RAIRO-Theor. Inf. Appl. 51 (2017) 205–216
https://doi.org/10.1051/ita/2017015

RAIRO - Theoretical Informatics and Applications
www.rairo-ita.org

A UNIFORM CUBE-FREE MORPHISM IS k-POWER-FREE FOR ALL INTEGERS $k \geq 4$

Francis Wlazinski*

Abstract. In the study of k-power-free morphisms, the case of 3-free-morphisms, *i.e.*, cube-free morphisms, often differs from other k-power-free morphisms. Indeed, cube-freeness is less restrictive than square-freeness. And a cube provides less equations to solve than any integer $k \geq 4$. Anyway, the fact that the image of a word by a morphism contains a cube implies relations that, under some assumptions, allow us to establish our main result: a cube-free uniform morphism is a k-power-free morphism for all integers $k \geq 4$.

Mathematics Subject Classification. 68R15

Accepted December 4, 2017.

1. Introduction

For an integer $k \geq 2$, a k-power is a repetition of k consecutive and identical sequences. A square and a cube respectively correspond to $k = 2$ and $k = 3$. An overlap is a word of the form $xuxux$ where x is a letter and u is a word. An infinite word without squares over a three-symbol alphabet and an infinite word without overlaps over a two-symbol alphabet were given by Thue [19, 20] (see also Berstel's translation [3]). These words are obtained by iteration of morphisms.

To find morphisms that generate k-power-free words, a method is to consider k-power-free morphisms. By definition, a k-power-free morphism maps a k-power-free word to a k-power-free word. Starting with a letter (so k-power-free), the word generated by the iteration of a k-power-free morphism is thus k-power-free. But non-k-power-free morphisms can generate k-power-free words. For instance, the Fibonacci morphism $\varphi : \{a, b\} \to \{a, b\}; a \mapsto ab; b \mapsto a$ generates the word $abaababaabaababaababa...$, which is 4-power-free [8]. Although φ is not 4-power-free since $\varphi(b^3a) = a^4b$.

Even if we know different ways to verify whether a morphism generates a k-power-free word [2, 6, 8, 16] and also whether a morphism is k-power-free [5, 7, 10, 11, 17, 18, 21], a simple question remains unanswered:

"Is a k-power-free morphism also a $(k + 1)$-power-free morphism?" (1)

It is difficult to give a general answer to this question. Some examples of morphisms are available. For instance, in [1], Bean, Ehrenfeucht, and McNulty gave the morphism h defined by $h(a) = abacbab$, $h(b) = cdabcabd$, $h(c) = cdacabcbd$ and $h(d) = cdacbcacbd$ that is square-free but not cube-free. An another example is given by the Thue-Morse morphism, $\mu : \{a, b\} \to \{a, b\}; a \mapsto ab; b \mapsto ba$. It is k-power-free for every integer $k > 2$ [4]. In restricted situations, some partial results have been given for $k = 2$ [12], for binary morphisms [10] and for

Keywords and phrases: Morphisms, repetition, cube, k-power.

L.A.M.F.A., UMR 7352 du CNRS, Université de Picardie Jules Verne, 33 rue Saint-Leu, 80039 Amiens Cedex 1, France.

* Corresponding author: wlazinski@u-picardie.fr

Article published by EDP Sciences

binary uniform morphisms [9]. In [22] and in this paper, we are interested in uniform morphisms from A^* to B^* for every A and B. When $k \geq 4$, we proved in [22] that a k-power-free uniform morphism is $(k+1)$-power-free. But one of the properties required to prove this result (Lem. 2.10) does not hold for $k = 3$. In this paper, we give a new result (Lem. 3.4) that allows us to conclude for cube-free uniform morphisms (Thm. 4.3): a cube-free uniform morphism from A^* to B^* is k-power-free for all integers $k \geq 4$. This gives a final answer to question (1) for uniform morphisms.

2. Preliminaries

We assume the reader is familiar (if not, see for instance [14, 15]) with basic notions on words and morphisms. Our notations largely come from [22].

2.1. Words

Let w be a non-empty word. For all $1 \leq i \leq j \leq |w|$, we denote by $w[i..j]$ the factor of w such that $w = pw[i..j]s$ for two words s and p with $|p| = i - 1$. When $i = j$, we also denote by $w[i]$ the factor $w[i..i]$, which is the i^{th} letter of w. In particular, $w[1]$ and $w[|w|]$ are respectively the first and the last letter of w. We denote by $\mathrm{Fcts}\,(w)$ the set of all factors of the word w.

Powers of a word u are defined inductively by $u^0 = \varepsilon$ and $u^n = uu^{n-1}$ for all integers $n \geq 1$. Given an integer $k \geq 2$, we call a k-power any word u^k with $u \neq \varepsilon$ (since the case ε^k is of little interest). Given an integer $k \geq 2$, a word is *k-power-free* if it does not contain any k-power as a factor. A *primitive* word is a word that is not a k-power of an another word whatever the integer $k \geq 2$. A (non-empty) k-power v^k is called *pure* if any proper factor of v^k is k-power-free.

The following proposition gives the well-known solutions (see [13]) to two elementary equations on words and will be widely used in the following sections:

Proposition 2.1. *Let A be an alphabet and u, v, w three words over A.*

1. *If $vu = uw$ and $v \neq \varepsilon$, then there exist two words r and s over A, and an integer n such that $u = r(sr)^n$, $v = rs$ and $w = sr$.*
2. *If $vu = uv$, then there exist a word w over A, and two integers n and p such that $u = w^n$ and $v = w^p$.*

We also need a property on words that is an immediate consequence of Proposition 2.1(2).

Lemma 2.2. [10, 12] *If a non-empty word v is an internal factor of vv,* i.e.*, if there exist two non-empty words x and y such that $vv = xvy$, then there exist a non-empty word t and two integers $i, j \geq 1$ such that $x = t^i$, $y = t^j$, and $v = t^{i+j}$.*

We also use the following result which is a corollary of a result of Fine and Wilf [13, 14].

Corollary 2.3. [10] *Let x and y be two words. If a power of x and a power of y have a common factor of length at least equal to $|x| + |y| - gcd(|x|, |y|)$, then there exist two words t_1 and t_2 such that x is a power of t_1t_2 and y is a power of t_2t_1 with t_1t_2 and t_2t_1 primitive words. Furthermore, if $|x| > |y|$ then x is not primitive.*

2.2. Morphisms

Let A and B be two alphabets. A *morphism* f from A^* to B^* is a mapping from A^* to B^* such that $f(uv) = f(u)f(v)$ for all words u, v over A.

Let $k \geq 2$ be an integer. A morphism f on A is *k-power-free* if $f(w)$ is k-power-free for all k-power-free words w over A.

A morphism f on A is called *prefix* (resp. *suffix*) if, for all letters a and b in A, the word $f(a)$ is not a prefix (resp. not a suffix) of $f(b)$. A prefix (resp. suffix) morphism f is non-erasing that is $f(a) \neq \varepsilon$ for all letters a. A morphism is *bifix* if it is prefix and suffix.

A morphism f from A^* to B^* is a *ps-morphism* if the equalities $f(a) = ps$, $f(b) = ps'$, and $f(c) = p's$ with $a, b, c \in A$ (possibly $c = b$) and p, s, p', and $s' \in B^*$ imply $b = a$ or $c = a$.

Let us recall some definitions and properties that will be used in the sequel. Lemma 2.4 derives directly from the definitions of a prefix or a suffix morphism. A detailed proof is left to the reader.

