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A NOTE ON UNIVOQUE SELF-STURMIAN NUMBERS *

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Abstract. We compare two sets of (infinite) binary sequences whose suffixes satisfy extremal conditions: one occurs when studying iterations of unimodal continuous maps from the unit interval into itself, but it also characterizes univoque real numbers; the other is a disguised version of the set of characteristic Sturmian sequences. As a corollary to our study we obtain that a real number β in $(1, 2)$ is univoque and self-Sturmian if and only if the β-expansion of 1 is of the form $1v$, where v is a characteristic Sturmian sequence beginning itself in 1.

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1. Introduction

The kneading sequences of a unimodal continuous map f from $[0, 1]$ into itself, with $f(1) = 0$ and $\sup f = 1$, are classically studied by first looking at the combinatorial properties of the kneading sequence of 1. Cosnard proved that, using a simple bijection on binary sequences (namely mapping the sequence $(x_n)_{n\geq 0}$ to $(y_n)_{n\geq 0}$, where $y_n := \sum_{0\leq j\leq n} x_j \bmod 2$), the set of kneading sequences of 1 for all maps f as above, maps to the set Γ defined by

$$\Gamma := \{u = (u_n)_{n\geq 0} \in \{0, 1\}^{\mathbb{N}},\ \forall k \geq 0,\ \overline{u} \leq S^k u \leq u\}$$

where $\overline{u} = (\overline{u}_n)_{n\geq 0}$ is the sequence defined by $\overline{u}_n := 1 - u_n$, where S^k is the kth iterate of the shift (*i.e.*, $S^k((u_n)_{n\geq 0}) := (u_{n+k})_{n\geq 0}$), and where $\leq$ is the lexicographical order on sequences induced by $0 < 1$. See [2,10], where the relevant set is actually $\Gamma \setminus \{(10)^\infty\}$. See also [1] for a detailed combinatorial study of the set Γ.

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A slight modification of the set Γ describes the expansions of 1 in the bases β, where β runs through the univoque numbers belonging to $(1,2)$. Recall that a number β is called *univoque* if 1 admits only one expansion in base β. More precisely, the set of expansions of 1 in all the univoque bases $\beta \in (1,2)$ is the set

$$\Gamma_1 := \{u = (u_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 0,\ \overline{u} < S^k u < u\}$$

(see [11], Rem. 1 p. 379; see also [4] and the bibliography therein).

Remark 1.1. Note that a binary sequence belongs to Γ_1 if and only if it belongs to Γ and is not purely periodic.

Other sequences can be defined by extremal properties of their suffixes: characteristic Sturmian sequences and Sturmian sequences. More precisely the following results can be found in several papers (see in particular [7,8,14–16]; see also the survey [6] and the discussion therein).

A binary sequence $u = (u_n)_{n\geq 0}$ is characteristic Sturmian if and only if it is not periodic and belongs to the set Ξ defined by

$$\Xi := \{u = (u_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 0,\ 0u \leq S^k u \leq 1u\}.$$

A binary sequence $u = (u_n)_{n\geq 0}$ is Sturmian if and only if it is not periodic and there exists a binary sequence $v = (v_n)_{n\geq 0}$ such that u belongs to Ξ_v, where

$$\Xi_v := \{u = (u_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}},\ \forall k \geq 0,\ 0v \leq S^k u \leq 1v\}.$$

The sequence v has the property that $1v = \sup_k S^k u$ and $0v = \inf_k S^k u$. This is the characteristic Sturmian sequence having the same slope as u.

Remark 1.2. The reader can find the essentials on Sturmian sequences in [13] Chapter 2. A hint for the proof of the two assertions above is that a sequence is Sturmian if and only if it is not periodic and for any binary (finite) word w, the words $0w0$ and $1w1$ cannot be simultaneously factors of the sequence. Furthermore a sequence u is characteristic Sturmian if and only if $0u$ and $1u$ are both Sturmian.

2. Comparing the sets Γ and Ξ

The analogy between the definitions of Γ and Ξ suggests the natural question whether any sequence can belong to their intersection. The disappointing answer is the following proposition.

