{\Omega} u_{2} \mathrm{~d} x\right) \nabla u_{2}\right)-\operatorname{div}\left(\mathcal{A}_{21} \nabla u_{1}+\mathcal{A}_{22} \nabla u_{2}+\mathcal{A}_{23} \nabla u_{3}\right)=G\left(u_{1}, u_{2}, u_{3}\right), \\
\partial_{t} u_{3}-\operatorname{div}\left(d_{3}\left(\int_{\Omega} u_{3} \mathrm{~d} x\right) \nabla u_{3}\right)-\operatorname{div}\left(\mathcal{A}_{32} \nabla u_{2}+\mathcal{A}_{33} \nabla u_{3}\right)=G\left(u_{1}, u_{2}, u_{3}\right),
\end{array}\right.
$$

where the cross-diffusion matrix $\mathcal{A}=\left(\mathcal{A}_{i j}\right)_{1 \leq i, j \leq 3}$ is defined by

$$
\mathcal{A}=\left(\begin{array}{ccc}
\alpha_{1} u_{1}+u_{2} & u_{1} & 0 \\
u_{2} & u_{1}+\alpha_{2} u_{2}+u_{3} & u_{2} \\
0 & u_{3} & u_{2}+\alpha_{3} u_{3}
\end{array}\right)
$$

Note that the special form of the cross-diffusion matrix (i.e. $\mathcal{A}_{13}=\mathcal{A}_{31}=0$ ) is due to the model studied in this paper is based on the food chain model where there is no interaction between preys and superpredators ( $u_{1}$ and $u_{3}$ ) and as we mention before cross-diffusion expresses the population fluxes of one species due to the presence of the other species.

The cross-diffusion matrix $\mathcal{A}$ is uniformly nonnegative. In fact, for all $u_{1}, u_{2}, u_{3} \geq 0$, using condition (1.5) and the inequality $a b \geq-\frac{a^{2}}{2}-\frac{b^{2}}{2}$ for all $a, b \in \mathbb{R}$ one gets that for any $\xi=\left(\xi_{1}, \xi_{2}, \xi_{3}\right) \in \mathbb{R}^{3}$,

$$
\begin{align*}
\xi^{T} \mathcal{A} \xi \geq & \left(\left(\alpha_{1}-\frac{1}{2}\right) u_{1}+\frac{u_{2}}{2}\right) \xi_{1}^{2}+\left(\frac{u_{1}}{2}+\left(\alpha_{2}-1\right) u_{2}+\frac{u_{3}}{2}\right) \xi_{2}^{2} \\
& +\left(\frac{u_{2}}{2}+\left(\alpha_{3}-\frac{1}{2}\right) u_{3}\right) \xi_{3}^{2} \\
\geq & c\left(\left(u_{1}+u_{2}\right) \xi_{1}^{2}+\left(u_{1}+u_{2}+u_{3}\right) \xi_{2}^{2}+\left(u_{2}+u_{3}\right) \xi_{3}^{2}\right) \tag{1.7}
\end{align*}
$$

for some constant $c>0$.
The next goal is to discretize our model. It is important to mention that there are very few works related to the numerical analysis of this type of model involving nonlocal and cross-diffusion terms. On the other hand, there are many finite volume schemes to tackle numerically a nonlinear reaction-diffusion system. One of them is the well-known finite volume method introduced by Eymard, Gallouët and Herbin in [10]. In [2-4,9] was used this idea by doing a convergence analysis of the method. We proposed a semi-implicit finite volume scheme based on [10], for the three interacting species model presented before. Our main contribution is related to the nonlocal and cross-diffusion terms in the scheme. We propose a semi-implicit finite volume scheme, specifically implicit for the cross-diffusion and reaction terms and explicit for the nonlocal diffusion terms, which makes less complicated the computer calculations; moreover, we do not need a CFL stability condition. Besides, we proved the existence and convergence of the discrete solution generated by the scheme.

The plan of this paper is as follows: in Section 2, we define weak solution to system (1.1)-(1.3). We introduce some notations for the finite volume method, we present our scheme and the main theorem of convergence. The proof of this convergence result is divided into Section 3 a priori estimates and existence of solution, Section 4 compactness for discrete solution and Section 5 convergence to a weak solution. Finally, in Section 6, we give some numerical examples to our model.

## 2. Finite volume approximation

### 2.1. Admissible mesh

In this work, we assume that $\Omega \subset \mathbb{R}^{\ell}, \ell=2$ (respectively, $\ell=3$ ) is an open bounded polygonal (resp., polyhedral) connected domain with boundary $\partial \Omega$. We consider a family $\mathcal{T}_{h}$ of admissible meshes of the domain


Figure 1. Control volumes, centers and diamonds (in dashed lines).
$\Omega$ consisting of disjoint open and convex polygons (resp., polyhedra) called control volumes. The parameter $h$ has the sense of an upper bound for the maximum diameter of the control volumes in $\mathcal{T}_{h}$. Whenever $\mathcal{T}_{h}$ is fixed, we will drop the subscript $h$ in the notation. Of course, the mesh should be admissible in the sense of [10].

A generic volume in $\mathcal{T}_{h}$ is denoted by $K$. For all $K \in \mathcal{T}_{h}$, we denote by $|K|$ the $\ell$-dimensional Lebesgue measure of $K$. For a given finite volume $K$, we denote by $N(K)$ the set of neighbors of $K$ which have a common interface with $K$; a generic neighbor of $K$ is often denoted by $L$. For all $L \in N(K)$, we denote by $\sigma_{K, L}$ the interface between $K$ and $L$; we denote by $\eta_{K, L}$ the unit normal vector to $\sigma_{K, L}$ outward to $K$. We have $\eta_{L, K}=-\eta_{K, L}$. For an interface $\sigma_{K, L},\left|\sigma_{K, L}\right|$ will denote its $(\ell-1)$-dimensional measure.

By saying that $\mathcal{T}_{h}$ is admissible, we mean that there exists a family $\left(x_{K}\right)_{K \in \mathcal{T}_{h}}$ such that the straight line $\overline{x_{K} x_{L}}$ is orthogonal to the interface $\sigma_{K, L}$. The point $x_{K}$ is referred to as the center of $K$. In the case where $\mathcal{T}_{h}$ is a simplicial mesh of $\Omega$ (a triangulation, in dimension $\ell=2$ ), one takes for $x_{K}$ the center of the circumscribed ball of $K$. We also require that $\eta_{K, L} \cdot\left(x_{L}-x_{K}\right)>0$ (in the case of simplicial meshes, this restriction amounts to the Delaunay condition, see e.g. Ref. [10]). The "diamond" constructed from the neighbor centers $x_{K}, x_{L}$ and the interface $\sigma_{K, L}$ is denoted by $T_{K, L} ; e . g$. in the case $x_{K} \in K, x_{L} \in L, T_{K, L}$ is the convex hull of $x_{K}, x_{L}$ and $\sigma_{K, L}$ (see Fig. 1). We have $\Omega=\cup_{K \in \mathcal{T}_{h}}\left(\cup_{L \in N(K)} \bar{T}_{K, L}\right)$.

We require local regularity restrictions on the family of meshes $\mathcal{T}_{h}$; namely,

$$
\begin{align*}
& \exists \gamma>0, \quad \forall h, \quad \forall K \in \mathcal{T}_{h}, \quad \forall L \in N(K), \quad \operatorname{diam}(K)+\operatorname{diam}(L) \leq \gamma d_{K, L}  \tag{2.1}\\
& \quad \exists \gamma>0, \quad \forall h, \quad \forall K \in \mathcal{T}_{h}, \quad \forall L \in N(K), \quad\left|\sigma_{K, L}\right| d_{K, L} \leq \gamma|K| \tag{2.2}
\end{align*}
$$

where $d_{K, L}$ is the distance between $x_{K}$ and $x_{L}$.
A discrete function on the mesh $\mathcal{T}_{h}$ is a set $\left(w_{K}\right)_{K \in \mathcal{T}_{h}}$. Whenever convenient, we identify it with the piecewise constant function $w_{h}$ on $\Omega$ such that $\left.w_{h}\right|_{K}=w_{K}$. Finally, the discrete gradient $\nabla_{h} w_{h}$ of a constant per control volume function $w_{h}$ is defined as the constant per diamond $T_{K, L}$ function, $\mathbb{R}^{\ell}$-valued, with the values

$$
\begin{equation*}
\left.\left(\nabla_{h} w_{h}\right)\right|_{T_{K, L}}=\nabla_{K, L} w_{h}:=\ell \frac{w_{L}-w_{K}}{d_{K, L}} \eta_{K, L} \tag{2.3}
\end{equation*}
$$

Remark 2.1. We do not need to distinguish between interior and exterior control volumes because we consider the zero-flux boundary condition, only inner interfaces between volumes are needed in order to formulate the scheme.