Lemma 2.4. *Let f be a bifix morphism on an alphabet A and let u, v, w, and t be words over A. The equality $f(u) = f(v)p$ where p is a prefix of $f(w)$ implies $u = vw'$ for a prefix w' of w such that $f(w') = p$. Symetrically, the equality $f(u) = sf(v)$ where s is suffix of $f(t)$ implies $u = t'v$ for a suffix t' of t such that $f(t') = s$.*

Lemma 2.5. [10, 12] *If f is not a ps-morphism then f is not a k-power-free morphism for all integers $k \geq 2$.*

Lemma 2.6. [22] *Let f be a ps-morphism from A^* to B^* and let u, v and w be words over A such that $f(u) = \delta\beta$, $f(v) = \alpha\beta$, and $f(w) = \alpha\gamma$ for some non-empty words α, β, γ, and δ over B. Then $v = v_1av_2$, $u = u_1bv_2$, and $w = v_1cw_2$ for some words v_1, v_2, u_1, and w_2, and some letters a, b, and c. Moreover, we have either $b = a$ or $c = a$.*

Furthermore, if $|\delta| < |f(u[1])|$, then $u_1 = \varepsilon$ and, if $|\gamma| < |f(w[|w|])|$, then $w_2 = \varepsilon$.

Assuming that $f(\overline{w}) = pu^ks$ for a factor $\overline{w}$ of a word w and a non-empty word u, and assuming that $\overline{w}$ contains a factor w_0 such that $|f(w_0)| = |u|$, Lemma 2.8 states that $\overline{w}$ necessarily contains a k-power w'^k such that $f(w')$ is a conjugate of u. We will say that $f(w)$ contains a synchronised k-power u^k. More precisely:

Definition 2.7. Let $k \geq 2$ be an integer. Let f be a morphism from A^* to B^*, w be a word over A, and u be a non-empty word over B such that $f(w)$ contains the k-power u^k. Let $\overline{w}$ be a shortest factor of w whose image by f contains u^k, *i.e.*, $f(\overline{w}) = pu^ks$ with $|p| < |f(\overline{w}[1])|$ and $|s| < |f(\overline{w}[|\overline{w}|])|$.

We say that $f(w)$ and u^k are synchronised if there exist three words w_0, w_1, and w_2 such that $|f(w_0)| = |u|$ and $\overline{w} = w_1w_0w_2$ with $p = \varepsilon$ if $w_1 = \varepsilon$, and $s = \varepsilon$ if $w_2 = \varepsilon$.

For instance, let f be the morphism from $\{a, b, c\}^*$ to $\{a, b, c, d\}^*$ defined by $f(a) = abcd$, $f(b) = ac$ and $f(c) = d$. If w is the word $abcbcbaa$, then we have $f(w) = ab(cda)^3\, c\, abcd\, abcd$. Taking $u = cda$, $\overline{w} = abcbcb$, $w_0 = bc$, we get that $f(w)$ and u^3 are synchronised.

The two following results, which will be used in this paper, were proved in [22].

Lemma 2.8. *Let $k \geq 2$ be an integer, let f be a ps-morphism, and let w be a word such that $f(w) = pu^ks$ with $|p| < |f(w[1])|$ and $|s| < |f(w[|w|])|$. If $f(w)$ contains a synchronised k-power then w starts or ends with a k-power w_0^k such that $f(w_0)$ and u are conjugated.*

The second recalled result is the main result in [22]:

Proposition 2.9. *Let A and B be two alphabets and let $k \geq 4$ be an integer. A k-power-free uniform morphism is a $(k+1)$-power-free morphism.*

The first goal is to give, for $k = 3$, a similar lemma to the one given in [22] for $k \geq 4$ that is:

Lemma 2.10. *Let $k \geq 4$ be an integer. Let f be a ps-morphism from A^* to B^*. Let v and T be non-empty words over A such that v^k is a pure k-power. Let us assume that $f(T) = \pi_1 f(v)^k \sigma_2$ with $|\pi_1| < |f(T[1])|$ and $|\sigma_2| < |f(T[|T|])|$. Then one of the following holds:*

- $(P.1)$ *: There exist a pure k-power x^k, a word y over A, and a word Z over B such that*
  $(P.1.1)$ *: $T = x^k y$, $|y| \leq 1$, $f(y) = \pi_1\sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$*
  $(P.1.2)$ *: or $T = yx^k$, $|y| = 1$, $f(y) = \pi_1\sigma_2$, $f(x) = Z\sigma_2$, and $f(v) = \sigma_2 Z$.*
- $(P.2)$ *: There exist a pure k-power x^k and a non-empty word y over A such that*
  $(P.2.1)$ *: $T = x^k y$ with $|f(x^{k-1})| < |\pi_1 f(v)|$*
  $(P.2.2)$ *: or $T = yx^k$ with $|f(x^{k-1})| < |f(v)\sigma_2|$.*
- $(P.3)$ *: f is not k-power-free.*

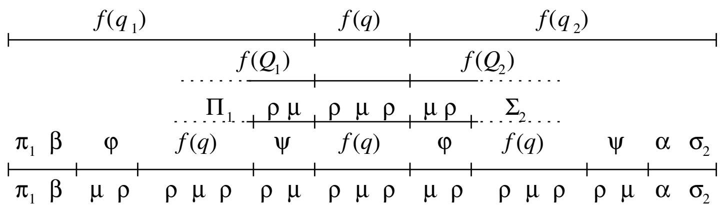

FIGURE 1. Image of $f(q_1qq_2)$.

3. About cube-free-morphisms

As mentioned in the introduction and in Section 2, Lemma 2.10 is no longer valid for $k = 3$. Even if parts of its proof can be extended to this case, some new problems appear. The following lemma is one of the situations we can obtain:

Lemma 3.1. *Let f be a ps-morphism from A^* to B^*. Let us assume that there exist ρ, μ, α, β, and θ words over B and q_1, q_2, and q words over A such that $q \neq \varepsilon$, $\rho = \alpha\beta$ is not the image of a word by f, $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(q) = \rho\mu\rho$, $f(q_1)$ ends with $\beta\theta$, $f(q_2)$ starts with $\theta\alpha$, and the words $q_1[2..|q_1|]\,q$ and $q\,q_2[1..|q_2|-1]$ are cube-free. Then f is not cube-free.*

The proof of Lemma 3.1 is simply done using iteratively Lemma 3.2. By induction, if f was not cube-free, we could find an infinite sequence $(\chi_i)_{i\geq 0}$ of non-empty words starting with $\chi_0 = q$ such that $|f(\chi_i)| > |f(\chi_{i+1})| > 0$; this is impossible.

Lemma 3.2. *Let f be a ps-morphism from A^* to B^*. Let us assume that there exist ρ, μ, α, β, and θ words over B and q_1, q_2, and q words over A such that $q \neq \varepsilon$, $\rho = \alpha\beta$ is not the image of a word by f, $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(q) = \rho\mu\rho$, $f(q_1)$ ends with $\beta\theta$, $f(q_2)$ starts with $\theta\alpha$, and the words $q_1[2..|q_1|]\,q$ and $q\,q_2[1..|q_2|-1]$ are cube-free.*

Then either f is not cube-free or there exist ρ', μ', α', β', and θ' words over B and q_1', q_2', and q' words over A such that $q' \neq \varepsilon$, $\rho' = \alpha'\beta'$ is not the image of a word by f, $\theta' = \mu'\rho'\rho'\mu'\rho'\rho'\mu'$, $f(q') = \rho'\mu'\rho'$, $f(q_1')$ ends with $\beta'\theta'$, $f(q_2')$ starts with $\theta'\alpha'$, and the words $q_1'[2..|q_1'|]\,q'$ and $q'\,q_2'[1..|q_2'|-1]$ are cube-free.

Moreover, we have $|f(q')| < |f(q)|$.

Proof. Let us first remark that the condition $q \neq \varepsilon$ is simply a consequence of the fact that ρ is not the image of a word by f. Indeed, it implies $\rho \neq f(\varepsilon) = \varepsilon$. Therefore, $f(q) = \rho\mu\rho \neq \varepsilon$ and so $q \neq \varepsilon$. Moreover, the fact that f is a ps-morphism implies that f is bifix and non-erasing.