Proposition 2.1. *A sequence $u \in \{0,1\}^{\mathbb{N}}$ belongs to $\Gamma \cap \Xi$ if and only if it is equal to 1^∞ or there exists $j \geq 1$ such that $u = (1^j 0)^\infty$.*

Proof. If the sequence u belongs to Γ, we have in particular $u \geq \overline{u}$. Hence $u = 1w$ for some binary sequence w. If u is not equal to 1^∞ (which clearly belongs to $\Gamma \cap \Xi$), let us write $u = 1^j 0z$ for some integer $j \geq 1$ and some binary sequence z. Since u belongs to Γ we have $S^{j+1}u \leq u$, *i.e.*, $z \leq u$. Now u belongs to Ξ, thus $S^j u \geq 0u$, *i.e.*, $0z \geq 0u$, hence $z \geq u$. This gives $z = u$. Hence $u = (1^j 0)^\infty$, which in turn clearly belongs to $\Gamma \cap \Xi$. □

The next question is whether a Sturmian sequence can belong to Γ. The answer is more interesting.

Proposition 2.2. *A (binary) Sturmian sequence u belongs to Γ if and only if there exists a characteristic Sturmian sequence v such that v begins in 1 and $u = 1v$.*

Proof. Let us first suppose that the Sturmian sequence u belongs to Γ. As above, since u belongs to Γ, u begins in 1. Hence $u = 1w$ for some binary sequence w. The inequalities $S^k u \leq u$ for all $k \geq 0$ imply that $\sup_k S^k u = u$ (the inequality $\geq$ is trivial since $S^0 u = u$). This can be written $\sup_k S^k u = 1w$. On the other hand u is Sturmian, hence there exists a characteristic Sturmian sequence v such that u belongs to Ξ_v. We also know that v is such that $1v = \sup_{k \geq 0} S^k u$. Hence $v = w$. Now $\inf_k S^k u = 0v = 0w$. But $S^k u \geq \overline{u}$ for all $k \geq 0$, since u belongs to Γ. Hence $0v = 0w \geq \overline{u} = 0\overline{w}$, thus $w \geq \overline{w}$, hence w begins in 1.

If, conversely, $u = 1v$ where v is a characteristic Sturmian sequence (which actually implies that u is Sturmian) beginning in 1, we first note that $0v \leq S^k v \leq 1v$ for all $k \geq 0$. Hence, immediately, $S^k u \leq 1v = u$ for all $k \geq 0$ (using that $S^{k+1} u = S^k v$ and that $S^0 u = u = 1v$). And also $S^k u \geq 0v \geq 0\overline{v} = \overline{u}$ (using furthermore that $v \geq \overline{v}$ since v begins in 1, and that $S^0 u = u \geq \overline{u}$ since u begins in 1). □

Remark 2.3. We see in particular that a Sturmian sequence belonging to Γ must begin in 11. This is not surprising since the only sequence belonging to Γ that begins in 10 is $(10)^\infty$. This claim is a particular case of a lemma in [1]: *if a sequence t belonging to Γ begins with $m\overline{m}$, where m is a (finite) nonempty binary word, then $t = (m\overline{m})^\infty$.*

3. Univoque self-Sturmian numbers

Several papers were devoted to univoque numbers having an extra property. For example:

- the smallest univoque number in $(1, 2)$ is determined in [12]; it is related to the celebrated Thue-Morse sequence and was proven transcendental in [3];
- univoque Pisot numbers belonging to $(1, 2)$ are studied in [5].

The notion of self-Sturmian numbers was introduced in [9]. These are the real numbers β such that the greedy β-expansion of 1 is a Sturmian sequence on some two-digit alphabet. It is tempting to ask which univoque numbers are self-Sturmian. We restrict the study to the numbers in $(1, 2)$ for simplicity.

Proposition 3.1. *The real self-Sturmian numbers in* $(1, 2)$ *that are univoque are exactly the real numbers* β *such that* $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$*, where* $u = (u_n)_{n \geq 0}$ *is a binary sequence of the form* $u = 1v$*, with* v *a characteristic Sturmian sequence beginning in* 1.

Proof. This is a rephrasing of Proposition 2.2. □

Remark 3.2.

– The equality $1 = \sum_{n \geq 1} \frac{u_n}{\beta^n}$, where $u = (u_n)_{n \geq 0}$ is a binary sequence, uniquely determines the real number β in $(1, 2)$.

– Self-Sturmian numbers correspond to Sturmian sequences of the form $u = 1v$, where v is any characteristic Sturmian sequence (see [9], Rem. p. 399). All self-Sturmian numbers are transcendental [9].

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