### 2.2. Approximation of the nonlocal cross-diffusion model and the main result

To discretize (1.1)-(1.3), we choose an admissible discretization of $\Omega_{T}$ denoted by $\mathcal{M}$, consisting of an admissible mesh $\mathcal{T}_{h}$ of $\Omega$ and of a time step size $\Delta t_{h}>0$; both $\Delta t_{h}$ and the size $\max _{K \in \mathcal{T}_{h}} \operatorname{diam}(K)$ tend to zero as $h \rightarrow 0$. We define $N_{h}>0$ as the smallest integer such that $\left(N_{h}+1\right) \Delta t_{h} \geq T$, and set $t^{n}:=n \Delta t_{h}$ for $n \in\left\{0, \ldots, N_{h}\right\}$. Whenever $\Delta t_{h}$ is fixed, we will drop the subscript $h$ in the notation.

Furthermore, we define

$$
\begin{align*}
& F_{K}^{n+1}=F\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right) \\
& G_{K}^{n+1}=G\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right) \\
& H_{K}^{n+1}=H\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right) \tag{2.4}
\end{align*}
$$

To approximate the diffusive terms, we introduce the terms $\mathcal{A}_{i j, K}^{n+1}$ for $i, j=1,2,3$. Herein, we make the choice

$$
\begin{equation*}
\mathcal{A}_{i j, K, L}^{n+1}:=\mathcal{A}_{i j}\left(\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}, \min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}, \min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\}\right) \tag{2.5}
\end{equation*}
$$

The computation starts from the initial cell averages

$$
\begin{equation*}
u_{i, K}^{0}=\frac{1}{|K|} \int_{K} u_{i, 0}(x) \mathrm{d} x, \quad \text { for } i=1,2,3 \tag{2.6}
\end{equation*}
$$

To advance the numerical solution from $t^{n}$ to $t^{n+1}=t^{n}+\Delta t$, we use the following implicit finite volume scheme: Determine $\left(u_{i, K}^{n+1}\right)_{K \in \mathcal{T}_{h}}$ for $i=1,2,3$ such that

$$
\begin{align*}
& |K| \frac{u_{1, K}^{n+1}-u_{1, K}^{n}}{\Delta t}-d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right) \\
& -\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{11, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{12, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right] \\
& =|K| F_{K}^{n+1} \text {, }  \tag{2.7}\\
& |K| \frac{u_{2, K}^{n+1}-u_{2, K}^{n}}{\Delta t}-d_{2}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{2, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right) \\
& -\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{21, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{22, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{23, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right] \\
& =|K| G_{K}^{n+1},  \tag{2.8}\\
& |K| \frac{u_{3, K}^{n+1}-u_{3, K}^{n}}{\Delta t}-d_{3}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{3, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right) \\
& -\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{32, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{33, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right] \\
& =|K| H_{K}^{n+1} \text {, } \tag{2.9}
\end{align*}
$$

for all $K \in \mathcal{T}_{h}$ and $n \in\left[0, N_{h}\right]$. Herein

$$
\begin{align*}
& \mathcal{A}_{11, K, L}^{n+1}:=\alpha_{1} \min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}+\min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}, \quad \mathcal{A}_{12, K, L}^{n+1}:=\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\} \\
& \mathcal{A}_{22, K, L}^{n+1}:=\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}+\alpha_{2} \min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}+\min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\} \\
& \mathcal{A}_{21, K, L}^{n+1}=\mathcal{A}_{23, K, L}^{n+1}:=\min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}, \quad \mathcal{A}_{13, K, L}^{n+1}=\mathcal{A}_{31, K, L}^{n+1}=0 \\
& \mathcal{A}_{33, K, L}^{n+1}:=\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}+\alpha_{3} \min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\}, \quad \mathcal{A}_{32, K, L}^{n+1}:=\min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\} \tag{2.10}
\end{align*}
$$

Note that the homogeneous Neumann boundary condition is taken into account implicitly. Indeed, the parts of $\partial K$ that lie in $\partial \Omega$ do not contribute to the $\sum_{L \in N(K)}$ terms, which means that the flux zero is imposed on the external edges of the mesh.

The set of values $\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right)_{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}$ satisfying (2.6)-(2.9) will be called a discrete solution. Whenever convenient, we will assimilate a discrete solution of the scheme at the time step $n$ to the triple $\mathbf{u}_{h}^{n+1}=\left(u_{1, h}^{n+1}, u_{2, h}^{n+1}, u_{3, h}^{n+1}\right)$ of piecewise constant on $\Omega$ functions given by

$$
\forall K \in \mathcal{T}_{h}, \quad \forall n \in\left[0, N_{h}\right],\left.\quad u_{i, h}^{n+1}\right|_{K}=u_{i, K}^{n+1}, \text { for } i=1,2,3
$$

We will write $\mathbf{u}_{h}=\left(u_{1, h}, u_{2, h}, u_{3, h}\right)$ for the discrete solution on $\Omega_{T}$, assimilated to the piecewise constant function

$$
\left(\sum_{\substack{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}} u_{1, K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K}, \sum_{\substack{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}} u_{2, K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K}, \sum_{\substack{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}} u_{3, K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K}\right)
$$

We will assume that the following mild time step condition is satisfied

$$
\begin{equation*}
\Delta t<\sup \left\{\frac{1}{2}, \frac{1}{2 L_{2} M_{2}}, \frac{1}{2 L_{3} M_{32}}\right\} \tag{2.11}
\end{equation*}
$$

which will be used to prove the existence of solutions to the scheme.
Before stating our main results, we give the definition of a weak solution of problem (1.1)-(1.3).
Definition 2.2. A triple $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ of nonnegative functions is a weak solution of (1.1)-(1.3) if $u_{1}, u_{2}, u_{3} \in L^{2}\left(0, T ; H^{1}(\Omega)\right)$ and for all test functions $\varphi, \psi, \xi \in \mathcal{D}([0, T) \times \bar{\Omega})$ :

$$
\begin{aligned}
& -\iint_{\Omega_{T}} u_{1} \partial_{t} \varphi \mathrm{~d} x \mathrm{~d} t+\int_{0}^{T} d_{1}\left(\int_{\Omega} u_{1} \mathrm{~d} x\right) \int_{\Omega} \nabla u_{1} \cdot \nabla \varphi \mathrm{~d} x \mathrm{~d} t \\
& +\iint_{\Omega_{T}}\left[\mathcal{A}_{11} \nabla u_{1}+\mathcal{A}_{12} \nabla u_{2}\right] \cdot \nabla \varphi \mathrm{d} x \mathrm{~d} t \\
= & \iint_{\Omega_{T}} F\left(u_{1}, u_{2}, u_{3}\right) \varphi \mathrm{d} x \mathrm{~d} t+\int_{\Omega} u_{1,0}(x) \varphi(0, x) \mathrm{d} x \\
& -\iint_{\Omega_{T}} u_{2} \partial_{t} \psi \mathrm{~d} x \mathrm{~d} t+\int_{0}^{T} d_{2}\left(\int_{\Omega} u_{2} \mathrm{~d} x\right) \int_{\Omega} \nabla u_{2} \cdot \nabla \psi \mathrm{~d} x \mathrm{~d} t \\
& +\iint_{\Omega_{T}}\left[\mathcal{A}_{21} \nabla u_{1}+\mathcal{A}_{22} \nabla u_{2}+\mathcal{A}_{23} \nabla u_{3}\right] \cdot \nabla \psi \mathrm{d} x \mathrm{~d} t \\
= & \iint_{\Omega_{T}} G\left(u_{1}, u_{2}, u_{3}\right) \xi \mathrm{d} x \mathrm{~d} t+\int_{\Omega} u_{2,0}(x) \psi(0, x) \mathrm{d} x, \\
& -\iint_{\Omega_{T}} u_{3} \partial_{t} \xi \mathrm{~d} x \mathrm{~d} t+\int_{0}^{T} d_{3}\left(\int_{\Omega} u_{3} \mathrm{~d} x\right) \int_{\Omega} \nabla u_{3} \cdot \nabla \xi \mathrm{~d} x \mathrm{~d} t
\end{aligned}
$$