We can write $f(q_1) = \pi_1\beta\theta$ and $f(q_2) = \theta\alpha\sigma_2$ for two words π_1 and σ_2 over B. Let $\varphi = \mu\rho$ and $\psi = \rho\mu$ so $\theta = \varphi f(q)\psi$.

Let Q_1 be the smallest suffix of q_1 whose image by f ends with ψ and Q_2 be the smallest prefix of q_2 whose image by f starts with φ. Since $\rho \neq \varepsilon$, Q_1 and Q_2 are not empty. There exist two words Π_1 and Σ_2 such that $f(Q_1) = \Pi_1\psi$, $f(Q_2) = \varphi\Sigma_2$, $|\Pi_1| < |f(Q_1[1])|$ and $|\Sigma_2| < |f(Q_2[|Q_2|])|$.

We have that $f(Q_1qQ_2) = \Pi_1\psi f(q)\varphi\Sigma_2 = \Pi_1(\rho\mu)^3\rho\Sigma_2$ and that Q_1qQ_2 is a factor of q_1qq_2.

The situation can be summed up by Figure 1.

If Q_1qQ_2 is cube-free, then f is not cube-free. Otherwise, Q_1qQ_2 contains a shortest cube q'^3; any proper factor of q'^3 is cube-free, *i.e.*, q'^3 is a pure cube. So we can write $Q_1qQ_2 = q_1'q'^3q_2'$ for some words q_1' and q_2'.

Let us remark that Q_1q is necessarily cube-free. Indeed, if $Q_1 \neq q_1$ then Q_1q is a suffix of $q_1[2..|q_1|]q$ which is cube-free by assumption. If $Q_1 = q_1$, by definition of Q_1, ψ is not a suffix of $f(q_1[2..|q_1|])$ and so $|f(q_1[2..|q_1|])| < |\rho\mu|$. It implies that $|f(Q_1[1])| = |f(q_1[1])| > |\pi_1\beta\varphi f(q)| \geq |f(q_1[2..|q_1|]q)|$. In particular, it means that the first letter of q_1, *i.e.*, $q_1[1] = Q_1[1]$ is not a letter of $q_1[2..|q_1|]q$. Since no cube appears in $q_1[2..|q_1|]q$, it follows that $q_1q = Q_1q$ is cube-free.

In the same way, we obtain that qQ_2 is cube-free.

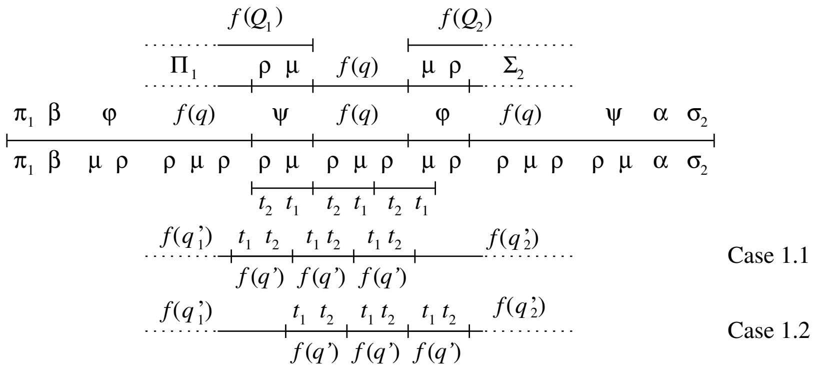

FIGURE 2. Case 1.

So q'^3 is neither a factor of Q_1q nor a factor of qQ_2. It follows that $|q'_1q'^3| > |Q_1q|$ and $|q'^3q'_2| > |qQ_2|$, that is, $|q'_2| < |Q_2|$ and $|q'_1| < |Q_1|$.

Let φ' be the greatest prefix of $\varphi = \mu\rho$ such that $f(Q_1q)\varphi'$ is a prefix of $f(q'_1q'^3)$. Let ψ' be the greatest suffix of $\psi = \rho\mu$ such that $\psi' f(qQ_2)$ is a suffix of $f(q'^3q'_2)$.

With these definitions, the word $\psi' f(q)\varphi' = \psi'\rho\mu\rho\varphi'$ is a common factor (not necessarily the greatest) of $f(q')^3$ and $(\rho\mu)^4$. In order to use Corollary 2.3, we have to study $|\psi' f(q)\varphi'|$. Note that the inequality $|\psi' f(q)\varphi'| \geq |f(q')| + |\rho\mu|$ is equivalent to $|f(q')| \leq |\psi'| + |\varphi'| + |\rho|$.

Case 1: $|f(q')| \leq |\psi'| + |\varphi'| + |\rho|$.

By Corollary 2.3, there exist two words t_1 and t_2, and two integers i and j such that $f(q') = (t_1t_2)^i$ and $\psi = \rho\mu = (t_2t_1)^j$ where t_1t_2 and t_2t_1 are primitive words. If $j \geq 2$, $f(Q_1q)$ ends with $\psi f(q) = (\rho\mu)^2\rho = (t_2t_1)^{2j}\rho$ where $2j > 3$. If $i \geq 2$, $f(q'^2) = (t_1t_2)^{2i}$ with q'^2 a proper prefix of q'^3 and $2i > 3$. In these two cases, the image by f of a cube-free word contains a cube; f is not cube-free. So $i = j = 1$.

We have $f(Q_1qQ_2) = \Pi_1(t_2t_1)^3\rho\Sigma_2 = f(q'_1q'^3q'_2) = f(q'_1)(t_1t_2)^3 f(q'_2)$. Since t_1t_2 is not an internal factor of $(t_1t_2)^2$ (otherwise, by Lemma 2.2, t_1t_2 would not be primitive), $|f(q'_1)| < |f(Q_1)| = |\Pi_1t_2t_1|$ and $|f(q'_2)| < |f(Q_2)| = |t_2t_1\Sigma_2|$, we have either $\Pi_1 = f(q'_1)t_1$ and $f(q'_2) = t_1\rho\Sigma_2$, or $\Pi_1t_2 = f(q'_1)$ and $t_2 f(q'_2) = \rho\Sigma_2$. This situation can be summed up by Figure 2.

Case 1.1: $\Pi_1 = f(q'_1)t_1$.

Since $|\Pi_1| < |f(Q_1[1])| = |f((q'_1q)[1])|$ and f is bifix, $q'_1 = \varepsilon$, $\Pi_1 = t_1$, and $f(Q_1) = \Pi_1\psi = t_1t_2t_1 = f(q')t_1$ where t_1 is a prefix of $f(q')$. Since f is a ps-morphism, and then a bifix morphism, by Lemma 2.4, we obtain that there exists a prefix x (possibly empty) of q' such that $f(x) = t_1$. From $f(q') = t_1t_2 = f(x)t_2$ with t_2 a prefix of $f(q)$ and f bifix, we obtain that there exists a prefix y of q such that $f(y) = t_2$.

From $f(q) = \rho\mu\rho = f(yx)\rho$ and f bifix, we obtain that ρ is the image of a word; a contradiction with the definition of ρ with the hypotheses of this lemma.

Case 1.2: $\Pi_1t_2 = f(q'_1)$.

This case is solved in the same way as Case 1.1.

From $f(Q_1) = \Pi_1t_2t_1 = f(q'_1)t_1$, we obtain that t_1 is the image of a word. From $f(q') = t_1t_2$, we obtain that t_2 is the image of a word. It follows that ρ is the image of a word; a contradiction with the definition of ρ.

Case 2: $|f(q')| > |\psi'| + |\varphi'| + |\rho|$.

If $q'_1 = \varepsilon$, *i.e.*, $Q_1qQ_2 = q'^3q'_2$ then, by definition of ψ', $\psi' = \psi = \rho\mu$. It follows that $|f(q')| > |\psi'| + |\varphi'| + |\rho| = |f(q)| + |\varphi'|$. Furthermore, $|f(q')^3| > |\Pi_1| + 2|f(q)| + 2|\varphi'| = |\Pi_1\psi| + |f(q)| + |\rho| + 2|\varphi'|$, *i.e.*, $|f(q')^3| > |f(Q_1q)| + |\rho| + 2|\varphi'|$. Since $|\rho| \neq 0$, we have $|\rho| + 2|\varphi'| > |\varphi'|$. If $q'_2 \neq \varepsilon$ then $|f(q'_2)| > |\Sigma_2|$ and, by definition of φ', we obtain $|f(q')^3| = |f(Q_1q)\varphi'|$; this is impossible. Hence, $q'_2 = \varepsilon$ and $\varphi' = \varphi = \mu\rho$. It follows that $|f(q')^2| =$

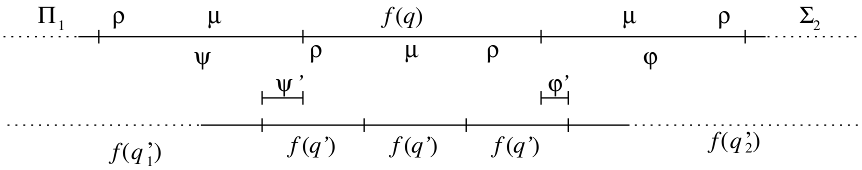

FIGURE 3. Case 2.

$|f(Q_1qQ_2)| - |f(q')| = |\Pi_1\rho\mu f(q)\mu\rho\Sigma_2| - |f(q')| < |\Pi_1\rho\mu\Sigma_2|$. If $|q'| \geq 2$ then $|f(q')f(q')| > |\Pi_1\Sigma_2| + |f(q')| > |\Pi_1\Sigma_2| + |\psi| + |\varphi| + |\rho| = |\Pi_1\Sigma_2| + |\rho\mu\rho\mu\rho|$ with $|\rho| \neq 0$; this conflicts with the previous inequality. Consequently, we have $|q'| = 1$ and $q'^3 = Q_1qQ_2$ with Q_1, q, and Q_2 non-empty words. Therefore, $Q_1 = q = Q_2 = q'$ with $\rho\mu\rho = f(q) = f(Q_1) = \Pi_1\rho\mu$, *i.e.*, $\mu\rho = \rho\mu$. By Proposition 2.1(2), there exist a non-empty word ω over A and two integers n, and p such that $\rho = \omega^n$ and $\mu = \omega^p$. Since $\rho \neq \varepsilon$, we obtain that $n \geq 1$. It follows that $f(Q_1q)$ contains ω^{3n+2p} with Q_1q cube-free and $3n + 2p \geq 3$; f is not cube-free.

In the same way, we obtain that either $q_2' \neq \varepsilon$ or f is not cube-free.

Since $Q_1qQ_2 = q_1'q'^3q_2'$, we have $|f(q_1'[1])| = |f(Q_1[1])| > |\Pi_1|$ and $|f(q_2'[|q_2'|])| = |f(Q_2[|Q_2|])| > |\Sigma_2|$. Hence, $|\psi'| < |\psi|$ and $|\varphi'| < |\varphi|$. It follows from the definitions of ψ' and φ' that $f(q')^3 = \psi' f(q)\varphi'$.

We have $|\psi'| + 2|\rho| + |\mu| + |\varphi'| = |\psi' f(q)\varphi'| = |f(q')^3| > 3(|\psi'| + |\varphi'| + |\rho|)$, that is, $|\mu| > 2|\psi'| + 2|\varphi'| + |\rho|$. It means that μ starts with φ' and ends with ψ'.

This situation can be summed up by Figure 3.

The word $f(q')$ starts with $\psi'\rho$ and ends with $\rho\varphi'$. There exist two words X and Y such that $f(q') = \psi'\rho X = Y\rho\varphi'$. Since $|f(q')| > |\psi'| + |\varphi'| + |\rho|$, we have $|X| > |\varphi'|$ and $|Y| > |\psi'|$. Therefore, there exist two non-empty words X' and Y' such that $X = X'\varphi'$, $Y = \psi'Y'$, and $f(q') = \psi'\rho X'\varphi' = \psi'Y'\rho\varphi'$. It follows that $\rho X' = Y'\rho$. By Proposition 2.1(1), there exist two words r and s and an integer i such that $\rho = r(sr)^i$, $X' = sr$, and $Y' = rs$. Let us also note that μ ends with Y' and starts with X'.

If $i \geq 1$ then $f(Q_1q)$ contains $\mu\rho\mu$ which contains $Y'\rho X' = (rs)^{2+i}r$ with Q_1q cube-free and $2 + i \geq 3$; f is not cube-free.

Let us now consider the case $i = 0$. We have $\rho = r$, $Y' = \rho s$, $X' = s\rho$, and $f(q') = \psi'\rho s\rho\varphi'$. From $f(q')^3 = \psi' f(q)\varphi' = \psi'\rho\mu\rho\varphi'$, we also obtain that $\mu = s\rho\varphi' f(q')\psi'\rho s$. Let us remark that μ starts and ends with s. But the word μ also starts with φ' and also ends with ψ'. In particular, the word $f(Q_1q)$ contains $\mu(\rho)\mu$ which contains $\psi'\rho s(\rho)s\rho\varphi'$.

If $|s| \leq |\varphi'|$ then s is a prefix of φ' and $\psi'\rho s\,\rho\, s\rho\varphi'$ contains the cube $(\rho s)^3$. If $|s| \leq |\psi'|$ then s is a suffix of ψ' and $\psi'\rho s\,\rho\, s\rho\varphi'$ contains the cube $(s\rho)^3$. If $|s| > |\varphi'|$, $|s| > |\psi'|$, and $|s| \leq |\varphi'| + |\psi'|$ then there exist three words a, b, and c such that $s = abc$, $\varphi' = ab$, and $\psi' = bc$. It follows that $\psi'\rho s\,\rho\, s\rho\varphi'$ contains the cube $(bc\rho a)^3$. In these three cases, $f(Q_1q)$ contains a cube with Q_1q cube-free; f is not cube-free.

The remaining case is $|s| > |\varphi'| + |\psi'|$; there exists a non-empty word μ' such that $s = \varphi'\mu'\psi'$ and we have $f(q') = \psi'\rho\varphi'\mu'\psi'\rho\varphi'$.

Let us denote $\alpha' = \psi'\rho$, $\beta' = \varphi'$, $\rho' = \psi'\rho\varphi' = \alpha'\beta'$, and $\theta' = \mu'\rho'\rho'\mu'\rho'\rho'\mu'$. We have $f(q') = \rho'\mu'\rho'$ and $\mu = s\rho\varphi' f(q')\psi'\rho s = \varphi'\mu'\rho' f(q')\rho'\mu'\psi' = \varphi'\theta'\psi'$.

Since $f(Q_1) = f(q_1')\psi'$ ends with μ, we obtain that $f(q_1')$ ends with $\varphi'\theta' = \beta'\theta'$. And, since $f(Q_2) = \varphi' f(q_2')$ starts with $\mu\rho$, we obtain that $f(q_2')$ starts with $\theta'\psi'\rho = \theta'\alpha'$. Moreover, by Lemma 2.4, ψ' and φ' are images of words by f.

Since f is bifix, if $\rho' = \psi'\rho\varphi'$ is the image of a word by f, then so is ρ; a contradiction with the hypotheses. So ρ' is not the image of a word by f.

Since $q_1'q'$ is a prefix of Q_1q and $q'q_2'$ is a suffix of qQ_2, the words $q_1'q'$ and $q'q_2'$ are cube-free.