$$
\begin{aligned}
& +\iint_{\Omega_{T}}\left[\mathcal{A}_{32} \nabla u_{2}+\mathcal{A}_{33} \nabla u_{3}\right] \cdot \nabla \xi \mathrm{d} x \mathrm{~d} t \\
= & \iint_{\Omega_{T}} H\left(u_{1}, u_{2}, u_{3}\right) \xi \mathrm{d} x \mathrm{~d} t+\int_{\Omega} u_{3,0}(x) \xi(0, x) \mathrm{d} x
\end{aligned}
$$

In this paper, the existence of a nonnegative weak solution for system (1.1)-(1.3) will be shown by proving convergence of the numerical scheme (2.6)-(2.9). On the other hand, it is possible to find works where the existence of weak solution is shown directly (see $[6,8]$ ).

Our main result is the following:
Theorem 2.3. Assume that $u_{i, 0} \in\left(L^{2}(\Omega)\right)^{+}$for $i=1,2,3$. Let $\mathbf{u}_{h}=\left(u_{1, h}, u_{2, h}, u_{3, h}\right)$ be the discrete solution generated by the finite volume scheme (2.6)-(2.9) on a family of meshes satisfying (2.1) and (2.2). Then, as $h \rightarrow 0, \mathbf{u}_{h}$ converges (along a subsequence) a.e. on $\Omega_{T}$ to a limit $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ that is a weak solution of the system (1.1)-(1.3).

The proof of the above theorem will follow by combining the results proved in the sequel.

## 3. A PRIORI ESTIMATES AND EXISTENCE

In this section, we will prove the nonnegativity of the discrete solution, some apriori estimates and the existence of the discrete solution.

Note that in order to ensure the positivity of the scheme, we will use the positive part of the discrete unknowns in the discretization of the diffusive and reactive terms. Herein we will use an intermediate scheme, the actual scheme (2.6)-(2.9) with the positive part of (2.4) and (2.5):

$$
\begin{align*}
& F_{K}^{n+1^{+}}=F\left(u_{1, K}^{n+1+}, u_{2, K}^{n+1+}, u_{3, K}^{n+1+}\right) \\
& G_{K}^{n+1^{+}}=G\left(u_{1, K}^{n+1+}, u_{2, K}^{n+1+}, u_{3, K}^{n+1+}\right) \\
& H_{K}^{n+1^{+}}=H\left(u_{1, K}^{n+1+}, u_{2, K}^{n+1+}, u_{3, K}^{n+1+}\right) \tag{3.1}
\end{align*}
$$

and for $i, j=1,2,3$,

$$
\begin{equation*}
\mathcal{A}_{i j, K, L}^{n+1^{+}}:=\mathcal{A}_{i j}\left(\min \left\{u_{1, K}^{n+1+}, u_{1, L}^{n+1+}\right\}, \min \left\{u_{2, K}^{n+1+}, u_{2, L}^{n+1+}\right\}, \min \left\{u_{3, K}^{n+1+}, u_{3, L}^{n+1+}\right\}\right) \tag{3.2}
\end{equation*}
$$

where $u_{i, J}^{n+1+}:=\max \left(0, u_{i, J}^{n+1}\right)$ for $i=1,2,3$ and $J=K, L$. Moreover, the choice of the minimum in the discretization of $\mathcal{A}_{i j, K, L}^{n+1^{+}}$for $i \neq j$ and $i, j=1,2,3$, is imposed to justify the non-negativity of our discrete solution. Moreover, the choice of the diagonal terms $\mathcal{A}_{i i, K, L}^{n+1}$ for $i=1,2,3$, is made in order to preserve, at the discrete level, the structure of the cross-diffusion matrix $\mathcal{A}$.

### 3.1. Nonnegativity

We have the following lemma to prove the nonnegativity of the discrete solution generated by the scheme.
Lemma 3.1. Let $\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right)_{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}$ be a solution of the finite volume scheme (2.6)-(2.9) and (3.1)-(3.2). Then, $\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right)_{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}$ is nonnegative.

Proof. We prove the nonnegativity by induction, that for all $n \in\left[0, N_{h}\right], \min \left\{u_{1, K}^{n+1}\right\}_{K \in \mathcal{T}_{h}} \geq 0$.

We multiply equation (2.7) by $-\Delta t u_{1, K}^{n+1^{-}}$and we sum the resulting equation over $K$ to deduce

$$
\begin{align*}
-\sum_{K \in \mathcal{T}_{h}}|K| u_{1, K}^{n+1^{-}}\left(u_{1, K}^{n+1}-u_{1, K}^{n}\right)= & -d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right) u_{1, K}^{n+1-} \\
& -\Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{11, K, L}^{n+1^{+}}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)\right. \\
& \left.+\mathcal{A}_{12, K, L}^{n+1^{+}}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right] u_{1, K}^{n+1-}-\Delta t \sum_{K \in \mathcal{T}_{h}}|K| F_{K}^{n+1^{+}} u_{1, K}^{n+1^{-}} \tag{3.3}
\end{align*}
$$

where $u_{1, K}^{n+1^{-}}=\max \left(0,-u_{1, K}^{n+1}\right)$. By the non-negativity of $\mathcal{A}_{11, K, L}^{n+1^{+}}$and $\mathcal{A}_{12, K, L}^{n+1^{+}}$, we get

$$
\begin{aligned}
& \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right)+\mathcal{A}_{11, K, L}^{n+1^{+}}\right)\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right) u_{1, K}^{n+1-} \\
& \quad=-\Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right)+\mathcal{A}_{11, K, L}^{n+1^{+}}\right)\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)\left(u_{1, L}^{n+1-}-u_{1, K}^{n+1-}\right) \geq 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{12, K, L}^{n+1^{+}}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right] u_{1, K}^{n+1-} \\
& \quad=-\Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{12, K, L}^{n+1^{+}}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right]\left(u_{1, L}^{n+1-}-u_{1, K}^{n+1-}\right)=0
\end{aligned}
$$

Herein, we have used

$$
\text { for all } a, b \in \mathbb{R},(a-b)\left(a^{-}-b^{-}\right) \leq 0 \text { and } \min \left(a^{+}, b^{+}\right)\left(a^{-}-b^{-}\right)=0
$$

Similarly, by the definition of $F_{K}^{n+1}$ we have

$$
\begin{equation*}
F_{K}^{n+1^{+}} u_{K}^{n+1^{-}}=\left(\left(1-\frac{u_{1, K}^{n+1+}}{k}\right) u_{1, K}^{n+1^{+}}-\frac{L_{2} M_{2} u_{1, K}^{n+1+} u_{2, K}^{n+1+}}{R_{0}+u_{1, K}^{n+1^{+}}}\right) u_{1, K}^{n+1^{-}}=0 . \tag{3.4}
\end{equation*}
$$

Finally, we use the identity $u_{1, K}^{n+1}=u_{1, K}^{n+1+}-u_{1, K}^{n+1^{-}}$and the nonnegativity of $u_{1, K}^{n}$ to deduce from (3.3) and (3.4) that $\sum_{K \in \mathcal{T}_{h}}|K|\left|u_{1, K}^{n+1-}\right|^{2}=0$. By induction in $n$, we infer that

$$
u_{1, K}^{n+1} \geq 0 \text { for all } n \in\left[0, N_{h}\right] \text { and } K \in \mathcal{T}_{h}
$$

Along the same lines as $u_{1, K}^{n+1}$, we obtain the nonnegativity of the discrete solution $u_{i, K}^{n+1}$ for all $K \in \mathcal{T}_{h}$ and $n \in\left[0, N_{h}\right]$ for $i=2,3$.

### 3.2. A priori estimates

The goal now is to establish several a priori (discrete energy) estimates for the finite volume scheme, which eventually will imply the desired convergence results. As we have established the nonnegativity of the solution to our numerical scheme in Lemma 3.4, we will prove the existence of a solution to the scheme without using the positive parts in the diffusive and reactive terms.

Proposition 3.2. Let $\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right)_{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}$, be a solution of the finite volume scheme (2.4)-(2.10). Then, there exist constants $C_{i}>0, i=1,2,3$, depending on $\Omega, T,\left\|u_{1,0}\right\|_{L^{2}(\Omega)},\left\|u_{2,0}\right\|_{L^{2}(\Omega)},\left\|u_{3,0}\right\|_{L^{2}(\Omega)}$, $L_{2}, L_{3}, M_{2}, M_{32}, C_{0}, R_{0}, a_{1}, a_{2}, a_{3}$ such that

$$
\begin{gather*}
\max _{\left[0, N_{h}\right]} \sum_{K \in \mathcal{T}_{h}}|K|\left|u_{1, K}^{n+1}\right|^{2}+\max _{\left[0, N_{h}\right]} \sum_{K \in \mathcal{T}_{h}}|K|\left|u_{2, K}^{n+1}\right|^{2}+\max _{\left[0, N_{h}\right]} \sum_{K \in \mathcal{T}_{h}}|K|\left|u_{3, K}^{n+1}\right|^{2} \leq C_{1},  \tag{3.5}\\
 \tag{3.6}\\
\sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2} \leq C_{2},
\end{gather*}
$$

and

$$
\begin{equation*}
\sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \overline{\mathcal{A}}_{i, K, L}^{n+1}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2} \leq C_{3} \tag{3.7}
\end{equation*}
$$

where,

$$
\begin{align*}
& \overline{\mathcal{A}}_{1, K, L}^{n+1}=\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}+\min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}, \\
& \overline{\mathcal{A}}_{2, K, L}^{n+1}=\min \left\{u_{1, K}^{n+1}, u_{1, L}^{n+1}\right\}+\min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}+\min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\}, \\
& \overline{\mathcal{A}}_{3, K, L}^{n+1}=\min \left\{u_{2, K}^{n+1}, u_{2, L}^{n+1}\right\}+\min \left\{u_{3, K}^{n+1}, u_{3, L}^{n+1}\right\} . \tag{3.8}
\end{align*}
$$

Proof. In order to carry out the proof we initially assume that the quantities are non-negative. Therefore, we consider in this first part a modified scheme replacing (2.4) by (3.1), and (2.5) by (3.2). Thus, we multiply (2.7), (2.8) and (2.9) by $\Delta t u_{1, K}^{n+1}, \Delta t u_{2, K}^{n+1}$ and $\Delta t u_{3, K}^{n+1}$, respectively, and add together the outcomes. Summing the resulting equation over $K$ and $n$ yields

$$
S_{1}+S_{2}+S_{3}+S_{4}=0
$$

where

$$
\begin{aligned}
S_{1}= & \sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}}|K|\left(\left(u_{1, K}^{n+1}-u_{1, K}^{n}\right) u_{1, K}^{n+1}+\left(u_{2, K}^{n+1}-u_{2, K}^{n}\right) u_{2, K}^{n+1}+\left(u_{3, K}^{n+1}-u_{3, K}^{n}\right) u_{3, K}^{n+1}\right) \\
S_{2}= & -\sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(d_{i}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{i, K_{0}}^{n}\right)\left(u_{i, L}^{n+1}-u_{i, K}^{n+1}\right) u_{i, K}^{n+1}\right) \\
S_{3}= & -\sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \\
& \times\left(\left[\mathcal{A}_{11, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{12, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right] u_{1, K}^{n+1}\right. \\
& +\left[\mathcal{A}_{21, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{22, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{23, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right] u_{2, K}^{n+1} \\
& \left.+\left[+\mathcal{A}_{32, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{33, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right] u_{3, K}^{n+1}\right), \\
S_{4}= & -\sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}}|K|\left(F_{K}^{n+1} u_{1, K}^{n+1}+G_{K}^{n+1} u_{2, K}^{n+1}+H_{K}^{n+1} u_{3, K}^{n+1}\right) .
\end{aligned}
$$

For the first term we have

$$
\begin{aligned}
S_{1} & =\sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}}|K|\left(\left(u_{1, K}^{n+1}-u_{1, K}^{n}\right) u_{1, K}^{n+1}+\left(u_{2, K}^{n+1}-u_{2, K}^{n}\right) u_{2, K}^{n+1}+\left(u_{3, K}^{n+1}-u_{3, K}^{n}\right) u_{3, K}^{n+1}\right) \\
& \geq \frac{1}{2} \sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}}|K|\left(\left|u_{1, K}^{n+1}\right|^{2}-\left|u_{1, K}^{n}\right|^{2}+\left|u_{2, K}^{n+1}\right|^{2}-\left|u_{2, K}^{n}\right|^{2}+\left|u_{3, K}^{n+1}\right|^{2}-\left|u_{3, K}^{n}\right|^{2}\right) \\
& =\frac{1}{2} \sum_{K \in \mathcal{T}_{h}}|K|\left(\left|u_{1, K}^{N_{h}+1}\right|^{2}-\left|u_{1, K}^{0}\right|^{2}+\left|u_{2, K}^{N_{h}+1}\right|^{2}-\left|u_{2, K}^{0}\right|^{2}+\left|u_{3, K}^{N_{h}+1}\right|^{2}-\left|u_{3, K}^{0}\right|^{2}\right),
\end{aligned}
$$

where we have used the inequality $a(a-b) \geq \frac{1}{2}\left(a^{2}-b^{2}\right)$. Gathering by edges, we obtain

$$
S_{2}=\sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \frac{d_{i}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{i, K_{0}}^{n}\right)}{2}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2}
$$

Next, using (1.7) where $u_{i}$ is replaced by $\min \left\{u_{i, K}^{n+1+}, u_{i, L}^{n+1^{+}}\right\}$for $i=1,2,3$, we deduce

$$
S_{3} \geq c \sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \overline{\mathcal{A}}_{i, K, L}^{n+1}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2}
$$

for some constant $c>0$ (recall that $\overline{\mathcal{A}}_{i, K, L}^{n+1}$ is defined in (3.8)). Now, we use the nonnegativity of $u_{i, K}^{n+1}$ for $i=1,2,3$, and the discrete expressions of $F, G, H$ given by (1.3) to deduce

$$
S_{4} \geq-\sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}}|K|\left(\left|u_{1, K}^{n+1}\right|^{2}+L_{2} M_{2}\left|u_{2, K}^{n+1}\right|^{2}+L_{3} M_{32}\left|u_{3, K}^{n+1}\right|^{2}\right)
$$

Collecting the previous inequalities we obtain

$$
\begin{align*}
& \frac{1}{2} \sum_{K \in \mathcal{T}_{h}}|K|\left(\left|u_{1, K}^{N_{h}+1}\right|^{2}-\left|u_{1, K}^{0}\right|^{2}+\left|u_{2, K}^{N_{h}+1}\right|^{2}-\left|u_{2, K}^{0}\right|^{2}+\left|u_{3, K}^{N_{h}+1}\right|^{2}-\left|u_{3, K}^{0}\right|^{2}\right) \\
& \quad+\sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \frac{d_{i}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{i, K_{0}}^{n}\right)}{2}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2} \\
& \quad+c \sum_{i=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \overline{\mathcal{A}}_{i, K, L}^{n+1}\left|u_{i, K}^{n+1}-u_{i, L}^{n+1}\right|^{2} \\
& \quad \leq \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}}|K|\left(\left|u_{1, K}^{n+1}\right|^{2}+L_{2} M_{2}\left|u_{2, K}^{n+1}\right|^{2}+L_{3} M_{32}\left|u_{3, K}^{n+1}\right|^{2}\right) . \tag{3.9}
\end{align*}
$$

By an application of the discrete Gronwall inequality, (3.5) follows from (3.9). Finally, the estimates (3.6) and (3.7) are the consequence of (3.9) . The proof of Proposition 3.2 is achieved using Lemma 3.1 and considering the original scheme (2.4)-(2.10).