Finally, since $3|f(q')| = |\psi' f(q)\varphi'| < |\psi f(q)\varphi| \leq 3|f(q)|$, we obtain that $|f(q')| < |f(q)|$. □

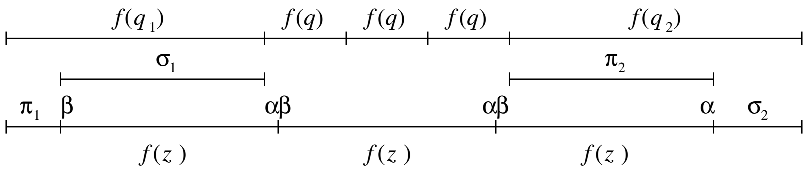

FIGURE 4. First decomposition.

When the image of a word w by a morphism f covers a cube, three factors of $f(w)$ are identical. This gives us three equations on words. To solve these equations when f is cube-free is the first part of the study. A first conclusion is given by the following lemma; a cube in w necessarily has a particular form.

Lemma 3.3. *Let f be a ps-morphism from A^* to B^*. Let q_1, q_2, and q be non-empty words over A and let $w = q_1q^3q_2$. Let us assume that $f(w) = \pi_1 f(z)^3\sigma_2$, $|\pi_1| < |f(q_1[1])|$, $|\sigma_2| < |f(q_2[|q_2|])|$, and z is a non-empty word over A such that z^3 is a pure cube.*

If $|q| \leq 2$, if $|z| \geq 2$, or if $2|f(q)| \geq |f(z)|$ then f is not cube-free.

Otherwise, either f is not cube-free or there exist two words X and Y such that $|Y| \leq 1$, $2|f(q)| < |f(X)| = |f(z)| < 3|f(q)|$, and $q_1q^3q_2 = X^3Y$ with q_1 a prefix of X, q_2 a suffix of XY, and $|f(Y)| < |f(z)\sigma_2|$ or $q_1q^3q_2 = YX^3$ with q_1 a prefix of YX, q_2 a suffix of X, and $|f(Y)| < |\pi_1 f(z)|$.

Proof. Let us first remark that we can assume that the image by f of any proper factor of z^3 is cube-free.

The hypotheses imply $|f(z)| > |f(q)|$. Furthermore, there exist a suffix σ_1 of $f(q_1)$ and a prefix π_2 of $f(q_2)$ such that $f(q_1) = \pi_1\sigma_1$ and $f(q_2) = \pi_2\sigma_2$. It means that $f(z)^3 = \sigma_1 f(q)^3\pi_2$.

If $|\sigma_1 f(q^3)| \leq |f(z)^2 f(z[1..|z|-1])|$ then $f(z)^2 f(z[1..|z|-1])$ contains the cube $f(q)^3$ with $z^2(z[1..|z|-1])$ a proper factor of z^3 and so cube-free. It ends the proof; f is not cube-free. Identically, if $|f(q^3)\pi_2| \leq |f(z[2..|z|])f(z)^2|$ then f is not cube-free. Consequently, we obtain that $|\pi_2| < |f(z[|z|])|$ and $|\sigma_1| < |f(z[1])|$; there exist two non-empty words α and β such that $f(z) = \sigma_1\alpha = \beta\pi_2$ and $f(q)^3 = \alpha f(z)\beta$.

This situation can be summed up by Figure 4.

Let us note that $\alpha \neq \varepsilon$ or $\beta \neq \varepsilon$. Otherwise, $f(z^2)$ contains $f(q)^3$. It ends the proof; f is not cube-free.

If $|f(q^2)| \geq |f(z)|$ then $|\alpha| + |\beta| = |f(q^3)| - |f(z)| \geq |f(q)|$ and the length of $\alpha f(z)\beta \in \mathrm{Fcts}\big(f(z)^3\big) \cap \mathrm{Fcts}\big(f(q)^3\big)$ is at least $|f(z)| + |f(q)|$. Let us note that this situation happens in particular when $|z| \geq 2$ because in this case $|\alpha| + |\beta| \geq |f(z[2..|z|])| + |f(z[1..|z|-1])| \geq |f(z)| > |f(q)|$. By Corollary 2.3, there exist two words z_1 and z_2, and two integers i and j such that $f(z) = (z_1z_2)^i$ and $f(q) = (z_2z_1)^j$. The inequality $|f(z)| > |f(q)| (> 0)$ implies $i > j \geq 1$. It follows that $f(z)^2 = (z_1z_2)^{2i}$ with $2i > 3$, that is, $f(z)^2$ contains a cube; f is not cube-free.

From now on, z is a letter and $|f(q^2)| < |f(z)|$, *i.e.*, $0 < |\alpha| + |\beta| < |f(q)|$. We obtain that $f(q)^3 = \alpha f(z)\beta$ starts and ends with $\alpha\beta$, *i.e.*, $\alpha\beta$ is a prefix and a suffix of $f(q)$. There exist two non-empty words φ and ψ such that $f(q) = (\alpha\beta)\varphi = \psi(\alpha\beta)$ and $f(z) = \beta\varphi f(q)\psi\alpha$. By Proposition 2.1(1), there exist two words ρ and μ, and an integer j such that $\alpha\beta = \rho(\mu\rho)^j$, $\varphi = \mu\rho$, and $\psi = \rho\mu$.

Since $f(z)^2$ contains $(\psi\alpha)(\beta\varphi) = \rho\mu\rho(\mu\rho)^j\mu\rho = \rho(\mu\rho)^{j+2}$ and $f(z)^2$ is cube-free, we necessarily have $j = 0$.

In this case, if we pose $\theta = \varphi f(q)\psi$, we have $\alpha\beta = \rho$, $f(q) = \rho\mu\rho$, $f(z) = \beta\mu\rho f(q)\rho\mu\alpha$. Thus we have $\theta = \mu\rho\rho\mu\rho\rho\mu$, $f(z) = \beta\theta\alpha$, $f(q_1) = \pi_1\beta\theta$ and $f(q_2) = \theta\alpha\sigma_2$.

Since f is a *ps*-morphism, and hence, is non-erasing and since $q \neq \varepsilon$, we have $\rho\mu \neq \varepsilon$. Since $f(z)$ is cube-free, we necessarily have $\mu \neq \varepsilon$ and $\rho \neq \varepsilon$. Otherwise, the factor $\theta = \mu\rho\rho\mu\rho\rho\mu$ of $f(z)$ would be equal to ρ^4 or μ^3.

We also necessarily have $q_1[2..|q_1|]\,q$ and $q\,q_2[1..|q_2|-1]$ cube-free. Otherwise, $f(q_1[2..|q_1|]\,q)$ or $f(q\,q_2[1..|q_2|-1])$ both factors of $f(z)^2$ would contain a cube.

By Lemma 3.1, if ρ is not the image of a word by f then f is not cube-free; it ends the proof.

Let us now assume that ρ is the image of a word and let $\dot{q}$ be the non-empty word such that $f(\dot{q}) = \rho = \alpha\beta(\neq \varepsilon)$. Since f is bifix, $f(q) = \rho\mu\rho$ and $\mu \neq \varepsilon$, there exists a non-empty word $\overline{q}$ such that $f(\overline{q}) = \mu$. It follows that $q = \dot{q}\overline{q}\dot{q}$ and necessarily $|q| \geq 3$. In particular, $|f(qq\dot{q}\overline{q})| = |f(z)|$ and $qq\dot{q}\overline{q}$ is an internal factor of w; $f(w)$ and $f(z)^3$ are synchronised.

Furthermore, if we denote $x = \overline{q}\dot{q}q\dot{q}\overline{q}$, we obtain that $f(x) = \theta$, $f(q_1) = \pi_1\beta f(x)$, and $f(q_2) = f(x)\alpha\sigma_2$. Since f is bifix, it follows, by Lemma 2.4, that $q_1 = W_1x$ for a non-empty word W_1 satisfying $f(W_1) = \pi_1\beta(\neq \varepsilon)$ and $q_2 = xW_2$ for a non-empty word W_2 satisfying $f(W_2) = \alpha\sigma_2(\neq \varepsilon)$.