Remark 3.3. Observe that from Lemma 3.1 and (3.7), we deduce easily

$$
\begin{align*}
& \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \min \left\{u_{i, K}^{n+1}, u_{i, L}^{n+1}\right\}\left|u_{1, K}^{n+1}-u_{1, L}^{n+1}\right|^{2} \leq C, \text { for } i=1,2 \\
& \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \min \left\{u_{i, K}^{n+1}, u_{i, L}^{n+1}\right\}\left|u_{2, K}^{n+1}-u_{2, L}^{n+1}\right|^{2} \leq C, \text { for } i=1,2,3, \\
& \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \min \left\{u_{i, K}^{n+1}, u_{i, L}^{n+1}\right\}\left|u_{3, K}^{n+1}-u_{3, L}^{n+1}\right|^{2} \leq C, \text { for } i=2,3 \tag{3.10}
\end{align*}
$$

for some constant $C>0$. This implies

$$
\begin{equation*}
\sum_{i, j=1}^{3} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}} \mathcal{A}_{i j, K, L}^{n+1}\left|u_{j, K}^{n+1}-u_{j, L}^{n+1}\right|^{2} \leq C_{3} \tag{3.11}
\end{equation*}
$$

### 3.3. Existence of a solution for the finite volume scheme

The existence of a solution for the finite volume scheme is given in the following proposition.
Proposition 3.4. Let $\mathcal{M}$ be an admissible discretization of $\Omega_{T}$ and assume that (2.11) holds. Then, the discrete problem (2.4)-(2.10) admits at least one solution $\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right)$ where $(K, n) \in \mathcal{T}_{h} \times\left[0, N_{h}\right]$.

Proof. As well as the proof of the previous proposition, we initially consider the modified scheme replacing (2.4) by (3.1), and (2.5) by (3.2). We will show the existence of $\mathbf{u}_{h}^{n}=\left(u_{1, h}^{n}, u_{2, h}^{n}, u_{3, h}^{n}\right)$ by induction on $n$. For a given $\mathbf{u}_{h}^{n}$, we deduce the existence of the solution $\mathbf{u}_{h}^{n+1}=\left(u_{1, h}^{n+1}, u_{2, h}^{n+1}, u_{3, h}^{n+1}\right)$. Now, we prove the existence of discrete solution $\mathbf{u}_{h}^{n+1}$.

Assume that $\mathbf{u}_{h}^{n}$ exists. We choose the componentwise product $[\cdot, \cdot]$ as the scalar product on $\mathbb{R}^{\mathcal{T}_{h}}$. We define the mapping $\mathcal{P}$ that associates to the vector $\mathcal{U}=\left(\mathbf{u}_{K}^{n+1}\right)_{K \in \mathcal{T}_{h}}$ the expression

$$
\mathcal{P}(\mathcal{U})=\left(\mathcal{P}_{1}(\mathcal{U}), \mathcal{P}_{2}(\mathcal{U}), \mathcal{P}_{3}(\mathcal{U})\right)
$$

where

$$
\begin{aligned}
\mathcal{P}_{1}(\mathcal{U})= & \left(|K| \frac{u_{1, K}^{n+1}-u_{1, K}^{n}}{\Delta t}-d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)\right. \\
& \left.-\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{11, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{12, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right]-|K| F_{K}^{n+1}\right)_{K \in \mathcal{T}_{h}}, \\
\mathcal{P}_{2}(\mathcal{U})= & \left(|K| \frac{u_{2, K}^{n+1}-u_{2, K}^{n}}{\Delta t}-d_{2}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{2, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right. \\
& -\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{21, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{22, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{23, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right] \\
& \left.-|K| G_{K}^{n+1}\right)_{K \in \mathcal{T}_{h}}, \\
\mathcal{P}_{3}(\mathcal{U})= & \left(|K| \frac{u_{3, K}^{n+1}-u_{3, K}^{n}}{\Delta t}-d_{3}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{3, K_{0}}^{n}\right) \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right.
\end{aligned}
$$

$$
\left.-\sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{32, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)+\mathcal{A}_{33, K, L}^{n+1}\left(u_{3, L}^{n+1}-u_{3, K}^{n+1}\right)\right]-|K| H_{K}^{n+1}\right)_{K \in \mathcal{T}_{h}}
$$

given by the equations in (2.7), (2.8) and (2.9). Now, using the definition of $F_{K}^{n+1}, G_{K}^{n+1}$ and $H_{K}^{n+1}$, estimate (2.11) and an application of Young's inequality to deduce

$$
[\mathcal{P}(\mathcal{U}), \mathcal{U}] \geq C|\mathcal{U}|^{2}-C^{\prime}|\mathcal{U}|-C^{\prime \prime} \geq 0, \quad \text { for }|\mathcal{U}| \text { large enough }
$$

for some constants $C, C^{\prime}, C^{\prime \prime}>0$. We deduce that

$$
[\mathcal{P}(\mathcal{U}), \mathcal{U}]>0, \quad \text { for }|\mathcal{U}| \text { large enough. }
$$

This implies (see for e.g. [15, 22]): there exists $\mathcal{U}$ such that

$$
\mathcal{P}(\mathcal{U})=(0,0,0)
$$

Then, by induction in $n$, we deduce the existence of at least one solution to the scheme (2.6)-(2.9). The proof is achieved using Lemma 3.1 and the original scheme (2.4)-(2.10).

## 4. Compactaness ARGUMENTS

In this section, we prove that the family $\mathbf{u}_{h}$ of discrete solutions constructed in Proposition 3.4 is relatively compact in $L^{1}\left(\Omega_{T}\right)$. With this aim, we will use the following lemma (see the proof in Appendix A in [5]).

Lemma 4.1. Let $\mathcal{T}_{h}$ be an admissible discretization of $\Omega$ satisfying the restriction (2.1), let $\Delta t_{h}$ be the associated time step. For all $h>0$, assume that discrete functions $\left(v_{h}^{n+1}\right)_{n \in\left[0, N_{h}\right]},\left(f_{h}^{n+1}\right)_{n \in\left[0, N_{h}\right]}$ and discrete fields $\left(\mathcal{F}_{h}^{n+1}\right)_{n \in\left[0, N_{h}\right]}$ satisfy the discrete evolution equations

$$
\begin{equation*}
\text { for } n \in\left[0, N_{h}\right], \quad \frac{v_{h}^{n+1}-v_{h}^{n}}{\Delta t}=\operatorname{div}_{h}\left[\mathcal{F}_{h}^{n+1}\right]+f_{h}^{n+1} \tag{4.1}
\end{equation*}
$$

with $v_{h}^{0}$ initial data. Assume that for all $\Omega^{\prime} \subset \Omega$, there exists a constant $M\left(\Omega^{\prime}\right)$ such that

$$
\begin{equation*}
\sum_{n=0}^{N_{h}} \Delta t\left\|v_{h}^{n+1}\right\|_{L^{1}\left(\Omega^{\prime}\right)}+\sum_{n=0}^{N_{h}} \Delta t\left\|f_{h}^{n+1}\right\|_{L^{1}\left(\Omega^{\prime}\right)}+\sum_{n=0}^{N_{h}} \Delta t\left\|\mathcal{F}_{h}^{n+1}\right\|_{L^{1}\left(\Omega^{\prime}\right)} \leq M\left(\Omega^{\prime}\right) \tag{4.2}
\end{equation*}
$$

and; moreover,

$$
\begin{equation*}
\sum_{n=0}^{N_{h}} \Delta t\left\|\nabla_{h} v_{h}^{n+1}\right\|_{L^{1}\left(\Omega^{\prime}\right)} \leq M\left(\Omega^{\prime}\right) \tag{4.3}
\end{equation*}
$$

assume also that $v_{h}^{0}$ is bounded in $L_{\mathrm{loc}}^{1}(\Omega)$. Then, there exists a measurable function $v$ on $\Omega_{T}$ such that, along a subsequence,

$$
\sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}} v_{K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K} \longrightarrow v \quad \text { in } L_{\mathrm{loc}}^{1}([0, T] \times \Omega) \quad \text { as } h \rightarrow 0
$$