Since $|\pi_1| < |f(q_1[1])| = |f(W_1[1])|$, $|\sigma_2| < |f(q_2[|q_2|])| = |f(W_2[|W_2|])|$, and $f(\dot{q}) = \rho = \alpha\beta$, by Lemma 2.6, we obtain that $\dot{q} = a$, $W_1 = b$, $W_2 = c$ for some letters a, b, c. Moreover, we have $b = a$ or $c = a$. It means that $q_1q^3q_2 = bx(\dot{q}\overline{q}\dot{q})^3xc = bxaxaxc$.

If $b = a$, let $X = ax$. It follows that $q_1q^3q_2 = X^3c$ with $|f(X)| = |f(ax)| = |f(\dot{q}x)| = |\beta f(x)\alpha| = |f(z)| = 3|f(q)| - |\rho|$ and $|f(c)| - |\sigma_2| = |\alpha| < |f(q)| < |f(z)|$.

In the same way, if $c = a$, let $X = xa$. We have $q_1q^3q_2 = bX^3$ with $|f(xa)| = |f(z)|$ and $|f(b)| < |\pi_1 f(z)|$. $\square$

We can now state a lemma for $k = 3$ which completes Lemma 2.10.

Lemma 3.4. *Let f be a ps-morphism from A^* to B^*. Let v and T be non-empty words over A such that v^3 is a pure cube. Let us assume that $f(T) = \pi_1 f(v)^3\sigma_2$ with $|\pi_1| < |f(T[1])|$, $|\sigma_2| < |f(T[|T|])|$, π_1 a suffix of the image by f of a shortest word v_1 and σ_2 a prefix of the image by f of a shortest word v_2.*
Then one of the following holds:

- $(P.1)$*: There exist a cube x^3, a word y over A, and a word Z over B such that*
  $(P.1.1)$*: $T = x^3y$, $|y| \leq 1$, $f(y) = \pi_1\sigma_2$, $f(x) = \pi_1 Z$, and $f(v) = Z\pi_1$*
  $(P.1.2)$*: or $T = yx^3$, $|y| = 1$, $f(y) = \pi_1\sigma_2$, $f(x) = Z\sigma_2$, and $f(v) = \sigma_2 Z$.*
- $(P.2)$*: There exist a pure cube x^3 and a non-empty word y over A such that*
  $(P.2.1)$*: $T = x^3y$ with $|f(x^2)| < |\pi_1 f(v)|$*
  $(P.2.2)$*: or $T = yx^3$ with $|f(x^2)| < |f(v)\sigma_2|$.*
- $(P.3)$*: $T = t^3$, $|v| \geq 3$, $|t| = 1$ (*i.e.*, t is a letter), $2|f(v)| < |f(t)| < 3|f(v)|$, and there exist two words $x \neq \varepsilon$ and y such that $|f(x)| = |f(t)|$ and*
  $(P.3.1)$*: $v_1v^3v_2 = x^3y$ with v_1 a prefix of x, v_2 a suffix of xy, and $|f(y)| < |f(t)\sigma_2'|$*
  $(P.3.2)$*: or $v_1v^3v_2 = yx^3$ with v_1 a prefix of yx, v_2 a suffix of x, and $|f(y)| < |\pi_1' f(t)|$*
  where π_1' and σ_2' are the words such that $f(v_1[1]) = \pi_1'\pi_1$ and $f(v_2[|v_2|]) = \sigma_2\sigma_2'$.
- $(P.4)$*: f is not cube-free.*

Proof. If T is cube-free, it ends the proof; f is not cube-free.

Let us suppose that T contains at least one cube. Among the cubes of T, we choose one whose image by f is a shortest; we can write $T = t_1t^3t_2$ where $|f(t)| = \min\{|f(t')| \text{ where } {t'}^3 \in \mathrm{Fcts}\,(T)\}$. By this definition, since f is bifix (as any ps-morphism) and so non-erasing, t^3 is a pure cube.

If $t_1 \neq \varepsilon$ and $t_2 \neq \varepsilon$ then, by Lemma 3.3, T satisfies condition $(P.1)$ or condition $(P.4)$.

As in the proof of Lemma 2.10 (see Lem. 3.1 in [22]), if a power of $f(t)$ and a power of $f(v)$ have a common factor of length at least $|f(t)| + |f(v)|$, we obtain that T satisfies condition $(P.1)$.

From now, let us assume the converse holds, *i.e.*, any common factor of $f(t)^3$ and $f(v)^3$ is of length at most $|f(t)| + |f(v)|$. It means that $|f(t)^3| - |\sigma_2| < |f(t)| + |f(v)|$ when $t_1 \neq \varepsilon$ and $t_2 = \varepsilon$, and that $|f(t)^3| - |\pi_1| < |f(t)| + |f(v)|$ when $t_1 = \varepsilon$ and $t_2 \neq \varepsilon$. Hence, T satisfies condition $(P.2)$ with $x = t$.

Let us now treat the case where $t_1 = t_2 = \varepsilon$. In this case, $f(v)^3$ is factor of $f(t^3) = f(T)$. Hence, $2|f(v)| < |f(t)|$ and $|\pi_1| + |\sigma_2| = 3|f(t)| - 3|f(v)| > 3|f(v)|$. If $\pi_1 = \varepsilon$ then $f(t^2)$ contains the cube $f(v)^3$ with t^2 cube-free; f is not cube-free. In the same way, if $\sigma_2 = \varepsilon$ then f is not cube-free. It follows that $v_1 \neq \varepsilon$ and $v_2 \neq \varepsilon$; there exist a prefix π_1' of $f(v_1[1])$ and a suffix σ_2' of $f(v_2[|v_2|])$ such that $f(v_1v^3v_2) = \pi_1'\pi_1 f(v)^3\sigma_2\sigma_2' = \pi_1' f(t)^3\sigma_2'$. By Lemma 3.3 with $q = v$, we obtain that either f is not cube-free (for instance if $|v| \leq 2$ or if $|t| \geq 2$) or T satisfies condition $(P.3)$. $\square$

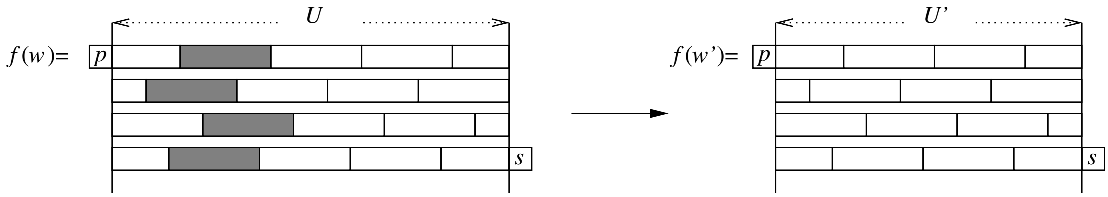

FIGURE 5. Tetris.

By Lemma 2.8, we immediately obtain:

Corollary 3.5. *With hypotheses and notations of Lemma 3.4, if $f(T)$ and $f(v)^3$ are synchronised then either f is not cube-free or T verifies $(P.1)$.*

4. The special case of uniform morphisms

If we represent the image of a letter by a morphism as a rectangle block, these blocks have the same length in the case of uniform morphisms. The main idea is to delete blocks as shown in Figure 5. Corollary 4.1 proved in [22] shows that, under some hypotheses, we can simplify the image of word by a uniform morphism in such a way.

Corollary 4.1. *Let $\kappa \geq 3$ and $\ell \geq 1$ be two integers, let α be an integer in $\{1, 2\}$ and let β be an integer in $\{\kappa - 1, \kappa\}$.*

Let f be a morphism from A^* to B^*. Let $(w_i)_{i=\alpha..\beta+1}$ and $(x_i)_{i=\alpha..\beta}$ be words over A such that $|f(x_i)| = |f(x_j)| \neq 0$ for all integers i, j in $[\alpha, \beta]$.

We denote by w the word $w_\alpha x_\alpha^\ell ... w_\beta x_\beta^\ell w_{\beta+1}$.