Denote by $\mathcal{A}^{h}$ the $3 \times 3$ matrix on $\Omega_{T}$ with the entries $\mathcal{A}_{i j}^{h}$ given by

$$
\mathcal{A}_{i j}^{h}:=\frac{1}{2} \sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \mathcal{A}_{i j, K, L}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times T_{K, L}} .
$$

We have the following convergence results along a subsequence:
Proposition 4.2. There exists a triple $\mathbf{u} \in\left(L^{10 / 3}\left(\Omega_{T}\right)\right)^{3} \cap L^{2}\left(0, T ; H^{1}(\Omega)\right)^{3}$ and a subsequence of $\mathbf{u}_{h}=$ $\left(u_{1, h}, u_{2, h}, u_{3, h}\right)$, not labelled, such that, as $h \rightarrow 0$,
(i) $\mathbf{u}_{h} \rightarrow \mathbf{u}$ strongly in $\left(L^{1}\left(\Omega_{T}\right)\right)^{3}$ and a.e. in $\Omega_{T}$,
(ii) $\nabla_{h} \mathbf{u}_{h} \longrightarrow \nabla \mathbf{u}$ weakly in $\left(L^{2}\left(\Omega_{T}\right)\right)^{3 \times 3}$,
(iii) $\mathcal{A}^{h} \nabla_{h} \mathbf{u}_{h} \longrightarrow \mathcal{A}(\mathbf{u}) \nabla \mathbf{u}$ weakly in $\left(L^{1}\left(\Omega_{T}\right)\right)^{3 \times 3}$,
(iv) $\left(F\left(\mathbf{u}_{h}\right), G\left(\mathbf{u}_{h}\right), H\left(\mathbf{u}_{h}\right)\right) \longrightarrow(F(\mathbf{u}), G(\mathbf{u}), H(\mathbf{u}))$ weakly in $\left(L^{1}\left(\Omega_{T}\right)\right)^{3}$.

Proof. In this proof we apply Lemma 4.1, using the estimates shown in Proposition 3.2.
Observe that we may consider that the evolution of the first component $\left(u_{1, h}^{n+1}\right)_{n \in\left[0, N_{h}\right]}$, the solution of (2.7), is governed by the system of discrete equations

$$
\begin{equation*}
\frac{u_{1, K}^{n+1}-u_{1, K}^{n}}{\Delta t}=\frac{1}{|K|} \sum_{L \in N(K)}\left|\sigma_{K, L}\right| \mathcal{F}_{K, L}^{n+1} \cdot \eta_{K, L}+f_{K}^{n+1}, \tag{4.4}
\end{equation*}
$$

where

$$
\begin{aligned}
f_{K}^{n+1}:= & F\left(u_{1, K}^{n+1}, u_{2, K}^{n+1}, u_{3, K}^{n+1}\right), \\
\mathcal{F}_{K, L}^{n+1}:= & d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \frac{u_{1, L}^{n+1}-u_{1, K}^{n+1}}{d_{K, L}} \eta_{K, L}+\mathcal{A}_{11, K, L}^{n+1} \frac{u_{1, L}^{n+1}-u_{1, K}^{n+1}}{d_{K, L}} \eta_{K, L} \\
& +\mathcal{A}_{12, K, L}^{n+1} \frac{u_{2, L}^{n+1}-u_{2, K}^{n+1}}{d_{K, L}} \eta_{K, L} \\
= & \frac{1}{\ell}\left[d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \nabla_{K, L} u_{1, h}^{n+1}+\mathcal{A}_{11, K, L}^{n+1} \nabla_{K, L} u_{1, h}^{n+1}+\mathcal{A}_{12, K, L}^{n+1} \nabla_{K, L} u_{1, h}^{n+1}\right],
\end{aligned}
$$

where in the last equality, we have used (2.3).
It is easy to see that equations (4.4) have the form (4.1) required in Lemma 4.1.
The next step is to check that the local $L^{1}$ bounds (4.2) and (4.3) are verified. Using the $L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$ estimate (3.5), the discrete $L^{2}\left(0, T ; H^{1}(\Omega)\right)$ estimate (3.6) and the estimate (3.7) (recall that (3.7) is exactly the $L^{2}\left(\Omega_{T}\right)$ estimate of the product $\sqrt{\left|\mathcal{A}^{h}\right|} \nabla_{h} \mathbf{u}_{h}$ ), we get the global $L^{1}\left(\Omega_{T}\right)$ uniform estimates on the discrete functions

$$
\begin{aligned}
u_{1, h} & :=\sum_{\substack{K \in \mathcal{T}_{h}, n \in\left[0, N_{h}\right]}} u_{1, K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K}, & \mathcal{F}^{h} & :=\frac{1}{2} \sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \mathcal{F}_{K, L}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times T_{K, L}}, \\
f_{h} & :=\sum_{\substack{K \in \tau_{h,} \\
n \in\left[0, N_{h}\right]}} f_{K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times K}, & \nabla_{h} u_{1, h} & :=\frac{1}{2} \sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \nabla_{K, L} u_{1, K}^{n+1} \mathbb{1}_{\left(t^{n}, t^{n+1}\right] \times T_{K, L}} .
\end{aligned}
$$

Note that the $L^{2}\left(\Omega_{T}\right)$ estimate on the discrete solutions derived from (3.5) and the quadratic growth of $F, G$ and $H$, ensure the $L^{1}\left(\Omega_{T}\right)$ bound on $f_{h}$. Combined with the Cauchy-Schwarz inequality, the same bound yields the $L^{1}\left(\Omega_{T}\right)$ estimate of $u_{1, h}$.

Using the critical discrete Sobolev embedding (see Prop. B. 1 from the Appendix B in [5]) and the interpolation between $L^{p_{t}}\left(0, T ; L^{p_{x}}(\Omega)\right)$ spaces, from the $L^{\infty}\left(0, T ; L^{2}(\Omega)\right)$ estimate (3.5) and the discrete $L^{2}\left(0, T ; H^{1}(\Omega)\right)$ estimate (3.6) we get a uniform $L^{10 / 3}\left(\Omega_{T}\right)$ bound on $u_{1, h}$ and a uniform $L^{1}\left(\Omega_{T}\right)$ bound on the terms $\mathcal{A}_{i j}^{h}$ (see [18]).

Moreover, the estimate (3.7) is exactly the $L^{2}\left(\Omega_{T}\right)$ estimate of the product $\sqrt{\left|\mathcal{A}^{h}\right|} \nabla_{h} \mathbf{u}_{h}$ (the square root of $\mathcal{A}^{h}$ is taken componentwise). The two latter bounds permit to control $\mathcal{A}^{h} \mathbf{u}_{h}$ in $L^{1}\left(\Omega_{T}\right)$.

Recall that $\mathcal{F}^{h}$ corresponds to the term $\frac{1}{\ell}\left[d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \nabla_{h} u_{1, h}+\mathcal{A}_{11}^{h} \nabla_{h} u_{1, h}+\mathcal{A}_{12}^{h} \nabla_{h} u_{1, h}\right]$, and $d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \nabla_{h} u_{1, h}$ is $L^{2}\left(\Omega_{T}\right)$ bounded by estimate (3.6). Then by the Cauchy-Schwarz inequality we deduce uniform $L^{1}\left(\Omega_{T}\right)$ estimate of $\mathcal{F}^{h}$ and also the one of $\nabla_{h} u_{1, h}$. This implies that (4.2) and (4.3) are verified. Furthermore the uniform $L^{1}(\Omega)$ bound on the initial data $u_{1,0}$ is also clear from (2.6). Then Lemma 4.1 can be applied to derive the $L^{1}\left(\Omega_{T}\right)$ compactness of $u_{1, h}$.