We assume that there exist U, p, s, $(X_i)_{i=\alpha..\beta}$, and $(Y_i)_{i=\alpha..\beta}$ words over B such that $f(w_i) = Y_{i-1}X_i$ for all integers i in $[1 + \alpha; \beta]$. Furthermore, we also assume that $f(w_\alpha) = pU^{\alpha-1}X_1$ and $f(w_{\beta+1}) = Y_\kappa U^{\kappa-\beta}s$ where $U = X_i f(x_i^\ell) Y_i (\neq \varepsilon)$ for all integers i in $[\alpha, \beta]$. It means that $f(w) = pU^\kappa s$.

Finally, we assume that there exists an integer q such that, for every integer i in $[\alpha, \beta]$, $0 \leq |X_q| - |X_i| \leq |X''_q|$ where X''_q is a common suffix of X_q and $f(x_q)$, $0 \leq |X_q| - |X_i| \leq |f(x_q)|$ when $\alpha = 2$, or $0 \leq |Y_i| - |Y_q| \leq |f(x_q)|$ when $\beta = \kappa - 1$.

Then, for every integer $0 \leq \phi < \ell$, the word $w' = w_\alpha x_\alpha^\phi ... w_\beta x_\beta^\phi w_{\beta+1}$ satisfies $f(w') = pU'^\kappa s$ with $U' = X_i f(x_i^\phi) Y_i$ for every integer i in $[1; \kappa]$.

In particular, $f(w')$ and U'^κ are synchronised only if $f(w)$ and U^κ are synchronised.

We need to specify the cases where the hypotheses of the corollary 4.1 are satisfied; More precisely, Lemma 4.2 proved in [22] describes sufficient conditions to remove blocks.

Again, in order to unify notations from the two papers, our notations come from [22]. For the next lemma and the rest of the paper, we use some specific variables. Without going into details, λ_{v} is the integer such that $f(v^3)$ starts in the $\lambda_{\mathrm{v}}{}^{\text{th}}$ occurrence of U; $d_{\mathrm{v}} = 1$ if the first occurrence of $f(v)$ from $f(v)^3$ is factor of U and $d_{\mathrm{v}} = 0$ if not; $c_{\mathrm{v}} = \min\{i \mid f(v)^3 \text{ is a factor of } U^i\}$; and D_{v} is a prefix of U such that $w = v_p v^3 v_s$ and $f(v_p v)$ ends with D_{v} or $D_{\mathrm{v}} f(v)$. Moreover, Condition (P.1) in Lemma 3.4 leads us to consider the sets $L_{j,v}$ (resp. $R_{j,v}$) of the words x^3 such that x^3 verifies Condition (P.1.1) (resp. Condition (P.1.2)) and $f(x^3)$ starts in the $(j - \max\{d_{\mathrm{v}}, d_{\mathrm{x}}\})^{\text{th}}$ (resp. $(j - \min\{d_{\mathrm{v}}, d_{\mathrm{x}}\})^{\text{th}}$) occurrence of U.

Lemma 4.2. *Let $k \geq 3$ be an integer and let $\kappa \in \{k; k+1\}$. Let f be a morphism from A^* to B^* and let w be a word over A such that $f(w) = pU^\kappa S$ for some words p, S and $U \neq \varepsilon$ over B such that $|p| < |f(w[1])|$. Moreover, we assume that $|S| < |f(w[|w|])|$ when $\kappa = k + 1$ and v^3 is a chosen factor of a pure k-power v^k.*

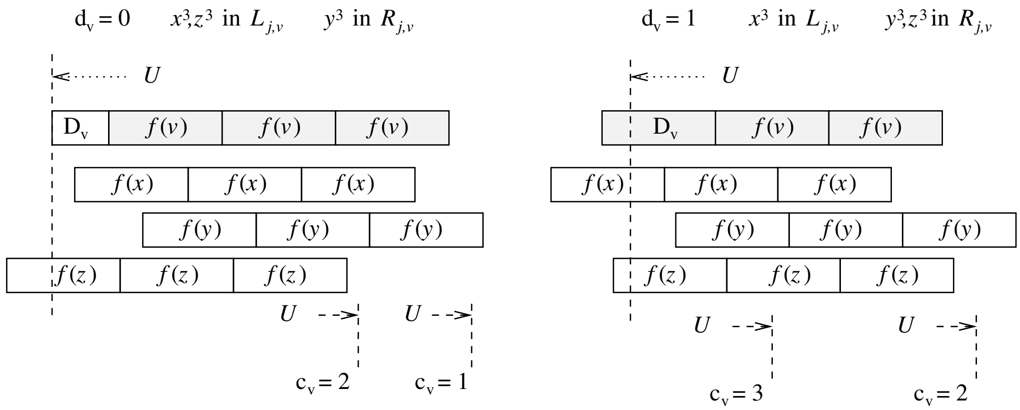

FIGURE 6. Examples of values for $\mathrm{d_v}$ and $\mathrm{c_v}$.

When one of the four following situations holds, there exists a word $\breve{w}$ such that $f(\breve{w}) = p'(U')^\kappa S'$ for some words p', S', and $U' \neq \varepsilon$ over B satisfying $|p'| < |f(\breve{w}[1])|$ and $0 < |U'| < |U|$; moreover $f(\breve{w})$ and $(U')^\kappa$ are synchronised only if $f(w)$ and U^κ are synchronised.

1. $\mathrm{d_v} = 1$, $|D_\mathrm{v} f(v)^2| < |U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [2, \kappa]$.
2. $\mathrm{d_v} = 1$, $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [2, \kappa - 1]$, *and there exists a positive integer* ϕ *such that* $w[n_\mathrm{v}..|w|]$ *starts with* $v^{\phi+2}$ *and* $\sup\left\{2|f(v)|; |D_\mathrm{v} f(v)^\phi|\right\} \leq |U| < |D_\mathrm{v} f(v)^{\phi+1}|$.
3. $\mathrm{d_v} = 0$, $|D_\mathrm{v} f(v)^2| \leq |U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [1, \kappa]$.
4. $\mathrm{d_v} = 0$, $|U| < |D_\mathrm{v} f(v)^2| < |D_\mathrm{v} U|$, *and* $L_{j,v} \cup R_{j,v} \neq \emptyset$ *for every integer* $j \in [1, \kappa - 1]$.

We say that we made a reduction of $f(w)$.

Using Lemma 3.4 and Lemma 4.2 with $\kappa = 4$, we can reduce a word whose image by a uniform morphism contains a cube. We state our main result.

Theorem 4.3. *Let A and B be two alphabets. A cube-free uniform morphism from A^* to B^* is k-power-free for all integers $k \geq 4$.*

It is a consequence of Proposition 2.9 (Prop. 4.1 in [22]) and Proposition 4.4:

Proposition 4.4. *Let A and B be two alphabets. A cube-free uniform morphism from A^* to B^* is 4-power-free.*

As previously said, we could not conclude in [22] for the case $k = 3$ due to the fact that Lemma 2.10 does not hold for this value.

Now, using Lemma 3.4, we only have to follow the proof of Proposition 2.9 (Prop. 4.1 in [22]). It would be unnecessary and redundant to give all the details of the notations that we have used in it. In fact, since the two proofs are almost the same, we only give the main ideas. And we only verify that all the steps are effectively checked for $k = 3$.

Proof. Let f be a uniform ps-morphism from A^* to B^*. We assume that f is not 4-power-free and we want to show that f is not cube-free.

Let w be a shortest 4-power-free word whose image by f contains a 4-power u^4 such that $f(w)$ and u^4 are not synchronised.

The central point of the proof is that, starting with w and u, we use iteratively the reduction of Corollary 4.1 on the word whose image contains a 4-power in such a way that it is not possible de reduce anymore. We obtain new words W et U such that $f(W) = pU^4s$ with p a proper prefix of $W[1]$, s a proper suffix of $W[|W|]$, and $f(W)$ and U^4 are not synchronised.

We show that either f is not cube-free, or $f(W)$ and U can again be reduced using Lemma 4.2 itself using Corollary 4.1; a contradiction with their definitions.

The first two steps are strictly identical to the first two steps in the proof of Proposition 2.9 (Prop. 4.1 in [22]); we only have to replace k by 3.

Step 1: For any pure cube v^3 of W, the words U^4 and $f(v)^3$ do not have any common factor of length at least $|U| + |f(v)|$.

Step 2: $W[2..|W|-1]$ contains a cube and so a pure-cube.

Step 3: For any pure cube $v^3 \in \text{Fcts}\,(W[2..|W|-1])$, the word $f(v)^3$ is an internal factor of U^3 and $|f(v^2)| < |U|$.

For any pure cube $v^3 \in \text{Fcts}\,(W[2..|W|-1])$, the word $f(v)^3$ is an internal factor of U^4. So $|f(v)^3| < |U| + |f(v)|$, *i.e.*, $|f(v)^2| < |U|$ and $|f(v)^3| < \frac{3}{2}|U|$. Hence, $f(v)^3$ is an internal factor of U^3. This implies that $c_{\mathrm{v}} = 1, 2$ or 3.

Let us recall that, for all integers $j \in [1; 5-c_{\mathrm{v}}]$, $f(v)^3$ is an internal factor of $p_j U^{c_{\mathrm{v}}} s_{j+c_{\mathrm{v}}}$. Consequently, if $\widehat{v}_j$ is the shortest factor of $W[i_j..i_{j+c_{\mathrm{v}}}]$ such that $f(\widehat{v}_j)$ contains $f(v)^3$ then, by Corollary 3.5, $\widehat{v}_j$ satisfies condition $(P.1)$ of Lemma 3.4 for all integers $j \in [1; 5-c_{\mathrm{v}}]$.

We are going to see that it implies that W can be reduced; a final contradiction.

Case 3.1: $c_{\mathrm{v}} = 3$

Since $|f(v^2)| < |U|$ and by definition of c_{v}, we necessarily have $d_{\mathrm{v}} = 1$ and $|D_{\mathrm{v}} f(v)| < |U| \leq |D_{\mathrm{v}} f(v^2)|$. For all integers $j \in [1; 2]$, if $\widehat{v}_j$ satisfies condition $(P.1.1)$ of Lemma 3.4 then ${x_{j,v}}^3 \in L_{j+1,v}$. And if $\widehat{v}_j$ satisfies condition $(P.1.2)$ of Lemma 3.4 then ${x_{j,v}}^3 \in R_{j+1,v}$. In other words, we have $L_{j+1,v} \cup R_{j+1,v} \neq \emptyset$ with $j+1 \in [2; 3]$; by Lemma 4.2(2) with $\phi = 1$, we can reduce W.

Case 3.2: $c_{\mathrm{v}} \neq 3$ and $d_{\mathrm{v}} = 1$

We necessarily have $c_{\mathrm{v}} = 2$, i.e, $5 - c_{\mathrm{v}} = 3$. For all integers $j \in [1; 3]$, if $\widehat{v}_j$ satisfies condition $(P.1.1)$ of Lemma 3.4 then ${x_{j,v}}^3 \in L_{j+1,v}$. And if $\widehat{v}_j$ satisfies condition $(P.1.2)$ of Lemma 3.4 then ${x_{j,v}}^3 \in R_{j+1,v}$. That is, $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [2; 4]$; by Lemma 4.2(1), a reduction can be done.

Case 3.3: $c_{\mathrm{v}} \neq 3$ and $d_{\mathrm{v}} = 0$

If $c_{\mathrm{v}} = 1$ then $|D_{\mathrm{v}} f(v)^2| \leq |U|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [1; 4]$. By Lemma 4.2(3), a reduction can be done.

If $c_{\mathrm{v}} = 2$ then $|U| < |D_{\mathrm{v}} f(v)^2|$ and $L_{j,v} \cup R_{j,v} \neq \emptyset$ for all integers $j \in [1; 3]$; by Lemma 4.2(4), a reduction can be done.

As previously said, these three cases lead to a reduction of $f(W)$: a contraction with its definition. So f is not cube-free. □

Acknowledgements. The author would like to thank Jean-Luc Chabert and the anonymous referees for useful remarks and suggestions that improved the presentation of this paper.

References

[1] D.R. Bean, A. Ehrenfeucht and G.F. McNulty, Avoidable patterns in strings of symbols. *Pac. J. Math.* **85** (1979) 261–294.

[2] J. Berstel, Mots sans carré et morphismes itérés. *Discret. Appl. Math.* **29** (1980) 235–244.

[3] J. Berstel, *Axel Thue's papers on repetition in words: a translation.* Technical Report 20. Laboratoire de Combinatoire et d'Informatique Mathématique, Université du Québec, Montréal (1995).

[4] F.-J. Brandenburg, Uniformly growing k-th power-free homomorphisms. *Theor. Comput. Sci.* **23** (1983) 69–82.

[5] M. Crochemore, Sharp characterizations of squarefree morphisms. *Theor. Comput. Sci.* **18** (1982) 221–226.

[6] M. Crochemore, *Régularités évitables* (thèse d'état), Technical Report 83-43. LITP (1983).

[7] J.D. Currie and N. Rampersad, There are k-uniform cubefree binary morphisms for all $k \geq 0$. *Discret. Appl. Math.* **157** (2009) 2548–2551.

[8] J. Karhumäki, On cube-free ω-words generated by binary morphisms. *Discret. Appl. Math.* **5** (1983) 279–297.
[9] V. Keränen, On k-repetition freeness of length uniform morphisms over a binary alphabet. *Discret. Appl. Math.* **9** (1984) 301–305.
[10] V. Keränen, On the k-freeness of morphisms on free monoids, in Annales Academiae Scientarium Fennicae 61, Series A. (1986).
[11] M. Leconte, A characterization of power-free morphisms. *Theor. Comput. Sci.* **38** (1985) 117–122.
[12] M. Leconte, *Codes sans répétition*. Ph.D. Thesis, LITP Université Paris 6 (1985).
[13] M. Lothaire, Combinatorics on Words. Vol. 17 of *Encyclopedia of Mathematics*. Addison-Wesley (1983). Reprinted in 1997 by Cambridge University Press in the Cambridge Mathematical Library, Cambridge, UK, 1997.
[14] M. Lothaire, Algebraic Combinatorics on Words. Vol. 90 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK (2002).
[15] M. Lothaire, Applied Combinatorics on Words. Vol. 105 of *Encyclopedia of Mathematics*. Cambridge University Press, Cambridge, UK (2005).
[16] G. Richomme and P. Séébold, Conjectures and results on morphisms generating k-power-free words. *Int. J. Found. Comput. Sci.* **15** (2004) 307–316.
[17] G. Richomme and F. Wlazinski, About cube-free morphisms, in STACS'2000. Vol. 1770 of *Lecture Notes in Computer Science*, edited by H. Reichel and S. Tison. Springer-Verlag (2000) 99–109.
[18] G. Richomme and F. Wlazinski, Some results on k-power-free morphisms. *Theor. Comput. Sci*. **273** (2002) 119–142.
[19] A. Thue, Uber unendliche zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv* **7** (1906) 1–22.
[20] A. Thue, Uber die gegenseitige Lage gleigher Teile gewisser Zeichenreihen. *Kristiania Videnskapsselskapets Skrifter Klasse I. Mat.-naturv* **1** (1912) 1–67.
[21] F. Wlazinski, A test-set for k-power-free binary morphisms. *TIA* **35** (2001) 437–452.
[22] F. Wlazinski, Reduction in non-$(k+1)$-power-free morphisms. Special issue dedicated to the 15th "Journées montoises d'informatique théorique". *RAIRO: ITA* **50** (2016) 3–20.