Along the same lines as $u_{1, h}$, we obtain the compactness of $u_{2, h}$ and $u_{3, h}$. Consequently, we can define the limit $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ of (a subsequence of) $\mathbf{u}_{h}$ and obtain the claim (i).

In the following step, it is easy to see that the claim (ii) is the consequence of the estimate (3.6). Observe that, one uses (3.6) to bound $\nabla_{h} u_{1, h}$ in $L^{2}\left(\Omega_{T}\right)$. Upon extraction of a further subsequence, we have e.g. $u_{1, h} \rightarrow u_{1}$ in $L^{2}\left(\Omega_{T}\right)$ and $\nabla_{h} u_{1, h} \rightarrow \chi$ in $\left(L^{2}\left(\Omega_{T}\right)\right)^{\ell}$, where $\chi$ has to be identified. For this purpose, one takes a smooth compactly supported vector-function $\phi$ on $\Omega_{T}$ and proves, using the definition (2.3) together with the discrete summation-by-parts and the consistency of the finite volume approximation of $\operatorname{div} \phi$, that

$$
\int_{0}^{T} \int_{\Omega} \nabla u_{1} \cdot \phi=-\int_{0}^{T} \int_{\Omega} u_{1} \operatorname{div} \phi
$$

This shows that $u_{1} \in L^{2}\left(0, T ; H^{1}(\Omega)\right)$ and that $\chi$ identifies with $\nabla u_{1}$. The proof for $\nabla_{h} u_{2, h}$ and $\nabla_{h} u_{3, h}$ are identical.

Finally, the claims (iii), (iv) follow because the uniform $L^{2}\left(\Omega_{T}\right)$ estimates of $\mathbf{u}_{h}$ and of $\sqrt{\left|\mathcal{A}_{i j}^{h}\right|}$. Using in addition the quadratic growth of $F$ and the a.e. convergence of $\mathbf{u}_{h}$ to $\mathbf{u}$, by the Vitali theorem we get (iv). Similarly, we get the strong $L^{2}\left(\Omega_{T}\right)$ convergence of $\sqrt{\left|\mathcal{A}_{i j}^{h}\right|}$ to $\sqrt{\left|\mathcal{A}_{i j}(\mathbf{u})\right|}$. Then, we pass to the limit first in $\sqrt{\left|\mathcal{A}_{i j}^{h}\right|} \nabla_{h} \mathbf{u}_{h}$ and then in $\mathcal{A}^{h} \nabla_{h} \mathbf{u}_{h}$; hence we get (iii).

## 5. Convergence analysis

Our final goal is to show that the limit functions $\mathbf{u}=\left(u_{1}, u_{2}, u_{3}\right)$ constructed in Proposition 4.2 constitute a weak solution of system (1.1)-(1.3). We start by passing to the limit in (2.7) to get the first equality in Definition 2.2 .

Let $\varphi \in \mathcal{D}([0, T) \times \bar{\Omega})$. Set $\varphi_{K}^{n+1}:=\varphi\left(t^{n+1}, x_{K}\right)$ for all $K \in \mathcal{T}_{h}$ and $n \in\left[0, N_{h}\right]$. We multiply the discrete equation (2.7) by $\Delta t \varphi_{K}^{n+1}$. Summing the result over $K \in \mathcal{T}_{h}$ and $n \in\left[0, N_{h}\right]$, yields

$$
S_{1}^{h}+S_{2}^{h}+S_{3}^{h}=S_{4}^{h}
$$

where

$$
\begin{aligned}
& S_{1}^{h}=\sum_{n=0}^{N_{h}} \sum_{K \in \mathcal{T}_{h}}|K|\left(u_{1, K}^{n+1}-u_{1, K}^{n}\right) \varphi_{K}^{n+1}, \\
& S_{2}^{h}=-\sum_{n=0}^{N_{h}} \Delta t d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right) \varphi_{K}^{n+1}, \\
& S_{3}^{h}=-\sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{\left|\sigma_{K, L}\right|}{d_{K, L}}\left[\mathcal{A}_{11, K, L}^{n+1}\left(u_{1, L}^{n+1}-u_{1, K}^{n+1}\right)+\mathcal{A}_{12, K, L}^{n+1}\left(u_{2, L}^{n+1}-u_{2, K}^{n+1}\right)\right] \varphi_{K}^{n+1}, \\
& S_{4}^{h}=\sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}}|K| F_{K}^{n+1} \varphi_{K}^{n+1} .
\end{aligned}
$$

Performing summation by parts in time and keeping in mind that $\varphi_{K}^{N_{h}+1}=0$ for all $K \in \mathcal{T}_{h}$, by classical arguments (see [10]), we get from Proposition 4.2 (i) the following convergence (along a subsequence)

$$
\lim _{h \rightarrow 0} S_{1}^{h}=-\int_{0}^{T} \int_{\Omega} u_{1} \partial_{t} \varphi-\int_{\Omega} u_{1,0} \varphi(0, \cdot)
$$

Gathering by edges and using the definition (2.3) of $\nabla_{h}$, we have

$$
\begin{aligned}
S_{2}^{h} & =\frac{1}{2} \sum_{n=0}^{N_{h}} \Delta t d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{1}{\ell}\left|\sigma_{K, L}\right| d_{K, L} \ell \frac{u_{1, L}^{n+1}-u_{1, K}^{n+1}}{d_{K, L}} \frac{\varphi_{L}^{n+1}-\varphi_{K}^{n+1}}{d_{K, L}} \\
& =\frac{1}{2} \sum_{n=0}^{N_{h}} \Delta t d_{1}\left(\sum_{K_{0} \in \mathcal{T}_{h}} m\left(K_{0}\right) u_{1, K_{0}}^{n}\right) \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)}\left|T_{K, L}\right|\left(\nabla_{K, L} u_{1, h}^{n+1} \cdot \eta_{K, L}\right)\left(\nabla \varphi\left(t^{n+1}, \overline{x_{K, L}}\right) \cdot \eta_{K, L}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
S_{3}^{h} & =\frac{1}{2} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)} \frac{1}{\ell}\left|\sigma_{K, L}\right| d_{K, L} \ell\left[\mathcal{A}_{11, K, L}^{n+1} \frac{u_{1, L}^{n+1}-u_{1, K}^{n+1}}{d_{K, L}}+\mathcal{A}_{12, K, L}^{n+1} \frac{u_{2, L}^{n+1}-u_{2, K}^{n+1}}{d_{K, L}}\right] \frac{\varphi_{L}^{n+1}-\varphi_{K}^{n+1}}{d_{K, L}} \\
& =\frac{1}{2} \sum_{n=0}^{N_{h}} \Delta t \sum_{K \in \mathcal{T}_{h}} \sum_{L \in N(K)}\left|T_{K, L}\right|\left[\mathcal{A}_{11, K, L}^{n+1} \nabla_{K, L} u_{1, h}^{n+1} \cdot \eta_{K, L}+\mathcal{A}_{12, K, L}^{n+1} \nabla_{K, L} u_{2, h}^{n+1} \cdot \eta_{K, L}\right] \nabla \varphi\left(t^{n+1}, \overline{x_{K, L}}\right) \cdot \eta_{K, L},
\end{aligned}
$$

where $\overline{x_{K, L}}$ is some point on the segment with the endpoints $x_{K}, x_{L}$. Since the values of $\nabla_{K, L}$ are directed by $\eta_{K, L}$, we have

$$
\left(\nabla_{K, L} u_{i, h}^{n+1} \cdot \eta_{K, L}\right)\left(\nabla \varphi\left(t^{n+1}, \overline{x_{K, L}}\right) \cdot \eta_{K, L}\right) \equiv \nabla_{K, L} u_{i, h}^{n+1} \cdot \nabla \varphi\left(t^{n+1}, \overline{x_{K, L}}\right)
$$

for $i=1,2$. Moreover, each term corresponding to $T_{K, L}$ appears twice in the above formula,

$$
S_{2}^{h}=\int_{0}^{T} d_{1}\left(\int_{\Omega} u_{1, h}(t, x) \mathrm{d} x\right) \int_{\Omega} \nabla_{h} u_{1, h} \cdot(\nabla \varphi)_{h},
$$

and

$$
\int_{0}^{T} \int_{\Omega}\left[\mathcal{A}_{11}^{h} \nabla_{h} u_{1, h}+\mathcal{A}_{12}^{h} \nabla_{h} u_{2, h}\right] \cdot \nabla(\varphi)_{h}
$$

where

$$
\left.(\nabla \varphi)_{h}\right|_{\left(t^{n}, t^{n+1}\right] \times T_{K, L}}:=\nabla \varphi\left(t^{n+1}, \overline{x_{K, L}}\right) .
$$

Observe that from the continuity of $\nabla \varphi$ we get $(\nabla \varphi)_{h} \rightarrow \nabla \varphi$ in $L^{\infty}\left(\Omega_{T}\right)$. Hence using (1.4), the strong $L^{p}$ convergence of $u_{1, h}$ to $u_{1}$ for $p<10 / 3$, the weak $L^{2}$ convergence of $\nabla_{h} u_{1, h}$ to $\nabla u_{1}$, and the weak $L^{1}$ convergence of $\mathcal{A}^{h} \nabla_{h} \mathbf{u}_{h}$ to $\mathcal{A}(\mathbf{u}) \nabla \mathbf{u}$, we pass to the limit in $S_{2}^{h}$ and $S_{3}^{h}$, as $h \rightarrow 0$.

Then, again along a subsequence, we have

$$
\begin{aligned}
\lim _{h \rightarrow 0} S_{2}^{h} & =\int_{0}^{T} d_{1}\left(\int_{\Omega} u_{1}(t, x) \mathrm{d} x\right) \int_{\Omega} \nabla u_{1} \cdot \nabla \varphi \\
\lim _{h \rightarrow 0} S_{3}^{h} & =\iint_{\Omega_{T}}\left(\mathcal{A}_{11}\left(u_{1}, u_{2}, u_{3}\right) \nabla u_{1}+\mathcal{A}_{12}\left(u_{1}, u_{2}, u_{3}\right) \nabla u_{2}\right) \cdot \nabla \varphi
\end{aligned}
$$

Note that our proof is slightly different from the classical one ( $c f$. reference [10]), adapted to the definition (2.3) of the discrete gradient and to the associated weak convergence statements of Proposition 4.2 items (ii) and


Figure 2. Initial condition for $u_{2}$ species
(iii). Let us put forward the arguments for the term $S_{2}^{h}$. In the proof of the convergence claim for $S_{3}^{h}$, we use Proposition 4.2 (iii) instead of Proposition 4.2 (ii).

Finally, using Proposition 4.2 (iv), we deduce that $S_{4}^{h}$ converges to $\iint_{\Omega_{T}} F\left(u_{1}, u_{2}, u_{3}\right) \varphi$ as $h \rightarrow 0$. Gathering the obtained results, we justify the first equality in Definition 2.2. Reasoning along the same lines as above, we conclude that also the second and the third equality in Definition 2.2 hold. This concludes the proof of Theorem 2.3.

## 6. Numerical RESULTS

In this section are presented some numerical results from our finite volume scheme refer to the system (1.1)-(1.3) given before.

To obtain the numerical results, we will reduce the number of the parameters in the model. For this reason, we nondimensionalize the system following Hasting and Powell in [13]. We choose

$$
U_{1}=\frac{u_{1}}{k} ; \quad U_{2}=\frac{u_{2}}{k} ; \quad U_{3}=\frac{u_{3}}{k}
$$

Making the substitution and simplifying, we obtain the following:

$$
\begin{align*}
F\left(U_{1}, U_{2}, U_{3}\right) & =\left(1-U_{1}\right) U_{1}-\frac{a_{1} U_{1}}{1+b_{1} U_{1}} U_{2} \\
G\left(U_{1}, U_{2}, U_{3}\right) & =\frac{a_{1} U_{1}}{1+b_{1} U_{1}} U_{2}-\frac{a_{2} U_{2}}{1+b_{2} U_{2}} U_{3}-c_{1} U_{2} \\
H\left(U_{1}, U_{2}, U_{3}\right) & =\frac{a_{2} U_{2}}{1+b_{2} U_{2}} U_{3}-c_{2} U_{3} \tag{6.1}
\end{align*}
$$

We now show some numerical experiments in two dimension, where the spatial domain corresponds to a simple square $\Omega=(-1,1) \times(-1,1)$. We considered an uniform mesh given by a Cartesian grid with $N_{x} \times N_{y}$ control volumes and we choose $N_{x}=N_{y}=256$ for the simulations. Obviously, it is possible to considered unstructured meshes, but we will restrain to an uniform mesh $\mathcal{T}_{h}=\left\{K_{i j} \subset \Omega / K_{i j}=\left(\frac{i-1}{N_{x}}, \frac{i}{N_{x}}\right) \times\left(\frac{j-1}{N_{y}}, \frac{j}{N_{y}}\right)\right\}$ for simplicity. The discretization in time is given by $N_{t}=500$ time steps for $T=0.5\left(\Delta t=T / N_{t}\right)$.

For the simulations, we take the following parameters used in [13]:

$$
a_{1}=5.0 ; \quad a_{2}=0.1 ; \quad b_{1}=2.0 ; \quad b_{2}=2.0 ; \quad c_{1}=0.4 ; \quad c_{2}=0.01
$$



Figure 3. Interaction of the three species at different times $t=1,5,10,20,30$.


Figure 4. Profiles for species $u_{1}, u_{2}$ and $u_{3}$ for different times, $t=1.0,1.5,2.0$. (a) nonlocal diffusion, (b) cross diffusion

For solving the corresponding nonlinear system arising from the implicit finite volume scheme, we have used the Newton method, where at each time step, only a few iterations are required to achieve convergence. In addition, the linear systems involved in Newton's method are solved by the GMRES method.

### 6.1. Example 1

For this example, the initial distribution for $u_{1}$ and $u_{3}$ species correspond to a constant $u_{1,0}=0.75, u_{3,0}=$ 0.215 , and the $u_{2}$ species concentrated in small pockets at two spatial points (see Fig. 2). The nonlocal diffusion

$\alpha_{1}=100, \alpha_{2}=100, \alpha_{3}=10$,


$$
\alpha_{1}=100, \alpha_{2}=100, \alpha_{3}=10000
$$



$$
\alpha_{1}=100, \alpha_{2}=300, \alpha_{3}=100
$$



Figure 5. Patterns of the three species with different cross diffusion parameters at $t=0.1$
terms are given by a simple choice of linear function, $d_{1}(s)=0.1 s$ and $d_{2}(s)=d_{3}(s)=0.01 s$, for all $s \in \mathbb{R}$. In other words, they depend linearly on the whole of each population in the domain. Finally, the cross diffusion parameters are given by $\alpha_{1}=10, \alpha_{2}=\alpha_{3}=1.5$.

In Figure 3, we can observe the behaviour of $u_{1}, u_{2}$ and $u_{3}$ for different times. Initially, we can observe the effect of the diffusion over the three populations. We notice the rapid movement of superpredators $\left(u_{3}\right)$ towards the regions occupied by predators $\left(u_{2}\right)$ and at the same time predators spread out to the areas where preys $\left(u_{1}\right)$ are located. As the time passes, we can observe the dynamics between the three species.

In Figure 4, we can observe a comparison of the behaviour of the three species, where we display profiles of the numerical solutions at different times in a 1D slice of the domain, namely the level $y=x$.


Figure 6. Patterns of the three species with different cross diffusion parameters at $t=0.1$

### 6.2. Example 2

As a second numerical example, we consider the system (1.1)-(1.3), where we have chosen the same biological parameters, nonlocal diffusion $d_{i}$, and the initial distributions for the species as in Example 1. In this case, the objective is to observe qualitatively and graphically, the sensitivity with respect to the cross diffusion by varying the parameters $\alpha_{i}$, with $i=1,2,3$.

In Figures 5 and 6 , it is clearly seen the effect of the cross diffusion to obtain spatial patterns. It is possible to observe different patterns when we change the values of $\alpha_{1}, \alpha_{2}$ and $\alpha_{3}$, all the results are obtained for $t=0.1$.

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